Quantum vacuum fluctuations in presence of dissipative bodies: Dynamical approach for nonequilibrium and squeezed states

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The present work contributes to the study of nonequilibrium aspects of the Casimir forces with the introduction of squeezed states in the calculations. Throughout this article two main results can be found, being both strongly correlated. Primarily, the more formal result involves the development of a first-principles canonical quantization formalism to study the quantum vacuum in the presence of different dissipative material bodies in completely general scenarios. For this purpose, we consider a one-dimensional quantum scalar field interacting with the volume elements' degrees of freedom of the material bodies, which are modeled as a set of composite systems consisting of quantum harmonic oscillators interacting with an environment (provided as an infinite set of quantum harmonic oscillators acting as a thermal bath). Solving the full dynamics of the composite system through its Heisenberg equations, we study each contribution to the field operator by employing general properties of the Green function. We deduce the long-time limit of the contributions to the field operator. In agreement with previous works, we show that the expectation values of the components of the energy-momentum tensor present two contributions, one associated to the thermal baths and the other one associated to the field's initial conditions. This allows the direct study of steady situations involving different initial states for the field (keeping arbitrary thermal states for the baths). This leads to the other main result, consisting of computing the Casimir force when the field is initially in thermal or continuum-single-mode squeezed states (the latter being characterized by a given bandwidth and frequency). Time averaging is required for the squeezed case, showing that both results can be given in a unified way, while for the thermal state, all the well-known equilibrium results can be successfully reproduced. Finally, we compared the initial conditions' contribution and the total force for each case, showing that the latter can be tuned in a wide range of values through varying the size of the bandwidth.

I. INTRODUCTION

The quantum nature of vacuum is still one of the most relevant features of quantum field theory (QFT) due to theoretical and also technological implications. From a conceptual point of view, understanding the physical properties of vacuum at the quantum level is unavoidable in a wide range of areas, from quantum optics [1] and condensed matter [2] to astrophysics and cosmology [3]. On the other hand, from a practical point of view, novel experiments measuring forces [4,5] and heat transfer at the nanoscale level [6] shows that, exploiting all these features, a new generation of technological improvements is coming [7,8].

For these reasons, the study of quantum vacuum fluctuations (going from Casimir–van der Waals interactions to quantum friction) is of main interest. In particular, Casimir forces arising from the physical adaptation of a quantum field to the presence of arbitrary-shaped objects acting as boundaries are still a source of abundant scientific research (see, for example, Ref. [9] for a review, including the study of multiple complicated geometries and also several applications in different fields). Since the foundational paper by Casimir [10], where the force between two perfect conductor plates was studied, numerous subsequent works pointed out a direction for studying different aspects of these vacuum phenomena more related to realistic experimental scenarios. The natural step, and one of the most remarkable works in this sense, was the work by Lifshitz [11], where dissipation and noise were included for the first time in the calculation of the Casimir force between dielectric plates at zero temperature through an approach based on stochastic electrodynamics, resulting in the celebrated Lifshitz formula for the force. This work also set the basis for the so-called fluctuational quantum electrodynamics (FQED) as it is currently known. A few years later, in Ref. [12], a formal approach based on QFT at finite temperature was developed to study thermal corrections on computing the force, arriving at a more sophisticated and complete form of the Lifshitz formula at thermal equilibrium. After these works, the study of different features of the Casimir effect had a significant growth going beyond fundamental physics and entering chemistry and biology [13]. Moreover, a new generation

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of extremely accurate experiments (see, for example, Refs. [14,15]) gave rise to the study of the forces between objects of different geometries, dynamical phenomena (such as the dynamical Casimir effect and quantum friction, which both involve particle creation by moving boundaries) and heat transfer at the nanoscale level [16]. Several discussions have taken place due to the introduction of models for real materials, exposing the discrepancies between the theoretical models and their contrasts with experiments, such as the Drude vs plasma controversy, first mentioned in Ref. [17] (see also Ref. [14] for a more recent review). In fact, some of these discussions are currently open yet and are awaiting a solution.

Beyond all these advances, addressing nonequilibrium scenarios from both theory and experiments was a significant unfinished task for a long time. In recent years, it gained great attention due to its potential applications [18]. One of the first and most remarkable works in the area was given by Antezza *et al.* [19], who studied the Casimir force between two slabs characterized by arbitrary frequency-dependent permittivity functions. However, the approach employed is more of an extension to nonequilibrium scenarios of the Lifshitz's method than a full quantization scheme, and then it takes part of the mentioned FQED approaches. As far as we know, a full quantum approach based on QFT for these situations was given subsequently in Refs. [20,21]. In these works, the Huttner-Barnett (HB) microscopic model [consisting of the $3 + 1$ electromagnetic (EM) field interacting with material polarizable bodies described by polarization degrees of freedom coupled to thermal baths in each point of space [22]] is fully quantized by the closed time path (CTP) or Schwinger-Keldysh functional formalism [23]. Implementing the influence functional to treat the field as an open quantum system, the field correlation can be calculated exactly and the result of Ref. [19] is recovered. Nevertheless, although both results agree, from a conceptual point of view, they present very subtle differences.

In Ref. [19] the steady (nonequilibrium) state is taken as an assumption and introduced in the calculations through implicitly assuming time invariance for the Heisenberg equations when giving its Fourier transform as the starting point. Considering that, in fact, the Heisenberg equations are always subjected to initial conditions, which are discarded here by the "steadiness" assumption, in this sense we can say that the scheme is a "steady quantization scheme." Moreover, the material bodies are described by the macroscopic frequency-dependent (complex) permittivity function and the fluctuations of the polarization sources inside, relating both by a fluctuation-dissipation relation containing the correct quantum thermal properties. As the relation is established separately for each point of each body, different temperatures for each body are easily introduced and the steadiness of the nonequilibrium scenario is quite reasonable and physically consistent. Then,

nonequilibrium forces between two half-spaces and between two slabs (finite width) are analyzed. In the first case, the force results in a sum of the contributions of each half-space; each one characterized by the respective temperature. For the two-slabs case, the force presents analog contributions (coming from each slab) but, since the bodies' configuration is surrounded by vacuum, the radiation (coming from distant sources) impinging on the external surfaces also contributes to the force experienced by each slab. Therefore, contributions from this radiation enter the force. For these terms, the same expressions deduced exclusively for dissipative materials are employed for the vacuum (dissipationless) external regions, taking advantage implicitly of the continuity of the dissipative result with the dissipationless one when the damping constant is taken to zero (as it is done also in Ref. [24]). Apart from the calculations, the entire framework and picture constitute a FQED scheme at a steady scenario rather than a purely quantum one.

On the other hand, in Refs. [20,21], the situation arises in another way. The CTP formalism implemented does not assume a steady situation. On the contrary, it is built to study the full time evolution from given initial conditions at time t_0, being closely related to the full solution of the Heisenberg equations. In this context, the steady situation emerges as the long-time limit of the full time evolution, i.e., by taking $t_0 \to -\infty$ in all the expressions. Regarding the materials, in these works, a microscopic quantum model (similar to the HB model) is introduced to describe the internal dynamics of the degrees of freedom and thermal baths at each point of the dielectric bodies. Before the initial time t_0, all the parts of the total system (field and materials) are not interacting. At time t_0, all the interactions begin and the system starts to evolve. Thus, the macroscopic EM properties of the material result from tracing out the internal degrees of freedom and the thermal baths during the interaction with the EM field. In this way, the steady situation is deduced and the fluctuation-dissipation relation results naturally from the open quantum system framework. In Ref. [20], a correspondence between this quantum model and a stochastic description is fully demonstrated, exposing the existing connection between FQED and a fully quantum theory. In principle, three contributions are shown to appear for every case, each one associated to each field's sources resulting from the interactions present. One is associated to the thermal baths, another one to the internal degrees of freedom and the last one to the initial conditions of the field. In Ref. [21], the formalism is applied to the study of the nonequilibrium half-space problem. The steady situation is deduced and it is shown that the only contribution present in this case is the one associated to the baths. Moreover, it gives exactly the same result as the one obtained from the FQED approach at Ref. [19]. However, related to the results of Ref. [25], it is suggested from the calculations that for the case of slabs (finite width), in addition to the baths, the initial conditions

would contribute to the long-time regime. This would be entirely related to the presence of infinite-size dissipationless regions, which causes the initial free field fluctuations overcome the damping in the dissipative material bodies and constitute the vacuum modified modes when reaching the steady situation. In other words, the modified modes would be the steady result of the dynamical adaptation process of the initial free field modes (without boundaries) after the appearance of the material boundaries, giving a nonvanishing initial conditions' contribution at the long-time regime. In this train of thought, for the case of half-spaces, there is no initial conditions' contribution since, on the contrary, dissipation overcomes free-field fluctuations. In fact, in the framework of Ref. [19], the initial conditions' contribution would match with the distant sources' contribution, although in this approach the radiation is related to the quantum fluctuations of the field already adapted to the boundaries. Moreover, it would also be matched with the homogeneous solution mentioned in Ref. [26], obtained from a "steady path integral quantization" scheme. On the other hand, in a "steady canonical quantization" scheme, this homogeneous solution is quantized in a specific Hilbert space with its own creation and annihilation operators for the vacuum modes. All in all, the results of Refs. [20,21,25] are suggesting, indeed, that these operators are defined at the initial time, i.e., they are the creation and annihilation operators of the free field, although they are used to quantize the homogeneous contribution when the boundaries are present.

In this sense, one of the main results of the present work is to show that all the mentioned suggestions and correspondences are true for arbitrary material configurations, validating the physical mechanism described above. For these purposes, we developed the canonical quantization version of Ref. [25], where the material model mentioned above is considered in interaction with a scalar field. Clearly, the approach presented here can be extended to the EM case but for simplicity in the calculations we considered a one-dimensional scalar field. We will proceed to solve the full dynamics of the total system, regarding the field correlation, which allows us to study the expectation values of the energy-momentum tensor and, consequently, the Casimir force. We obtain the three mentioned contributions for every time and deduce steady expressions by taking the long-time limit $t_0 \to -\infty$. At this point, all the results are given for a general configuration. In this way, we are able to study the scalar field dynamics at any nonequilibrium scenario for this composite system. As far as we know, this kind of approach to nonequilibrium situations is completely new and its advantage is to be very suitable for computing the force and every quantity that can be obtained from the field operator. This is the main contribution of this work from a purely conceptual point of view.

Nevertheless, one of the practical advantages of this approach is that it gives the chance to study the quantities for nonclassical initial states of the field. In quantum optics and cavity QED, squeezed states are the most common nonclassical field states considered [27,28]. For example, in quantum communications using this state, quantum fluctuations of one of its quadrature components can be lower than those of the corresponding part in a coherent state, while the other component is higher. This gives the possibility of tuning the signal-to-noise ratios of its quadrature components, giving the chance to use one of the quadratures to absorb the quantum noise, while the other can be implemented to transmit an extremely well-defined signal. In several applications, the quantum EM field inside a cavity is in a squeezed state. The squeezing can be generated through interacting with atoms inside the cavity [29], which also opens the way to study other different quantum phenomena as decoherence and teleportation [30–33]. Hence, understanding the physics of the interactions pertaining to the field inside a cavity and involving nonclassical states is necessary for the development of new quantum technologies. Although the cavity is made of a good (but not perfect) conductor, it is common in the analysis to consider it as made of perfect materials, disregarding any corrections due to dissipation and nonequilibrium thermal effects [34]. It is clear that the effectiveness of this approximation depends on the experimental conditions achieved. Therefore, it is of great interest to include these effects accurately. In this work we shed light on how to address these calculations. We calculate the Casimir force between two dissipative material slabs (playing the role of a cavity) when the field is in a squeezed state and each slab has its own temperature. This greatly improves the results found in a previous work [35], where the Casimir force between two perfect conductor plates is calculated when the field is in a squeezed state. Moreover, our present result could be also implemented to enter the discussion given in Ref. [36] about the spatial properties of the squeezing of vacuum, including realistic details as thermal imbalance between the plates and the effect of dissipation, which escapes the scope of this work.

With the aim of focusing the main text of the work and the calculations on the mentioned results, we have left large calculations and deductions to several appendixes at the end of the paper. The paper is organized as follows: in the next section we describe the model and write the field equation with its solution. The full derivation of the field equation is left to Appendix A. In Sec. III, we study the long-time limit of the contributions to the field operator, with the main features of each one and the relation to other works. The analytical technique (including a general discussion of different configurations) along with the derivation of the long-time limit for each contribution are left to Appendixes B, C, and D. Then, in Sec. IV, it is shown that two contributions take part in the expectation values of the energy-momentum tensor at the steady state and each contribution is worked out. For the baths' contribution, thermal states are considered. For the initial conditions' contribution, the expectation values are computed both for the thermal and also squeezed initial state

of the field with the implementation of the time average. Section V is devoted to the calculation of the total Casimir force for the finite width plates configuration, deriving general expressions for each contribution to the force. In Appendix E we give the homogeneous solutions and the Green function needed for the calculations of the section, while in Appendix F we show how our result recovers the dissipationless limit and the Lifshitz formula at thermal equilibrium. In Sec. VI, we present a comparison between the Casimir forces obtained by taking a thermal state for the field and a squeezed one, when the baths keep the same temperature. Finally, Sec. VII summarizes our findings.

Throughout the paper, for simplicity, we have set $\hbar = k_B = c = 1$.

II. LAGRANGIAN DENSITY AND FIELD EQUATION

With the aim of including effects of dissipation and noise in the evaluation of the Casimir energy or force, we will use the theory of open quantum systems, keeping in mind the paradigmatic example of quantum Brownian motion (QBM) [37].

The model is a simplified version of the HB model, consisting of a system composed of two parts: a massless scalar field and dielectric material which, in turn, are described by their internal degrees of freedom (a set of harmonic oscillators). Both subsystems constitute a composite system that is coupled to a second set of harmonic oscillators (playing the role of an external environment or termal bath). For simplicity we will work in $1+1$ dimensions. In our toy model the massless field represents the electromagnetic field, and the first set of harmonic oscillators directly coupled to the scalar field represents the polarizable volume elements of the material.

Considering the usual interaction term between the electromagnetic field and the ordinary polarizable matter, the coupling between the field and the volume elements of the material will be taken as a current-type one, where the field couples to the velocity of variation of the volume elements' degrees of freedom. The coupling constant for this interaction is the electric charge e. We will also assume that there is no direct coupling between the field and the thermal bath. The Lagrangian density is therefore given by

$$
\begin{aligned}
\mathcal{L} &= \mathcal{L}_\phi + \mathcal{L}_S + \mathcal{L}_{\phi-S} + \mathcal{L}_B + \mathcal{L}_{S-B} \\
&= \frac{1}{2}\partial_\mu\phi\partial^\mu\phi + 4\pi\eta\left(\frac{1}{2}m\dot{r}_x^2(t) - \frac{1}{2}m\omega_0^2 r_x^2(t)\right) \\
&\quad + 4\pi\eta e\phi(x,t)\dot{r}_x(t) \\
&\quad + 4\pi\eta\sum_n\left(\frac{1}{2}m_n\dot{q}_{n,x}^2(t) - \frac{1}{2}m_n\omega_n^2 q_{n,x}^2(t)\right) \\
&\quad - 4\pi\eta\sum_n\lambda_n q_{n,x}(t)r_x(t),
\end{aligned}
\tag{1}
$$

where we have stressed the fact that r and q_n have a dependence on position as a label identifying the point of space at which they are located but without being a dynamical variable (as it happens for the scalar field). It is clear that each atom interacts with a thermal bath placed at the same position. We have denoted by η the density of the degrees of freedom of the volume elements. The constants λ_n are the coupling constants between the volume elements and the bath oscillators. It is implicitly understood that Eq. (1) represents the Lagrangian density inside the material, while outside the Lagrangian is given by the free field one.

The quantization of the theory is straightforward. It should be noted that the full Hilbert space of the model H, where the quantization is performed, is not only the field Hilbert space H_ϕ (as is considered in others works where the field is the only relevant degree of freedom), but also includes the Hilbert spaces of the volume elements' degrees of freedom H_A and the bath oscillators H_B, in such a way that $H = H_\phi \otimes H_A \otimes H_B$. We will assume, as frequently done in the context of QBM, that for $t < t_0$ the three parts of the systems are uncorrelated and not interacting. Interactions are turned on at $t = t_0$. Therefore, the initial conditions for the operators $\hat{\phi}$, $\hat{r}$ must be given in terms of operators acting in each part of the Hilbert space. The interactions are such that initial operators become operators over the whole space H. The initial density matrix of the total system is of the form

$$
\hat{\rho}(t_0) = \hat{\rho}_{IC}(t_0) \otimes \hat{\rho}_A(t_0) \otimes \hat{\rho}_B,
\tag{2}
$$

so, in principle, each part can be in any state.

Once the model of the interaction between the field and the matter is properly described, the equations of motion can be obtained. Solving the respective equations for the bath oscillators and the volume elements, an equation of motion for the field can be deduced. In Appendix A, this work is done in the context of the open quantum system's framework, showing that the field equation is given by

$$
\begin{aligned}
\Box\hat{\phi} &+ \frac{\partial^2}{\partial t^2}\left[\int_{t_0}^t d\tau \chi_x(t-\tau)\hat{\phi}(x,\tau)\right] \\
&= 4\pi\eta e C(x)\left[\dot{G}_1(t-t_0)\hat{r}_x(t_0) + \dot{G}_2(t-t_0)\frac{\hat{p}_x(t_0)}{m}\right. \\
&\quad \left. + \int_{t_0}^t d\tau \dot{G}_2(t-\tau)\frac{\hat{F}_x(\tau-t_0)}{m}\right],
\end{aligned}
\tag{3}
$$

where $\chi_x(t) = \omega_{Pl}^2 G_{2,x}(t)C(x)$ is the susceptibility function with $\omega_{Pl}^2 = \frac{4\pi\eta e^2}{m}$ the plasma frequency. It is worth noting that we have included a spatial label denoting the straightforward generalization to inhomogeneous media, where each point of the material can have different properties. Beyond this dependence, the boundaries of the material

bodies enter through the spatial material distribution function C, which is zero in free space points. The regions filled (and the contours) with real material are defined by this function. This is clearly essential for the determination of the field's boundary conditions.

This equation (like all Heisenberg equations) is clearly subjected to initial conditions, in this case, free field conditions:

$$\hat{\phi}(x,t_0) = \int dk \left[\frac{1}{\omega_k}\right]^{\frac{1}{2}} (\hat{a}_k(t_0)e^{i(kx-\omega_k t_0)} + \hat{a}_k^\dagger(t_0)e^{-i(kx-\omega_k t_0)}),$$
(4)

$$\dot{\hat{\phi}}(x,t_0) = \int dk \left[\frac{1}{\omega_k}\right]^{\frac{1}{2}} i\omega_k(-\hat{a}_k(t_0)e^{i(kx-\omega_k t_0)}$$
$$+ \hat{a}_k^\dagger(t_0)e^{-i(kx-\omega_k t_0)}),$$
(5)

where $\hat{a}_k(t_0)$ and $\hat{a}_k^\dagger(t_0)$ are the annihilation and creation operators for the free field at the initial time, and $\omega_k = |k|$.

To solve the equation, the retarded Green function can be employed, in such a way that the associated equation for $t > 0$ reads

$$\Box\mathfrak{G}_{\text{Ret}} + \frac{\partial^2}{\partial t^2}\left(\int_0^t \chi_x(t-\tau)\mathfrak{G}_{\text{Ret}}(x,x',\tau)d\tau\right) = 0, \quad (6)$$

subjected to the following initial conditions:

$$\mathfrak{G}_{\text{Ret}}(x,x',t=0) = 0, \quad \dot{\mathfrak{G}}_{\text{Ret}}(x,x',t=0) = -\delta(x-x'),$$
(7)

in such a way that the field can be written as

$$\hat{\phi}(x,t) = -\int dx' \dot{\mathfrak{G}}_{\text{Ret}}(x,x',t-t_0)\hat{\phi}(x',t_0) - \int dx' \mathfrak{G}_{\text{Ret}}(x,x',t-t_0)\frac{\partial\hat{\phi}}{\partial t}(x',t_0)$$
$$- \int_{t_0}^t dt' \int dx' \mathfrak{G}_{\text{Ret}}(x,x',t-t')4\pi\eta eC(x')\left(\dot{G}_1(t'-t_0)\hat{r}_{x'}(t_0) + \dot{G}_2(t'-t_0)\frac{\hat{p}_{x'}(t_0)}{m}\right)$$
$$- \int_{t_0}^t dt' \int dx' \mathfrak{G}_{\text{Ret}}(x,x',t-t')4\pi\eta eC(x')\int_{t_0}^{t'} d\tau\dot{G}_2(t'-\tau)\frac{\hat{F}_{x'}(\tau-t_0)}{m},$$
(8)

which is the general solution for the field operator of the full time-dependent problem from given initial conditions.

As is expected, the field operator presents three parts, each one consisting of an operator acting in one of the three Hilbert spaces of the total Hilbert space.

III. CONTRIBUTIONS TO THE FIELD OPERATOR

As it was recently stressed, the last equation of the preceding section means that the field operator begins at the initial time t_0 as an operator on H_ϕ. Nevertheless, the switching-on of the interactions causes the field operator to become an operator on the full Hilbert space H during the time evolution:

$$\hat{\phi}(x,t) = \hat{\phi}_{\text{IC}}(x,t) \otimes \mathbb{I}_A \otimes \mathbb{I}_B + \mathbb{I}_\phi \otimes \hat{\phi}_A(x,t) \otimes \mathbb{I}_B$$
$$+ \mathbb{I}_\phi \otimes \mathbb{I}_A \otimes \hat{\phi}_B(x,t).$$
(9)

As we are interested in evaluating the Casimir force in nonequilibrium but steady situations, we have to investigate the long-time limit ($t_0 \to -\infty$) of these three contributions. Although the field operator will remain as an operator in the full Hilbert space H, as we will see, the Casimir force does not necessarily contain contributions associated to each part of H. This will depend on the internal dynamics of the material and the initial state $\hat{\rho}(t_0)$ but also strongly on the boundaries' configuration considered, as mentioned in Refs. [21,25].

A. Long-time limit of initial conditions' contribution

Let us consider the field operator's contribution associated to the initial conditions. Since the initial conditions are written in terms of the creation and annihilation operators of the free field through Eqs. (4) and (5), the contribution can also be written in terms of these operators. Hence, the contribution splits into $\hat{\phi}_{\text{IC}}(x,t) = \hat{\phi}_{\text{IC}}^{(+)}(x,t) + \hat{\phi}_{\text{IC}}^{(-)}(x,t)$, with $\hat{\phi}_{\text{IC}}^{(-)}(x,t) = (\hat{\phi}_{\text{IC}}^{(+)}(x,t))^\dagger$ due to the fact that the retarded Green function $\mathfrak{G}_{\text{Ret}}$ is real and the Hermiticity of the initial free-field operators. Therefore, $\hat{\phi}_{\text{IC}}^{(+)}(x,t)$ is associated to the free-field annihilation operator $\hat{a}_k(t_0)$ and $\hat{\phi}_{\text{IC}}^{(-)}(x,t)$ to the free-field creation operator $\hat{a}_k^\dagger(t_0)$. Moreover, given Eqs. (6) and (7), the initial conditions problem can be solved by a Laplace transform. Therefore, the equation of motion for the retarded Green function's Laplace transform $\tilde{\mathfrak{G}}_{\text{Ret}}(x,x',s)$ is given by

$$\frac{\partial^2\tilde{\mathfrak{G}}_{\text{Ret}}}{\partial x^2} - s^2 n_x^2(s)\tilde{\mathfrak{G}}_{\text{Ret}}(x,x',s) = \delta(x-x'), \quad (10)$$

with $n_x(s)$ the refraction index at the point x:

$$n_x^2(s) \equiv 1 + \omega_{\text{Pl}}^2 \tilde{G}_{2,x}(s)C(x). \quad (11)$$

The last equation is valid for every spatial dependence on the material properties and also for every configuration of boundaries. For a given material's distribution $C(x)$, the boundary conditions are determined through integrating the

equation. [It is worth noting that the Laplace transform $\tilde{\mathfrak{G}}_{\mathrm{Ret}}$ turns out to be the Green function associated to the operator $\frac{\partial^2}{\partial x^2} - s^2 n_x^2(s)$.]

Once we have obtained $\tilde{\mathfrak{G}}_{\mathrm{Ret}}$ in the s-space, we can go back to the coordinate retarded Green function $\mathfrak{G}_{\mathrm{Ret}}$ via the Laplace antitransform (or Mellin's formula; see Refs. [38,39]).

Therefore, the field operator can be written in terms of the Laplace transform of the retarded Green function:

$$\hat{\phi}_{\mathrm{IC}}^{(+)}(x,t) = -\int dk \left(\frac{1}{\omega_k}\right)^{\frac{1}{2}} \hat{a}_k(t_0) e^{-i\omega_k t_0}$$
$$\times \int_{l-i\infty}^{l+i\infty} \frac{ds}{2\pi i} e^{s(t-t_0)} (s - i\omega_k)$$
$$\times \int dx' \tilde{\mathfrak{G}}_{\mathrm{Ret}}(x,x',s) e^{ikx'}. \qquad (12)$$

To find the Green function $\tilde{\mathfrak{G}}_{\mathrm{Ret}}$, we use the technique found in Ref. [40], where the Green function of the Sturm-Liouville differential equation can be obtained from two solutions of the associated homogeneous equation, each one satisfying the boundary conditions on each side of the interval where the variable takes values. Then, we perform a spatial integration for the general case of an arbitrary number of interfaces. Finally, analyzing the complex-analytical properties given by the poles configuration, the long-time (steady) operator for this contribution can be worked out exactly by assuming causality as the only physical requirement. This entire laborious task can be found in Appendix B, where it is shown that the final and most general expression for the long-time operator is given by

$$\hat{\phi}_{\mathrm{IC}}^{(+)}(x,t) \to \hat{\phi}_{\mathrm{IC}}^{(+),\infty}(x,t) = \int dk \left[\frac{1}{\omega_k}\right]^{\frac{1}{2}} \hat{a}_k(-\infty) [e^{-ikt}\Theta(k)\Phi_{-ik}^{>}(x) + e^{ikt}\Theta(-k)(\Phi_{-ik}^{<}(x))^*]$$
$$+ [\text{time- and space-independent oscillatory term}]. \qquad (13)$$

The first term of this expression is exactly the one suggested as an ansatz in the steady situation in Ref. [41], based on the solution obtained for the dissipationless material case in Ref. [42]. This also includes and confirms the result shown in Ref. [25] for the initial condition contribution in the case of a single delta plate (which verifies the ansatz of Ref. [41]). It also includes the case of one thick plate analyzed in Ref. [39]. In fact, this demonstration proves the general case based on the canonical quantization scheme, extending to a wide variety of situations. The functions $\Phi_{-ik}^{>}$ correspond to the modified modes for positive frequencies while $(\Phi_{-ik}^{<})^*$ are the modes for the negative ones. The dynamical appearance and physics of these modes from a transient stage to a steady situation were mentioned briefly in Ref. [25] and examined more deeply in Ref. [21], although in the $3+1$ electromagnetic Lifshitz problem (two parallel half-spaces separated by a distance) analyzed in that work there were no modified modes at all. Here we have the same result for the Lifshitz problem of a $1+1$ scalar field. Clearly, the physics are the same. The modified modes appear in situations where there is an infinite-size dissipationless region, since the free fluctuation of the quantum field prevails over the dissipation in the finite regions occupied by materials, achieving a nonvanishing steady contribution at the long-time limit. Inversely, if there is no infinite-size dissipationless region, the initial conditions' contribution vanishes at the long-time limit, since dissipation overcomes free

fluctuation in finite regions and the contribution is damped.

The second term, which is oscillatory and also time (and space) independent, will have no relevance on the calculation of the energy-momentum tensor expectation values since it involves time and space derivatives of the field operator.

As a final comment for this section, it is worth noting that the present demonstration is valid for every material represented by a refraction index n, which enters the Laplace transform of the retarded Green function via Eq. (10). Then, the deduction only stands on the Green function's properties, but without any restriction on the material model in addition to causality and physical consistency (which implies poles with nonpositive real parts). The information about the material is indeed contained in the specific form of the refraction index, which is the result of the interaction between the materials and the quantum field. Here, as it happens in Ref. [39], this is related to the definition of the susceptibility function χ, obtained from solving the Heisenberg equations of motion for the material's degrees of freedom (volume elements plus thermal baths). In the CTP-integral formulation of Ref. [25] (and Ref. [20] for the electromagnetic version), the refraction index is directly related to the dissipation kernel generated by the material. However, as we have seen, the fact that the initial conditions' contribution does not vanish in the steady regime is more related to the existence of the infinite-size dissipationless regions rather than the material properties.

B. Long-time limit of volume elements' contribution

Now, we consider the field operator's contribution associated to the volume elements. Since this contribution contains the volume elements' initial conditions $\{\hat{r}_x(t_0), \hat{p}_x(t_0)\}$, the operator of this part always acts on the volume elements' Hilbert spaces. These operators can always be expressed in terms of the annihilation and creation operators:

$$\hat{r}_x(t_0) = \frac{1}{\sqrt{2m\omega_0}}(\hat{b}_{0,x}^\dagger(t_0) + \hat{b}_{0,x}(t_0)),$$

$$\hat{p}_x(t_0) = i\sqrt{\frac{m\omega_0}{2}}(\hat{b}_{0,x}^\dagger(t_0) - \hat{b}_{0,x}(t_0)). \tag{14}$$

We can write this contribution as split in terms of the annihilation and creation operators of each volume element, $\hat{\phi}_A(x, t) = \hat{\phi}_A^{(+)}(x, t) + \hat{\phi}_A^{(-)}(x, t)$. From Eq. (8), we have

$$\hat{\phi}_A^{(+)}(x, t) = -\int dx' \frac{4\pi\eta e C(x')}{\sqrt{2m\omega_0}} \hat{b}_{0,x'}(t_0)$$
$$\times \int_{t_0}^{t} dt' \mathfrak{G}_{\text{Ret}}(x, x', t - t')(\dot{G}_1(t' - t_0)$$
$$- i\omega_0 \dot{G}_2(t' - t_0)), \tag{15}$$

where, for simplicity, we have omitted the spatial labels on the volume elements' properties, such as the parameters (frequency, mass and density), or Green functions of the material $(G_{1,2})$. However, all the calculations will be valid considering this spatial dependence.

Again, the crucial point is to deduce the long-time limit for this operator contribution. The Green function can be written in terms of its Laplace transform and analyzing the poles configuration. The steady expression can be obtained by only involving assumptions related to causality. This work is realized at Appendix C. Therefore, the long-time limit $(t_0 \to -\infty)$ of the operator is

$$\hat{\phi}_A^{(+)}(x, t) \to \hat{\phi}_A^{(+),\infty} = -\frac{1}{2}\int dx' \frac{4\pi\eta e C(x')}{\sqrt{2m\omega_0}} \hat{b}_{0,x'}(-\infty), \tag{16}$$

which is a time- and space-independent operator.

At first glance, it is clear that the presence of the material distribution $C(x')$ causes the integration to be over the regions containing material.

However, since the energy-momentum tensor is constructed from expectation values of binary products of the field operator derivatives, the time and space independence of the long-time limit of the field operator means that the volume elements do not contribute in the steady situation for the physical dynamical quantities of interest. This is clearly in accordance with the results obtained in

Refs. [20,21,25] for different situations. This also shows how the approximation considered in Refs. [41,43], in which this contribution was discarded, is not correct at the operator level but turns out to be valid when calculating energy-momentum tensor expectation values.

As a final comment, it should be noted that the behavior of the volume elements' contribution and the fact that it does not join the steady situation are directly related to the material model considered. The dissipative dynamics of each volume element, considered effectively as a quantum Brownian particle, makes the contribution vanish at the long-time limit. Nevertheless, if the material model for the volume elements were taken to be quantum systems with nondissipative dynamics, then a contribution in the steady situation would be present. This will be the case for the contribution of the baths in the next section. Moreover, this is also the case if, for example, we set the dissipation parameter γ_0 equal to zero, i.e., if we set every coupling constant between the baths and the volume elements equal to zero. The permittivity will be plasmalike and, in addition to the pole at $s = 0$, two more poles with zero real part $(s = \pm i\omega_0)$ will contribute to the volume elements' field operator at the steady situation. For the Drude model, the situation goes back to the case in which $s = 0$ is the only pole, although the expression of the long-time limit of the field operator changes slightly.

C. Long-time limit of thermal baths' contribution

From Eq. (8), the baths' contribution is given by

$$\hat{\phi}_B(x, t) = -\int_{t_0}^{t} dt' \int dx' \mathfrak{G}_{\text{Ret}}(x, x', t - t')4\pi\eta e C(x')$$
$$\times \int_{t_0}^{t'} d\tau \dot{G}_2(t' - \tau)\frac{\hat{F}_{x'}(\tau - t_0)}{m}. \tag{17}$$

In contrast with how we proceeded for the other contributions, the full expression can be worked out now, instead of considering the annihilation contribution separately. However, the same methodological approach as for the other two contributions can be implemented. The crucial point is that, due to the dissipationless dynamics of the harmonic oscillators of the baths, the long-time limit of the solution is an operator that depends on both time and space. This work is shown in detail in Appendix D, with the conclusion that the long-time contribution in this case reads

$$\hat{\phi}_B(x, t) \to \hat{\phi}_B^\infty(x, t) = \int dx' \frac{4\pi\eta e C(x')}{m}$$
$$\times \int_{-\infty}^{+\infty} \frac{d\omega}{2\pi} e^{-i\omega t} i\omega \overline{G}_2(\omega)$$
$$\times \overline{\mathfrak{G}}_{\text{Ret}}(x, x', \omega)\hat{\overline{F}}_{x'}^\infty(\omega), \tag{18}$$

which has exactly the same form as the expression achieved in Ref. [19] for the EM field using a stochastic electrodynamics framework for the (quantum) Lifshitz problem.

It is also clear that the presence of the matter distribution C means that the spatial integration is carried out over the regions occupied by the material bodies, which in fact can be inhomogeneous since the local properties of the material can change in each point of space. This is why the integration over x' contains every factor in the rhs, to eventually consider inhomogeneous materials. This last expression also has the time and space dependence suggested in Refs. [41,43] for the so-called Langevin contribution associated to the baths and consisting in outgoing waves from the material bodies. This last feature is verified through the time dependence, while the space dependence is correctly supported by the spatial dependence of the field's retarded Green function transform $\overline{\mathcal{G}}_{\text{Ret}}$.

As a final comment to this section, it is clear that the transform of the stochastic force operator at the long-time limit $\hat{\overline{F}}_{x'}^{\infty}(\omega)$ contains a limit on t_0 that seems to be oscillatory; however this will not enter the correlation expectation values of this operator, which are governed by the QBM theory.

IV. CONTRIBUTIONS TO THE ENERGY-MOMENTUM TENSOR

Finally, we have determined the long-time expressions for each part of the field operator, given in Eqs. (13), (16), and (18). Therefore, for $t_0 \to -\infty$, the field operator reads

$$\hat{\phi}(x,t) \to \hat{\phi}^{\infty}(x,t) = \hat{\phi}_{\text{IC}}^{\infty}(x,t) \otimes \mathbb{I}_A \otimes \mathbb{I}_B + \mathbb{I}_{\phi} \otimes \hat{\phi}_A^{\infty} \otimes \mathbb{I}_B$$
$$+ \mathbb{I}_{\phi} \otimes \mathbb{I}_A \otimes \hat{\phi}_B^{\infty}(x,t), \quad (19)$$

where we have stressed the fact that the volume elements' field operator in the steady situation does not depend on the spatial or temporal coordinates.

It is worth noting that a similar separation was considered, without rigorous demonstration, in Refs. [41,43] about the field operator in the steady situation. In both

works, the field operator does not contain any contribution from the volume elements. Based on the dissipative dynamics of the volume elements and the relaxation of its degrees of freedom, it is assumed that in the steady situation $\hat{\phi}_A^{\infty} \equiv 0$. Here we show that this is not true and what happens is that $\hat{\phi}_A^{\infty}$ is in fact independent of the spacetime coordinates. However, this has no direct implications on the calculation of the energy-momentum tensor expectation values and forces, as we shall see, so the results of Refs. [41,43] are correct. As we certainly have $\partial_{\mu}\hat{\phi}_A^{\infty} \equiv 0$, this implies that $\partial_{\mu}\hat{\phi}^{\infty} = \partial_{\mu}\hat{\phi}_{\text{IC}}^{\infty} \otimes \mathbb{I}_A \otimes \mathbb{I}_B + \mathbb{I}_{\phi} \otimes \mathbb{I}_A \otimes \partial_{\mu}\hat{\phi}_B^{\infty}$. It is worth noting that the derivative of the initial conditions' contribution to the last expression makes the time- and space-independent terms contained in $\hat{\phi}_{\text{IC}}^{\infty}$ vanish [see Eq. (13)].

Then, the expectation value of the components of the energy-momentum tensor operator at the steady situation will not contain any contribution of the volume elements independently of its initial state, which is in agreement with the results obtained by putting $\hat{\phi}_A^{\infty} \equiv 0$ from the very beginning.

In the quantum theory, the expectation values of the energy-momentum tensor involve the correlation of the derivatives, which are the expectation values of symmetrized products. Therefore, for the components of the energy-momentum tensor operator we have

$$\hat{T}_{\mu\nu}(x_1^{\sigma}, t_0) \equiv \left(\delta_{\mu}^{\gamma}\delta_{\nu}^{\alpha} - \frac{1}{2}\eta_{\mu\nu}\eta^{\gamma\alpha} \right) \frac{1}{2} (\partial_{\gamma}\hat{\phi}(x_1^{\sigma})\partial_{\alpha}\hat{\phi}(x_1^{\sigma})$$
$$+ \partial_{\alpha}\hat{\phi}(x_1^{\sigma})\partial_{\gamma}\hat{\phi}(x_1^{\sigma})), \quad (20)$$

where we have stressed the dependence of these quantities on the initial time t_0.

Therefore, the long-time limit of the components is straightforward:

$$\hat{T}_{\mu\nu}(x_1^{\sigma}, t_0) \to \hat{T}_{\mu\nu}^{\infty}(x_1^{\sigma}) = \hat{T}_{\mu\nu}^{\text{IC},\infty}(x_1^{\sigma}) \otimes \mathbb{I}_A \otimes \mathbb{I}_B + \mathbb{I}_{\phi} \otimes \mathbb{I}_A \otimes \hat{T}_{\mu\nu}^{\text{B},\infty}(x_1^{\sigma})$$
$$+ \left(\delta_{\mu}^{\gamma}\delta_{\nu}^{\alpha} - \frac{1}{2}\eta_{\mu\nu}\eta^{\gamma\alpha} \right) \frac{1}{2} [\partial_{\gamma}\hat{\phi}_{\text{IC}}^{\infty}(x_1^{\sigma}) \otimes \mathbb{I}_A \otimes \partial_{\alpha}\hat{\phi}_B^{\infty}(x_1^{\sigma}) + \partial_{\alpha}\hat{\phi}_{\text{IC}}^{\infty}(x_1^{\sigma}) \otimes \mathbb{I}_A \otimes \partial_{\gamma}\hat{\phi}_B^{\infty}(x_1^{\sigma})$$
$$+ \partial_{\alpha}\hat{\phi}_{\text{IC}}^{\infty}(x_1^{\sigma}) \otimes \mathbb{I}_A \otimes \partial_{\gamma}\hat{\phi}_B^{\infty}(x_1^{\sigma}) + \partial_{\gamma}\hat{\phi}_{\text{IC}}^{\infty}(x_1^{\sigma}) \otimes \mathbb{I}_A \otimes \partial_{\alpha}\hat{\phi}_B^{\infty}(x_1^{\sigma})]. \quad (21)$$

As the derivatives of the baths' contribution to the field operator are obviously linear in the annihilation and creation operators of the baths and we are considering a thermal initial density matrix for the baths, we have that its expectation values are zero, i.e., $\langle \partial_{\gamma}\hat{\phi}_B^{\infty}(x_1^{\sigma}) \rangle_B \equiv \text{Tr}_B(\hat{\rho}_B \otimes \hat{T}_{\mu\nu}^{\text{B},\infty}(x_1^{\sigma})) = 0$. This makes the expectation values of the second line of the last equation zero, independently of the field's initial state

considered. Hence, the expectation values of the components of the energy-momentum tensor read

$$\langle \hat{T}_{\mu\nu}^{\infty}(x_1^{\sigma}) \rangle = \langle \hat{T}_{\mu\nu}^{\text{IC},\infty}(x_1^{\sigma}) \rangle_{\phi} + \langle \hat{T}_{\mu\nu}^{\text{B},\infty}(x_1^{\sigma}) \rangle_B, \quad (22)$$

where $\langle ... \rangle$ on the lhs is the quantum expectation value over the total Hilbert space H, while $\langle ... \rangle_{\phi}$ and $\langle ... \rangle_B$ on the rhs are the quantum expectation values on the

parts of the total Hilbert space associated to the field (H_ϕ) and to the baths (H_B), respectively.

Moreover, the last equation constitutes a generalization of the expression considered in Ref. [43] for the calculation of the pressure and also is in agreement with the separation of contributions deduced in Refs. [20,25] for different specific situations studied through a functional integral approach. It is worth noting that in this case the thermal state of the baths ensures separation, regardless of the field's initial state. However, the same splitting can be achieved if the field has an initial thermal state, regardless of the state of the baths.

It is clear that the agreement between the calculations in Refs. [41,43] (which mistakenly assume no contribution from the volume elements to the field operator) and the present ones relies on the fact that the physical quantities of interest are constructed from derivatives of the field operator. In this sense, if the field correlation could be measured directly, both approaches would differ due to the presence of the terms independent of the coordinates that would enter the field correlation, making the latter finally also depend on the volume elements' initial state.

As a final comment, it is important to note that here we are obtaining the splitting in the long-time regime, i.e., as a result of (and after) the dynamical transient evolution from an uncorrelated and noninteracting initial situation. Thus, the steady state achieved is closely related with the establishment of a nonequilibrium energy exchange at a constant (time-independent) rate between the different parts

of the total system (see Ref. [44] for a similar analysis on a system characterized by a Langevin equation of motion). In this sense, an analog splitting in a steady situation was found in Ref. [45] by assuming a stationary regime in the implementation of a steady canonical quantization scheme. However, this is not deduced from an initial conditions' problem, because the steadiness assumption is crucial for the calculations and limits this approach to the case of thermal states. On the other hand, our present approach, without assuming a steady situation, gives the correct framework to include arbitrary quantum states and address more general scenarios in a consistent physical way.

A. Thermal baths' contribution to the energy-momentum tensor

We start by calculating the contribution to the expectation values of the energy-momentum tensor associated to the thermal baths, given by the second term in Eq. (22):

$$\langle \hat{T}^{B,\infty}_{\mu\nu}(x_1^\sigma) \rangle_B \equiv \left(\delta_\mu{}^\gamma \delta_\nu{}^\alpha - \frac{1}{2}\eta_{\mu\nu}\eta^{\gamma\alpha} \right)$$
$$\times \frac{1}{2}\langle \{\partial_\gamma \hat{\phi}^\infty_B(x_1^\sigma), \partial_\alpha \hat{\phi}^\infty_B(x_1^\sigma)\} \rangle_B, \quad (23)$$

where $\{\hat{A},\hat{B}\} = \hat{A}\hat{B} + \hat{B}\hat{A}$ is the anticommutator of the operators $\hat{A}$ and $\hat{B}$.

Considering Eq. (18), the derivative can be written as

$$\partial_\mu \hat{\phi}^\infty_B(x,t) = \int dx' \frac{4\pi\eta e C(x')}{m} \int_{-\infty}^{+\infty} \frac{d\omega'}{2\pi} i\omega' \overline{G}_2(\omega') e^{-i\omega' t} \left[\delta_\mu{}^0(-i\omega')\overline{\mathscr{G}}_{\text{Ret}}(x,x',\omega') + \delta_\mu{}^1 \partial_x \overline{\mathscr{G}}_{\text{Ret}}(x,x',\omega') \right] \hat{\overline{F}}^\infty_{x'}(\omega'). \quad (24)$$

Therefore, the expectation value of the product of derivatives involves the correlation of the stochastic force operators. These expectation values can be obtained from the definitions of the noise kernel and the stochastic force operator in Eqs. (A7) and (A9), as done in Ref. [43]:

$$\langle \{\hat{\overline{F}}^\infty_{x'}(\omega'), \hat{\overline{F}}^\infty_{x''}(\omega'')\} \rangle_B = (2\pi)^2 \delta(x' - x'') \frac{J(\omega')}{2\eta} \coth\left(\frac{\beta_B \omega'}{2} \right) \delta(\omega' + \omega''). \quad (25)$$

Due to the delta functions, we obtain for the expectation value in the rhs of Eq. (23)

$$\langle \{\partial_\mu \hat{\phi}^\infty_B(x,t), \partial_\alpha \hat{\phi}^\infty_B(x,t)\} \rangle_B = \int dx' \omega^2_{\text{Pl}} C(x') \int_{-\infty}^{+\infty} d\omega \omega^2 |\overline{G}_2(\omega)|^2 2\pi \frac{J(\omega)}{m} \coth\left(\frac{\beta_B \omega}{2} \right)$$
$$\times \left[\delta_\gamma{}^0(-i\omega)\overline{\mathscr{G}}_{\text{Ret}}(x,x',\omega) + \delta_\gamma{}^1 \partial_x \overline{\mathscr{G}}_{\text{Ret}}(x,x',\omega) \right]$$
$$\times \left[\delta_\alpha{}^0 i\omega \overline{\mathscr{G}}^*_{\text{Ret}}(x,x',\omega) + \delta_\alpha{}^1 \partial_x \overline{\mathscr{G}}^*_{\text{Ret}}(x,x',\omega) \right]. \quad (26)$$

Considering Eq. (A6), it can be easily proved that $\omega\overline{\gamma}(\omega) = \frac{2\pi}{m}J(\omega)$. Moreover, from the definition of the refractive index below Eq. (10) and given a cutoff function without poles for the spectral density $J(\omega)$, it can be shown that

$$\omega^2_{\text{Pl}}\omega\overline{\gamma}(\omega)|\overline{G}_2(\omega)|^2 = 2\text{Re}(n)\text{Im}(n), \quad (27)$$

which is valid for every odd spectral density for any type of environment, in agreement with the results found in Ref. [43].

Finally, for the expectation value of the components of the energy-momentum tensor, we have

$$
\langle \hat{T}_{\mu\nu}^{\text{B},\infty}(x) \rangle_{\text{B}} = \int dx' C(x') \int_{-\infty}^{+\infty} d\omega 2\omega^2 \text{Re}(n_{x'}) \text{Im}(n_{x'}) \coth\left(\frac{\beta_{\text{B},x'}\omega}{2}\right)
$$

$$
\times \left([\delta_\mu{}^0(-i\omega)\overline{\mathcal{G}}_{\text{Ret}}(x,x',\omega) + \delta_\mu{}^1\partial_x\overline{\mathcal{G}}_{\text{Ret}}(x,x',\omega)][\delta_\nu{}^0 i\omega\overline{\mathcal{G}}_{\text{Ret}}^*(x,x',\omega) + \delta_\nu{}^1\partial_x\overline{\mathcal{G}}_{\text{Ret}}^*(x,x',\omega)] \right.
$$

$$
\left. - \frac{1}{2}\eta_{\mu\nu}[\omega^2|\overline{\mathcal{G}}_{\text{Ret}}(x,x',\omega)|^2 - |\partial_x\overline{\mathcal{G}}_{\text{Ret}}(x,x',\omega)|^2] \right), \tag{28}
$$

which does not depend on the time coordinate. On the other hand, there is still a spatial dependence in principle. Moreover, it should be noted that in the last expression we have included spatial labels for the material properties, denoting that the result is also valid for inhomogeneous materials.

This contribution of the baths to the energy-momentum tensor is in fact the $1+1$ scalar version and also the generalization (in terms of boundaries and inhomogeneity properties) of the expressions found in Refs. [19–21,24, 25,39,41,43], but this time deduced from a full canonical quantum procedure.

B. Initial conditions' contribution to the energy-momentum tensor

We can now calculate the contribution to the energy-momentum tensor resulting from the initial conditions. With the aim of calculating the expectation values of the products of derivatives of the field operator, for simplicity we rewrite Eq. (13) as

$$
\hat{\phi}_{\text{IC}}^\infty(x,t)
$$

$$
= \int dk \left[\frac{1}{\omega_k}\right]^{\frac{1}{2}} [\hat{a}_k(-\infty)e^{-i\omega_k t}\Phi_k(x) + \hat{a}_k^\dagger(-\infty)e^{i\omega_k t}(\Phi_k(x))^*]
$$

$$
+ [\text{time- and space-independent oscillatory term}], \tag{29}
$$

where we have to consider that $\Phi_k(x) = \Phi_{-ik}^>(x)$ for $k > 0$, $\Phi_k(x) = (\Phi_{-ik}^<(x))^*$ for $k < 0$, and $\omega_k = |k|$.

This way, the derivative of the field operator is obtained in a straightforward fashion by considering that $\partial_\mu[e^{-i\omega_k t}\Phi_k(x)] = e^{-i\omega_k t}(\delta_\mu{}^0(-i\omega_k)\Phi_k(x) + \delta_\mu{}^1\Phi_k'(x))$:

$$
\partial_\mu\hat{\phi}_{\text{IC}}^\infty(x^\sigma) = \int dk\left[\frac{1}{\omega_k}\right]^{\frac{1}{2}}[\hat{a}_k(-\infty)e^{-i\omega_k t}(\delta_\mu{}^0(-i\omega_k)\Phi_k(x)
$$

$$
+ \delta_\mu{}^1\Phi_k'(x)) + \hat{a}_k^\dagger(-\infty)e^{i\omega_k t}(\delta_\mu{}^0 i\omega_k(\Phi_k(x))^*
$$

$$
+ \delta_\mu{}^1(\Phi_k'(x))^*)]. \tag{30}
$$

Considering that $\langle\{\hat{a}_k^\dagger(-\infty),\hat{a}_{k'}^\dagger(-\infty)\}\rangle_\phi = \langle\{\hat{a}_k(-\infty), \hat{a}_{k'}(-\infty)\}\rangle_\phi^* = 2\langle\hat{a}_k(-\infty)\hat{a}_{k'}(-\infty)\rangle_\phi^*$ and also that $\langle\{\hat{a}_k^\dagger(-\infty), \hat{a}_{k'}(-\infty)\}\rangle_\phi = 2\langle\hat{a}_k^\dagger(-\infty)\hat{a}_{k'}(-\infty)\rangle_\phi + \delta(k-k') = \langle\{\hat{a}_k(-\infty), \hat{a}_{k'}^\dagger(-\infty)\}\rangle_\phi^*$, we obtain the expectation values of the components of the energy-momentum tensor:

$$
\langle\hat{T}_{\mu\nu}^{\text{IC},\infty}(x_1^\sigma)\rangle_\phi = \langle\hat{T}_{\mu\nu}^{\text{IC},\infty}(\mathbf{x}_1)\rangle_\phi^{\text{Vac}} + \mathcal{T}_{\mu\nu}^{\text{State}}(x_1^\sigma), \tag{31}
$$

where $\langle\hat{T}_{\mu\nu}^{\text{IC},\infty}(\mathbf{x}_1)\rangle_\phi^{\text{Vac}}$ corresponds to the contribution associated entirely to vacuum fluctuations at zero temperature, which is always present, state and (at least) time independent, and is given by

$$
\langle\hat{T}_{\mu\nu}^{\text{IC},\infty}(\mathbf{x}_1)\rangle_\phi^{\text{Vac}} = \int dk\frac{1}{\omega_k}\text{Re}[(\delta_\mu{}^0(-i\omega_k)\Phi_k + \delta_\mu{}^1\Phi_k')(\delta_\nu{}^0 i\omega_k)(\Phi_k)^* + \delta_\nu{}^1(\Phi_k')^*) - \frac{\eta_{\mu\nu}}{2}(\omega_k^2|\Phi_k|^2 - |\Phi_k'|^2)], \tag{32}
$$

while $\mathcal{T}_{\mu\nu}^{\text{State}}(x_1^\sigma)$ corresponds to the specific contribution for the given initial state we consider for the field:

$$
\mathcal{T}_{\mu\nu}^{\text{State}}(x_1^\sigma) = \int dk \int dk'\left[\frac{1}{\omega_k\omega_{k'}}\right]^{\frac{1}{2}}2\text{Re}\left[\langle\hat{a}_k(-\infty)\hat{a}_{k'}(-\infty)\rangle_\phi e^{-i(\omega_k+\omega_{k'})t}[(\delta_\mu{}^0(-i\omega_k)\Phi_k + \delta_\mu{}^1\Phi_k')(\delta_\nu{}^0(-i\omega_{k'})\Phi_{k'} + \delta_\nu{}^1\Phi_{k'}')\right.
$$

$$
+ \frac{1}{2}\eta_{\mu\nu}(\omega_k\omega_{k'}\Phi_k\Phi_{k'} + \Phi_k'\Phi_{k'}')] + \langle\hat{a}_k^\dagger(-\infty)\hat{a}_{k'}(-\infty)\rangle_\phi e^{i(\omega_k-\omega_{k'})t}
$$

$$
\left. \times [(\delta_\mu{}^0 i\omega_k\Phi_k + \delta_\mu{}^1\Phi_k')(\delta_\nu{}^0(-i\omega_{k'})(\Phi_{k'})^* + \delta_\nu{}^1(\Phi_{k'}')^*) - \frac{1}{2}\eta_{\mu\nu}(\omega_k\omega_{k'}(\Phi_k)^*\Phi_{k'} - (\Phi_k')^*\Phi_{k'}')]\right], \tag{33}
$$

which in principle depends on all the spacetime coordinates.

All in all, depending on the initial state considered for the field, the different expectation values of the components of the energy-momentum tensor, which are expressed as the sum of a state-independent expectation value (corresponding to the vacuum fluctuations at zero temperature) and a state-dependent term. For this last term, two important cases to consider are thermal and squeezed initial states.

1. Thermal and continuum-single-mode squeezed states

By considering a thermal initial state for the field, characterized by a temperature $\beta_\phi = 1/T_\phi$, the expectation values of the products of annihilation and creation operators that appear in Eq. (33) can be calculated straightforwardly:

$$\langle \hat{a}_k(-\infty)\hat{a}_{k'}(-\infty)\rangle_\phi = 0,$$
$$\langle \hat{a}_k^\dagger(-\infty)\hat{a}_{k'}(-\infty)\rangle_\phi = N(\omega_k)\delta(k-k'), \quad (34)$$

where $N(\omega_k) = \frac{1}{e^{\beta_\phi\omega_k}-1}$ is the boson occupation number, i.e., the Bose-Einstein distribution.

In this case, $\mathcal{T}_{\mu\nu}^{\text{State}}(x_1^\sigma)$ simplifies to give the same integral as $\langle \hat{T}_{\mu\nu}^{\text{IC},\infty}(\mathbf{x}_1)\rangle_\phi^{\text{Vac}}$ but containing $2N(\omega_k)$ in the integrand. In other words, $\mathcal{T}_{\mu\nu}^{\text{State}}(x_1^\sigma)$ turns out to be the thermal correction for the vacuum fluctuations at zero temperature, which is the expected result. Therefore, considering that $\coth(\frac{\beta_\phi\omega_k}{2}) = 1 + 2N(\omega_k)$, we have for Eq. (31)

$$\langle \hat{T}_{\mu\nu}^{\text{IC},\infty}(x_1^\sigma)\rangle_\phi$$
$$\equiv \langle \hat{T}_{\mu\nu}^{\text{IC},\infty}(\mathbf{x}_1)\rangle_\phi^{T_\phi}$$
$$= \int dk \frac{1}{\omega_k}\coth\left(\frac{\beta_\phi\omega_k}{2}\right)\text{Re}\left[(\delta_\mu^{\ 0}(-i\omega_k)\Phi_k + \delta_\mu^{\ 1}\Phi'_k)\right.$$
$$\left. \times (\delta_\nu^{\ 0}i\omega_k(\Phi_k)^* + \delta_\nu^{\ 1}(\Phi'_k)^*) - \frac{\eta_{\mu\nu}}{2}(\omega_k^2|\Phi_k|^2 - |\Phi'_k|^2)\right],$$
$$(35)$$

which is, at least, a time-independent expression also. The dependence on the spatial coordinate is determined for each configuration through the introduction of the appropriate mode functions Φ_k.

Considering a continuum-single-mode squeezed state for the field characterized by a squeezing parameter $\xi(k) = |\xi(k)|e^{i\varphi(k)}$ (see Refs. [1,46]),

$$|\xi(k)\rangle = e^{-\frac{1}{2}\int_{-\infty}^{+\infty} dk(\xi(k)\hat{a}_k^{\dagger 2} - \xi^*(k)\hat{a}_k^2)}|0\rangle, \quad (36)$$

and the expectation values of the products of annihilation and creation operators, we have

$$\langle \hat{a}_k(-\infty)\hat{a}_{k'}(-\infty)\rangle_\phi = -e^{i\varphi(k)}\sinh(|\xi(k)|)$$
$$\times \cosh(|\xi(k)|)\delta(k-k'), \quad (37)$$

$$\langle \hat{a}_k^\dagger(-\infty)\hat{a}_{k'}(-\infty)\rangle_\phi = \sinh^2(|\xi(k)|)\delta(k-k'). \quad (38)$$

In this case, the term of $\mathcal{T}_{\mu\nu}^{\text{State}}(x_1^\sigma)$ associated to $\langle \hat{a}^\dagger\hat{a}\rangle$ combines with $\langle \hat{T}_{\mu\nu}^{\text{IC},\infty}(\mathbf{x}_1)\rangle_\phi^{\text{Vac}}$ because of the delta function $\delta(k-k')$. For the other term, $\delta(k-k')$ simplifies the expression, but the results are both space and time dependent.

Considering that $1 + 2\sinh^2(|\xi(k)|) = \cosh(2|\xi(k)|)$ and $2\sinh(|\xi(k)|)\cosh(|\xi(k)|) = \sinh(2|\xi(k)|)$, we finally have

$$\langle \hat{T}_{\mu\nu}^{\text{IC},\infty}(x_1^\sigma)\rangle_\phi = \langle \hat{T}_{\mu\nu}^{\text{IC},\infty}(\mathbf{x}_1)\rangle_\phi^{\text{Squeezed}} + \mathcal{T}_{\mu\nu}^{\text{Squeezed}}(x_1^\sigma), \quad (39)$$

where the (at least) time-independent expectation value is given by

$$\langle \hat{T}_{\mu\nu}^{\text{IC},\infty}(\mathbf{x}_1)\rangle_\phi^{\text{Squeezed}} = \int dk \frac{1}{\omega_k}\cosh(2|\xi(k)|)\text{Re}\left[(\delta_\mu^{\ 0}(-i\omega_k)\Phi_k + \delta_\mu^{\ 1}\Phi'_k)(\delta_\nu^{\ 0}i\omega_k(\Phi_k)^* + \delta_\nu^{\ 1}(\Phi'_k)^*) - \frac{\eta_{\mu\nu}}{2}(\omega_k^2|\Phi_k|^2 - |\Phi'_k|^2)\right],$$
$$(40)$$

and the second term is

$$\mathcal{T}_{\mu\nu}^{\text{Squeezed}}(x_1^\sigma) = -\int dk \frac{1}{\omega_k}\sinh(2|\xi(k)|)\text{Re}\left[e^{i\varphi(k)}e^{-i2\omega_k t}[(\delta_\mu^{\ 0}(-i\omega_k)\Phi_k + \delta_\mu^{\ 1}\Phi'_k)(\delta_\nu^{\ 0}(-i\omega_k)\Phi_k + \delta_\nu^{\ 1}\Phi'_k)\right.$$
$$\left. + \frac{1}{2}\eta_{\mu\nu}(\omega_k^2\Phi_k^2 + \Phi'^2_k)]\right].$$
$$(41)$$

This last expression can be taken one step further through time averaging. For these steady quantities defined at the long-time limit, the time average for a quantity A is given by

$$\langle\langle A\rangle\rangle_t = \lim_{\tau\to+\infty}\frac{1}{\tau}\int_{-\tau/2}^{\tau/2}d\tau' A(\tau'). \qquad (42)$$

Therefore, for the time dependence in the integrand of $\mathcal{T}_{\mu\nu}^{\text{Squeezed}}$, we have $\langle\langle e^{-i2\omega_k t}\rangle\rangle_t = \lim_{\tau\to+\infty}\frac{\sin(\omega_k\tau)}{\omega_k\tau}=0$ for $k\neq 0$ and $\langle\langle 1\rangle\rangle_t = 1$ for $k=0$. However, for this last case, the rest of the integrand of $\mathcal{T}_{\mu\nu}^{\text{Squeezed}}$ vanishes since Φ_k is given by sums of $e^{\pm ikx}$. Finally, we obtain

$$\langle\langle\mathcal{T}_{\mu\nu}^{\text{Squeezed}}\rangle\rangle_t = 0. \qquad (43)$$

It should be noted that this result is due to the oscillatory time dependence for every k. If the modes have another dependence, this last time average could be different from zero. However, for boundary conditions on the spatial coordinate only, the time dependence is in general oscillatory for a field in dielectric media and, therefore, the time average vanishes.

All in all, for both cases, after taking the time average, the expectation value of the energy-momentum tensor can be written as

$$\langle\langle\langle\hat{T}_{\mu\nu}^{\text{IC},\infty}(x_1^\sigma)\rangle_\phi\rangle\rangle t$$
$$= \int dk\frac{1}{\omega_k}\mathfrak{F}(k)\text{Re}\Big[(\delta_\mu{}^0(-i\omega_k)\Phi_k + \delta_\mu{}^1\Phi_k')$$
$$\times (\delta_\nu{}^0 i\omega_k(\Phi_k)^* + \delta_\nu{}^1(\Phi_k')^*)$$
$$-\frac{\eta_{\mu\nu}}{2}(\omega_k^2|\Phi_k|^2 - |\Phi_k'|^2)\Big], \qquad (44)$$

having $\mathfrak{F}(k) = \coth(\frac{\beta_\phi\omega_k}{2})$ for a thermal state and $\mathfrak{F}(k) = \cosh(2|\xi(k)|)$ for a squeezed state. It should be noted that other initial states could provide another time and space dependence for the expectation values of the components of the energy-momentum tensor, complicating the time-average procedure and opening a new type of formula for the force. The case of continuum-single-mode squeezed states, given the expectation values for annihilation and creation operators, turns out to be very simple as we will see in the next section. Due to the time average, the forces for thermal and continuum-single-mode squeezed states can be written as particular cases of the last expression. Similar results can be obtained for the case of continuum-two-mode squeezed states characterized by a frequency Ω (see Ref. [1] for the quantum state), which after time averaging the expressions results in basically the same result. On the other hand, other initial states, as continuum-coherent states for example, would present more complicated expressions, including double integration over the frequency, since the expectation values for the products of annihilation and creation operators do not include a Dirac delta function $\delta(k-k')$.

V. CASIMIR FORCE FOR FINITE WIDTH PLATES CONFIGURATION

Once we have given expressions for the expectation values of the components of the energy-momentum operator, we proceed to calculate the Casimir force between two homogeneous plates of finite width d and different materials (n_L and n_R for the left and right plates, respectively) separated by a distance a. With this aim, to calculate the force over one of the plates we subtract the field's pressures on each side of the plate. In our case, the pressure is given by the expectation value of the xx component of the energy-momentum tensor operator. Moreover, as this expectation value splits into two contributions, the same is true for the Casimir force (as it happens in Refs. [20,21,24,25,37,39,41,43]). Therefore, we have

$$F_C = \langle\hat{T}_{xx}\rangle^{\text{Ext}} - \langle\hat{T}_{xx}\rangle^{\text{Int}}$$
$$= \langle\hat{T}_{xx}^{\text{IC}}\rangle^{\text{Ext}} + \langle\hat{T}_{xx}^{\text{B}}\rangle^{\text{Ext}} - \langle\hat{T}_{xx}^{\text{IC}}\rangle^{\text{Int}} - \langle\hat{T}_{xx}^{\text{B}}\rangle^{\text{Int}}$$
$$= F_C^{\text{IC}} + F_C^{\text{B}}, \qquad (45)$$

where the superscript "Int" denotes the region of the vacuum gap between the plates and "Ext" denotes the region outside the plates' configuration adjacent to the respective plate under consideration. It is clear that in the last expression, for the initial conditions' contribution to the energy-momentum tensor, we are considering the time-averaged expression given in Eq. (44).

Considering Eqs. (28) and (44), for the full calculation of each contribution, we need both the mode functions (or homogeneous solutions) Φ and the transform of the Green function $\mathfrak{G}_{\text{Ret}}$ for the two-plates configuration. These expressions are given in Appendix E. Then, we can easily calculate the contribution to the Casimir force acting on the left plate when the field is in a thermal or continuum-single-mode squeezed state through Eqs. (44) and (45), obtaining

$$F_C^{\text{IC}}[a,d,\mathfrak{F}] = \int_0^{+\infty}dk k\mathfrak{F}(k)[1 + |R_{-ik}^>|^2 + |T_{-ik}|^2$$
$$- |C_{-ik}^>|^2 - |D_{-ik}^>|^2 - |C_{-ik}^<|^2 - |D_{-ik}^<|^2], \qquad (46)$$

which is an extension of the results found in Refs. [41–43, 47]. The explicit dependence on $\mathfrak{F}$ denotes that the same expression is valid for both thermal and squeezed states. Nevertheless, while for the former, $\mathfrak{F}(k)$ is always an even function, for the latter, an even $\mathfrak{F}(k)$ is required for an even $\xi(k)$, which was assumed for obtaining the last result.

It should be noted that due to the dissipation present in this scenario, $|R_{-ik}^>|^2 + |T_{-ik}|^2 \neq 1$ and $|r_i|^2 + |t_i|^2 \neq 1$.

For the contribution of the baths, employing the last expressions and replacing them in Eq. (28) to obtain the contribution as Eq. (45) states, it is straightforward to obtain

$$
\begin{aligned}
F_C^B[a, d, \beta_{B,L}, \beta_{B,R}] = &\int_0^{+\infty} d\omega \frac{\omega}{8} |T_{-i\omega}|^2 \left[\coth\left[\frac{\beta_{B,L}\omega}{2}\right] \frac{|n_L + 1|^2}{|n_L|^2} \frac{1}{|t_R|^2 |t_L|^2} (\text{Re}(n_L)(1 - e^{-2\omega \text{Im}(n_L)d}) \right. \\
&\times [|(1 - r_L r_R e^{i2\omega a})(1 - r_{n_L} r_L) - r_{n_L} r_R t_L^2 e^{i2\omega a}|^2 - |(1 - r_L r_R e^{i2\omega a})(r_L - r_{n_L}) + r_R t_L^2 e^{i2\omega a}|^2 \\
&+ |t_L|^2 (1 + |r_R|^2)(1 - |r_{n_L}|^2)] + 2\text{Im}(n_L)\text{Im}[(e^{i2\omega \text{Re}(n_L)d} - 1)([(1 - r_L r_R e^{i2\omega a})(1 - r_{n_L} r_L) \\
&- r_{n_L} r_R t_L^2 e^{i2\omega a}][(1 - r_L^* r_R^* e^{-i2\omega a})(r_L^* - r_{n_L}^*) + r_R^* t_L^{*2} e^{-i2\omega a}] + |t_L|^2 (1 + |r_R|^2)r_{n_L})]) \\
&+ \coth\left[\frac{\beta_{B,R}\omega}{2}\right] \frac{|n_R + 1|^2}{|n_R|^2} \left[1 - \frac{1 + |r_L|^2}{|t_L|^2}\right] (\text{Re}(n_R)(e^{2\omega \text{Im}(n_R)d} - 1)[1 - |r_{n_R}|^2] - 2\text{Im}(n_R) \\
&\left. \times \text{Im}[r_{n_R}(e^{i2\omega \text{Re}(n_R)d} - 1)]) \right].
\end{aligned}
\tag{47}
$$

Therefore, considering the last expression and Eq. (46), the nonequilibrium force experienced by the left plate of a Casimir configuration is given by

$$
F_C[a, d, \mathfrak{F}, \beta_{B,L}, \beta_{B,R}] = F_C^{IC}[a, d, \mathfrak{F}] + F_C^B[a, d, \beta_{B,L}, \beta_{B,R}].
\tag{48}
$$

All in all, this is the Casimir force for a nonequilibrium scenario consisting of two plates of finite width d and different materials. It is, in fact, the generalization of the results found in Refs. [41,43]. As it happens in those situations, it is expected that our new result is regularized by the dependence of the reflection coefficient on k, which ensures convergence by including a natural cutoff in the model considered.

Moreover, two important limit cases can be recovered from this general expression. One is the force for the case of materials without dissipation (real frequency-independent refractive indexes) and the other one is the Lifshitz formula, which is the force between two half-spaces (infinite width) at thermal equilibrium. Appendix F is devoted to showing how these results can be recovered from our general nonequilibrium expressions.

VI. COMPARISON BETWEEN CASIMIR FORCES FOR THERMAL AND SQUEEZED STATES

We have calculated the Casimir force between two finite width plates in Eqs. (46), (47), and (48) for both situations, when the initial state for the field is thermal or squeezed, while the baths at each point of the plates are always characterized by a proper temperature. Therefore, the comparisons between both cases and between our result and previous ones are mandatory. Although we have general formulas valid even for the case of different temperatures in each slab, in the present work we focus on the comparison between different states of the field, while keeping the same temperature on both plates (and equal to the temperature of the field when considering the thermal state for it). For simplicity, analyzing full non-equilibrium scenarios, including different temperatures

between the parts and squeezed states, is left as pending future work.

In a previous work [35], the Casimir force between perfect conductor plates was calculated when the EM field was in a squeezed state. Given the perfect material, the plates enter as boundary conditions on the quantum field. As the material of the plates does not present any internal dynamics, the system of interest (the field) is not an open system for this case.

Therefore, the quantization of the system is based directly on quantizing the modes of the field, confined to the space between the plates. This implies that only a Hilbert space for the field is required for the corresponding quantum theory.

As the field is confined in the transverse direction to the plates, the transverse component of the wave vector of each mode is discretized. Then, this is inherited by the eigenfrequencies of the problem. As in our calculation, the squeezed state enters when computing the expectation values of products of the creation and annihilation operators. This results in the factor $\cosh(2|\xi_m|)$, equivalent to ours but logically discretized due to the allowed modes for this idealized case (with the subscript m as the label of the discrete modes here). This was done in Ref. [35] in which, for simplicity, one of the cases analyzed was to take a constant squeezing for all the modes, $\xi_m = \xi$ for every m. Therefore, the factor $\cosh(2|\xi_m|)$ is a constant and the force turns out to be the well-known Casimir's result times $\cosh(2|\xi|)$.

In our case, we are considering a one-dimensional scalar field instead of a full EM field. However, the respective limit of perfect conductors can be addressed from our general formulas. The first step is to erase any internal dynamics in order to remove the dissipation from the result. As mentioned in the first section of Appendix F, this is achieved by setting $\bar{\gamma}_{L,R}(\omega) \equiv 0$ (which gives $F_C^B \equiv 0$) and taking the zeroth order of the permittivity functions. Therefore, the Casimir force is given by the initial conditions' contribution only through Eq. (F4). For the case of an initial squeezed state, we have to put $\mathfrak{F}(k) = \cosh(2|\xi(k)|)$. If we also impose a constant squeezing for all the modes, we obtain

$$F_C[a, d, \xi = \text{const}]|_{\text{No Diss}} = \cosh(2|\xi|) \int_0^{+\infty} dk k \left[2 - \frac{[|t_L|^2(1 + |r_R|^2) + |t_R|^2(1 + |r_L|^2)]}{|1 - r_L r_R e^{i2\omega a}|^2} \right]$$

$$= \cosh(2|\xi|) F_C^{T_\phi=0}[a, d]|_{\text{No Diss}}, \tag{49}$$

where $F_C^{T_\phi=0}[a, d]|_{\text{No Diss}}$ is the Casimir force between two slabs of materials without dissipation at zero temperature, which is in agreement with Ref. [47].

This result is analog to the one obtained in Ref. [35] but to the case of two slabs of materials without dissipation. Moreover, for the case of perfect-conductor plates, following the procedure given in Ref. [42] for the respective limit (analyzing the behavior of the integrand), it can be easily shown that $F_C^{T_\phi=0}[a, d]|_{\text{No Diss}}$ can be reduced to the corresponding expression for the Casimir force between perfect conductors. Therefore, the one-dimensional scalar version of the result given in Ref. [35] is fully achieved.

Nevertheless, considering constant squeezing for all the modes of the field is too idealized. On the one hand, in cavity QED, it is usual to consider that in perfect conductor cavities, only one of the discrete modes of the field is squeezed [30–33]. On the other hand, in our case, the cavity is formed by dielectric slabs, so it is reasonable to expect that the squeezing is not limited to a unique mode. If we want to squeeze one mode of frequency Ω_0, we expect to effectively squeeze this mode but also the modes around it contained in a bandwidth σ centered in Ω_0. We assume for simplicity that the squeezing parameter $\xi(k)$ for all the modes contained in the bandwidth (for $\Omega_0 - \frac{\sigma}{2} < k < \Omega_0 + \frac{\sigma}{2}$) is the same and given by $1/\sigma$ (there are no significant changes in the analysis if we choose for the squeezing const/σ, such that const is any real number). This choice is made because we want the squeezing parameter ξ, as a function of k, to recover a squeezed state

only for the mode Ω_0 when the bandwidth σ approaches 0. In other words, we are demanding that $\xi(k) \rightarrow \delta(k - \Omega_0)$ when $\sigma \rightarrow 0$, in such a way that Eq. (36) results in a squeezed state only for the frequency Ω_0. Accordingly, comparing the initial conditions' contributions for a squeezed state with this squeezing distribution $\xi_{\sigma, \Omega_0}(k)$ and for a thermal state with $T_\phi = 300K$ (room temperature), and considering ohmic thermal baths in both cases, we obtain Fig. 1(a) for different values of Ω_0 as a function of σ (see Ref. [37]).

As it can be seen, the curves in Fig. 1(a) are always below 1, implying that the contribution for the squeezed state is always greater than for the thermal state for any values of σ and Ω_0. This is expected because the presence of the factor $\cosh(2|\xi(k)|)$ in the integrand, combined with the chosen squeezing distribution, makes the integration unfold, but all the frequencies keep contributing to the value of the integral. For the frequencies outside the bandwidth σ centered at Ω_0 we have that $\cosh(2|\xi(k)|) \equiv 1$ since $\xi(k) \equiv 0$, while for the frequencies belonging to the bandwidth, $\cosh(2|\xi(k)|)$ takes a constant value greater than 1. Therefore, the final value of the initial conditions' contribution for the squeezed state is greater than that for the thermal state.

However, as the bandwidth becomes larger, the value of the squeezing given by $1/\sigma$ becomes smaller. Therefore, for large σ, the squeezing approaches zero and the value of the contribution for the squeezed state gets closer to the thermal value. At first glance, this seems natural since the initial
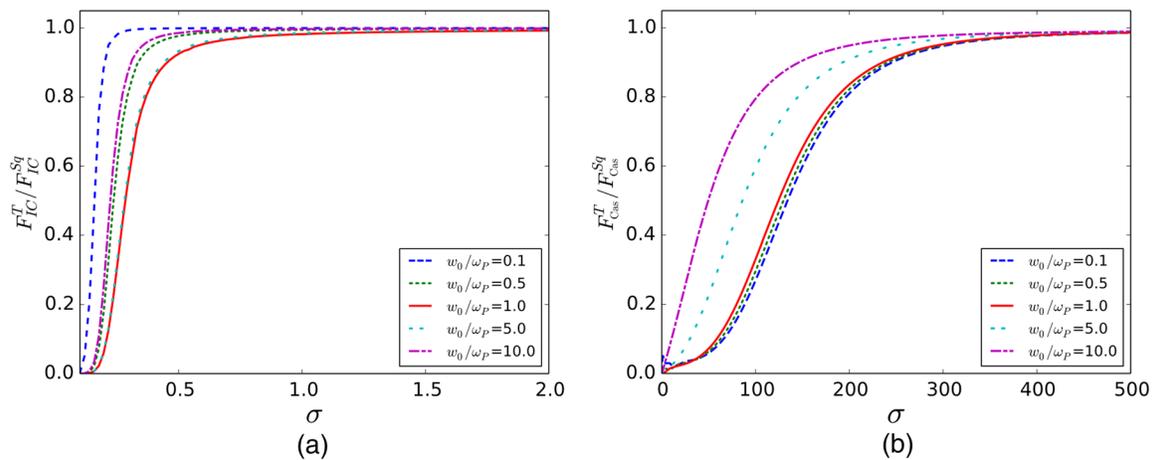

FIG. 1.   (a) Ratio of the initial conditions' contributions for a thermal state with $T_\phi = 300K$ and a continuum-single-mode squeezed state as a function of the bandwidth σ for different values of the center frequency Ω_0 (compared with the plasma frequency ω_{Pl}). (b) Ratio of the total forces for a thermal state with $T_\phi = 300K$ and a continuum-single-mode squeezed state as a function of σ for different Ω_0's. Parameters are $\gamma_{L,R} = 10^{-1}/a$; $\omega_{0,i} = 10/a$; $\omega_{\text{Pl},i} = 10/a$; $d/a = 10^2$; $a = 100 \text{ nm}$, and σ is given in units of a.

conditions' contribution at zero temperature results from setting $\mathfrak{F} \equiv 1$ for every k. But, in our case, we are comparing the contribution for a squeezed state with the value for a thermal one with $T_\phi = 300K$ instead of zero temperature. However, the integrand of the initial conditions' contribution does not change significantly for different values of temperature because $k[1 + |R^{>}_{-ik}|^2 + |T_{-ik}|^2 - |C^{>}_{-ik}|^2 - |D^{>}_{-ik}|^2 - |C^{<}_{-ik}|^2 - |D^{<}_{-ik}|^2]$ in Eq. (46) is different from zero when the thermal factor $\coth(\frac{\beta_\phi k}{2})$ is close to 1. This means that the initial conditions' contribution is insensitive to the chosen temperature. Thus, the ratio between the initial conditions' contributions for a thermal state with $T_\phi = 300K$ and for a squeezed state with large bandwidth σ (which approaches the result with $T_\phi = 0K$) is logically close to 1.

A similar argument also explains the fact that the curves in Fig. 1(a) are not ordered according to their values of Ω_0 for the values of σ shown in the figure. The point is that for a fixed σ, different values of Ω_0 enhance (through the squeezing factor) different parts of the spectrum of the integrand. Therefore, the final value of the contribution to the force strongly depends on the chosen Ω_0 for a narrow bandwidth σ. However, for large σ, the enhanced parts of the integrand are very similar, independently of the chosen value of Ω_0, and the final value of the contributions approach it and get ordered according to it. This can be seen at the end of the curves. This was checked also for higher values of σ.

On the other hand, the behavior of the curves for very small σ is explained by another feature of the squeezing distribution $\xi(k)$. As the bandwidth becomes narrower, the contribution for the modes inside it gets greatly enhanced since the squeezing parameter is given by $1/\sigma$. Therefore, although the number of enhanced modes is lower, its contributions to the integral grow strongly. Therefore, the final contributions to the force also grow, getting numerically divergent values when the bandwidth approaches zero. Therefore, the ratio for the contributions of the force in the different situations tends to zero. However, it should be noted that this is in fact a numerical limitation instead of a correct result. The limit of $\sigma \to 0$ is well defined analytically. As we mentioned before, the choice of the squeezing distribution was to describe an imperfect squeezing of a mode of frequency Ω_0 characterized by a bandwidth σ, in such a way that the perfect squeezing on the mode Ω_0 only could be obtained through the limit of null bandwidth. If we like to obtain this result, we should go to Eq. (36) and set $\xi(k) = \delta(k - \Omega_0)$. Then, the full calculation of the initial conditions' contribution is straightforward, obtaining that it corresponds to the evaluation of the integrand at Eq. (46) on the chosen frequency Ω_0, which a well-defined finite result. It is clear that this value strongly depends on the chosen frequency; however it remains always finite.

Beyond all these features, although the final value of the initial conditions' contribution to the force seems to vary significantly, it does for a small interval of σ (approximately until 0,75 in units of a for the given curves). Then, it seems to be insensitive to the changes of σ for a wide range of values. This is because we are comparing the initial conditions' contributions only. However, while the baths' contributions to the total force (based on the expelled field by the materials of the plates) tends to separate the plates, the initial conditions' contribution tends to attract them. Then, the total force results from a subtraction which is more sensitive to the changes of σ. This is observed in Fig. 1(b), where the approach to an asymptotic value (also from below) of the ratio of the total forces for the squeezed and thermal states occurs for values of σ four orders of magnitude larger (until 350 in units of a approximately).

It should be noted that for the baths in each plate, the temperatures were all equal to $300K$. Thus, for the thermal case, the total force is the one at thermal equilibrium.

Contrary to what we have when comparing the initial conditions' contributions separately, the curves are ordered by the value of Ω_0. For the large-σ behavior, the asymptotic value is close to 1 but it is not equal to it. This shows that the ratio of the total forces is sensitive to the chosen temperature. On the other hand, the behavior of the curves for small σ presents the same troubles that were mentioned for the other figure. Again, the limit $\sigma \to 0$ can be studied analytically.

All in all, the ratio between the total forces when the initial conditions' contributions are different seems to allow tuning the force by modifying the squeezing properties (the bandwidth σ and the frequency Ω_0) of the field state in a wide range of values. As the ratio is smaller than 1, the tuning could intensify the (always attractive) total force from a minimum value defined by the temperature of the plates.

VII. FINAL REMARKS

In this work we have studied several aspects of the Casimir forces in different out-of-equilibrium situations, analyzing the magnitude and elucidating the steady dynamics at different nontrivial scenarios for the chosen model.

Specifically, from a conceptual point of view, we have achieved a well-defined and consistent approach for the study of the Casimir forces at completely general scenarios. We developed a first-principles canonical quantization formalism for the study of the interaction between a quantum scalar field and dissipative-material polarizable bodies. These are locally modeled as complex quantum systems in each point of space, formed by quantum harmonic oscillators describing the volume elements of the polarizable body which are coupled to its own thermal bath, as commonly used since Ref. [22]. In analogy to the results obtained through the CTP integral formulation given in Refs. [20,25], we have obtained a full equation of motion

for the field interacting effectively with the material bodies. However, as in Ref. [39], the equation stands for the quantum field operator and (as every Heisenberg equation) is subjected to initial conditions, which for simplicity we took as free field operators. The solution for this equation is given in terms of the Green function and all the sources included in the problem that generate field due to the interactions between the different parts of the total system. It is important to remark that the interactions are switched on at a given initial time t_0, being a sudden start analog to the time evolution considered in problems involving quantum quenches. For arbitrary time, we obtain three contributions to the field operator, each one associated to a specific part of the total system. One is associated to the volume elements, another one is associated to the baths and, finally, there is a contribution entirely related to the field initial conditions. As the parts of the total system are initially free, the field operator is separate in three contributions, each one acting on the respective Hilbert space ($H_\phi \otimes H_A \otimes H_B$). This splitting is critical for studying the long-time limit ($t_0 \to -\infty$) of each contribution separately.

We achieved expressions for the long-time contribution to the field operator of each part of the total system, showing that at the steady situation only two of them contribute to the expectation values of the energy-momentum tensor components. There is no long-time contribution from the volume elements. Moreover, we showed that the baths' contribution to the long-time field operator matches with the one considered in Ref. [19] from a FQED approach, while the one associated to the initial conditions has the form of the homogeneous solutions considered in previous works for other steady quantization schemes. In fact, this contribution also agrees with the result obtained in Ref. [25] and the mechanism described in Ref. [21], where this was suggested but not fully demonstrated. Here, we describe clearly how the existence of infinite-size regions without dissipation for the field are directly related to a nonvanishing long-time contribution of the initial conditions. All in all, as in Refs. [20,25] but this time from a canonical quantization scheme, our general result closes a conceptual hole between the full quantum theory and the FQED approaches. The matching between both and the way that FQED is related to quantum fluctuations are clearly explained.

Once we explored the general features of the developed approach, the other main result of the work is based on taking advantage of the results obtained for the expectation values of the energy-momentum tensor components of general material configurations. We proceeded to fully calculate the force between two slabs of different materials and temperatures but the same width. The initial conditions' contribution was evaluated for two types of states, thermal and continuum single squeezed. For the latter, a time average was required to determine the value of the force and write the two cases in a unified way. Then, for the squeezed case, we showed how to recover the results in

Ref. [35] of the force between perfect conductor plates. However, our result is the out-of-equilibrium generalized version to dissipative materials and squeezed states for the field. Therefore, we proceeded to compare this case with the result at thermal equilibrium, taking $T_\phi = 300 K$. From Fig. 1(a), we showed that the initial conditions' contribution is sensitive to the change on the bandwidth of squeezed modes σ only for small values of the bandwidth, regardless of the centering frequency Ω_0. Moreover, the contribution is insensitive to changes in temperature. Nevertheless, in Fig. 1(b), we showed that when comparing the total force for each case, the sensitivity is increased by four orders of magnitude when changing the values of σ. We also showed that the total force is also sensitive when changing the temperature. All in all, considering a constant squeezing for given σ and Ω_0, we showed that the change of σ allows us to tune the value of the (always attractive) Casimir force from a minimum value defined by the temperature of the plates. This could be of great interest for application in technological improvements.

Moreover, out-of-thermal-equilibrium forces and heat transfer can be studied from the present results. Also the situation of suddenly changing the distance between the plates, which is related to the dynamical Casimir effect. This can be achieved by employing the long-time field operator for a given configuration of the plates as the initial condition for a new configuration. This is left as pending future work.

As a final comment, it should be noted that these results can be easily extended to the three-dimensional scalar field, where two kinds of modes enter, the evanescent and the propagating. For our case of a one-dimensional field, we only deal with propagating modes. On the other hand, addressing the extension for the EM case could be in principle a nontrivial issue, but is in any case achievable. It is clear that the main complication to adapt this canonical quantization approach to the EM case will be related to the difficulties associated to quantizing the EM field, which forces us to deal with its gauge invariance and vectorial nature at a quantum framework (as it happens on a path integral formulation in Ref. [20]). However, as it was shown from the CTP approach (Refs. [20,25]), the conclusions obtained here for the scalar case will remain broadly valid for the EM field. Including squeezed states together with the vectorial case and the different kinds of modes could present particular behaviors, especially when studying the long- and short-distances regimes, where the different modes become dominant. However, all these features are out of the scope of the present work and are also left as pending work.

ACKNOWLEDGMENTS

APPENDIX A: DEDUCING THE FIELD EQUATION

In this section, we show how Eq. (3) is rigorously obtained given the chosen model. Starting from the Lagrangian (1), it is easy to derive the Heisenberg equations of motion for the different operators. They are given by

$$\hat{p}_{n,x} = m_n \dot{\hat{q}}_{n,x}, \qquad \dot{\hat{p}}_{n,x} = -m_n \omega_n^2 \hat{q}_{n,x} + \lambda_n \hat{r}_x, \quad \text{(A1)}$$

$$\hat{p}_x = m \dot{\hat{r}}_x + e \hat{\phi}, \qquad \dot{\hat{p}}_x = -m \omega_0^2 \hat{r}_x + \sum_n \lambda_n \hat{q}_{n,x}, \quad \text{(A2)}$$

$$\Box \hat{\phi} = 4\pi \eta e \dot{\hat{r}}_x, \quad \text{(A3)}$$

where the operators $\hat{p}_x$ and $\hat{p}_{n,x}$ are the conjugate momentum operators associated to the operators $\hat{r}_x$ and $\hat{q}_{n,x}$, respectively.

Combining the first two equations, we get

$$m_n \ddot{\hat{q}}_{n,x} + m_n \omega_n^2 \hat{q}_{n,x} - \lambda_n \hat{r}_x = 0. \quad \text{(A4)}$$

As usual in the context of QBM, we solve the equations for the operators $\hat{q}_{n,x}$, taking $\hat{r}_x$ as a source, and replace the solutions into the pair of Eqs. (A2). In this way, the microscopic degrees of freedom in the material satisfy a Langevin-like equation of the form

$$\dot{\hat{p}}_x = -m \omega_0^2 \hat{r}_x - m \frac{d}{dt} \left(\int_{t_0}^t d\tau \gamma(t - \tau) \hat{r}_x(\tau) \right) + \hat{F}_x(t - t_0), \quad \text{(A5)}$$

where the damping kernel γ and the stochastic force operator $\hat{F}_x$ are the same as the ones of QBM (see [37] for a general and complete view of QBM and also [48] for an extensive study of Langevin-like equations in different contexts). They are given by

$$\gamma(t) = \frac{2}{m} \int_0^{+\infty} d\omega \frac{J(\omega)}{\omega} \cos(\omega t), \quad \text{(A6)}$$

$$\hat{F}_x(t - t_0) = \sum_n \frac{\lambda_n}{\sqrt{2 m_n \omega_n}} \left(e^{-i\omega_n(t - t_0)} \hat{b}_{n,x}(t_0) \right.$$
$$\left. + e^{i\omega_n(t - t_0)} \hat{b}_{n,x}^\dagger(t_0) \right). \quad \text{(A7)}$$

Here $\hat{b}_{n,x}(t_0)$ and $\hat{b}_{n,x}^\dagger(t_0)$ are the annihilation and creation operators associated to $\hat{q}_{n,x}(t_0)$, and $J(\omega)$ is the spectral density that characterizes the environment, which gives the number of oscillators in each frequency for given values of the coupling constants λ_n (see Refs. [37,43] for more details).

At equilibrium, the stochastic force operator in Eq. (A7) and the damping kernel γ in Eq. (A6) are not independent. The statistical properties of the stochastic force operator are given by the dissipation and noise kernels:

$$D(t - t') \equiv i \langle [\hat{F}(t); \hat{F}(t')] \rangle = i[\hat{F}(t); \hat{F}(t')]$$
$$= -2 \int_0^{+\infty} d\omega J(\omega) \sin(\omega(t - t')), \quad \text{(A8)}$$

$$D_1(t - t') \equiv \langle \{ \hat{F}(t); \hat{F}(t') \} \rangle$$
$$= 2 \int_0^{+\infty} d\omega J(\omega) \coth \left(\frac{\beta_B \omega}{2} \right) \cos(\omega(t - t')), \quad \text{(A9)}$$

which are the formal quantum open systems generalization of the relations employed in Ref. [39] for general environments and arbitrary temperature. Note that only the noise kernel D_1 involves the environmental temperature $\beta_B = 1/T_B$ as a parameter. Considering Eq. (A6) and Eq. (A8), it is easy to show that

$$\frac{d}{dt} (\gamma(t - s)) = -\frac{1}{m} D(t - s), \quad \text{(A10)}$$

which relates the damping kernel γ to the statistical properties of the stochastic force operator $\hat{F}$.

All in all, the set of equations to solve now are Eqs. (A2), (A3), and (A5).

It is possible to obtain a formal solution for the operators $\hat{r}_x(t)$ by considering the field ϕ as a source for the equation. This solution generalizes the crude approximation made in Ref. [39] for the evolution of the microscopic degrees of freedom in the mirrors. It is given by

$$\hat{r}_x(t) = G_1(t - t_0) \hat{r}_x(t_0) + G_2(t - t_0) \dot{\hat{r}}_x(t_0)$$
$$+ \frac{1}{m} \int_{t_0}^t d\tau G_2(t - \tau) (\hat{F}_x(\tau - t_0) - e \dot{\hat{\phi}}(x, \tau)), \quad \text{(A11)}$$

where $G_{1,2}$ are the Green functions associated to the QBM equation that satisfy

$$G_1(0) = 1, \qquad \dot{G}_1(0) = 0, \quad \text{(A12)}$$

$$G_2(0) = 0, \qquad \dot{G}_2(0) = 1, \quad \text{(A13)}$$

for which the Laplace transforms are given by

$$\tilde{G}_n(z) = \frac{z^{2-n}}{z^2 + \omega_0^2 + z \tilde{\gamma}(z)}, \quad \text{(A14)}$$

with $n = 1, 2$ and where $\tilde{\gamma}$ is the Laplace transform of the damping kernel. Note that, given these conditions, one can prove that $G_1(t) = \dot{G}_2(t)$.

Inserting this solution into Eq. (A3), we obtain the following equation for the field operator given in Ref. [43]:

$$\Box\hat{\phi} + \frac{4\pi\eta e^2}{m}\int_{t_0}^{t} d\tau G_1(t-\tau)\dot{\hat{\phi}}(x,\tau)$$

$$= 4\pi\eta e[\dot{G}_1(t-t_0)\hat{r}_x(t_0) + G_1(t-t_0)\dot{\hat{r}}_x(t_0)$$

$$+ \frac{1}{m}\int_{t_0}^{t} d\tau G_1(t-\tau)\hat{F}_x(\tau-t_0)], \tag{A15}$$

which can also be rewritten as in Ref. [39], finally arriving at Eq. (3).

APPENDIX B: STEADY INITIAL CONDITIONS' CONTRIBUTION FROM COMPLEX ANALYSIS

This section is devoted to show how Eq. (13) is obtained for the long-time operator of the initial conditions' contribution.

We begin by considering Eq. (12) as a starting point. Following Ref. [40], we can write the desired Green function as

$$\tilde{\mathfrak{G}}_{\text{Ret}}(x,x',s) = \frac{\Phi_s^<(x_<)\Phi_s^>(x_>)}{W_s(x')}, \tag{B1}$$

where $x_>$ ($x_<$) is the bigger (smaller) value between x and x', $\Phi_s^<$ ($\Phi_s^>$) is the homogeneous solution associated to Eq. (10) satisfying only the boundary condition on the left (right) limit of the variable value's interval and $W_s(x) = \Phi_s^<(x)\frac{d\Phi_s^>}{dx} - \frac{d\Phi_s^<}{dx}\Phi_s^>(x)$ is the Wronskian of the solutions (which has to be independent of x).

At this point, we go a step further and give a demonstration for the long-time limit of this operator for a completely general case. We will consider the case of $N-1$ interfaces separating different materials characterized by refractive index n_α, with $\alpha = 1, 2, ..., N-1, N$ for a given positive integer N. However, beyond the regions imposed by the $N-1$ interfaces, the construction of the Green function $\tilde{\mathfrak{G}}_{\text{Ret}}(x,x',s)$ always contains a jolt associated to the relation between the field and source points (x, x', respectively). Given that the field point x is chosen in

the m region, the spatial integral over the source point x' in Eq. (12) separates in $N+1$ integrals, $N-1$ integrals associated to each region not containing x, and two more associated to the splitting of the m region due to the jolt in the relation between x and x'. Therefore,

$$\int dx' \rightarrow \left[\int_{-\infty}^{x_1} + \int_{x_1}^{x_2} + \cdots + \int_{x_{m-2}}^{x_{m-1}} + \int_{x_{m-1}}^{x} + \int_{x}^{x_m}\right.$$

$$\left. + \int_{x_m}^{x_{m+1}} + \cdots + \int_{x_{N-1}}^{x_N} + \int_{x_N}^{+\infty}\right] dx', \tag{B2}$$

where $\{x_i\}$ are the positions of the N interfaces.

Then, considering the construction of the Green function through Eq. (B1), the first m integrals in Eq. (B2) are of the form

$$\int_{x_{\alpha-1}}^{x_\alpha} dx'\tilde{\mathfrak{G}}_{\text{Ret}}(x,x',s)e^{ikx'} = \frac{\Phi_s^>(x)}{W_s}\int_{x_{\alpha-1}}^{x_\alpha} dx'\Phi_s^<(x')e^{ikx'}, \tag{B3}$$

for every $\alpha = 1, ..., m$ by considering $x_0 = -\infty$ for the lower limit in the integral with $\alpha = 1$ and $x_m = x$ in the upper limit of the last integral ($\alpha = m$).

The last $N - m + 2$ integrals in Eq. (B2) are of the form

$$\int_{x_{\alpha-1}}^{x_\alpha} dx'\tilde{\mathfrak{G}}_{\text{Ret}}(x,x',s)e^{ikx'} = \frac{\Phi_s^<(x)}{W_s}\int_{x_{\alpha-1}}^{x_\alpha} dx'\Phi_s^>(x')e^{ikx'}, \tag{B4}$$

with $\alpha = m, ..., N + 1$, by considering $x_{m-1} = x$ for the first integral and $x_{N+1} = +\infty$ for the upper limit of the last integral.

By considering that the homogeneous solutions in each region can be, in general (except for the first and last regions as we will see below), written as the superposition of waves traveling to the right and to the left, i.e., $\Phi_s^\lessgtr(x) = \sum_{j=+,-} K_s^\lessgtr(j,\alpha)e^{jsn_\alpha(s)x}$, where $K_s^\lessgtr(j,\alpha)$ are the coefficients resulting from the appropriate boundary conditions, we therefore have

$$\int dx'\tilde{\mathfrak{G}}_{\text{Ret}}(x,x',s)e^{ikx'} = \frac{\Phi_s^>(x)}{W_s}\sum_{j=+,-}j\left[\sum_{\alpha=1}^{m-1}\frac{K_s^<(j,\alpha)}{(sn_\alpha(s)+jik)}\left(e^{(jsn_\alpha(s)+ik)x_\alpha} - e^{(jsn_\alpha(s)+ik)x_{\alpha-1}}\right)\right.$$

$$+ \frac{K_s^<(j,m)}{(sn_m(s)+jik)}\left(e^{(jsn_m(s)+ik)x} - e^{(jsn_m(s)+ik)x_{m-1}}\right)\Bigg]$$

$$+ \frac{\Phi_s^<(x)}{W_s}\sum_{j=+,-}j\left[\frac{K_s^>(j,m)}{(sn_m(s)+jik)}\left(e^{(jsn_m(s)+ik)x_m} - e^{(jsn_m(s)+ik)x}\right)\right.$$

$$+ \sum_{\alpha=m+1}^{N+1}\frac{K_s^>(j,\alpha)}{(sn_\alpha(s)+jik)}\left(e^{(jsn_\alpha(s)+ik)x_\alpha} - e^{(jsn_\alpha(s)+ik)x_{\alpha-1}}\right)\Bigg]. \tag{B5}$$

After introducing this into Eq. (12), the next step is to perform the complex integration over s. For this purpose, the residue theorem has to be employed considering a complex contour, including the line in the complex space $l - i\Omega$, with $\Omega \in \mathbb{R}$ and closing to the left. It is important to consider that, by definition, the Laplace integration is such that the poles lie at the left of the line $l - i\Omega$. Therefore, solving the full time evolution of the contribution to the field operator of Eq. (12) implies knowing the poles' configuration of all of the terms of Eq. (B5). However, not all of the poles will contribute to the long-time limit ($t_0 \to -\infty$). In fact, as it was pointed out in Ref. [21], the steady situation will be determined by the poles with zero real part, i.e., purely imaginary poles. Nevertheless, the causality property implies that the poles of the Laplace transform of the retarded Green function $\tilde{\mathfrak{G}}_{\mathrm{Ret}}(x, x', s)$ have a negative real part. The only pole with zero real part is the one at $s = 0$ provided by the Wronskian. Therefore, the long-time limit will be defined by the poles resulting from the spatial integration. These poles are provided by $sn_\alpha(s) + jik$ in each region.

In the first place, it turns out that if a given region α is filled with a dissipative material $[n_\alpha(s) \neq 1]$, then $sn_\alpha(s) + jik$ has no poles with zero real part. Therefore, those terms associated to filled regions will not contribute to the steady state; i.e., the result vanishes in the long-time limit.

Thus, in principle, only regions filled with dissipationless materials may contribute to the steady situation. In particular, we have the vacuum regions, where $n_\alpha \equiv 1$, and the denominator reads $s + jik$, providing its roots as a candidate to the poles ($s = -jik$). However, if an intermediate region of finite length is considered ($\alpha = 2, \ldots, N$), the corresponding term after spatial integration contains a factor $e^{(js+ik)A} - e^{(js+ik)B}$, with A, B the corresponding limit of integration of the given term. This factor cancels out when $s = -jik$ in such a way that the limit of $(e^{(js+ik)A} - e^{(js+ik)B})/(s + jik)$ goes to zero (by l'Hôpital's rule), giving that there is no pole at $s = -jik$ for these terms. Clearly, this means that these terms do not contribute to the steady state.

Considering the analyzed cases at this point, the last possibility is the case where the dissipationless regions are not intermediate, having α equal to 1 or $N + 1$. Given the convergence of $\tilde{\mathfrak{G}}_{\mathrm{Ret}}(x, x', s)$ for great values of x, x', for $\alpha = 1$ we have $K_s^<(-, 1) \equiv 0$, while for $\alpha = N + 1$ we have $K_s^>(+, N + 1) \equiv 0$. This means that in these regions, outgoing waves are the only solution there. At the same time, for $\alpha = 1$, given that $x_0 = -\infty$ and that $s = l - i\Omega$ (with $l > 0$), then $e^{(s+ik)x_0} \to 0$ in the corresponding term in Eq. (B5). Analogously, for $\alpha = N + 1$, given that $x_{N+1} = +\infty$ and that $s = l - i\Omega$ (with $l > 0$), we have $e^{(-s+ik)x_{N+1}} \to 0$.

All in all, for dissipationless regions with $\alpha = 1, N + 1$, we can write the Laplace integration of Eq. (B5) as

$$\int_{l-i\infty}^{l+i\infty} \frac{ds}{2\pi i} e^{s(t-t_0)}(s - i\omega_k) \int dx' \tilde{\mathfrak{G}}_{\mathrm{Ret}}(x, x', s) e^{ikx'}$$
$$= \int_{l-i\infty}^{l+i\infty} \frac{ds}{2\pi i} e^{s(t-t_0)}(s - i\omega_k) \Phi_s^>(x) \frac{K_s^<(+, 1)}{W_s} \frac{e^{(s+ik)x_1}}{(s + ik)}$$
$$+ \int_{l-i\infty}^{l+i\infty} \frac{ds}{2\pi i} e^{s(t-t_0)}(s - i\omega_k) \Phi_s^<(x) \frac{K_s^>(-, N+1)}{W_s} \frac{e^{(-s+ik)x_N}}{(s - ik)} + [\text{terms of intermediate regions}]. \quad (B6)$$

One last simplification can be done related to the explicit calculation of the Wronskian. Given that it is independent of the spatial coordinate, the Wronskian can be calculated in every region, giving equalities between the different coefficients in each region. If we calculate it in the first region ($\alpha = 1$) by considering $\Phi_s^<(x') = K_s^<(+, 1)e^{sx'}$ and $\Phi_s^>(x') = e^{-sx'} + K_s^<(+, 1)e^{sx'}$ for each homogeneous solution, we obtain $W_s = -2sK_s^<(+, 1)$. Analogously, calculating in the last region ($\alpha = N + 1$), we obtain $W_s = -2sK_s^>(-, N+1)$. Then, using the first expression of the Wronskian for the first term in the rhs of Eq. (B6), and the second expression for the second term of the same equation, we can write

$$\int_{l-i\infty}^{l+i\infty} \frac{ds}{2\pi i} e^{s(t-t_0)}(s - i\omega_k) \int dx' \tilde{\mathfrak{G}}_{\mathrm{Ret}}(x, x', s) e^{ikx'} = -\int_{l-i\infty}^{l+i\infty} \frac{ds}{2\pi i} e^{s(t-t_0)}(s - i\omega_k) \Phi_s^>(x) \frac{e^{(s+ik)x_1}}{2s(s + ik)}$$
$$- \int_{l-i\infty}^{l+i\infty} \frac{ds}{2\pi i} e^{s(t-t_0)}(s - i\omega_k) \Phi_s^<(x) \frac{e^{(-s+ik)x_N}}{2s(s - ik)}$$
$$+ [\text{terms of intermediate regions}]. \quad (B7)$$

Taking into account that the long-time limit of this integral will be dominated by the poles with zero real parts, each of the first two terms on the rhs of the last equation can be written as the sum of the residue at $s = 0$, $s = \pm ik$, and the residue that

will not contribute to the steady situation. It is worth noting that the pole at $s = 0$ makes the parameter l that defines the Laplace antitransform necessarily positive. Thus, the new poles with zero real part resulting from the spatial integration are always located at the left of the line $l + i\Omega$ and then inside the contour of integration, as is needed.

Now, as we mentioned before, in general (except for the first and last regions) the homogeneous solutions in the region α are given by $\Phi_s^{\lessgtr}(x) = \sum_{j=+,-} K_s^{\lessgtr}(j, \alpha) e^{jsn_\alpha(s)x}$, and then each exponential contributes to different causality restrictions for each term.

The first integral splits into two, each one containing the exponential factors $e^{s(t-t_0+jn_\alpha(s)x+x_1)}$ for each j. The exponents define the convergence of the integral and in which direction the contour has to be closed to use the residue theorem. It is clear that closing to the right will make the integral equal to zero due to the lack of poles inside the contour. Therefore, the integral is different from zero when the contours close to the left. To obtain the restriction in

terms of $t, t_0, x,$ and x_1 resulting in each integral, as in Refs. [39,49], we analyze the case of large $|s|$. In that case, $n_\alpha(s) \to 1$ and the exponents tends to $s(t - t_0 + jx + x_1)$. It turns out that the contour must be closed to the left when $t - t_0 + jx + x_1 > 0$. Although these results are obtained for large $|s|$, they are valid for the entire integral, and express the natural mathematical manifestation of causality and the retardation effects that take place in the transient stage.

For the second integral the situation is analogous, and the restrictions are $t - t_0 + jx - x_N > 0$. It is clear that the terms of intermediate regions in Eq. (B7) also give causality restrictions but, as those terms will not contribute to the steady situation, we do not analyze them.

The generalization of this deduction for the first and last regions ($\alpha = 1$ or $N + 1$, respectively) is straightforward. However, for the demonstration we follow the general situation of intermediate regions.

Therefore, we have

$$
\int_{l-i\infty}^{l+i\infty} \frac{ds}{2\pi i} e^{s(t-t_0)}(s - i\omega_k) \int dx' \tilde{\mathfrak{G}}_{\text{Ret}}(x, x', s) e^{ikx'}
$$

$$
= -\sum_{j=+,-} \Theta(t - t_0 + jx + x_1) \left(\text{Res}\left[e^{s(t-t_0)}(s - i\omega_k) K_s^>(j, \alpha) e^{jsn_\alpha(s)x} \frac{e^{(s+ik)x_1}}{2s(s + ik)}, 0 \right] \right.
$$

$$
+ \text{Res}\left[e^{s(t-t_0)}(s - i\omega_k) K_s^>(j, \alpha) e^{jsn_\alpha(s)x} \frac{e^{(s+ik)x_1}}{2s(s + ik)}, -ik \right] \right) + [\text{residue with vanishing long-time limit}]
$$

$$
- \sum_{j=+,-} \Theta(t - t_0 + jx - x_N) \left(\text{Res}\left[e^{s(t-t_0)}(s - i\omega_k) K_s^<(j, \alpha) e^{jsn_\alpha(s)x} \frac{e^{(-s+ik)x_N}}{2s(s - ik)}, 0 \right] \right.
$$

$$
+ \text{Res}\left[e^{s(t-t_0)}(s - i\omega_k) K_s^<(j, \alpha) e^{jsn_\alpha(s)x} \frac{e^{(-s+ik)x_N}}{2s(s - ik)}, ik \right] \right)
$$

$$
+ [\text{residue with vanishing long-time limit}] + [\text{residue of terms of intermediate regions}]. \tag{B8}
$$

It turns out that an explicit calculation of the poles at $s = 0$ results in time- and space-independent quantities [after assuming that $K_0^<(+, \alpha) = 1 = K_0^>(-, \alpha)$, while $K_0^<(-, \alpha) = 0 = K_0^>(+, \alpha)$, as it happens in general for the zero frequency solutions due to the scattering properties of the coefficients for every α]. The calculation of the other residue at $s = \pm ik$ is straightforward, giving

$$
\int_{l-i\infty}^{l+i\infty} \frac{ds}{2\pi i} e^{s(t-t_0)}(s - i\omega_k) \int dx' \tilde{\mathfrak{G}}_{\text{Ret}}(x, x', s) e^{ikx'}
$$

$$
= \frac{1}{2} \frac{\omega_k}{k} [\Theta(t - t_0 + x + x_1) e^{ikx_1} - \Theta(t - t_0 - x - x_N) e^{ikx_N}]
$$

$$
- \frac{1}{2} \sum_{j=+,-} \Theta(t - t_0 + jx + x_1) e^{-ik(t-t_0)} \left(1 + \frac{\omega_k}{k} \right) K_{-ik}^>(j, \alpha) e^{-jikn_\alpha(-ik)x}
$$

$$
- \frac{1}{2} \sum_{j=+,-} \Theta(t - t_0 + jx - x_N) e^{ik(t-t_0)} \left(1 - \frac{\omega_k}{k} \right) K_{ik}^<(j, \alpha) e^{jikn_\alpha(ik)x}
$$

$$
+ [\text{residue with vanishing long-time limit}] + [\text{residue of terms of intermediate regions}], \tag{B9}
$$

where the first two terms correspond to the poles at $s = 0$ of both integrals.

The crucial point is that, when considering the long-time limit ($t_0 \to -\infty$), all the Heaviside functions present go to 1. Thus, on one hand, the first two terms (associated to the pole at $s = 0$) become time and space independent and, as we shall see, they will not contribute to the calculations relative to the energy-momentum tensor expectation value. On the other hand, the sums over j in the third and fourth terms add up to the homogeneous solutions $\Phi^{\lessgtr}$ again. The rest terms vanish in the long-time limit. Ignoring the oscillatory dependence on t_0 of the third and fourth terms and that $\frac{\omega_k}{k} \equiv sgn(k)$, we can write

$$
\int_{l-i\infty}^{l+i\infty} \frac{ds}{2\pi i} e^{s(t-t_0)}(s - i\omega_k) \int dx' \tilde{\mathfrak{G}}_{\text{Ret}}(x, x', s) e^{ikx'}
$$
$$
\to \frac{1}{2} sgn(k) [e^{ikx_1} - e^{ikx_N}] - \frac{1}{2} e^{-ik(t-t_0)} \left(1 + \frac{\omega_k}{k}\right) \Phi^{>}_{-ik}(x)
$$
$$
- \frac{1}{2} e^{ik(t-t_0)} \left(1 - \frac{\omega_k}{k}\right) \Phi^{<}_{ik}(x). \tag{B10}
$$

Considering that $1 \pm \frac{\omega_k}{k} \equiv 2\Theta(\pm k)$ and assuming that the homogeneous solution satisfies $(\Phi_{-ik}(x))^* = \Phi_{ik}(x)$ [or analogously, that $\Phi_s(x)$ is real for real s], inserting this into Eq. (12) and taking the long-time limit ($t_0 \to -\infty$) by considering all the properties of each term mentioned before, we finally obtain Eq. (13).

APPENDIX C: STEADY VOLUME ELEMENTS' CONTRIBUTION FROM COMPLEX ANALYSIS

In this section, we follow the same approach as in Appendix B, but to address the long-time limit of the field operator for the volume elements' contribution given in Eq. (16).

Therefore, our starting point is Eq. (15). Writing the Green function in terms of its Laplace transform and making the substitution $\tau = t' - t_0$, the contribution to the field operator reads

$$
\hat{\phi}_{\text{A}}^{(+)}(x, t) = -\int dx' \frac{4\pi \eta e C(x')}{\sqrt{2m\omega_0}} \hat{b}_{0,x'}(t_0)
$$
$$
\times \int_{l-i\infty}^{l+i\infty} \frac{ds}{2\pi i} \tilde{\mathfrak{G}}_{\text{Ret}}(x, x', s)
$$
$$
\times \int_0^{t-t_0} d\tau e^{s(t-t_0-\tau)} (\dot{G}_1(\tau) - i\omega_0 \dot{G}_2(\tau)). \tag{C1}
$$

It is worth noting that the integral over τ is a convolution. Its Laplace transform is a product of the Laplace transform of e^{st} and $\dot{G}_1(t) - i\omega_0 \dot{G}_2(t)$. Therefore, the convolution can be written as an antitransform too, and the contribution to the field operator can be written as

$$
\hat{\phi}_{\text{A}}^{(+)}(x, t) = -\int dx' \frac{4\pi \eta e C(x')}{\sqrt{2m\omega_0}} \hat{b}_{0,x'}(t_0)
$$
$$
\times \int_{l-i\infty}^{l+i\infty} \frac{ds}{2\pi i} \tilde{\mathfrak{G}}_{\text{Ret}}(x, x', s)
$$
$$
\times \int_{l'-i\infty}^{l'+i\infty} \frac{dz}{2\pi i} \frac{e^{z(t-t_0)}}{(z-s)} (z(z - i\omega_0) \tilde{G}_2(z) - 1), \tag{C2}
$$

where $l' > l$ to have a well-defined Laplace transform of e^{st}, ensuring the convergence of the integral and the causality property at once.

This expression allow us to write the convolution in terms of the residue of the poles of the integrand. Due to the causality property, $\tilde{G}_2$ has poles with nonpositive real parts. Then, as $l' > l$, the integral over z can be written as the residuum at $z = s$ [associated to $1/(z - s)$] plus all the residue with nonpositive real parts that will give terms that vanish at the long-time limit ($t_0 \to -\infty$). Thus, we obtain

$$
\hat{\phi}_{\text{A}}^{(+)}(x, t) = -\int dx' \frac{4\pi \eta e C(x')}{\sqrt{2m\omega_0}} \hat{b}_{0,x'}(t_0)
$$
$$
\times \int_{l-i\infty}^{l+i\infty} \frac{ds}{2\pi i} e^{s(t-t_0)} \tilde{\mathfrak{G}}_{\text{Ret}}(x, x', s)
$$
$$
\times (s(s - i\omega_0) \tilde{G}_2(s) - 1)
$$
$$
+ [\text{terms with vanishing long-time limit}]. \tag{C3}
$$

Again, the causality of the retarded Green functions $\mathfrak{G}_{\text{Ret}}, G_2$ gives that the integrand in the integral over s has poles with nonpositive real parts. The only pole having zero real part is the one at $s = 0$ provided by the Wronskian contained in $\tilde{\mathfrak{G}}_{\text{Ret}}$, which is the only one that will not vanish at the long-time limit. It should be noted that only the second term of $s(s - i\omega_0) \tilde{G}_2(s) - 1$ will have $s = 0$ as a pole, since for the first term the denominator that provides the pole cancels with s in the numerator.

Considering the expression of the Green function of Eq. (B1) and the general form for the homogeneous solutions $\Phi^{\lessgtr}_s$, the Green function can be written as

$$
\mathfrak{G}_{\text{Ret}}(x, x', s) = \frac{1}{W_s} \sum_{j,k=+,-} [\Theta(x-x') K^<_s(j, \alpha) K^>_s(k, \beta)
$$
$$
+ \Theta(x'-x) K^<_s(k, \beta) K^>_s(j, \alpha)] e^{s(jn_\alpha(s)x' + kn_\beta(s)x)}, \tag{C4}
$$

for arbitrary x in the α region and x' in the β region.

Now, by taking into account the properties of the coefficients $K^{\lessgtr}_s$ at $s = 0$ and the analysis associated to the causality of the transient stage, we can calculate the residuum at $s = 0$, obtaining for every x

$$\hat{\phi}_{\mathrm{A}}^{(+)}(x,t) = -\int dx' \frac{4\pi\eta e C(x')}{\sqrt{2m\omega_0}} \hat{b}_{0,x'}(t_0)$$
$$\times \frac{\Theta(t-t_0+x'-x)\Theta(t-t_0-x'+x)}{2}$$
$$+ [\text{terms with vanishing long-time limit}].$$
$$\text{(C5)}$$

Thus, taking the long-time limit $(t_0 \to -\infty)$, we easily obtain Eq. (16).

APPENDIX D: STEADY BATHS' CONTRIBUTION FROM COMPLEX ANALYSIS

The present section shows how Eq. (18) can be obtained by taking Eq. (17) as the starting point. The approach will be the same as in the last two sections for the other two contributions, by separating the steady contribution by analyzing the poles configuration.

The first step consists of writing the field's retarded Green function as a Laplace antitransform, rewriting Eq. (17) as

$$\hat{\phi}_{\mathrm{B}}(x,t) = -\int dx' 4\pi\eta e C(x') \int_{l-i\infty}^{l+i\infty} \frac{ds}{2\pi i} \tilde{\mathfrak{G}}_{\mathrm{Ret}}(x,x',s)$$
$$\times \int_{t_0}^{t} dt' e^{s(t-t')} \int_{t_0}^{t'} d\tau \dot{G}_2(t'-\tau) \frac{\hat{F}_{x'}(\tau-t_0)}{m}.$$
$$\text{(D1)}$$

The resulting integrals over t' and τ are basically a double convolution and we can write it in terms of its Laplace transform:

$$\int_{t_0}^{t} dt' e^{s(t-t')} \int_{t_0}^{t'} d\tau \dot{G}_2(t'-\tau) \frac{\hat{F}_{x'}(\tau-t_0)}{m}$$
$$= \int_{l'-i\infty}^{l'+i\infty} \frac{dz}{2\pi i} e^{z(t-t_0)} \frac{z}{(z-s)} \tilde{G}_2(z) \frac{\hat{\tilde{F}}_{x'}(z)}{m}, \quad \text{(D2)}$$

where, as for the volume elements' contribution, we have to take $l' > l$ in order to verify the convergence requirement of the function e^{st}. Given this, we can rewrite the double convolution in terms of its residue associated to the poles of the integrand in the rhs of the last expression. These are the poles at $z = s$, the poles provided by the Laplace transform of the stochastic force operator and the poles associated to $\tilde{G}_2$, which gives terms that go to zero in the long-time limit:

$$\int_{t_0}^{t} dt' e^{s(t-t')} \int_{t_0}^{t'} d\tau \dot{G}_2(t'-\tau) \frac{\hat{F}_{x'}(\tau-t_0)}{m}$$
$$= e^{s(t-t_0)} s\tilde{G}_2(s) \frac{\hat{\tilde{F}}_{x'}(s)}{m} + [\text{residue of the poles of } \hat{\tilde{F}}_{x'}]$$
$$+ [\text{terms with vanishing long-time limit}]. \quad \text{(D3)}$$

Now, by considering Eq. (A7), the Laplace transform of the stochastic force operator is given by

$$\hat{\tilde{F}}_{x'}(z) = \sum_n \frac{\lambda_n}{\sqrt{2m_n\omega_n}} \left(\frac{1}{(z+i\omega_n)} \hat{b}_{n,x'}(t_0) + \frac{1}{(z-i\omega_n)} \hat{b}_{n,x'}^\dagger(t_0) \right), \quad \text{(D4)}$$

which presents poles at $z = \pm i\omega_n$ for each term, and therefore the second term of the rhs of Eq. (D3) can be written as

$$[\text{Residue of the poles of } \hat{\tilde{F}}_{x'}] = \mathrm{Res}\left[e^{z(t-t_0)} \frac{z\tilde{G}_2(z)}{(z-s)} \frac{\hat{\tilde{F}}_{x'}(z)}{m}, -i\omega_n \right] + \mathrm{Res}\left[e^{z(t-t_0)} \frac{z\tilde{G}_2(z)}{(z-s)} \frac{\hat{\tilde{F}}_{x'}(z)}{m}, i\omega_n \right]$$
$$= \frac{1}{m} \sum_n \frac{\lambda_n i\omega_n}{\sqrt{2m_n\omega_n}} \left[\frac{\tilde{G}_2(-i\omega_n)}{(s+i\omega_n)} e^{-i\omega_n(t-t_0)} \hat{b}_{n,x'}(t_0) - \frac{\tilde{G}_2(i\omega_n)}{(s-i\omega_n)} e^{i\omega_n(t-t_0)} \hat{b}_{n,x'}^\dagger(t_0) \right]. \quad \text{(D5)}$$

At this point, we can write the last expression as an integral over a continuous frequency ω by introducing Dirac delta functions:

$$[\text{Residue of the poles of } \hat{\tilde{F}}_{x'}] = \int_{-\infty}^{+\infty} \frac{d\omega}{2\pi} e^{-i\omega t} \frac{i\omega\tilde{G}_2(-i\omega)}{m(s+i\omega)} 2\pi \sum_n \frac{\lambda_n}{\sqrt{2m_n\omega_n}} [\delta(\omega-\omega_n) e^{i\omega_n t_0} \hat{b}_{n,x'}(t_0) + \delta(\omega+\omega_n) e^{-i\omega_n t_0} \hat{b}_{n,x'}^\dagger(t_0)]$$
$$= \int_{-\infty}^{+\infty} \frac{d\omega}{2\pi} e^{-i\omega t} \frac{i\omega\overline{G}_2(\omega)\hat{\tilde{F}}_{x'}(\omega)}{(s+i\omega)} \frac{1}{m}, \quad \text{(D6)}$$

where we have used that $\tilde{f}(-i\omega) = \overline{f}(\omega)$ for every causal function as the retarded Green function G_2, $\overline{f}(\omega)$ being the Fourier transform of the given causal function $f(t)$ (which also admits a Laplace transform). Moreover, $\hat{\overline{F}}_{x'}(\omega)$ turns out to be the Fourier transform of the stochastic force operator, considering the time variable in Eq. (A7) extended to the entire real domain.

Considering the last expression and replacing Eq. (D3) in the field's contribution, we have

$$\hat{\phi}_{\mathrm{B}}(x,t) = -\int dx' 4\pi\eta eC(x') \int_{l-i\infty}^{l+i\infty} \frac{ds}{2\pi i} e^{s(t-t_0)} s\tilde{G}_2(s)\tilde{\mathfrak{G}}_{\mathrm{Ret}}(x,x',s)\frac{\hat{\tilde{F}}_{x'}(s)}{m}$$

$$-\int dx' 4\pi\eta eC(x') \int_{-\infty}^{+\infty} \frac{d\omega}{2\pi} e^{-i\omega t} i\omega\overline{G}_2(\omega)\frac{\hat{\overline{F}}_{x'}(\omega)}{m} \int_{l-i\infty}^{l+i\infty} \frac{ds}{2\pi i} \frac{\tilde{\mathfrak{G}}_{\mathrm{Ret}}(x,x',s)}{(s+i\omega)}$$

$$+ \,[\text{terms with vanishing long-time limit}]. \tag{D7}$$

Nevertheless, the integral over s of the second term of the last equation can be worked out further. Considering it as a limit of a Laplace antitransform, we have

$$\int_{l-i\infty}^{l+i\infty} \frac{ds}{2\pi i} \frac{\tilde{\mathfrak{G}}_{\mathrm{Ret}}(x,x',s)}{(s+i\omega)} = \lim_{t'\to t_0} \int_{l-i\infty}^{l+i\infty} \frac{ds}{2\pi i} e^{s(t'-t_0)} \frac{\tilde{\mathfrak{G}}_{\mathrm{Ret}}(x,x',s)}{(s+i\omega)}, \tag{D8}$$

where the rhs results in the antitransform of a convolution of the functions associated to the transforms $\tilde{\mathfrak{G}}_{\mathrm{Ret}}(x,x',s)$ and $1/(s+i\omega)$. Therefore, we can write the convolution and take the limit on the integral, which clearly vanishes due to the fact that both integration limits become the same:

$$\int_{l-i\infty}^{l+i\infty} \frac{ds}{2\pi i} \frac{\tilde{\mathfrak{G}}_{\mathrm{Ret}}(x,x',s)}{(s+i\omega)} = \lim_{t'\to t_0} \int_{t_0}^{t'} d\tau \mathfrak{G}_{\mathrm{Ret}}(x,x',t'-\tau)e^{-i\omega(\tau-t_0)} \equiv 0. \tag{D9}$$

Hence, the contribution of the baths to the field operator reads

$$\hat{\phi}_{\mathrm{B}}(x,t) = -\int dx' 4\pi\eta eC(x') \int_{l-i\infty}^{l+i\infty} \frac{ds}{2\pi i} e^{s(t-t_0)} s\tilde{G}_2(s)\tilde{\mathfrak{G}}_{\mathrm{Ret}}(x,x',s)\frac{\hat{\tilde{F}}_{x'}(s)}{m} + [\text{terms with vanishing long-time limit}]. \tag{D10}$$

Again, for the Laplace antitransform of the first term in the rhs, we can proceed as before, and write it in terms of the residue of the integrand. This time, we have the poles of $\hat{\tilde{F}}_{x'}$ (located at $s = \pm i\omega_n$) and the ones with negative real part provided by $\tilde{G}_2$ and $\tilde{\mathfrak{G}}_{\mathrm{Ret}}$, but excluding the pole at $s = 0$ due to the factor s present in the integrand, which prevents it. As we mentioned before, the poles with negative real part will result in terms that vanish at the long-time limit.

We can proceed as before to write the terms in the last expression as an integral over ω, finally obtaining for the contribution of the baths to the field operator

$$\hat{\phi}_{\mathrm{B}}(x,t) = -\int dx' \frac{4\pi\eta eC(x')}{m} \int_{-\infty}^{+\infty} \frac{d\omega}{2\pi} e^{-i\omega t} (-i\omega) \frac{\overline{G}_2(\omega)}{W_{-i\omega}} \sum_{j,k=+,-} \Theta(t-t_0+jx'+kx)(\Theta(x-x')K_{-i\omega}^{<}(j,\alpha)$$

$$\times K_{-i\omega}^{>}(k,\beta) + \Theta(x'-x)K_{-i\omega}^{<}(k,\beta)K_{-i\omega}^{>}(j,\alpha))e^{-i\omega(jn_\alpha(-i\omega)x'+kn_\beta(-i\omega)x)}\hat{\overline{F}}_{x'}(\omega)$$

$$+ \,[\text{terms with vanishing long-time limit}]. \tag{D11}$$

By taking the long-time limit of this expression, it is straightforward to finally obtain Eq. (18).

APPENDIX E: GREEN FUNCTION FOR TWO FINITE WIDTH PLATES CONFIGURATION

To calculate each contribution to the total force through Eqs. (28) and (44), we need the modified modes $\Phi_s^{\lessgtr}$ (and consequently, the Green function) obtained from Eq. (10) for the specific problem. This section is devoted to giving those expressions.

For the present problem of two different slabs of width d separated by a distance a, the modified modes are given by

$$\Phi_s^<(x) = \begin{cases} T_s^< e^{sx}, & \text{for } x < -d - \frac{a}{2} \\ E_s^< e^{sn_L x} + F_s^< e^{-sn_L x}, & \text{for } -d - \frac{a}{2} < x < -\frac{a}{2} \\ C_s^< e^{sx} + D_s^< e^{-sx}, & \text{for } -\frac{a}{2} < x < \frac{a}{2} \\ A_s^< e^{sn_R x} + B_s^< e^{-sn_R x}, & \text{for } \frac{a}{2} < x < d + \frac{a}{2} \\ e^{sx} + R_s^< e^{-sx}, & \text{for } d + \frac{a}{2} < x \end{cases} \tag{E1}$$

$$\Phi_s^>(x) = \begin{cases} e^{-sx} + R_s^> e^{sx}, & \text{for } x < -d - \frac{a}{2} \\ A_s^> e^{-sn_L x} + B_s^> e^{sn_L x}, & \text{for } -d - \frac{a}{2} < x < -\frac{a}{2} \\ C_s^> e^{-sx} + D_s^> e^{sx}, & \text{for } -\frac{a}{2} < x < \frac{a}{2} \\ E_s^> e^{-sn_R x} + F_s^> e^{sn_R x}, & \text{for } \frac{a}{2} < x < d + \frac{a}{2} \\ T_s^> e^{-sx}, & \text{for } d + \frac{a}{2} < x \end{cases}, \tag{E2}$$

where the coefficients for each homogeneous solution can be found in Ref. [47]:

$$R_s^> = \left[r_L + \frac{r_R t_L^2 e^{-2sa}}{1 - r_L r_R e^{-2sa}} \right] e^{s(a+2d)}, \qquad T_s^> = \frac{t_R t_L e^{2sd}}{1 - r_L r_R e^{-2sa}}, \tag{E3}$$

$$C_s^> = e^{-sd} \frac{T_s^>}{t_R}, \qquad D_s^> = e^{-s(a+d)} \frac{r_R}{t_R} T_s^>, \tag{E4}$$

$$A_s^> = \frac{(n_L + 1)}{2n_L} e^{s(1-n_L)(\frac{a}{2}+d)} \left[1 - r_{n_L} R_s^> e^{-s(a+2d)}\right], \qquad B_s^> = \frac{(n_L + 1)}{2n_L} e^{s(1+n_L)(\frac{a}{2}+d)} \left[R_s^> e^{-s(a+2d)} - r_{n_L}\right], \tag{E5}$$

$$E_s^> = \frac{(n_R + 1)}{2n_R} e^{s(n_R-1)(\frac{a}{2}+d)} T_s^>, \qquad F_s^> = \frac{(n_R - 1)}{2n_R} e^{-s(n_R+1)(\frac{a}{2}+d)} T_s^>, \tag{E6}$$

where we have given all the coefficients in terms of the reflection and transmission coefficients of the two-plates configuration ($R_s^>$ and $T_s^>$). Moreover, $r_{L,R}$ and $t_{L,R}$ are the reflection and transmission coefficients for the left and right plates, respectively:

$$r_i = \frac{r_{n_i}(1 - e^{-2sn_i d})}{(1 - r_{n_i}^2 e^{-2sn_i d})}, \qquad t_i = \frac{4n_i}{(n_i + 1)^2} \frac{e^{-sn_i d}}{(1 - r_{n_i}^2 e^{-2sn_i d})}, \tag{E7}$$

with $r_{n_i} = \frac{1-n_i}{1+n_i}$, the reflection coefficient of a surface of refractive index n_i.

It should be noted that the $<$ coefficients are obtained from the given ones by the interchange of L and R in the expressions. Considering this, it turns out that $T_s^> = T_s^<$, so the superscript for this coefficient can be omitted.

For the given configuration of finite width plates, the boundary conditions on the modes were a continuity of the mode and its spatial derivative at the interfaces between the material slabs and the surrounding vacuum.

On the other hand, to calculate the contribution of the baths to the Casimir force, we also need the Green function (or its Laplace transform) with the field point x in the regions outside and between the plates.

Therefore, taking $x < -d - \frac{a}{2}$, through Eq. (B1), the Laplace transform of the retarded Green function reads

$$\tilde{\mathfrak{G}}_{\mathrm{Ret}}(x,x',s) = -\frac{1}{2s} \begin{cases} e^{sx'}(e^{-sx} + R_s^> e^{sx}), & \text{for } x' < x < -d - \frac{a}{2} \\ (e^{-sx'} + R_s^> e^{sx'})e^{sx}, & \text{for } x < x' < -d - \frac{a}{2} \\ (A_s^> e^{-sn_L x'} + B_s^> e^{sn_L x'})e^{sx}, & \text{for } x < -d - \frac{a}{2} < x' < -\frac{a}{2} \\ (C_s^> e^{-sx'} + D_s^> e^{sx'})e^{sx}, & \text{for } x < -d - \frac{a}{2} < -\frac{a}{2} < x' < \frac{a}{2} \\ (E_s^> e^{-sn_R x'} + F_s^> e^{sn_R x'})e^{sx}, & \text{for } x < -d - \frac{a}{2} < \frac{a}{2} < x' < d + \frac{a}{2} \\ T_s e^{-sx'} e^{sx}, & \text{for } x < -d - \frac{a}{2} < d + \frac{a}{2} < x' \end{cases}$$

(E8)

For the case where $-\frac{a}{2} < x < \frac{a}{2}$, we have

$$\tilde{\mathfrak{G}}_{\mathrm{Ret}}(x,x',s) = -\frac{1}{2s} \begin{cases} T_s e^{sx'} \frac{e^{-sd}}{t_R}(e^{-sx} + r_R e^{-sa}e^{sx}), & \text{for } x' < -d - \frac{a}{2} < -\frac{a}{2} < x < \frac{a}{2} \\ (E_s^< e^{sn_L x'} + F_s^< e^{-sn_L x'})\frac{e^{-sd}}{t_R}(e^{-sx} + r_R e^{-sa}e^{sx}), & \text{for } -d - \frac{a}{2} < x' < -\frac{a}{2} < x < \frac{a}{2} \\ (C_s^< e^{sx'} + D_s^< e^{-sx'})\frac{e^{-sd}}{t_R}(e^{-sx} + r_R e^{-sa}e^{sx}), & \text{for } -\frac{a}{2} < x' < x < \frac{a}{2} \\ (C_s^> e^{-sx'} + D_s^> e^{sx'})\frac{e^{-sd}}{t_L}(e^{sx} + r_L e^{-sa}e^{-sx}), & \text{for } -\frac{a}{2} < x < x' < \frac{a}{2} \\ (E_s^> e^{-sn_R x'} + F_s^> e^{sn_R x'})\frac{e^{-sd}}{t_L}(e^{sx} + r_L e^{-sa}e^{-sx}), & \text{for } -\frac{a}{2} < x < \frac{a}{2} < x' < d + \frac{a}{2} \\ T_s e^{-sx'} \frac{e^{-sd}}{t_L}(e^{sx} + r_L e^{-sa}e^{-sx}), & \text{for } -\frac{a}{2} < x < \frac{a}{2} < d + \frac{a}{2} < x' \end{cases}$$

(E9)

APPENDIX F: LIMIT CASES: DISSIPATIONLESS MATERIAL AND THE LIFSHITZ FORMULA

In this section, we show how the general expressions for the force given in Eqs. (46), (47), and (48) allow us to recover two well-known results for the Casimir force. One of the limit cases is to obtain the Casimir force existing between two slabs of arbitrary thickness made of materials without dissipation. The other one is to obtain the force between two half-spaces (infinite width) of dissipative materials at thermal equilibrium, which is known as the Lifshitz formula.

1. Material without dissipation

The first limit case to verify is the Casimir force between two plates of finite width and different materials without dissipation. Considering the definition of the refractive index in each point of the material, given by Eq. (11), setting dissipation equal to zero implies taking $\bar{\gamma}_{\mathrm{L,R}}(\omega) \equiv 0$, i.e., setting a damping kernel equal to zero for each plate. Therefore, the refractive indices (and consequently the permittivity function) simplify to

$$n_i^2(-i\omega) = \varepsilon_i(\omega) = 1 + \frac{\omega_{\mathrm{Pl},i}^2}{(\omega_{0,i}^2 - \omega^2)}, \qquad (\mathrm{F1})$$

which is a real function of ω.

However, real frequency-dependent permittivities violate the Kramers-Kronig relations and, consequently, are forbidden in order to respect physical causality.

Hence, setting dissipation equal to zero is not enough to obtain a well-defined physical model without dissipation. To finally obtain the correct model, as done in Ref. [25], we have to take the zeroth order of this last expression by setting $\omega = 0$. In this case, we obtain

$$\varepsilon_{i,\mathrm{ND}} = 1 + \frac{\omega_{\mathrm{Pl},i}^2}{\omega_{0,i}^2}, \qquad (\mathrm{F2})$$

which is a real but frequency-independent function. These permittivities verify the Kramers-Kronig relation trivially, since a null imaginary part implies a frequency-independent real part (see Ref. [49]).

All in all, taking the limit case of materials without dissipation is achieved by putting $\bar{\gamma} \equiv 0$ plus taking the zeroth order of the resulting permittivity.

Once we have considered how to take the limit of materials without dissipation, in any case, we have $\mathrm{Im}(n_i) \equiv 0$. This also implies that $(\pm 1)(e^{\pm 2\omega \mathrm{Im}(n_i)d} - 1) \equiv 0$. Therefore, the direct consequence is that the baths' contribution trivially vanishes:

$$F_{\mathrm{C}}^{\mathrm{B}}[a,d,\beta_{\mathrm{B,L}},\beta_{\mathrm{B,R}}]|_{\mathrm{No\ Diss}} \equiv 0. \qquad (\mathrm{F3})$$

Nevertheless, the contribution of the initial conditions to the Casimir force $F_{\mathrm{C}}^{\mathrm{IC}}[a,d,\mathfrak{F}]$ does not vanish. In fact, in this case, it is responsible for the total Casimir force.

Moreover, once the dissipation is suppressed, both identities $|R_{-ik}^>|^2 + |T_{-ik}|^2 = 1$ and $|r_i|^2 + |t_i|^2 = 1$ are also recovered.

Accordingly, the total Casimir force reads

$$F_C[a,d,\mathfrak{F}]|_{\text{No Diss}}$$
$$\equiv F_C^{\text{IC}}[a,d,\mathfrak{F}]|_{\text{No Diss}}$$
$$= \int_0^{+\infty} dk k \mathfrak{F}(k)[2 - |C_{-ik}^>|^2 - |D_{-ik}^>|^2 - |C_{-ik}^<|^2 - |D_{-ik}^<|^2]$$
$$= \int_0^{+\infty} dk k \mathfrak{F}(k)\left[2 - \frac{[|t_L|^2(1+|r_R|^2) + |t_R|^2(1+|r_L|^2)]}{|1 - r_L r_R e^{i2\omega a}|^2}\right].$$
(F4)

For the case of considering the same material for both plates, we have that $r_L = r_R = r$ and $t_L = t_R = t$, and the last expression straightforwardly simplifies to

$$F_C[a,d,\mathfrak{F}]|_{\text{No Diss}}^{\text{1-Mat}} = 2\int_0^{+\infty} dk k \mathfrak{F}(k)\left[1 - \frac{1 - |r|^4}{|1 - r^2 e^{i2\omega a}|^2}\right],$$
(F5)

which is the Casimir force between two finite plates of finite width and the same material without dissipation. If we consider an equilibrium scenario, $\mathfrak{F}(k) = \coth(\frac{\beta_\phi \omega_k}{2})$, and the result is in agreement with the result found in Ref. [42] for a formally equivalent situation.

Moreover, from this point of view for the thermal case, Eq. (F4) is the generalization of the result of Ref. [42] for the case of plates of different materials without dissipation. Nevertheless, the result obtained is an equilibrium situation ensured by the equilibrium state for the field. Considering the result obtained for this limit case, the calculation of the Casimir force between material bodies without dissipation is formally achieved through a quantum field described in a Hilbert space that here we can match to the Hilbert space of a free field. This is in fact what is done in well-known literature to obtain the Casimir force between dielectric

plates without dissipation and in equilibrium (see Refs. [50,51]). In previous works, the result including dissipative materials is often obtained as an extension of this simplified scenario through letting the real and frequency-independent permittivity in this case to be replaced by a complex frequency-dependent function in the final result. This extension procedure in fact works in equilibrium situations because the results including and disregarding dissipation are formally the same. However, as we have seen before, nonequilibrium scenarios critically require the introduction of degrees of freedom for the materials bodies since the extension procedure loses physical sense in successfully introducing nonequilibrium features through a particular state for the field.

2. Lifshitz formula

The other important limit case is the Lifshitz formula. Starting from Eqs. (46) and (47), the formal procedure to recover the well-known Lifshitz result consists of taking the limit of infinite width ($d \to +\infty$) in the expressions and then setting thermal equilibrium between the baths and the field at the initial time [therefore, for the initial conditions' contribution, we will be considering $\mathfrak{F}(k) = \coth(\frac{\beta_\phi \omega_k}{2})$ in this section].

To successfully take the limit of infinite width on the contributions to the total force, from Eq. (E7), we have that $r_i \to r_{n_i}$, $t_i \to 0$ but $|t_i|^2 e^{2\omega \text{Im}(n_i)d} \to \frac{16|n_i|^2}{|n_i+1|^4}$. Therefore, considering the definitions for the different coefficients given in Eqs. (E3) and (E4) and that $1 - |r_{n_i}|^2 = \frac{2\text{Re}(n_i)}{|n_i+1|^2}$, we get each contribution to the force:

$$F_C^{\text{IC}}[a, d \to +\infty, \beta_\phi] = \int_0^{+\infty} dk \coth\left(\frac{\beta_\phi k}{2}\right)[1 + |r_{n_L}|^2],$$
(F6)

$$F_C^B[a, d \to +\infty, \beta_{B,L}, \beta_{B,R}] = \int_0^{+\infty} d\omega \omega \left[\coth\left(\frac{\beta_{B,L}\omega}{2}\right)[1 - |r_{n_L}|^2]\left(1 - \frac{[1 + |r_{n_R}|^2]}{|1 - r_{n_L} r_{n_R} e^{i2\omega a}|^2}\right)\right.$$
$$\left. - \coth\left(\frac{\beta_{B,R}\omega}{2}\right)\frac{[1 - |r_{n_R}|^2][1 + |r_{n_L}|^2]}{|1 - r_{n_L} r_{n_R} e^{i2\omega a}|^2}\right].$$
(F7)

Setting thermal equilibrium between the baths and the field ($\beta_{B,L} = \beta_{B,R} = \beta_\phi \equiv \beta$), we obtain the total force as

$$F_C[a, d \to +\infty, \beta, \beta, \beta] = F_C^{\text{IC}}[a, d \to +\infty, \beta] + F_C^B[a, d \to +\infty, \beta, \beta]$$
$$= 4\int_0^{+\infty} dk k \coth\left(\frac{\beta k}{2}\right)\frac{[|r_{n_R}|^2|r_{n_L}|^2 - \text{Re}(r_{n_L} r_{n_R} e^{i2\omega a})]}{|1 - r_{n_L} r_{n_R} e^{i2\omega a}|^2}$$
$$= -\int_{-\infty}^{+\infty} dk k \coth\left(\frac{\beta k}{2}\right)\text{Re}\left[\frac{r_{n_L}(-ik) r_{n_R}(-ik) e^{i2ka}}{1 - r_{n_L}(-ik) r_{n_R}(-ik) e^{i2ka}}\right],$$
(F8)

where we have used the fact that the integrand is even and stressed the explicit dependence on k related to the definition of the coefficients given in Eq. (E7).

In order to calculate the integral through the residue theorem, we can consider the expansion of the thermal factor in terms of the Matsubara poles:

$$\coth\left(\frac{\beta k}{2}\right) = \frac{2}{\beta}\left[\frac{1}{k} + \sum_{l=1}^{+\infty}\left(\frac{1}{k + i\xi_l} + \frac{1}{k - i\xi_l}\right)\right], \quad (F9)$$

with $\xi_l = \frac{2\pi l}{\beta}$ the Matsubara frequencies.

Assuming that $k\mathrm{Re}\left[\frac{r_{n_L}r_{n_R}e^{i2\omega a}}{1 - r_{n_L}r_{n_R}e^{i2\omega a}}\right]$ has no poles on the upper half of the complex plane, the Cauchy theorem can be applied. If we consider a contour closing in the upper half,

the only poles that contribute to the integral are $i\xi_l$ (the denominator $1/k$ for the first term of the last equation is canceled, giving no pole at $k = 0$). Therefore, we obtain

$$\begin{aligned}&F_C[a, d \to +\infty, \beta, \beta, \beta]\\&= \frac{4\pi}{\beta}\sum_{l=1}^{+\infty}\xi_l\frac{r_{n_L}(\xi_l)r_{n_R}(\xi_l)e^{-2\xi_l a}}{(1 - r_{n_L}(\xi_l)r_{n_R}(\xi_l)e^{-2\xi_l a})},\quad (F10)\end{aligned}$$

where from Eq. (E7) we have that $r_{n_i}(s)$ is real for real s.

The last equation corresponds exactly to the finite-temperature Lifshitz formula for two plates of different dissipative materials (being the generalization of the result found in Ref. [43]).

[1] S. M. Barnett and P. M. Radmore, *Methods in Theoretical Quantum Optics* (Oxford University Press, Oxford, 2002).

[2] E. S. Fradkin, D. M. Gitman, and S. M. Shvartsman, *Quantum Electrodynamics with Unstable Vacuum* (Springer-Verlag, Berlin, 1991).

[3] E. Calzetta and L. B. Hu, Phys. Rev. D **35**, 495 (1987).

[4] G. Bimonte, D. Lopez, and R. S. Decca, Phys. Rev. B **93**, 184434 (2016).

[5] J. N. Munday, F. Capasso, and V. A. Parsegian, Nature (London) **457**, 170 (2009).

[6] K. Kloppstech, N. Könne, S.-A. Biehs, A. W. Rodriguez, L. Worbes, D. Hellmann, and A. Kittel, arXiv:1510.06311.

[7] F. M. Serry, D. Walliser, and G. J. Maclay, J. Appl. Phys. **84**, 2501 (1998).

[8] E. Buks and M. L. Roukes, Phys. Rev. B **63**, 033402 (2001).

[9] M. Bordag, G. L. Klimchitskaya, U. Mohideen, and V. M. Mostepanenko, *Advances in the Casimir Effect* (Oxford University Press, Oxford, 2009).

[10] H. B. G. Casimir, Proc. Kon. Nederland. Akad. Wetensch. **51**, 793 (1948).

[11] E. M. Lifshitz, Sov. Phys. JETP **2**, 73 (1956).

[12] I. E. Dzyaloshinskii, E. M. Lifshitz, and L. P. Pitaevskii, Phys. Usp. **4**, 153 (1961).

[13] M. Kardar and R. Golestanian, Rev. Mod. Phys. **71**, 1233 (1999).

[14] G. L. Klimchitskaya, U. Mohideen, and V. M. Mostepanenko, Rev. Mod. Phys. **81**, 1827 (2009).

[15] S. K. Lamoreaux, Rep. Prog. Phys. **68**, 201 (2005).

[16] A. I. Volokitin and B. N. J. Persson, Rev. Mod. Phys. **79**, 1291 (2007).

[17] V. B. Bezerra, G. L. Klimchitskaya, V. M. Mostepanenko, and C. Romero, Phys. Rev. A **69**, 022119 (2004).

[18] G. Bimonte, T. Emig, M. Kardar, and M. Krüger, arXiv:1606.03740.

[19] M. Antezza, L. P. Pitaevskii, S. Stringari, and V. B. Svetovoy, Phys. Rev. A **77**, 022901 (2008).

[20] A. E. Rubio Lopez and F. C. Lombardo, Eur. Phys. J. C **75**, 93 (2015).

[21] F. C. Lombardo, F. D. Mazzitelli, A. E. Rubio Lopez, and G. J. Turiaci, Phys. Rev. D **94**, 025029 (2016).

[22] B. Huttner and S. M. Barnett, Phys. Rev. A **46**, 4306 (1992).

[23] E. A. Calzetta and B. L. Hu, *Nonequilibrium Quantum Field Theory* (Cambridge University Press, Cambridge, 2008).

[24] M. F. Maghrebi, R. L. Jaffe, and M. Kardar, Phys. Rev. A **90**, 012515 (2014).

[25] A. E. Rubio Lopez and F. C. Lombardo, Phys. Rev. D **89**, 105026 (2014).

[26] A. Bechler, J. Mod. Opt. **46**, 901 (1999).

[27] V. V. Dodonov, J. Opt. B **4**, R1 (2002).

[28] V. V. Dodonov, Adv. Chem. Phys. **119**, 309 (2001).

[29] S. B. Zheng, Opt. Commun. **273**, 460 (2007).

[30] C. J. Villas-Boas, N. G. de Almeida, R. M. Serra, and M. H. Y. Moussa, Phys. Rev. A **68**, 061801 (2003).

[31] R. Guzman, J. C. Retamal, E. Solano, and N. Zagury, Phys. Rev. Lett. **96**, 010502 (2006).

[32] Z. Jin, T. T. Han, T. Y. Zheng, L. Zheng, and H. Yang, Int. J. Theor. Phys. **53**, 2275 (2014).

[33] X. H. Cai and L. M. Kuang, arXiv:quant-ph/0206163.

[34] F. Belgiorno, S. Liberati, M. Visser, and D. W. Sciama, Phys. Lett. A **271**, 308 (2000).

[35] T. Y. Zheng and L. Zheng, Commun. Theor. Phys. **40**, 407 (2003).

[36] S. Weigert, Phys. Lett. A **214**, 215 (1996).

[37] H.-P. Breuer and F. Petruccione, *The Theory of Open Quantum Systems* (Oxford University Press, New York, USA, 2002).

[38] J. L. Schiff, *The Laplace Transform: Theory and Applications* (Springer-Verlag, New York, USA, 1999).

[39] L. Knöll and U. Leonhardt, J. Mod. Opt. **39**, 1253 (1992).

[40] R. E. Collin, *Field Theory of Guided Waves*, 2nd ed. (IEEE Press, New York, USA, 1990).

[41] D. Kupiszewska, Phys. Rev. A **46**, 2286 (1992).

[42] D. Kupiszewska and J. Mostowski, Phys. Rev. A **41**, 4636 (1990).

[43] F. C. Lombardo, F. D. Mazzitelli, and A. E. Rubio Lopez, Phys. Rev. A **84**, 052517 (2011).

[44] X. L. Li, G. W. Ford, and R. F. O'Connell, Phys. Rev. E **48**, 1547 (1993).

[45] F. S. S. Rosa, D. A. R. Dalvit, and P. W. Milonni, Phys. Rev. A **81**, 033812 (2010).

[46] K. J. Blow, R. Loudon, S. J. D. Phoenix, and T. J. Shepherd, Phys. Rev. A **42**, 4102 (1990).

[47] D. Kupiszewska, J. Mod. Opt. **40**, 517 (1993).

[48] G. W. Ford, J. T. Lewis, and R. F. O'Connell, Phys. Rev. A **37**, 4419 (1988).

[49] J. D. Jackson, *Classical Electrodynamics* (Wiley, New York, 1975), 2nd ed.

[50] P. W. Milonni, *The Quantum Vacuum: An Introduction to Quantum Electrodynamics* (Academic Press, San Diego, 1994).

[51] M. Bordag, U. Mohideen, and V. M. Mostepanenko, Phys. Rep. **353**, 1 (2001).