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# Large deviations for multiple ergodic averages

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**Abstract**

The main purpose of this work is to estimate how multiple ergodic averages appart from a given quantity. This problem can be studied by describing a large deviation process for empirical measures as obtained by using the contraction principle. The case of single ergodic averages for empirical measures was already studied by Pfister and Sullivan [Nonlinarity, 10 (2005) 237-261]. To have a more complete picture on empirical measures and $V-$ statistics, we estimate the size of the sets $G_K = \{x : L_r(x) \subset K\}$, where $L_r(x)$ is the limit-point set of the sequence of empirical measures and $K$ is a compact subset of $\mathcal{M}(X^r)$ with $\mathcal{M}(X)$ the set of measures on $X$. In pasrticular, we obtain a variational formula for the topological entropy of $G_k$. The result of this work about the dimension of the sets $G_k$ can be compared with the one recently circulated by Fan, Schemeling and Wu [arXiv:1206.3214v1 (2012)].

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## 1. Introduction

Let $(X, f)$ be a topological dynamical system, with $X$ a compact metric space and $f$ a continuous map. Let $X^r = X \times ... \times X$ be the product of $r-$ copies of $X$ with $r \geq 1$. A dimension theory in $X^r$ has been formulated by Fan and collaborators [4], [5], [6]. The main problems studied in those articles were related with the description of multifractal spectra for multiple ergodic averages like

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$$V_\Phi(n,x) = \frac{1}{n^r} \sum_{0 \le i_1 \le \ldots \le i_r \le n-1} \Phi\left(f^{i_1}(x), \ldots, f^{i_r}(x)\right), \tag{1}$$

where $\Phi : X^r \to \mathbf{R}$ is a continuous map. These kind of avarages are called $V$-statistics. The problems about the description of multifractal spectra of $V$-statistics were motivated by the study of the convergence of multiple ergodic averages. In particular ergodic limits of the form

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \Phi\left(f^{i}(x), \ldots, f^{ir}(x)\right),$$

were studied among others by Furstenberg[7], Bergelson[1] and Bourgain[2]. The multiple ergodic averages are of interest because they can be taken as an interplay between Dynamical Systems and Number Theory.

Fan, Schemeling and Wu[4] have recently obtained a variational expression for the size of the multifractal sets

$$E_\Phi(\alpha) = \left\{x : \lim_{n\to\infty} V_\Phi(n,x) = \alpha\right\}.$$

In [9] we study the irregular, or historic set of the multifractal spectra of $V$-statistics, i.e. the set of points $x$ for which $\lim_{n\to\infty} V_\Phi(n,x)$ does not exist. It was proved that the historic set for $V$-statistics has full topological entropy for any $r \geq 1$.

In connection with the multiple ergodic averages (1), empirical measures on $X^r$ can be introduced:

$$\mathcal{E}_{n,r}(x) = \frac{1}{n^r} \sum_{1 \le i_1, \ldots, i_r \le n} \delta_{\left(f^{i_1}(x), \ldots, f^{i_r}(x)\right)} \tag{2}$$

with $r \geq 1$ and where $\delta$ is the Dirac point mass measure.

Besides Dimension Theory of Dynamical Systems and Multifractal Analysis, other discipline in which appears problems involving scaled limits is Large Deviations Theory. The main issue in this area is to estimate the average of sequences like $\{\nu_n(A)\}$, where $\nu_n$ are measures in a topological space $\mathcal{X}$ and $A$ is an open or closed subset of $\mathcal{X}$. To describe a large deviation process it must be found a function

$$I : \mathcal{X} \to [0, \infty)$$

such that

$$\limsup_{n\to\infty} \frac{1}{n} \log \nu_n(F) \le \sup\{-I(x) : x \in F, F \subset \mathcal{X} \text{ closed}\}$$

and

$$\liminf_{n\to\infty} \frac{1}{n} \log \nu_n(U) \ge -\inf\{I(x) : x \in U, U \subset \mathcal{X} \text{ open}\}.$$

We shall analyze a process of large deviations for the case $\mathcal{X} = \mathcal{M}(X^r)$ and the sequences

$$\nu_n = \nu\left(\left\{x : \mathcal{E}_{n,r}(x) \in A\right\}\right),$$

with $\nu \in \mathcal{M}(\mathcal{X})$ and $A$ an open or closed set. By the contraction principle for large deviations can be estimated the rate of convergence of the $V-$statistics, i.e. the convergence of the measure of sets $\left\{x : V_\Phi(n,x) \in A\right\}$, with $A$ an open or closed interval. The case $r = 1$, i.e. large deviations for the classical ergodic averages $S_n(\varphi)(x) = \frac{1}{n}\sum_{i=0}^{n} \varphi\left(f^i(x)\right)$ was studied in the relavant articles by Kifer[8] and Young [11].

We will also estimate the size of sets

$$G_K = \{x : L_r(x) \subset K\},$$

where $K$ is a compact subset of $\mathcal{M}(X^r)$ and $L_r(x)$ denotes the set of limit points of the sequence $\left\{\mathcal{E}_{n,r}(x)\right\}$. We obtain a variational formula for the the topological entropy of $G_K$, which can be compared with the result of [4]. The condition required for the dynamics will be the specification property. For $r = 1$, a variational expression was achieved by Pfister and Sullivan, but under a weaker condition than specification, the almost property product.

## 2. Preliminary definitions

Let us begin by recalling the Bowen definition of topological entropy of sets: Let $f : X \to X$ with $X$ a compact metric space, the dynamical metric, or Bowen metric, is defined, for each for $n \ge 1$ as

$d_n(x,y) = \max\left\{d\left(f^i(x), f^i(y)\right) : i = 0,1,\ldots,n-1\right\}$. We denote by $B_{n,\varepsilon}(x)$ the ball of centre $x$ and radius ε in the metric $d_n$. Let $Z \subset X$ and let $\mathcal{C}(n,\varepsilon,Z)$ be the collection of finite or countable coverings of the set $Z$ by balls $B_{n,\varepsilon}(x)$ with $m \geq n$. Let

$$M(Z,s,n,\varepsilon) = \inf_{\mathcal{B}\in\mathcal{C}(n,\varepsilon,Z)} \sum_{B_{m,\varepsilon}(x)\in\mathcal{B}} \exp(-sm),$$

and set

$$M(Z,s,\varepsilon) = \lim_{n\to\infty} M(Z,s,n,\varepsilon).$$

This limits does exist since $M(Z,s,n,\varepsilon)$ is a non-decresing function of $n$. There is an unique number $\overline{s}$ such that $M(Z,s,\varepsilon)$ jumps from $+\infty$ to 0. Let

$$H(Z,\varepsilon) = \overline{s} = \sup\left\{s : M(Z,s,\varepsilon) = +\infty\right\} = \inf\left\{s : M(Z,s,\varepsilon) = 0\right\}$$

and

$$h_{top}(Z) = \lim_{\varepsilon\to 0} H(Z,\varepsilon).$$

This limit exists [3]. The number $h_{top}(Z)$ is the *topological entropy* of $Z$.

**Definition:** A set $E \subset X$ is $(n, \varepsilon)$ – separated if for any $x, y \in E$, $x \neq y$ holds $d\left(f^i(x), f^i(y)\right) > \varepsilon$, $i = 0,1,\ldots,n-1$. The $(n, \varepsilon)$ – separated sets are finite when $X$ is compact.

**Definition:** A dynamical system $(X, f)$ has the *specification property* if the following condition holds: for $\varepsilon > 0$, there is an integer $M(\varepsilon)$ such that for any finite disjoint collection of integer intervals $I_1 = [a_1, b_1], \ldots, I_k = [a_k, b_k]$, of length $\geq M(\varepsilon)$ and for any points $x_1, x_2, \ldots, x_k \in X$, there is a point $z \in X$ which ε–shadows the sequence $\{x_1, x_2, \ldots, x_k\}$, i.e. $d\left(f^{a_j+n}(z), f^n(x_j)\right) \leq \varepsilon$, for any $n = 0,..,b_j - a_j$ and $j = 0,1,\ldots,k$.

By $\mathcal{M}(X)$ we denote the space of measures in $X$, $M_{inv}(X, f)$ will denote the space of $f$–invariant measures on $X$ and $\mathcal{M}_E(X, f)$ the set of $f$–invariant ergodic measures. The space $\mathcal{M}(X)$ can be endowed with the following metric

$$D(\mu,\nu)=\sum_{n=1}^{\infty}\frac{\left|\int\varphi_n d\mu-\int\varphi_n d\nu\right|}{2^n\left\|\varphi_n\right\|_\infty},$$

where $\{\varphi_n\}$ is a dense set in $C(X)$. We denote by $B_R(\mu)$ the ball of center μ and radius $R$ in the above metric. The topology induced by this metric is the weak $*-$ topology, and if $X$ is compact then $\mathcal{M}(X)$ is compact in the weak topology. The weak convergence is the convergence in the metric which induces the weak topology.

Recall that by $L_r(x)$ is denoted the set of weak $*-$ limit points of the sequence of empirical measures $\mathcal{E}_{n,r}(x)=\frac{1}{n^r}\sum_{1\le i_1,\dots,i_r\le n}\delta_{\left(f^{i_1}(x),\dots,f^{i_r}(x)\right)}$, since $X$ is compact, $L_r(x)\neq\varnothing$, for any $r\geq 1$. If μ is a measure on $X$ then a point $x\in X$ is μ–generic if $L_1(x)=\{\mu\}$, by $G_\mu$ is denoted the set of μ–generic points.

The following result is due to Bowen

**Lemma**[3] *Let $t\geq 0$ and let $h_\mu(f)$ be the measure-theoretic entropy of ∝ If*

$$B(t)=\left\{x:\text{there is a }\mu\in L_1(x)\text{ such that }h_\mu(f)\leq t\right\}$$

then

$$h_{top}(B(t))\leq t.$$

Therefore since $G_\mu\subset B(h_\mu(f))$ holds

$$h_{top}(G_\mu)\leq h_\mu(f)$$

If $\mu$ is ergodic then, by the ergodic theorem, $\mu(G_\mu)=1$ and by theorem 1 of [3] results

$$h_{top}(G_\mu)=h_\mu(f).$$

For dynamical systems with the specification property the equality holds for any invariant measure, this equality was proved in[6].

## 3. A variational formula for the topological entropy of the sets $G_K$.

The result to be proved is:

**Theorem 1:** *Let (X, f) be a topological dynamical system with specification, and let* $K \subset \mathcal{M}\left(X^r\right)$ *compact. Then*

$$h_{top}\left(G_K\right) = \sup\left\{h_\mu\left(f\right) : \mu^{\otimes r} \in K\right\}.$$

Here $\mu^{\otimes r}$ means $\mu \times \mu \times \ldots \times \mu$ $r-$times.

***Proof:*** We shall firstly see that for any $x \in G_K$ there is a $\mu \in L_1(x)$ such that $\mu^{\otimes r} \in K$. Let $\Phi : X^r \to \mathbf{R}$ continuous, in [4] was proved that for any $\varepsilon > 0$ there is a map $\widetilde{\Phi}: X^r \to \mathbf{R}$ of the form

$$\widetilde{\Phi} = \sum_j \varphi_j^{(1)} \otimes \ldots \otimes \varphi_j^{(r)}, \tag{3}$$

with $\varphi_j^{(i)} \in C(X)$, such that $\left\|\Phi - \widetilde{\Phi}\right\|_\infty < \varepsilon$. Since $L_r(x) \neq \varnothing$, for any $r$, in particular there exists a measure $\mu$ such that for some sequence $(n_k)$

$$w^* - \lim_{k\to\infty} \mathcal{E}_{n_k,r}\left(x\right) = \mu.$$

We have

$\int \widetilde{\Phi}\, d\, \mathcal{E}_{n_k,r}\left(x\right) = \sum_j \prod_{i=1}^{r} \frac{1}{n} S_{n_k}\left(\varphi_j^{(i)}\right)\left(x\right)$ where $S_n\left(\varphi\right)\left(x\right) = \sum_{k=0}^{n-1} \varphi\left(f^i(x)\right)$. Since $\left\{\mathcal{E}_{n_k,r}\left(x\right)\right\}$ weakly converges to $\mu$, holds

$$\lim_{k\to\infty} \sum_j \prod_{i=1}^{r} \frac{1}{n} S_{n_k}\left(\varphi_j^{(i)}\right)\left(x\right) = \sum_j \prod_{i=1}^{r} \int_X \varphi_j^{(i)}\, d\,\mu = \int_{:X^r} \widetilde{\Phi}\, d\,\mu^{\otimes r}. \tag{4}$$

Besides

$$\left|\int \Phi\, d\, \mathcal{E}_{n_k,r}\left(x\right) - \int \widetilde{\Phi}\, d\, \mathcal{E}_{n_k,r}\left(x\right)\right| < \varepsilon,$$

then

$$\lim_{k\to\infty} \int \Phi\, d\mathcal{E}_{n_k,r}\left(x\right) = \int_{:X^r} \Phi\, d\,\mu^{\otimes r}.$$

Since $x \in G_K$ is $\mu^{\otimes r} \in K$.

In this way is proved that for any $x \in G_K$ there is a $\mu \in L_r(x)$ with $\mu^{\otimes r} \in K$. This leads to

$$G_K \subset B\left(\sup\left\{h_\mu(f) : \mu^{\otimes r} \in K\right\}\right). \tag{5}$$

Therefore by the Bowen lemma

$$h_{top}(G_K) \leq \sup\left\{h_\mu(f) : \mu^{\otimes r} \in K\right\}.$$

To prove the opposite inequality, let $x \in G_\mu$ and $\mu \in \mathcal{M}(X)$, with $\mu^{\otimes r} \in K$. Analogously than before

$$\lim_{n\to\infty}\int \widetilde{\Phi}\, d\, \mathcal{E}_{n,r}(x) = \int \widetilde{\Phi}\, d\, \mu^{\otimes r}, \text{ thus for } \varepsilon > 0$$

$$\varliminf_{n\to\infty}\int \Phi\, d\, \mathcal{E}_{n,r}(x) \leq \varliminf_{n\to\infty}\int \widetilde{\Phi}\, d\, \mathcal{E}_{n,r}(x) + \varepsilon = \int \widetilde{\Phi}\, d\mu^{\otimes r} + \varepsilon.$$

Since ε is arbitrary we have

$$w^* - \lim_{n\to\infty} \mathcal{E}_{n,r}(x) = \mu^{\otimes r} \in K.$$

Therefore $x \in G_K$ and so $G_\mu \subset G_K$. Thus, by a property of the topological entropy and since the system has specification property

$$h_{top}(G_K) \geq h_{top}(G_\mu) = h_\mu(f) \text{ for } \mu^{\otimes r} \in K.$$

Finally

$$h_{top}(G_K) \geq \sup\left\{h_\mu(f) : \mu^{\otimes r} \in K\right\}. \qquad \square$$

The main result by Fan, Schemeling and Wu in [4] is the variational formula

$$h_{top}(E_\Phi(\alpha)) = \sup\left\{h_\mu(f) : \int \Phi d\, \mu^{\otimes r} = \alpha\right\},$$

where $E_\Phi(\alpha) = \left\{ x : \lim_{n\to\infty} \frac{1}{n^r} \sum_{1\le i_1,\ldots,i_r\le n} \Phi\left(f^{i_1}(x),\ldots,f^{i_r}(x)\right) = \alpha \right\}$. To compare this result with the theorem 1, notice that the sequence $\{\mathcal{E}_{n,r}(x)\}$ has all limit points in the set $\mathcal{F}_\Phi(\alpha) = \left\{ \nu : \int \Phi d\nu = \alpha \right\}$ if and only if $\lim_{n\to\infty} \frac{1}{n^r} \sum_{1\le i_1,\ldots,i_r\le n} \Phi\left(f^{i_1}(x),\ldots,f^{i_r}(x)\right) = \alpha$. Let $G_{\mathcal{F}_\Phi(\alpha)} = \{x : L_r(x) \subset \mathcal{F}_\Phi(\alpha)\}$, so that by the theorem 1 we have

$$h_{top}(G_{\mathcal{F}_\Phi(\alpha)}) = \sup\left\{h_\mu(f) : \mu^{\otimes r} \in \mathcal{F}_\Phi(\alpha)\right\},$$

leading to the result of [4].

## 4. Large deviations

A general *level* – 2 large deviation process is described in the following way: let $\mathcal{X}$ be a Hausdorff topological vector space a let $I : \mathcal{X} \to [0,+\infty]$ be a lower semi-continuous map, a sequence $\{\nu_n\}_{n\ge1}$ of measures in $\mathcal{X}$ satisfies a *level* – 2 *large deviation process* with rate function *I* if holds

$$\limsup_{n\to\infty} \frac{1}{n} \log \nu_n(F) \le \sup\{-I(x) : x \in F, F \subset \mathcal{X} \text{ closed}\}$$

and

$$\liminf_{n\to\infty} \frac{1}{n} \log \nu_n(U) \ge -\inf\{I(x) : x \in U, U \subset \mathcal{X} \text{ open}\}.$$

The *contraction principle* says: let $\{\nu_n\}_{n\ge1}$ be a sequence of measures in $\mathcal{X}$ satisfying a level – 2 large deviation process with rate function *I*, let $\mathcal{Y}$ be a topological space and $\tau : \mathcal{X} \to \mathcal{Y}$ a continuous map, then $\{\tau_*(\nu_n)\}_{n\ge1}$ satisfies a large deviation principle in $\mathcal{Y}$ with rate function. $G(y) := \inf\{I(x) : x \in \mathcal{X}, \tau(x) = y\}$. Here $\tau_*(\nu)$ denotes the pushforward of the measure ν. A large deviation process obtained by the contraction principle is called a *level*– 1 large deviation process.

In our case consider $\mathcal{X} = \mathcal{M}(X^r), \mathcal{Y} = \mathbf{R}, \nu_n(A) = \nu\left(\{x : \mathcal{E}_{n,r}(x) \in A\}\right)$. Thus, our problem in large deviations will be to estimate the rate of convergence of the the empirical measures $\mathcal{E}_{n,r}(x)$ by analyzing the the convergence of the measure of sets and $\{x:\mathcal{E}_{n,r}(x)\in A\}$, $A$ open or closed subset of $\mathcal{M}(X^r)$.

Let $\Psi: X^r \to \mathbf{R}$ be a continuous function and let $\widehat{\Psi}:\mathcal{M}(X^r)\to \mathbf{R}$ the map defined by $\widehat{\Psi}(\rho)=\int \Psi d\rho.$ We apply the contraction principle for $\tau=\widehat{\Psi}$, therefore the sequence $\left\{\widehat{\Psi}(\nu_n)\right\}_{n\geq 1}$ will satisfiy a large deviation principle in $\mathbf{R}$. If $J\subset \mathbf{R}$ then $\widehat{\Psi}(\nu_n)(J)=\nu_n\left(\widehat{\Psi}^{-1}(J)\right)=\nu\left(\{x:V_\Psi(n,x)\in J\}\right)$, and so large deviations for $V$–statistics can be estimated.

**Definition:** A function $\Phi_\nu : X^r \to \mathbf{R}^+$, assigned to a measure $\nu\in\mathcal{M}(X)$, is a *lower energy function* if it is upper-semi continuous and if for $r\geq 1$

$$\lim_{\varepsilon\searrow 0}\liminf_{n\to\infty}\left(\inf_{x\in X}\frac{1}{n^r}\log\nu\left(B_{n,\varepsilon}(x)\right)+\int\Phi_\nu d\mathcal{E}_{n,r}(x)\right)\geq 0. \tag{6}$$

A map $\Phi_\nu : X^r \to \mathbf{R}^+$, is an *upper energy function* if it is upper-semi continuous and if for $r\geq 1$

$$\lim_{\varepsilon\searrow 0}\limsup_{n\to\infty}\left(\sup_{x\in X}\frac{1}{n^r}\log\nu\left(B_{n,\varepsilon}(x)\right)+\int\Phi_\nu d\mathcal{E}_{n,r}(x)\right)\leq 0. \tag{7}$$

The following lemma is a direct extension of the proposition 2.1 in [10]. Nevertheless we display the proof in detail for completeness.

**Lemma 1:** *Let* $\mu$ *be an ergodic measure and* $\overline{h}<h_\mu(f)$, *there exists* $\overline{\varepsilon}>0$, *such that for any neighborhood* $F$ *of* $\mu$, *there is a number* $N=N_F$ *such that for any* $n\geq N$, $r\geq 1$ *there is a* $(n^r,\overline{\varepsilon})-$ *separated set* $\Gamma_{n,r}\subset X_{n,F^r}:=\{x:\mathcal{E}_{n,r}(x)\in F^r\}$ *and with* $card(\Gamma_{n,r})\geq \exp(n^r\overline{h})$.

***Proof:*** Let $0<\overline{h}<h_\mu(f)$, let $\mathcal{A}=\{A_1,A_2,\ldots,A_k\}$ be a partition of $X$ such that

$$h_\mu(f,\mathcal{A})>h'>h'' \text{ with } \overline{h}<h''<h'<h_\mu(f)$$

and let
$$\phi_{n^r}:X\to\{1,\ldots,k\}^{n^r}$$

be the map which assigns to any point $x\in X$ its name of length $n^r$, i.e. $\phi_{n^r}(x)=w=\left(w_1,w_2,\ldots,w_{n^r}\right)$ accordingly $f^i(x)\in A_{w_i}$, $i=1,2,\ldots,n^r$. Let $Y_{n^r}=\phi_{n^r}(X)$, and let $\phi: X\to\{1,\ldots,k\}^{\mathbf{Z}}$ be the map defined like $\phi_n$ but assigning the name of bi-infinite length. For $\mu\in\mathcal{M}(X)$, let us denote $\tilde{\propto}$ the push-forward of $\mu$ by $\phi$, i.e. for any $w\in Y_{n^r}$

$\tilde{\mu}(w) = \mu(\phi^{-1}(w)) = \mu\left(\bigcap_{i=0}^{n^r-1} f^{-i}(A_{w_i})\right)$. If $h <' h_\mu(f)$ then there is a $N = N_{\mathcal{A}}$, such that for any $n \geq N$ holds

$$-\sum_{w \in Y_{n^r}} \tilde{\mu}(w) \log \tilde{\mu}(w) > n^r h.'$$

For any $\delta > 0$ there is a compact $B_j^\delta \subset A_j$ such that $\mu\left(B^\delta := \bigcup_{j=1}^{k} B_j^\delta\right) > 1 - \delta/2$. Let $F$ be a weak $*-$ neighborhood of $\mu$, and so $F^r$ is neighborhood of $\mu^{\otimes r}$, let

$$X_{n,F^r}^{B^\delta} = X_{n,F^r} \cap \left\{x : V_{I_{B^\delta}}(n,x) > 1 - \delta\right\},$$

where $I_A$ is the characteristic function of $A$ and $V_{I_A}$ is the multiergodic average like in (1) for the map $I_A$ Let

$$\tilde{\mu}_{n,\delta}(w) = \frac{\mu\left(\bigcap_{i=0}^{n^r-1} f^{-i}(A_{w_i}) \cap X_{n,F^r}^{B^\delta}\right)}{\mu\left(X_{n,F^r}^{B^\delta}\right)}.$$

We have $H(m, \mathcal{A}) := -\sum_{i=1}^{k} m(A_i) \log m(A_i) \leq n^r \log k$, for any $m \in \mathcal{M}(X)$, then there is a $N = N_{F,\delta}$ such that for $n \geq N_{F,\delta}$ is

$$\log\left(\left\{w \in Y_n : \tilde{\mu}_{n,\delta}(w) > 0\right\}\right) \geq -\sum_{w \in Y_n} \tilde{\mu}(w) \log \tilde{\mu}(w) > n^r\ h.''$$

Now define $E_{n,r} \subset X$ in this way, let $w \in Y_{n^r}$ with $\tilde{\mu}_{n,\delta}(w) > 0$, now pick a $x_{n,\delta} \in X_{n,F^r}^{B^\delta}$ with $\phi_n(x_{n,\delta}) = w$, so each $x_{n,\delta}$ has name $w$. Then consider a subset $\Gamma_{n,r}$ of $E_{n,r}$ in such a way that satisfy

$x \neq x \in' \Gamma_{n,r} \Rightarrow d_{n^r}^H(\phi_n(x), \phi_n(x)) > 3\delta n^r$. Here $d_{n^r}^H$ is the Hamming distance defined by $d_{n^r}^H(w,w) = card\left\{j : w_j \neq w_j\right\}$.

Since each $B_j^\delta$ is compact there is a $\varepsilon_\delta > 0$ such that if $x \in B_j^\delta$, $x \in' B_j^{\delta'}$ $(j \neq j)'$ then $x \neq x \in' E_{N_{F,\delta},r}$ If $x \neq x \in' E_{N_{F,\delta},r}$ then

$card\left\{i \in \{0,1,\ldots,n^r-1\} : f^i(x) \notin B^\delta \text{or } f^i(x) \notin B^\delta\right\} \leq 2\delta n^r$. Therefore if $x \neq x \in' \Gamma_{n,r}$ then

$$card\left\{\begin{matrix} i\in\{0,1,\dots,n^r-1\}: f^i(x)\notin B^\delta \text{ or } f^i(x)\notin B^\delta \\ \text{and } d\left(f^i(x),f^i(x)\right)>\varepsilon_\delta \end{matrix}\right\}\geq \delta n^r.$$

Thus $\Gamma_{n,r}$ is $\left(n^r,\varepsilon_\delta\right)-$ separated and

$$card\left(\Gamma_{n,r}\right)\geq \frac{\exp(n^r h)''}{\exp(H(3\delta)n^r)(k-1)^{n^r 3\delta}}, \tag{8}$$

with $H(t)=-t\log t-(1-t)\log(1-t)$.

Let $\bar{\delta}$ such that $H(3\delta)+3\bar{\delta}\log(k-1)<h-''\bar{h}$, so we have that if $\bar{h}<h_\mu(f)$ then exists a $\bar{\delta}>0$ such that for any neighborhood $F$ of $\mu$ there is a natural $N=N_{F,\delta}$ such that for any $N\geq N_{F,\delta}$ there is a set $\Gamma_{n,r}$ which is $\left(N^r_{F,\delta},\varepsilon_\delta\right)-$ separated and with $card\left(\Gamma_{n,r}\right)\geq \exp(n^r\bar{h})$. □

Let $\mathcal{M}_0\subset\mathcal{M}(X^r)$, non empty, and let $\widehat{\mathcal{M}_0}$ be the smallest closed, convex set containing $\mathcal{M}_0$. By $S\left(n,\varepsilon\mathcal{M}_0\right)$ is denoted the $\left(n^r,\varepsilon\right)-$ separated, subset of $X_{n,\mathcal{M}_0}$ of maximal cardinality. In similar way than in [10] can be proved that if $\mu\to h_\mu(f)$ is upper-semi continuous

$$S\left(\varepsilon,\mathcal{M}_0\right)\leq \sup\left\{h_\mu(f):\mu^{\otimes r}\in\mathcal{M}_0\right\},$$

where $S\left(\varepsilon,\mathcal{M}_0\right):=\limsup_{n\to\infty}\frac{1}{n^r}\log S\left(n,\varepsilon,\mathcal{M}_0\right)$.

Next we present our large deviations result, we follow [10]

**Theorem 2:**

1. Let $U\subset\mathcal{M}\left(X^r\right)$ open, $\nu\in\mathcal{M}(X)$ for any lower energy function $\Phi_\nu: X^r\to\mathbf{R}^+$ holds

$$\begin{aligned}&\liminf_{n\to\infty}\frac{1}{n^r}\log\nu\left(\left\{x:\mathcal{E}_{n,r}(x)\in U\right\}\right)\\ &\geq \inf\left\{h_\mu(f)-\int_{X^r}\Phi_\nu d\mu^{\otimes r}:\mu\in\mathcal{M}_E(X,f)\text{ and }\mu^{\otimes r}\in U\right\}.\end{aligned}$$

2. Let $F\subset\mathcal{M}\left(X^r\right)$ closed, convex, $\nu\in\mathcal{M}(X)$, for any upper energy function $\Phi_\nu: X^r\to\mathbf{R}^+$ holds

$$\limsup_{n\to\infty} \frac{1}{n^r} \log \nu\left(\left\{x : \mathcal{E}_{n,r}(x) \in F\right\}\right)$$
$$\leq \sup\left\{h_\mu(f) - \int_{X^r} \Phi_\nu d\mu^{\otimes r} : \mu \in \mathcal{M}_{inv}(X,f) \text{ and } \mu^{\otimes r} \in F\right\}.$$

***Proof:***

1. $U_{\nu,\delta} = U \cap \left\{m \in \mathcal{M}\left(X^r\right) : \int_{:X^r} \Phi_\nu dm - \int_{:X^r} \Phi_\nu d\mu^{\otimes r} < \delta\right\}$,

so that $U_{\nu,\delta}$ is open. Let $F$ be a weak $*-$ neighborhood of $\mu$, with $F^r \subset U_{\nu,\delta}$. The neighborhoods of a measure $\mu$ in the topology considered are of the form

$$F^{(\alpha)}(\mu) = F^{(\alpha)}\left(\mu; \varphi_1, \ldots, \varphi_{k;} \varepsilon_1, \ldots, \varepsilon_{k;}\right) =$$

$$= \left\{\rho \in \mathcal{M}(X) : \left|\int \varphi_i d\rho - \int \varphi_i d\mu\right| < \varepsilon_i, i = 1, 2, \ldots, k\right\}, \text{with } \|\varphi_i\| < 1.$$

If $\bar{h} < h_\mu(f)$ then, by the lemma1, there is a natural $\overline{N}$ and $\bar{\varepsilon} > 0$ such that for $n \geq \overline{N}$ there is $\left(n^r, \bar{\varepsilon}\right)-$ separated set $\Gamma_{n,r} \subset X_{n,\left(F^{1/2}\right)^r}$ with $card\left(\Gamma_{n,r}\right) \geq \exp(n^r \bar{h})$. Let $\bar{\delta} > 0$ such that $d(x,y) < \bar{\delta} \Rightarrow \|\varphi_i(x) - \varphi_i(y)\| < \varepsilon_i$. Let $\varepsilon^* = \min\left(\bar{\varepsilon}/2, \bar{\delta}\right)$, so that occurs that for any $n \geq \overline{N}$, $0 < \varepsilon < \varepsilon^*$ if $\Gamma_{n,r}$ is $\left(n^r, \bar{\varepsilon}\right)-$ separated then $\Gamma_{n,r}$ is $\left(n^r, 2\bar{\varepsilon}\right)-$ separated. Then holds
$x \neq y \in \Gamma_{n,r}$ then $B_{n,\varepsilon}(x) \cap B_{n,\varepsilon}(y) = \varnothing$, and $card\left(\Gamma_{n,r}\right) \geq \exp(n^r \bar{h})$.
We also have that $\bigcup_{x\in\Gamma_{n,r}} B_{n,\varepsilon}(x) \subset X_{n,F^r}$.

Let $n \geq \overline{N}$, $0 < \varepsilon < \varepsilon^*$, $F^r \subset U$, so that

$$\log \nu\left(\left\{x : \mathcal{E}_{n,r}(x) \in U\right\}\right) \geq card\left(\Gamma_{n,r}\right) \times \inf_{x\in\Gamma_{n,r}} \left\{\frac{1}{n^r} \log \nu\left(B_{n,\varepsilon}(x)\right)\right\}.$$

Therefore

$$\frac{1}{n^r} \log\log \nu\left(\left\{x : \mathcal{E}_{n,r}(x) \in U\right\}\right) \geq \frac{1}{n^r} \log card\left(\Gamma_{n,r}\right) +$$
$$\inf_{x\in\Gamma_{n,r}} \left\{\frac{1}{n^r} \log \nu\left(B_{n,\varepsilon}(x)\right) + \int \Phi_\nu d\mathcal{E}_{n,r}(x) - \int_{:X^r} \Phi_\nu d\mu^{\otimes r} - \delta\right\}.$$

Because $\Phi_\nu$ is an energy function we have

$$\limsup_{n\to\infty}\frac{1}{n^r}\log\nu\left(\left\{x:\mathcal{E}_{n,r}(x)\in U\right\}\right)\geq\bar{h}-\int_{:X^r}\Phi_\nu d\mu^{\otimes r}-\delta.$$

Since δ is arbitrary small the result follows.

2. The map $\Phi_\nu$ is upper-semi continuous and so bounded. Let δ > 0 and let us consider the partition $0=a_0<a_1<\ldots<a_n$ with $a_j=\sup\left\{\Phi_\nu(x):a_i-a_{i-1}<\delta, i=1,\ldots,j\right\}$. Let

$$F_j=\left\{\mu:\mu^{\otimes r}\in F:\int\Phi_\nu d\mu^{\otimes r}\in\left[a_{j-1},a_j\right]\right\}.$$

We have

$$\lim\sup_{n\to\infty}\frac{1}{n^r}\log\nu\left(X_{n,F}\right)=\max_{j=1,\ldots,k}\lim\sup_{n\to\infty}\frac{1}{n^r}\log\nu\left(X_{n,Fj}\right)$$

For ε > 0 there is a natural $N$ such that for any $n\geq N$ and for any $x\in X$ $\frac{1}{n^r}\log\nu\left(B_{n,\varepsilon}(x)\right)+\int\Phi_\nu d\mathcal{E}_{n,r}(x)<\delta.$

If $E$ is a $\left(n^r,\varepsilon\right)$– separated, subset of $X_{n,F}$ with maximal cardinality then $E\subset\bigcup_{x\in X_{n,F}}B_{n,\varepsilon}(x)$, thus for $n\geq N$

$$\lim\sup_{n\to\infty}\frac{1}{n^r}\log\nu\left(X_{n,Fj}\right)\leq\lim\sup_{n\to\infty}\frac{1}{n^r}\log S\left(n,\varepsilon,F_j\right)+\sup_{x\in X_{n,F_j}}\nu\left(B_{n,\varepsilon}(x)\right),$$

therefore

$$\lim\sup_{n\to\infty}\frac{1}{n^r}\log\nu\left(X_{n,Fj}\right)\leq\lim\sup_{n\to\infty}\frac{1}{n^r}\log S\left(n,\varepsilon,F_j\right)-\inf_{\mu:\mu^{\otimes r}\in F_j}\int_{:X^r}\Phi_\nu d\mu^{\otimes r}+\delta.$$

Since each $F_j$ is contained in $F$ we have

$$\lim\sup_{n\to\infty}\frac{1}{n^r}\log\nu\left(X_{n,Fj}\right)\leq\sup_{\mu\mathcal{M}_{inv}(X,f):\mu^{\otimes r}\in F_j}\left\{h_\mu(f)-a_{j-1}\right\}+\delta.$$

For any measure $\mu \in \mathcal{M}_{inv}(X,f)$ with $\mu^{\otimes r} \in F_j$ holds

$$\int_{:X^r} \Phi_\nu d\mu^{\otimes r} \leq a_{j-1} + 2\delta.$$

Finally

$$\limsup_{n\to\infty} \frac{1}{n^r} \log \nu\left(X_{n,F}\right) \leq \sup\left\{h_\mu(f) - \int_{X^r} \Phi_\nu d\mu^{\otimes r} : \mu \in \mathcal{M}_{inv}(X,f) \text{ and } \mu^{\otimes r} \in F\right\}$$

□

Let $\Psi : X^r \to \mathbf{R}$ and let $G_\Psi : \mathbf{R} \to \mathbf{R}$ be defined by $G_\Psi(t) = \inf\left\{h_\mu(f) - \int_{X^r} \Phi_\nu d\mu^{\otimes r} : \mu \in \mathcal{M}_{inv}(X,f) \text{ and } \int \Psi d\mu^{\otimes r} = t\right\}$. By applying the contraction principle to $\tau = \widehat{\Psi}$ with $\widehat{\Psi} : \mathcal{M}\left(X^r\right) \to \mathbf{R}$ the map defined by $\widehat{\Psi}(\rho) = \int \Psi d\rho$, we obtain

**Proposition 1:** Let $\Psi : X^r \to \mathbf{R}$ be a continuous map and $V_\Psi(n,x)$ the $V$–statistic for $\Psi$. If $\nu \in \mathcal{M}(X)$, and $J$ is an open real interval, then the following large deviation description holds

$$\liminf_{n\to\infty} \frac{1}{n^r} \log \nu\left(\left\{x : V_\Psi(n,x) \in J\right\}\right) \geq \inf_{t\in J}\left\{ G_\Psi(t)\right\}.$$

If $K$ is a closed real interval then

$$\limsup_{n\to\infty} \frac{1}{n^r} \log \nu\left(\left\{x : V_\Psi(n,x) \in K\right\}\right) \leq \sup_{t\in K}\left\{ G_\Psi(t)\right\}.$$ □

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