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Fermion zero modes in a Z_2 vortex background

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In this paper we study the zero energy solutions of the Dirac equation in the background of a Z_2 vortex of a non-Abelian gauge model with three charged scalar fields. We determine the number of the fermionic zero modes giving their explicit form for two specific *Ansätze*.

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I. INTRODUCTION

Programa Nacional de Pós Doutorado da CAPES The spectrum of Dirac-like operators in the presence of topologically nontrivial backgrounds has attracted the attention of physicists since the early work of Jackiw and Rebbi [1] discussing the cases $d = 1$ and $d = 3$ soliton backgrounds (kinks and monopoles) as well as $d = 4$ instanton backgrounds. In particular, 't Hooft found a solution of a notorious problem in high energy physics, the so-called U(1) problem in QCD [2–5] taking into account the contribution of the Dirac operator zero modes in a topologically nontrivial gauge background of instanton configurations.

Later on Jackiw and Rossi [6] considered the case in which the topological background is provided by vortexlike configurations and explicitly constructed the zero modes of the Dirac operator in $d = 2$ spatial dimensions. The result suggested that also in two-dimensional noncompact spaces the index theorem is valid, as was afterward proven in [7]. Interestingly enough, the Jackiw and Rossi zero modes can be chosen to be eigenmodes of a particle conjugation operator and hence considered as Majorana zero modes (see [8] and references therein).

The physical implications of zero modes are very surprising. Apart from their QCD application mentioned above, they are at the basis of charge fractionalization and cosmic string superconductivity, just to name some examples ([1,9,10]).

Concerning planar physics, a more recent wave of interest started after the realization that Majorana quasiparticles can appear in some solid states systems—the topological superconductors—and they could play an important role in building topological protected qubits [11].

As mentioned before, in $d = 2$ dimensional systems, the existence of zero modes is linked to the presence of a vortexlike background. The original work of Jackiw and Rossi [3] was concerned with zero modes of electrons moving in the background of a Nielsen-Olesen vortex. Many generalizations are possible. For instance, the case in which the vortex background is the one arising in a non-Abelian theory was considered in [12,13]. Zero modes for the case of a Chern-Simons vortex background were studied in [14–15] and more recently in the context of models having hidden sectors [16] that could be relevant in connection to superconductivity [17].

Recently a new type of Z_2 vortices in non-Abelian gauge theories was presented in [18]. This type of configuration is a local generalization of magnetic vortices that appear in some triangular lattices of antiferromagnetic materials [19]. It corresponds to a non-Abelian $SU(2)$ gauge theory with *three* scalar Higgs fields in the adjoint representation. We analyze in this paper the existence of fermionic zero modes under such backgrounds by constructing them explicitly.

The paper is organized as follows: In Sec. II, we briefly review the Z_2 vortices in non-Abelian gauge theories coupled to three scalar triplets [18] that will be taken as a background of the Dirac equation defining the zero mode problem. Then in Sec. III we introduce the Lagrangian for fermions minimally coupled to the non-Abelian gauge field background and also include a scalar-fermion coupling inspired in the one introduced in [12] for studying the zero mode problem in the background of the Z_N vortices discussed in [20]. After proposing an axially symmetric *Ansatz*, we are able to decouple the gauge field thanks to the existence of a charge conjugation operator that reduces the zero mode equations to ordinary radial differential equations in the scalar field background. Solving these equations we find the explicit form and number of the zero modes. We present in Sec. IV a summary of our results and a discussion of possible applications.

II. THE VORTEX BACKGROUND

As a background for the Dirac fermion equation, we consider the vortex solutions found in [18] for a $SU(2)$ gauge theory coupled to three scalar fields in the adjoint representation. The $2 + 1$ dimensional Lagrangian leading to vortex configurations reads

$$L = -\frac{1}{4}\vec{F}_{\mu\nu}\vec{F}^{\mu\nu} + \frac{1}{2}D_\mu\vec{\Phi}_a D^\mu\vec{\Phi}_a - V(\vec{\Phi}_a). \quad (1)$$

Here the gauge fields A_μ take values in the Lie algebra of $SU(2)$, $A_\mu = \vec{A}_\mu \cdot \vec{\sigma}/2$ while the scalars in the adjoint representation are written as $\Phi_a = \vec{\Phi}_a \cdot \vec{\sigma}/2$ ($a = 1, 2, 3$ and $\vec{\sigma}$ are the Pauli matrices). Field strengths and covariant derivatives are defined as

$$\vec{F}_{\mu\nu} = \partial_\mu\vec{A}_\nu - \partial_\nu\vec{A}_\mu + e\vec{A}_\mu \times \vec{A}_\nu, \quad (2)$$

$$D_\mu\vec{\Phi}_a = \partial_\mu\vec{\Phi}_a + e\vec{A}_\mu \times \vec{\Phi}_a. \quad (3)$$

As for the potential, one has

$$V(\vec{\Phi}_a) = \lambda_1(\vec{\Phi}_1\cdot\vec{\Phi}_1 - \eta^2)^2 + \lambda_2(\vec{\Phi}_2\cdot\vec{\Phi}_2 - \eta^2)^2 + \lambda_3(\vec{\Phi}_3\cdot\vec{\Phi}_3 - \eta^2)^2 + V_{\mathrm{mix}}(\vec{\Phi}_a) \quad (4)$$

where

$$V_{\mathrm{mix}}(\vec{\Phi}_a) = \mu^2(\vec{\Phi}_1 + \vec{\Phi}_2 + \vec{\Phi}_3)^2 + \lambda_4(\vec{\Phi}_1 + \vec{\Phi}_2 + \vec{\Phi}_3)^4. \quad (5)$$

It is clear that if we take $\lambda_i > 0$ and $\mu^2 > 0$ then the vacuum corresponds to

$$\vec{\Phi}_a \cdot \vec{\Phi}_a = \eta^2, \quad (6)$$

$$\vec{\Phi}_1 + \vec{\Phi}_2 + \vec{\Phi}_3 = 0. \quad (7)$$

Note that the condition (7) corresponds to a 120° configuration of the triplet of scalars, which in the antiferromagnetic model defined in a triangular lattice corresponds to spins arranged as in the "Mercedes-Benz" logo.

Concerning V_{mix}, the first term is the continuum analogue of the Heisenberg interaction in antiferromagnets (the term with λ_4 coupling constant is included because it is compatible with renormalization).

Two different *Ansätze* were shown to lead to topologically nontrivial axially symmetric vortexlike solutions [18]. Written in polar coordinates they read

(i) *Ansatz* I:

$$\begin{aligned}
\vec{\Phi}_1 &= f(r)(-\sin n\varphi, \cos n\varphi, 0),\\
\vec{\Phi}_2 &= f(r)\left(-\sin\left(n\varphi + \frac{2\pi}{3}\right), \cos\left(n\varphi + \frac{2\pi}{3}\right), 0\right),\\
\vec{\Phi}_3 &= f(r)\left(-\sin\left(n\varphi + \frac{4\pi}{3}\right), \cos\left(n\varphi + \frac{4\pi}{3}\right), 0\right),\\
\vec{A}_\varphi &= -\frac{1}{e}\left(0, 0, \frac{a(r)}{r}\right).
\end{aligned} \quad (8)$$

(ii) *Ansatz* II:

$$\begin{aligned}
\vec{\Phi}_1 &= (0, 0, \eta),\\
\vec{\Phi}_2 &= \frac{1}{2}(-\sqrt{3}f(r)\sin(n\varphi), \sqrt{3}f(r)\cos(n\varphi)), -\eta),\\
\vec{\Phi}_3 &= \frac{1}{2}(\sqrt{3}f(r)\sin(n\varphi), -\sqrt{3}f(r)\cos(n\varphi)), -\eta),\\
\vec{A}_\varphi &= -\frac{1}{e}\left(0, 0, \frac{a(r)}{r}\right),
\end{aligned} \quad (9)$$

with $n \in \mathbb{Z}$. Notice that both *Ansätze* satisfy Eq. (7). The conditions to ensure finite energy configurations are

$$\begin{aligned}
&\lim_{r\to 0} f(r) \sim r^{|n|} \qquad a(0) = 0\\
&\lim_{r\to\infty} f(r) = \eta \qquad \lim_{r\to\infty} a(r) = -n.
\end{aligned} \quad (10)$$

The field equations derived from Lagrangian (1) reduce to the radial equation

$$f'' + \frac{1}{r}f' - \frac{1}{r^2}(n + a)^2 f = 4\lambda f(r)(f^2 - \eta^2) \quad (11)$$

which apart from a numerical factor coincides with the radial equation for the Abelian Higgs model equation of motion for the complex scalar if one shifts λ according to $\lambda \to \lambda/3$ in the case of *Ansatz* I and $\lambda \to 8\lambda/9$ for *Ansatz* II.

III. THE DIRAC EQUATION

As mentioned above, inspired by the zero-mode analysis presented in [12] extending to the non-Abelian case in the Jackiw-Rossi Abelian construction [3], we shall consider the following $SU(2)$ gauge invariant Dirac Lagrangian:

$$L = \int d^3x\bar{\psi}(i\gamma^\mu\partial_\mu \times I + e\gamma^\mu \times A_\mu - g_a I \times \Phi_a)\psi. \quad (12)$$

Here γ^μ are the 2×2 gamma matrices and the background fields A_μ and Φ_a are those discussed in the previous section. The $\times$ symbol denotes the tensor product with the first factor acting in the spinorial indices and the second one in $SU(2)$ ones.

Fermion ψ is in the fundamental representation of $SU(2)$ and will be written in the form

$$\psi = \begin{pmatrix} \psi_1^U \\ \psi_2^U \\ \psi_1^D \\ \psi_2^D \end{pmatrix} \tag{13}$$

with spinorial indices U, D and $SU(2)$ ones 1,2. The fermion-scalar couplings g_a have the same dimensions as the gauge coupling e, $[g_a] = [e] = m^{1/2}$. Note that the scalar-fermion interaction is gauge invariant.

Lagrangian (12) leads to the fermion field equation

$$(i\alpha^j \partial_j \times I + e\alpha^j \times A_j - g_a \beta \times \Phi_a)\psi = -i\partial_t \psi \tag{14}$$

where $\gamma^0 = \beta, \gamma^j = \beta\alpha^j$ and $j = 1, 2$ are the spatial indices. We choose the Dirac matrices α^j, β in the form

$$\alpha^j = \sigma^j, \qquad j = 1, 2, \quad \beta = \sigma_3, \tag{15}$$

where σ^j, σ^3 are the Pauli matrices.

Following [12] we shall introduce the transformation $\psi \to L_3 \psi$

$$L_3 = \beta \times \sigma^3 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{16}$$

which will play an important role in finding and classifying zero modes, which is one of the main purposes of this work.

From Eq. (14), the zero-energy solutions satisfy

$$(i\sigma^j \partial_j \times 1 + e\sigma^j \times A_j - g_a \beta \times \Phi_a)\psi = 0. \tag{17}$$

To find zero mode solutions we start by considering *Ansatz* I. It will be convenient to write the gauge field in the form

$$A_j(\mathbf{r}) = \frac{1}{e} \epsilon_{ji} \partial^i k(r) \sigma^3 \tag{18}$$

where

$$k = -\int_0^r \frac{a(\rho)}{\rho} d\rho. \tag{19}$$

We now make the following change on the fermion field:

$$\psi(\mathbf{r}) = T(r) X(\mathbf{r}) \tag{20}$$

with

$$T(r) = \exp\left(k(r) L_3\right) \tag{21}$$

so that the gauge field in Eq. (17) decouples and we are left with

$$(i\sigma^j \partial_j \times I - g_a \beta \times \Phi_a) X(\mathbf{r}) = 0. \tag{22}$$

The decoupling was possible because, for *Ansatz* I, the operator L_3 anticommutes with the zero-mode Dirac operator in Eq. (17). Indeed, concerning Dirac matrices, L_3 anticommutes with the first two terms in Eq. (17) and commutes with the third one while for the $SU(2)$ generators, they commute with the first two terms and anticommute with the last one. Then as a result L_3 anticommutes with the zero-mode Dirac operator.

Written in components, Eq. (22) reads

$$\begin{aligned} &(i\partial_1 + \partial_2) X_1^D + i f(r) G_1 X_2^U = 0, \\ &(i\partial_1 - \partial_2) X_2^U + i f(r) G_2 X_1^D = 0, \end{aligned} \tag{23}$$

$$\begin{aligned} &(i\partial_1 + \partial_2) X_2^D - i f(r) G_2 X_1^U = 0, \\ &(i\partial_1 - \partial_2) X_1^U - i f(r) G_1 X_2^D = 0, \end{aligned} \tag{24}$$

where

$$\begin{aligned} G_1 &= (g_1 e^{-in\phi} + g_2 e^{-i(n\phi + 2\pi/3)} + g_3 e^{-i(n\phi + 4\pi/3)})/2 \\ &= e^{-in\phi}(g_1 + g_2 e^{-i2\pi/3} + g_3 e^{-i4\pi/3})/2 \equiv A e^{-in\phi} \\ &= |A| e^{i\alpha} e^{-in\phi}, \end{aligned} \tag{25}$$

$$\begin{aligned} G_2 &= (g_1 e^{in\phi} + g_2 e^{i(n\phi + 2\pi/3)} + g_3 e^{i(n\phi + 4\pi/3)})/2 \\ &= e^{in\phi}(g_1 + g_2 e^{i2\pi/3} + g_3 e^{i4\pi/3})/2 \equiv A^* e^{in\phi} \\ &= |A| e^{-i\alpha} e^{in\phi}. \end{aligned} \tag{26}$$

In view of the cylindrical symmetry, it is convenient to use polar coordinates for which Eqs. (23) and (24) become

$$\begin{aligned} &e^{-i\phi}\left(i\partial_r + \frac{1}{r}\partial_\phi\right) X_1^D + i f(r) |A| e^{i\alpha} e^{-in\phi} X_2^U = 0, \\ &e^{i\phi}\left(i\partial_r - \frac{1}{r}\partial_\phi\right) X_2^U + i f(r) |A| e^{-i\alpha} e^{in\phi} X_1^D = 0, \end{aligned} \tag{27}$$

and

$$\begin{aligned} &e^{-i\phi}\left(i\partial_r + \frac{1}{r}\partial_\phi\right) X_2^D - i f(r) |A| e^{-i\alpha} e^{in\phi} X_1^U = 0, \\ &e^{i\phi}\left(i\partial_r - \frac{1}{r}\partial_\phi\right) X_1^U - i f(r) |A| e^{i\alpha} e^{-in\phi} X_2^D = 0. \end{aligned} \tag{28}$$

We now propose for the first two equations the *Ansatz*

$$X_1^D = \chi_1^D e^{i(m-n+1)\phi},$$
$$X_2^U = \chi_2^U e^{im\phi}. \quad (29)$$

As a result, the angular dependence factorizes and the zero-mode equations for Eq. (27) reduce to ordinary differential equations

$$\left(\partial_r + \frac{(m-n+1)}{r}\right)\chi_1^D + f(r)\chi_2^U = 0,$$
$$\left(\partial_r - \frac{m}{r}\right)\chi_2^U + f(r)|A|^2\chi_1^D = 0, \quad (30)$$

where we have redefined $A\chi_2^U$ as χ_2^U.

Similarly, for the third and forth equations we write

$$X_1^U = \chi_1^U e^{-im\phi},$$
$$X_2^D = \chi_2^D e^{-i(m-n-1),\phi} \quad (31)$$

so that Eqs. (28) become

$$\left(\partial_r - \frac{(m-n-1)}{r}\right)\chi_2^D - f(r)|A|^2\chi_1^U = 0,$$
$$\left(\partial_r + \frac{m}{r}\right)\chi_1^U - f(r)\chi_2^D = 0, \quad (32)$$

where we have redefined $A\chi_2^D$ as χ_2^D. Without loss of generality we choose χ_1^U, χ_2^U and χ_1^D, χ_2^D real.

In view of conditions (10) for the vortex background, in order to have well-behaved zero modes near the origin the first set of equations, Eq. (30), implies

$$\chi_1^D {}_{\text{small r}}^{\tilde{}} r^{n-m-1}, r^{m+|n|+1},$$
$$\chi_2^U {}_{\text{small r}}^{\tilde{}} r^{n-m+|n|}, r^m. \quad (33)$$

Compatibility of behaviors (33) implies

$$n - 1 \geq m \geq 0. \quad (34)$$

These conditions imposes n to be a positive vortex number.

The second set of equations, Eq. (32), imposes

$$\chi_1^U {}_{\text{small r}}^{\tilde{}} r^{m-n+|n|}, r^{-m},$$
$$\chi_2^D {}_{\text{small r}}^{\tilde{}} r^{m-n-1}, r^{-m+|n|+1}. \quad (35)$$

Following the same procedure as above, we get in this case the following condition from Eq. (35):

$$n + 1 \leq m \leq 0. \quad (36)$$

These conditions correspond to a negative vortex number.

In summary, both for positive and negative values of n we conclude that there are $|n|$ zero modes.

Using the explicit form of L_3

$$\exp(k(r)L_3) = \begin{pmatrix} \exp(k(r)) & 0 & 0 & 0 \\ 0 & \exp(-k(r)) & 0 & 0 \\ 0 & 0 & -\exp(k(r)) & 0 \\ 0 & 0 & 0 & -\exp(-k(r)) \end{pmatrix} \quad (37)$$

zero-energy eigenfunctions for *Ansatz* I where $n - 1 \geq m \geq 0$ are

$$\psi_{n>0}(\vec{r}) = \begin{pmatrix} 0 \\ e^{-k(r)}A^{-1}\chi_2^U e^{im\phi} \\ -e^{k(r)}\chi_1^D e^{i(m-n+1)\phi} \\ 0 \end{pmatrix}. \quad (38)$$

For the interval $n + 1 \leq m \leq 0$, the zero modes are

$$\psi_{n<0}(\vec{r}) = \begin{pmatrix} e^{k(r)}\chi_1^U e^{-im\phi} \\ 0 \\ 0 \\ -e^{-k(r)}A^{-1}\chi_2^D e^{-i(m-n-1)\phi} \end{pmatrix}. \quad (39)$$

Notice that the factors $\exp(\pm k(r))$ do not affect normalizability of zero modes since $k(0) = 0$ and $k(r) \to \pm n \log r$ when $r \to \infty$ and the $\chi' s$ are exponentially decreasing functions.

It is important to stress that L_3 classifies zero modes according to

$$L_3\psi_{n\gtrless 0}(\vec{r}) = \mp\, \psi_{n\gtrless 0}(\vec{r}). \quad (40)$$

The analysis for the case in which the background corresponds to a vortex obeying *Ansatz* II goes similarly. Instead of Eq. (24) we now have

$$(i\partial_1 + \partial_2)X_1^D + if(r)H_1 e^{-in\phi}X_2^U - H_2X_1^U = 0,$$
$$(i\partial_1 - \partial_2)X_2^U + if(r)H_1 e^{in\phi}X_1^D - H_2X_2^D = 0,$$
$$(i\partial_1 + \partial_2)X_2^D - if(r)H_1 e^{in\phi}X_1^U + H_2X_2^U = 0,$$
$$(i\partial_1 - \partial_2)X_1^U - if(r)H_1 e^{-in\phi}X_2^D + H_2X_1^D = 0, \quad (41)$$

where

$$H_1 = \frac{\sqrt{3}}{4}(g_2 - g_3), \tag{42}$$

$$H_2 = \eta\frac{(2g_1 - g_2 - g_3)}{4}. \tag{43}$$

Notice that because of the particular scalar-fermion interaction in Lagrangian (12) there is, in the case of *Ansatz* II, an effective coupling of fermions with the components of the scalar fields in the third direction of the group leading to the last terms in the left-hand side of each equation in Eq. (41). Such terms are analogous to the type of perturbation introduced by Haldane in [21] and discussed in [22]. They correspond to the term proportional to L_3 in the Hamiltonian. Now, because of the presence of such terms, the Dirac operator does not anticommute with L_3, in analogy to what happens concerning chiral invariance in $3+1$ dimensions when fermions are massive and the Dirac operator does not commute with γ_5. As in the $3+1$ dimensional case, in the present case the existence of zero modes requires anticommutation of L_3 with the Dirac operator which otherwise would have a nonzero determinant. Since our aim is to find zero modes associated with *Ansatz* II, we impose a condition ensuring $H_2 = 0$, this implies that the following relation between coupling constants should hold:

$$2g_1 - g_2 - g_3 = 0. \tag{44}$$

Once condition (44) is adopted, Eq. (41) becomes

$$\begin{aligned}(i\partial_1 + \partial_2)X_1^D + if(r)H_1e^{-in\phi}X_2^U &= 0,\\ (i\partial_1 - \partial_2)X_2^U + if(r)H_1e^{in\phi}X_1^D &= 0,\\ (i\partial_1 + \partial_2)X_2^D - if(r)H_1e^{in\phi}X_1^U &= 0,\\ (i\partial_1 - \partial_2)X_1^U - if(r)H_1e^{-in\phi}X_2^D &= 0,\end{aligned} \tag{45}$$

or, in polar coordinates (r,ϕ)

$$\begin{aligned}e^{-i\phi}\left(i\partial_r + \frac{1}{r}\partial_\phi\right)X_1^D + if(r)H_1e^{-in\phi}X_2^U &= 0,\\ e^{i\phi}\left(i\partial_r - \frac{1}{r}\partial_\phi\right)X_2^U + if(r)H_1e^{in\phi}X_1^D &= 0,\\ e^{-i\phi}\left(i\partial_r + \frac{1}{r}\partial_\phi\right)X_2^D - if(r)H_1e^{in\phi}X_1^U &= 0,\\ e^{i\phi}\left(i\partial_r - \frac{1}{r}\partial_\phi\right)X_1^U - if(r)H_1e^{-in\phi}X_2^D &= 0.\end{aligned} \tag{46}$$

The adequate phase *Ansatz* for X_1^D, X_2^U is now

$$\begin{aligned}X_1^D &= \chi_1^D e^{-im\phi},\\ X_2^U &= \chi_2^U e^{i(-m+n-1)\phi},\end{aligned} \tag{47}$$

leading to

$$\begin{aligned}\left(\partial_r - \frac{m}{r}\right)\chi_1^D + f(r)\chi_2^U &= 0,\\ \left(\partial_r - \frac{(-m+n-1)}{r}\right)\chi_2^U + f(r)H_1^2\chi_1^D &= 0,\end{aligned} \tag{48}$$

and for the other two components

$$\begin{aligned}X_2^D &= \chi_2^D e^{im\phi},\\ X_1^U &= \chi_1^U e^{i(m-n-1)\phi},\end{aligned} \tag{49}$$

leading in this case to

$$\begin{aligned}\left(\partial_r + \frac{m}{r}\right)\chi_2^D - f(r)\chi_1^U &= 0,\\ \left(\partial_r - \frac{(-n+m-1)}{r}\right)\chi_1^U - f(r)H_1^2\chi_2^D &= 0,\end{aligned} \tag{50}$$

where we have shifted $H_1X_2^U \to X_2^U$ and $H_1X_1^U \to X_1^U$.

From the first set of equations we find that the appropriate behavior at the origin ensuring zero-mode regularity is

$$\begin{aligned}&\chi_{1\ \mathrm{small\,r}}^D \tilde{\ } r^m,\ r^{-m+n+|n|},\\ &\chi_{2\ \mathrm{small\,r}}^U \tilde{\ } r^{m+|n|+1},\ r^{-m+n-1},\end{aligned} \tag{51}$$

and from the second,

$$\begin{aligned}&\chi_{1\ \mathrm{small\,r}}^U \tilde{\ } r^{-m+|n|+1},\ r^{-n+m-1},\\ &\chi_{2\ \mathrm{small\,r}}^D \tilde{\ } r^{-m},\ r^{-n+m+|n|}.\end{aligned} \tag{52}$$

All the solutions to these equations are regular as long as the following inequalities hold for the first set of equations:

$$n - 1 \geq m \geq 0 \tag{53}$$

or

$$n + 1 \leq m \leq 0 \tag{54}$$

for the second one, which are exactly the same conditions found in the case of *Ansatz* I. Therefore, there are also $|n|$ zero modes for *Ansatz* II.

The explicit form of zero-energy eigenfunctions in this case is

$$\psi_{n>0}(\vec{r}) = \begin{pmatrix} 0 \\ e^{-k(r)}H^{-1}\chi_2^U e^{i(-m+n-1)\phi} \\ -e^{k(r)}\chi_1^D e^{-im\phi} \\ 0 \end{pmatrix} \tag{55}$$

for positive vortex numbers. Concerning negative vortex numbers, we obtain the following zero mode:

$$\psi_{n<0}(\vec{r}) = \begin{pmatrix} e^{k(r)}H^{-1}\chi_1^U e^{i(-n+m-1)\phi} \\ 0 \\ 0 \\ -e^{-k(r)}\chi_2^D e^{im\phi} \end{pmatrix}. \tag{56}$$

Also for this *Ansatz*, L_3 classifies the zero modes as

$$L_3\psi_{n\gtrless 0}(\vec{r}) = \mp\, \psi_{n\gtrless 0}(\vec{r}). \tag{57}$$

Note that the relation between the signs of L_3 eigenvalues and vortex number is inverted with respect to that arising for *Ansatz* I, Eq. (40).

We end this section by analyzing explicitly the only existing zero mode for the case $n = 1$. For *Ansatz* I it takes the form

$$\psi_1(\vec{r}) = \begin{pmatrix} 0 \\ e^{-k(r)}A^{-1}\chi_2^U \\ -e^{k(r)}\chi_1^D \\ 0 \end{pmatrix} \tag{58}$$

where χ_2^U and χ_1^D satisfy Eq. (30). Concerning *Ansatz* II we have

$$\psi_1(\vec{r}) = \begin{pmatrix} 0 \\ e^{-k(r)}H^{-1}\chi_2^U \\ -e^{k(r)}\chi_1^D \\ 0 \end{pmatrix} \tag{59}$$

where χ_2^U and χ_1^D satisfy Es. (51).

IV. SUMMARY AND DISCUSSION

In this work we have been able to construct explicit zero modes of the Dirac equation in the gauge and scalar fields background of the Z_2 vortices recently introduced in [18]. We have constructed the zero modes in two different *Ansätze*. While for *Ansatz* I, zero modes exist for generic value of the scalar-fermion coupling constants, in the case of *Ansatz* II an explicit relation between coupling constants is required [see Eq. (44)].

As discussed in [18], from an energetic point of view, vortices of type *Ansatz* II are favored against those of type *Ansatz* I. Also, although vortices with arbitrary n are possible, their energy increases with n and as a result only vortices with $|n| = 1$ are topologically protected. So, the zero modes of the type given by Eqs. (58) and (59) are those that are relevant as well as the analogous ones with $n = -1$. One then concludes that in this $SU(2)$ gauge invariant theory there is only one zero mode with no angular dependence associated with an $|n| = 1$ vortex.

It can be easily seen that a simple change of the fermion basis transforms the Hamiltonian H associated with our problem to a Hamiltonian $\tilde{H}$ which is of the same type of the one considered recently by Schuster *et al.* [23] (see also [24–26]).

$$\tilde{H} = \begin{pmatrix} 0 & -i\nabla_- - eA_- & g\Delta^* & 0 \\ -i\nabla_+ - eA_+ & 0 & 0 & g\Delta^* \\ g\Delta & 0 & 0 & i\nabla_- - eA_- \\ 0 & g\Delta & i\nabla_+ - eA_+ & 0 \end{pmatrix} \tag{60}$$

where Δ is related to the scalar fields of our *Ansätze* and can be written in the form $|\Delta(r)|\exp(in\phi)$. The main difference is that our backgrounds are those arising in a non-Abelian gauge theory, and they correspond to regular solutions of finite energy. Also, as explained before, in our non-Abelian case topology selects automatically the $|n| = 1$ sector, leaving us with a single zero mode.

The vortex backgrounds considered in [18] were inspired by global magnetic vortices appearing in antiferromagnetic materials in the triangular lattice. In solving the zero mode problem, the gauge potential does not play a central role as it is in fact *decoupled* by the transformation given in Eq. (20). It would be interesting to explore in such systems if excitations coupled to the magnetization in a similar way as in the fermion-scalar field coupling considered here do exist. Non-Abelian gauge fields also naturally arise in systems with spin-orbit interactions and cold atoms [27]. It would be interesting to analyze if nontrivial field configurations could be explicitly realized in such systems. We hope to work on these issues in a future work.

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