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Multifractal Spectrum for Barycentric Averages

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Abstract Let (X, ν) and Y be a measured space and a $CAT(0)$ space, respectively. If $\mathcal{M}_2(Y)$ is the set of measures on Y with finite second moment then a map $bar : \mathcal{M}_2(Y) \rightarrow Y$ can be defined. Also, for any $x \in X$ and for a map $\varphi : X \rightarrow Y$, a sequence $\left\{\mathcal{E}_{N,\varphi}(x)\right\}$ of empirical measures on Y can be introduced. The sequence $\left\{bar\left(\mathcal{E}_{N,\varphi}(x)\right)\right\}$ replaces in $CAT(0)$ spaces the usual ergodic averages for real valuated maps. It converges in Y (to a map $\overline{\varphi}(x)$) almost surely for any $x \in X$ (Austin J Topol Anal. 2011;3: 145–152). In this work, we shall consider the following multifractal decomposition in X :

$$K_{y,\varphi} = \left\{x : \lim_{N\rightarrow\infty} bar\left(\mathcal{E}_{N,\varphi}(x)\right) = y\right\},$$

and we will obtain a variational formula for this multifractal spectrum.

Keywords Multifractal analysis · Barycenter map · CAT(0)-spaces

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1 Introduction

An important subject in the area of Dimension Theory of Dynamical Systems is the *Multifractal Analysis*. It was originated in physics to study the behavior of measures supported on strange attractors. When chaotic dynamical behaviors are analyzed, invariant sets with a

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complex mathematical structure can be found. The analysis of these attractors can be done by a *fractal decomposition* of such invariant sets.

The general formulation of the Multifractal Analysis can be presented as follows. Let X be a set and $f : X \to [-\infty, +\infty]$; X can be partitioned in level sets:

$$K_\alpha = K_\alpha(f) = \{x : f(x) = \alpha\}.$$

Let G be a function defined on sets, and let $F(\alpha) = G(K_\alpha)$, the map F is called *the multifractal spectrum* specified by the pair (f, G). An important example is when

$$f(x) = D_\mu(x) := \lim_{r\to 0} \frac{\log(\mu(B_r(x)))}{-\log r},$$

the *pointwise dimension of the measure* μ, and $F(\alpha) = \dim_H K_\alpha$, the Hausdorff dimension. This is called the *pointwise dimension spectrum.*

Dynamical examples are (with T a map $T : X \to X$):

Local entropies spectrum. In this case, we have

$$f(x) = h_\mu(T, x) \quad \text{and} \quad F(\alpha) = h_{top}(T, K_\alpha),$$

where $h_\mu(T, x)$ is the pointwise entropy of the measure μ and $h_{top}(T, .)$ is the topological entropy defined by Bowen [2] (here the underlying set need not to be neither compact nor invariant).

Ergodic averages. Here,

$$f(x) = \lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n} \varphi\left(T^i(x)\right), \text{ with } \varphi : X \to \mathbf{R}.$$

The problem of describing these spectra has been extensively studied. For dynamical systems satisfying special conditions, the description can be given by a map which is the Legendre transform of the multifractal map $F(\alpha)$. For the spectrum of ergodic averages, a variational description is given in [7]:

$$h_{top}(T, K_\alpha) = \sup\left\{h_\mu(T) : \int \varphi d\mu = \alpha\right\},$$

where $h_\mu(T)$ is the measure-theoretic entropy.

Here, we propose to study spectra specified by a map f valuated in more general spaces. Let X be a compact metric space with a Lebesgue measure ν, $T : X \to X$, and let $\varphi : X \to Y$, where Y is a complete, separable, $CAT(0)$ space. Empirical measures on Y across the image by $\varphi : X \to Y$ of the orbits of T can be defined by

$$\mathcal{E}_{N,\varphi}(x) := \frac{1}{N} \sum_{n=0}^{N-1} \delta_{\varphi(T^n(x))}, \tag{1}$$

where δ is the point mass measure.

Let $\mathcal{M}_2(Y)$ be the set of measures on Y with finite second moment, a map $bar : \mathcal{M}_2(Y) \to Y$, called the *barycenter map*, can be defined. With $bar(\mu)$ is denoted the barycenter of the measure μ. It holds the contraction property

$$d(bar(\mu_1), bar(\mu_2)) \le W_2(\mu_1, \mu_2),$$

where W_2 is the 2−Wasserstein metric in $\mathcal{M}_2(Y)$. This important result was initially proved by Sturm [6] and extended by Navas [4] to Buseman spaces. In the work of Navas, a new definition of barycenter, where the map *bar* is applied to measures with finite first moment and controlled by 1−Wasserstein metric, was introduced.

For maps in the class $L^2(X, Y, \nu)$ (see next section for the definition), the sequence $bar(\mathcal{E}_{N,\varphi}(x))$ converges in Y, almost surely for any x, to a map $\overline{\varphi}(x)$ which is constant almost surely when the action is ergodic [1]. In fact, this result was established in the more general setting of amenable, locally compact groups Γ, provided for this general situation the existence of adequate sequences $(F_n) \subset \Gamma$. This barycentric convergence can be seen as the $CAT(0)$ version of the Birkhoff ergodic average convergence, which corresponds to $Y = \mathbf{R}$. In his above mentioned work, Navas extended the result by Austin to L^1 maps valuated in non-positively curved spaces

This article is inspired in the work by Austin, and so we work in the L^2 setting. We believe that the main result can be extended to the L^1 setting and maps valuated in non-positively curved spaces.

We consider the multifractal decomposition

$$K_{y,\varphi} = \left\{ x : \lim_{N\to\infty} bar(\mathcal{E}_{N,\varphi}(x)) = y \right\}$$

and we describe the corresponding multifractal spectrum. We obtain a variational formula like Takens and Verbitski but with a contraction on the set

$$\{\nu : bar\,(\varphi_*\,(\nu)) = y\}\,,$$

where $\varphi_*\,(\nu)$ is the pushforward of the measure ν by φ.

For the dynamics, we shall impose the conditions of uniform separation and the $g-$almost product property ($g - APP$), which is weaker than specification. Also, we shall consider a class of maps $\varphi : X \to Y$ with the following distortion property: for any $\varepsilon > 0$, there exist a number $\eta > 0$ such that for any N holds that $z \in B_{N,\varepsilon}\,(g, x) \Longrightarrow \varphi\left(T^i\,(z)\right) \in B_{N,\varepsilon}\left(T^i\,(x)\right)$ for any $i \in \Lambda_N \subset \{0, 1, ..., N-1\}$. The definitions of $B_{N,\varepsilon}\,(g, x)$ and Λ_N are remembered in next section as well as the $g-$almost product property.

The main result to be proved in this work reads:

Theorem 1 *Let $T : X \to X$ with the $g-$almost product property and with the uniform separation property. Let $\varphi : X \to Y$ be a map with Y a geodesic, complete, separable, $CAT(0)-$space, and satisfying the bounded distortion property. Then*

$$h_{top}\left(T, K_{y,\varphi}\right) = \sup\{h_\nu\,(T) : \varphi_*\,(\nu) \quad \in \mathcal{M}_2\,(Y)\,,\; bar\,(\varphi_*\,(\nu)) = y\}\,.$$

To proof it, we shall go along the lines of reference [5], where the variational result Takens and Verbitski gave in [7] is generalized.

2 Preliminary Definitions

A geodesic space (Y, d) is a $CAT(0)$–space if for every geodesic triangle Δ in Y there is a comparison triangle $\overline{\Delta}$ in $\mathbf{R}^2$, i.e., a triangle with sides of the same length as the sides of Δ, such that distances between points on Δ are less than or equal to the distances between corresponding points on $\overline{\Delta}$.

Let $\mathcal{M}(Y)$ be the space of measures on Y, This space is endowed with the metric

$$d(\mu_1, \mu_2) = \sum_{n=0}^{\infty} 2^{-n} \left| \int \varphi_n d\mu_1 - \int \varphi_n d\mu_2 \right|,$$

where $\{\varphi_n\}$ is a dense subset of $C(X)$ with $0 \leq \varphi_n \leq 1$. The topology induced by this distance is known as the $*-$weak topology.

Let $\mathcal{M}_2(Y)$ be the set of all the measures in $\mathcal{M}(Y)$ with finite second moment, i.e., the measures μ which satisfy

$$\int_Y d(y, z)^2 d\mu(z) < \infty, \quad \text{for any } y \in Y.$$

A map $\varphi : X \to Y$ belongs to the class $L^2(X, Y, \nu)$, where ν is a measure on X, if

$$\int_X d(\varphi(x), y)^2 d\nu(x) < \infty \quad \text{for any } y \in Y.$$

The space $L^2(X, Y, \nu)$ can be endowed with the metric

$$d_2(\varphi, \psi) := \sqrt{\int_X d(\varphi(x), \psi(x))^2 \, d\nu(x)}. \tag{2}$$

The barycenter map $bar : \mathcal{M}_2(Y) \to Y$ is defined in the following way: for any $\mu \in \mathcal{M}_2(Y)$, there is an unique $y \in Y$ which minimizes $\int_Y d(y, z)^2 d\mu(z)$ [3], thus is defined $bar(\mu) = y$, and the value y is defined as the *barycenter of the measure* μ.

A *coupling* of two measures $\mu_1, \mu_2 \in \mathcal{M}(Y)$ is a measure $m \in \mathcal{M}(Y \times Y)$ that projects into μ 1 and μ 2 on the first and the second factor, respectively. The $2-$ *Wasserstein metric* is defined as

$$W_2(\mu_1, \mu_2) = \inf_{\substack{m \text{ coupling of} \\ \mu_1, \mu_2 \in \mathcal{M}(Y)}} \sqrt{\int_{Y \times Y} d(y, z)^2 \, dm(y, z)}. \tag{3}$$

We recall that the map bar is controlled by W_2 [6]: if $\mu_1, \mu_2 \in \mathcal{M}_2(Y)$ then

$$d(bar(\mu_1), bar(\mu_2)) \leq W_2(\mu_1, \mu_2). \tag{4}$$

The dynamical ball for $T : X \to X$ is

$$B_{n,\varepsilon}(x) = \left\{ z : \max \left\{ d\left(T^i(x), T^i(z)\right) : i = 0, 1, .., n \right\} < \varepsilon \right\}$$

Let $g : \mathbf{N} \to \mathbf{N}$ be a non-decreasing, non-bounded function such that

$$\frac{g(n)}{n} < 1 \text{ and } \frac{g(n)}{n} \to 0 \text{ as } n \to \infty.$$

The dynamic ball for T and g is defined as:

$$\begin{aligned} B_{n,\varepsilon}(g, x) = \{ & z : \text{there is a } \Lambda_n \subset \{0, 1, ..., n-1\} \text{ with} \\ & card(\{0, 1, \ldots, n-1\} - \Lambda_n) \leq g(n) \text{ and} \\ & \max \left\{ d\left(T^i(x), T^i(z)\right) : i \in \Lambda_n \right\} < \varepsilon \right\}. \end{aligned}$$

Definition 1 A map $T : X \to X$ has the*specification property* if: for any $\varepsilon > 0$, there is an integer $M(\varepsilon)$ such that for any collection of intervals $I_j = \left[a_j, b_j\right] \subset \mathbb{Z}^+$, $j = 0, \cdots, k-1$ such that $a_j - b_{j-1} \geq M(\varepsilon)$, $j = 1, \cdots, k-1$ and for any $x_0, \cdots, x_{k-1} \in X$ there is a $x \in X$ such that

$$d\left(T^{a_j+n}(x), T^n(x_j)\right) < \varepsilon, \quad \text{for } 0 \leq n \leq b_j - a_j, \quad j = 0, 1, 2.,,, k-1.$$

Definition 2 A map $T : X \to X$ satisfies the $g-$*almost product property* (APP), with g a function as above, if there exists a map $m : \mathbf{R}^+ \to \mathbf{N}$ such that for any points $x_1, x_2, ..., x_k \in X$, for any $\varepsilon_1 > 0, \varepsilon_2 > 0, ..., \varepsilon_k > 0$ and for any numbers $n_i \geq m(\varepsilon_i)$, $i = 1, 2, .., k$ holds

$$\bigcap_{j=1}^{k} T^{n_{j-1}}\left(B_{n_j,\varepsilon_j}\left(g, x_j\right)\right) \neq \varnothing.$$

The specification property implies APP, but there are systems with APP that do not fulfil specification [5].

Definition 3 Two points x, z are $(n, \varepsilon)-$*separated* if $d\left(T^j(x), T^j(z)\right) > \varepsilon$ holds for some $j = 0, 1, \ldots, n$. A set $E \subset X$ is $(n, \varepsilon)-$*separated* if all points of E are $(n, \varepsilon)-$separated. A pair of points x, z are $(\delta, n, \varepsilon)-$*separated* if $card\left\{j = 0, 1, ..., n-1 : d\left(T^j(x), T^j(z)\right) > \varepsilon\right\} \geq \delta n$. A set $E \subset X$ is $(\delta, n, \varepsilon)-$*separated* if all points of E are $(\delta, n, \varepsilon)-$separated.

Let $\varphi : X \to Y$ with Y a geodesic, complete, separable, $CAT(0)$ space, if $F \subset \mathcal{M}(Y)$ then define

$$X_{N,F,\varphi} := \left\{x : \mathcal{E}_{N,\varphi}(x) \in F\right\}.$$

By $R_{N,\varepsilon,F,\varphi}$ will be denote the maximal cardinality of $(N, \varepsilon)-$separated sets contained in $X_{N,F,\varphi}$ and by $R_{\delta,N,\varepsilon,F,\varphi}$ the maximal cardinality of $(\delta, N, \varepsilon)-$separated sets contained in $X_{N,F,\varphi}$.

Let $\nu \in \mathcal{M}(X)$ and let $\mathcal{F}_{\varphi_*(\nu)}$ be the filter of neighborhoods of $\varphi_*(\nu)$ in the weak topology in $\mathcal{M}(Y)$. We consider the following entropies:

$$\overline{S}(\nu, \varepsilon, \varphi) = \inf_{F \in \mathcal{F}_{\varphi_*(\nu)}} \limsup_{N \to \infty} \frac{1}{N} \log R_{N,\varepsilon,F,\varphi},$$
$$\underline{S}(\nu, \varepsilon, \varphi) = \inf_{F \in \mathcal{F}_{\varphi_*(\nu)}} \liminf_{N \to \infty} \frac{1}{N} \log R_{N,\varepsilon,F,\varphi},$$

and

$$\overline{S}(\nu, \varphi) = \lim_{\varepsilon \to 0} \overline{S}(\nu, \varepsilon, \varphi),$$
$$\underline{S}(\nu, \varphi) = \lim_{\varepsilon \to 0} \underline{S}(\nu, \varepsilon, \varphi).$$

If $h_\nu(T)$ is the measure-theoretic entropy of the measure ν and the map T, then we shall see that $\overline{S}(\nu, \varepsilon, \varphi) \leq h_\nu(T)$, for any φ, where the equality holds when ν is ergodic.

Finally, we recall the Bowen definition of topological entropy of sets. Let $Z \subset X$ and let $\mathcal{C}(n, \varepsilon, Z)$ be the collection of finite or countable coverings of the set Z by balls $B_{m,\varepsilon}(x)$ with $m \geq n$. Let

$$M(Z, s, n, \varepsilon) = \inf_{\mathcal{B} \in \mathcal{C}(n,\varepsilon,Z)} \sum_{B_{m,\varepsilon}(x) \in \mathcal{B}} \exp(-sm),$$

and set

$$M(Z, s, \varepsilon) = \lim_{n \to \infty} M(Z, s, n, \varepsilon).$$

This limit does exist since $M(Z, s, n, \varepsilon)$ is a non-decreasing function of n. There is an unique number $\overline{s}$ such that $M(Z, s, \varepsilon)$ jumps from $+\infty$ to 0. Let

$$H(Z, \varepsilon) = \overline{s} = \sup\{s : M(Z, s, \varepsilon) = +\infty\} = \inf\{s : M(Z, s, \varepsilon) = 0\}$$

and [2]

$$h_{top}(T, Z) = h_{top}(Z) = \lim_{\varepsilon \to 0} H(Z, \varepsilon).$$

The number $h_{top}(Z)$ is the *topological entropy* of Z.

3 Description of the Barycentric Averages Multifractal Spectrum

As we have mentioned, we want to analyze the multifractal spectrum for the decomposition

$$K_{y,\varphi} = \left\{x : \lim_{N \to \infty} bar(\mathcal{E}_{N,\varphi}(x)) = y\right\}.$$

Recall that we are considering the class of maps $\varphi : X \to Y$ with the bounded distortion property as defined above.

Definition 4 The map $T : X \to X$ has the *uniform separation property* if the following condition is satisfied: for any $\gamma > 0$, there are numbers $\overline{\delta} > 0$, $\overline{\varepsilon} > 0$ such that for any ergodic measure ν and for any $F \in \mathcal{F}_{\varphi_*(\nu)}$ there is a natural $\overline{N} = \overline{N}(F, \varphi, \nu)$ such that for any $N \geq \overline{N}$

$$R_{\overline{\delta},N,\overline{\varepsilon},F,\varphi} \geq \exp(N(h_\nu(T) - \gamma)).$$

Definition 5 A subset $\mathcal{M}_0$ of $\mathcal{M}(X)$ is *entropy dense* if for any $\nu \in \mathcal{M}(X)$, any $F \in \mathcal{F}_{\varphi_*(\nu)}$ and any $\overline{h} > h_\nu(T)$ there is a $\rho \in \mathcal{M}_0$ such that $\varphi_*(\nu) \in F$ and $\overline{h} > h_\rho(T)$.

Proposition 1 *Let us consider a dynamical system (X, T) having the $g-$almost product property. Let $x_1, x_2, ..., x_k \in X$, $\varepsilon_1 > 0, \varepsilon_2 > 0, ..., \varepsilon_k > 0$ and $N_j \geq m(\varepsilon_j)$, $j = 1, 2, .., k$, be given. Let us assume that for $\rho_1, \rho_2, ..., \rho_k \in \mathcal{M}(Y)$ holds $\mathcal{E}_{N_j,\varphi}(x_j) \in B_{r_j}(\rho_j)$. Then for any $z \in \bigcap_{J=1}^{k} T^{-M_{j-1}}\left(B_{N_j,\varepsilon_j}(g, x_j)\right)$ and for any probability measure ρ holds*

$$d_{\mathcal{M}}\left(\mathcal{E}_{M_k,\varphi}(z),\mu\right) \leq \frac{1}{M_k}\sum_{j=1}^{k} N_j\left[\frac{1}{N_j}\left(g\left(N_j\right)+\eta_j\left(N_j+g\left(N_j\right)\right)+r_j+d_{\mathcal{M}}\left(\rho_j,\mu\right)\right)\right], \tag{5}$$

with $M_j = N_1 + N_2 + ... + N_j$. The numbers η_j are those that correspond to ε_j in the bounded distortion property of φ.

Proof We have

$$\mathcal{E}_{N_j,\varphi}\left(T^{M_{j-1}}(z)\right) = \frac{1}{N_j}\sum_{i=0}^{N_j-1}\delta_{\varphi\left(T^{M_{j-1}}(z)\right)}, \tag{6}$$

then

$$\mathcal{E}_{M_k,\varphi}(z) = \frac{1}{M_k}\sum_{j=1}^{k} N_j\mathcal{E}_{N_j,\varphi}\left(T^{M_{j-1}}(z)\right). \tag{7}$$

Therefore,

$$d_{\mathcal{M}}\left(\mathcal{E}_{N_j,\varphi}\left(x_j\right),\mathcal{E}_{N_j,\varphi}\left(T^{M_{j-1}}(z)\right)\right) \leq \frac{1}{N_j}\sum_{i=0}^{N_j-1} d\left(\varphi\left(T^i\left(x_j\right)\right),\varphi\left(T^{M_{j-1+i}}(z)\right)\right).$$

Since

$$T^{M_{j-1}}(z) \in B_{N_j,\varepsilon_j}\left(g,x_j\right), \text{ for any } j = 1, 2, ..., k,$$

by the bounded distortion property of the map φ and the metric considered in Y, we obtain

$$\begin{aligned} d_{\mathcal{M}}\left(\mathcal{E}_{N_j,\varphi}\left(x_j\right),\mathcal{E}_{N_j,\varphi}(T^{M_{j-1}}(z))\right) &\leq \frac{1}{N_j} card\left(\left\{0,1,...,N_{j-1}-1\right\}\right)-\Lambda_{Nj}) \\ &+\eta_j\left(\frac{N_j-g(N_j)}{N_j}\right) \leq \\ &\frac{1}{N_j}\left[g(N_j)+\eta_j\left(N_j-g(N_j)\right)\right]. \end{aligned}$$

Let $\mu \in \mathcal{M}(Y)$, so

$$d_{\mathcal{M}}\left(\mathcal{E}_{M_k,\varphi}(z),\mu\right) = d_{\mathcal{M}}\left(\frac{1}{M_k}\sum_{j=1}^{k} N_j\mathcal{E}_{N_j,\varphi}\left(T^{M_{j-1}}(z)\right),\mu\right) \leq$$

$$\frac{1}{M_k}\sum_{j=1}^{k} N_j\left[d_{\mathcal{M}}\left(\mathcal{E}_{N_j,\varphi}\left(x_j\right),\mathcal{E}_{N_j,\varphi}\left(T^{M_{j-1}}(z)\right)\right)+d_{\mathcal{M}}\left(\mathcal{E}_{N_j,\varphi}\left(x_j\right),\rho_j\right)+d_{\mathcal{M}}\left(\rho_j,\mu\right)\right]\leq$$

$$\frac{1}{M_k}\sum_{j=1}^{k} N_j\left[\frac{1}{N_j}\left[g(N_j)+\eta_j\left(N_j-g(N_j)\right)\right]+r_j+d_{\mathcal{M}}\left(\rho_j,\mu\right)\right]$$

□

Next, we consider some results about the entropy maps $\nu \longmapsto \overline{S}(\nu,\varphi)$ and $\nu \longmapsto \underline{S}(\nu,\varphi)$.

Lemma 1 *For any measure $\nu \in \mathcal{M}(X)$ and $\varphi : X \to Y$ holds*

$$\overline{S}\,(\nu,\varphi) \leq h_\nu\,(T)\,.$$

Proof Let us suppose that $\overline{S}\,(\nu,\varphi) = \lim_{\varepsilon\to 0} \inf_{F\in\mathcal{F}_{\varphi*(\nu)}} \lim_{N\to\infty} \sup \frac{1}{N} \log R_{N,\varepsilon,F,\varphi} > h_\nu\,(T)$, so there are numbers $\overline{\varepsilon} > 0$, $\delta > 0$ such that for $\overline{\varepsilon} \geq \varepsilon$

$$\inf_{F\in\mathcal{F}_{\varphi*(\nu)}} \lim_{N\to\infty} \sup \frac{1}{N} \log R_{N,\varepsilon,F,\varphi} \geq h_\nu\,(T) + 2\delta. \tag{8}$$

Let us consider a sequence $\{D_N\}$ of closed, convex sets in $\mathcal{M}(X)$ with

$$\bigcap_{N\geq 1} D_N = \{\nu\}\,.$$

If $\{E_N\}$ be a sequence of (N,ε) −separated sets, then, like in the demonstration of the variational principle for the entropy (Theorem 8.6 in [8]) it can be proved that when the sequence of measures

$$\nu_N := \frac{1}{card\,E_N} \sum_{x\in E_N} \frac{1}{N} \sum_{n=0}^{N-1} \delta_{T^n(x)} \in D_N. \tag{9}$$

weakly converges to a measure ν holds

$$\lim_{N\to\infty} \frac{1}{N} \log card\,E_N \leq h_\nu\,(T)\,.$$

Let us consider a family of sets(N,ε) −separated sets $\{E_N\}$, of maximal cardinality in $X_{N,C_N,\varphi}$, with

$$C_N = \overline{\varphi_*\,(D_N)}.$$

Since $\{\nu_N\}$ converges weakly to ν and φ_* is continuous, we have that $\{\varphi_*\,(\nu_N)\}$ converges to $\varphi_*\,(\nu)$, and so $C_N \in \mathcal{F}_{\varphi_*(\nu)}$, for any $N \geq 1$.

Now

$$\lim_{N\to\infty} \sup \frac{1}{N} \log R_{N,\varepsilon,C_N,\varphi} \geq h_\nu\,(T) + 2\delta,$$

but if $x \in E_N$ then

$$\mathcal{E}_{N,\varphi}(x) \in C_N$$

and $x \in X_{N,C_{N,\varphi}}$.Therefore, if the E_N are of maximal cardinality in $X_{N,C_N,\varphi}$, then

$$h_\nu\,(T) \geq \lim_{N\to\infty} \frac{1}{N} \log card\,E_N = \lim_{N\to\infty} \frac{1}{N} \log R_{N,\varepsilon,F,\varphi}$$

contradicting (8). □

Lemma 2 *If $T : X \to X$ has the uniform separation property and the set of ergodic measures is entropy-dense then holds $\overline{S}\,(\nu,\varphi) = h_\nu\,(T)$.*

Proof Let $\gamma > 0$ and $F \in \mathcal{F}_{\varphi_*(\nu)}$, when ν is ergodic for $\gamma/2$ there are $\overline{\delta} > 0$, $\overline{\varepsilon} > 0$ such that

$$R_{\overline{\delta},N,\overline{\varepsilon},F,\varphi} \geq \exp\left[N\,(\,h_\nu\,(T)) - \gamma/2)\right],$$

for $n \geq N = N(F, \gamma/2, \mu)$. Let ρ be non-ergodic, and let $\overline{h} := h_\rho(T) - \gamma/2 < h_\rho(T)$, then there exists an ergodic measure ν with

$$h_\nu(T) - h_\rho(T) < \gamma/2.$$

Thus, there is a natural $\overline{N} = \overline{N}(F, \gamma/2, \nu)$ such that

$$R_{\overline{\delta},N,\overline{\varepsilon},F,\varphi} \geq \exp\left[N\left(h_\nu(T) - \gamma/2\right)\right] > \exp\left[N\left(h_\rho(T) - \gamma\right)\right]$$

for $N \geq \overline{N}$. Then

$$R_{\overline{\delta},N,\overline{\varepsilon},F,\varphi} \geq \exp\left[N\left(h_\rho(T) - \gamma\right)\right],$$

for $N \geq \overline{N_1}(F, \eta, \rho) := \overline{N}(F, \eta/2, \nu)$.

Therefore,

$$h_\nu(T) \leq \lim_{\varepsilon\to 0}\lim_{\delta\to 0}\inf_{F\in\mathcal{F}_{\varphi_*(\nu)}}\lim_{n\to\infty}\inf\frac{1}{N}\log\ R_{\delta,N,\varepsilon,F,\varphi} \leq \underline{S}(\nu,\varphi) \leq \overline{S}(\nu,\varphi).$$

By Lemma 1 is valid that $\overline{S}(\nu,\varphi) \leq h_\nu(T)$, for any measure ν. Thus,

$$S(\nu,\varphi) = h_\nu(T).$$

□

The Theorem 1 will be derived from a more general result. Let $V_\varphi(x)$ be the set of limit points of the sequence $\left\{\mathcal{E}_{N,\varphi}(x)\right\}$ and let Ω be a compact, connected subset of $\mathcal{M}_2(Y)$. Now set

$$G_{\Omega,\varphi} = \left\{x : V_\varphi(x) = \Omega\right\}.$$

The multifractal spectrum appears as a particular case, which corresponds to

$$\Omega = A(y) := \{\mu : bar(\mu) = y\}.$$

Notice that $\lim_{N\to\infty} bar(\mathcal{E}_{N,\varphi}(x)) = y$ is equivalent, by the continuity of *bar*, to that the sequence $\left\{\mathcal{E}_{N,\varphi}(x)\right\}$ has all its limit points in the set $A(y)$

Before proving the *Theorem 1*, we find the upper bound for $h_{top}(G_{\Omega,\varphi})$.

Proposition 2 *It holds:*

$$h_{top}(G_{\Omega,\varphi}) \leq \inf\{h_\nu(T) : \varphi_*(\nu) \in \Omega\}$$

Proof Let $s := \sup\{h_\nu(T) : \varphi_*(\nu) \in \Omega\}$ and let $\overline{s}$ be such that $\overline{s} - s = 2\delta > 0$. For $F \in \mathcal{F}_{\varphi_*(\nu)}$ we have

$$\inf_{F\in\mathcal{F}_{\varphi_*(\nu)}}\lim_{N\to\infty}\sup\frac{1}{N}\log\ R_{N,\varepsilon,F,\varphi} \leq h_\nu(T).$$

For any $\varepsilon > 0$, there is a neighborhood $F(\nu,\varepsilon)$ of $\varphi_*(\nu)$ and a natural $\overline{N} = \overline{N}(F(\nu,\varepsilon))$ such that

$$\frac{1}{N}\log\ R_{N,\varepsilon,F(\nu,\varepsilon),\varphi} \leq h_\nu(T) + \delta,$$

for $N \geq \overline{N}$.

Let E be a $(N, \varepsilon)-$separated set in $X_{N,F(\nu,\varepsilon)}$ which maximal cardinality. If it is considered the covering $\left\{B_{N,\varepsilon}(x) : x \in E\right\}$ of $X_{n,F(\mu,\varepsilon)}$ then results

$$M\left(X_{N,F(\mu,\varepsilon)}, \overline{s}, N, \varepsilon\right) \leq \exp(-n\overline{s})\, R_{N,\varepsilon,F(\nu,\varepsilon),\varphi} \leq$$
$$\exp(-N(\delta - s)) \exp(N h_\nu(T)) \leq \exp(-N\delta).$$

Let

$$g_{\Omega,\varphi} = \left\{x : \left\{\mathcal{E}_{N,\varphi}(x)\right\} \text{ has a limit point in } \Omega\right\}.$$

Let us cover Ω with sets $\left\{F\left(\nu_j, \varepsilon\right)\right\}_{j=1,\ldots,k_\varepsilon}$, so if $\left\{\mathcal{E}_{N,\varphi}(x)\right\}$ has a limit point in Ω then $x \in X_{N,F(\nu_j,\varepsilon)}$ for $N \geq \overline{N}\left(F\left(\nu_j, \varepsilon\right)\right)$ and for some $1 \leq j \leq k_\varepsilon$. Thus, if

$$N \geq \max\left\{\overline{N}\left(F\left(\nu_j, \varepsilon\right)\right) : 1 \leq j \leq k_\varepsilon\right\}$$

then

$$M\left(g_{\Omega,\varphi}, \overline{s}, N, \varepsilon\right) \leq k_\varepsilon \sum_{m \geq N} \exp(-m\delta),$$

and so

$$M\left(g_{\Omega,\varphi}, \overline{s}, N, \varepsilon\right) < \infty.$$

Hence,

$$h_{top}(g_{\Omega,\varphi}) \leq s = \sup\{h_\nu(T) : \varphi_*(\nu) \in \Omega\}.$$

Let

$$G_{\nu,\varphi} = \left\{x : V_\varphi(x) = \{\varphi_*(\nu)\}\right\},$$

therefore $h_{top}(G_{\nu,\varphi}) \leq h_\nu(T)$ and $G_{\Omega,\varphi} \subset G_{\nu,\varphi}$, for any $\nu \in \mathcal{M}(X)$. So that

$$h_{top}(G_{\Omega,\varphi}) \leq \inf\{h_\nu(T) : \varphi_*(\nu) \in \Omega\} \tag{10}$$

□

Remark As seen, to obtain the above result, the conditions of uniform separation and almost product property are not used. They are used for getting the lower bound for $h_{top}\left(G_{\Omega,\varphi}\right)$.

Proof of the Theorem 1: Let $\gamma > 0$ and $\overline{h} := \inf\{h_\nu(T) : \varphi_*(\nu) \in \Omega\} - \gamma$, we construct, following Pfister1, a set G contained in $G_{\Omega,\varphi}$ and with $h_{top}(G) \geq \overline{h}$. Let $\varepsilon > 0$, there is a sequence of measures $\{\rho_1, \rho_2, ..., \rho_n\} \subset \Omega$ such that if $\mu \in \Omega$ then $d_{\mathcal{M}}(\mu, \rho_i) < \varepsilon$, $i = 1, 2, ..., n$. Then can be found an infinite sequence $\{\rho_1, \rho_2, ..., \rho_n, ...\} \subset \Omega$ with $d_{\mathcal{M}}(\rho_n, \rho_{n+1}) \to 0$, as $n \to \infty$ and such that $\overline{\{\rho_i : i > n\}} = \Omega$. There exist $\overline{\delta} > 0, \overline{\varepsilon} > 0$ such that for $F \in \mathcal{F}_{\varphi_*(\nu)}$, there is a natural $\overline{N} = \overline{N}(F, \varphi, \nu)$ such that for any $N \geq \overline{N}$

$$R_{\overline{\delta},N,\overline{\varepsilon},F,\varphi} \geq \exp\left[N\left(h_\nu(T) - \gamma\right)\right].$$

Let $\{r_k\}$, $\{\varepsilon_k\}$ be sequences with $r_k \searrow 0$, $\varepsilon_k \searrow 0$ and such that if $\{N_k\}$ are the corresponding of $\{\varepsilon_k\}$ in the bounded distortion property of φ then $\eta_k \searrow 0$. We can now consider a sequence of measures $\{\rho_k\}$ such that $B_{r_k}(\rho_k) \in \mathcal{F}_{\varphi_*(\nu)}$. Thus, for $\varepsilon_1 < \overline{\varepsilon}$, there is a sequence $\{N_k\}$ and a family of $\left(\overline{\delta}, N_k, \overline{\varepsilon}\right)-$separated sets $\{E_k\} \subset X_{N_k,B_{r_k}(\rho_k)}$ such that

$$card\, E_k \geq \exp\left[N\,\overline{h}\right].$$

So that if $x \in E_k$ then $d_{\mathcal{M}}\left(\mathcal{E}_{N_k,\varphi}(x), \rho_k\right) < r_k$. It can be assumed that

$$\overline{\delta} N_k > 2g(N_k) + 1 \quad \text{and} \quad \frac{g(N_k)}{N_k} < \varepsilon_k.$$

It holds that if $x \in E_k$, $z \in B_{N_k,\varepsilon_k}(g, x)$ then $\mathcal{E}_{N_k,\varphi}(z) \in B_{r_k+2\varepsilon_k}(\rho_k)$. Let us choose an increasing sequence $\{L_k\}$, $L_k \in \mathbf{N}$, such that

$N_{k+1} \leq r_k \sum_{j=1}^{k} L_j N_j$ and $\sum_{j=1}^{k-1} L_j N_j \leq r_k \sum_{j=1}^{k} L_j N_j$. Now let us consider the sequences $\{N'_j\}$, $\{\varepsilon'_j\}$, $\{E'_j\}$ defined as follows:

for $j = L_1 + ... + L_{k-1} + r, \quad 1 \leq r \leq L_k$ let $N'_j = N_k, \quad \varepsilon'_j = \varepsilon_k$ and $E'_j = E_k$.

Let $M_j = \sum_{l=1}^{j} N'_l$ and let

$$G_m = \left\{ z : T^{M_{j-1}}(z)) \in \bigcup_{x_j \in E'_j} B_{N'_j,\varepsilon'_j}\left(g, x_j\right), \quad x_j \in E'_j, \, j = 1, .., m \right\}. \tag{11}$$

Now set

$$G := \bigcap_{m \geq 1} G_m.$$

Any element of G can be labeled by a sequence $x_1 \, x_2 ...$, with $x_j \in E'_j$. Also is valid: let $x_j, y_j \in E'_j, x_j \neq y_j$, if $x \in B_{N'_j,\varepsilon'_j}\left(g, x_j\right)$, $y \in B_{N'_j,\varepsilon'_j}\left(g, y_j\right)$ then

$$\max \left\{ d\left(T_k(x), T_k(y)\right) : k = 0, ..., N_j - 1 \right\} > 2\varepsilon,$$

with $\varepsilon < \overline{\varepsilon}/4$. Let us see now that

$$G \subset G_{\Omega,\varphi}.$$

For this let $\{\rho'_m\}$ be the sequence of measures in Y defined by:

$$\rho'_m = \rho_k if \sum_{j=1}^{k-1} L_j N_j \leq m \leq \sum_{j=1}^{k} L_j N_j.$$

We may suppose that $\sum_{l=1}^{j} N_l L_l \leq M_k \leq \sum_{l=1}^{j+1} N_l L_l$, so that $\rho'_{M_k} = \rho_{j+1}$. Thus

$$\begin{aligned} d_{\mathcal{M}}\left(\mathcal{E}_{M_k,\varphi}(z), \rho'_{M_k}\right) &\leq \frac{1}{M_k} \sum_{l=1}^{j} N_l L_l \times d_{\mathcal{M}}\left(\mathcal{E}_{\sum_{l=1}^{j} N_l L_l,\varphi}(z), \rho'_{M_k}\right) \\ &+ \frac{N_j L_j}{M_k}\left[\rho_j + 2K_j + d_{\mathcal{M}}\left(\rho_j, \rho_{j+1}\right)\right] \\ &+ \frac{M_k - \sum_{l=1}^{j} N_l L_l}{M_k}\left[\rho_{j+1} + 2K_{j+1}\right] \\ &\leq 2r_j + 2K_j + d_{\mathcal{M}}\left(\rho_j, \rho_{j+1}\right) + r_{j+1} + 2K_{j+1}. \end{aligned}$$

Therefore, if $\sum_{l=1}^{j} N_l L_l \leq M_k \leq \sum_{l=1}^{j+1} N_l L_l$, then $d_{\mathcal{M}}\left(\mathcal{E}_{M_k,\varphi}(z), \rho'_{M_k}\right) \to 0$, as $j \to \infty$, so that the sequence $\{\mathcal{E}_{M_k,\varphi}(z)\}$, $z \in G$, has the same limit points of $\{\rho'_m\}$, which is considered to have all its limit points in $G_{\Omega,\varphi}$. This proves that $G \subset G_{\Omega,\varphi}$ and so $h_{top}(G_{\Omega,\varphi}) \geq h_{top}(G) \geq \overline{h}$.

Let $s < \overline{h}$, the set G is closed, and so it is compact, let us consider a finite covering $\mathcal{U}$ by balls $B_{m,\varepsilon}(x)$ having non-empty intersection with G. Now

$$M(G, s, N, \varepsilon) = \inf_{\mathcal{U}\in\mathcal{C}(n,\varepsilon,G)} \sum_{B_{m,\varepsilon}(x)\in\mathcal{U}} \exp(-sm),$$

it can considered balls $B_{M_r,\varepsilon}(x)$, instead of $B_{m,\varepsilon}(x)$, with $M_r \leq m \leq M_{r+1}$, and let $\mathcal{U}_0$ be the covering formed by these such a balls. Thus

$$M(G, s, N, \varepsilon) = \inf_{\mathcal{U}\in\mathcal{C}(n,\varepsilon,G)} \sum_{B_{m,\varepsilon}(x)\in\mathcal{U}} \exp(-sm) \geq \inf_{\mathcal{U}\in\mathcal{C}(n,\varepsilon,G)} \sum_{B_{M_r,\varepsilon}\in\mathcal{U}_0} \exp(-sM_{r+1}).$$

Let $\mathcal{U}_0$ be a covering by balls $B_{M_r,\varepsilon}(z)$ and let $m = \max\left\{r : B_{M_r,\varepsilon}(z) \in \mathcal{U}_0\right\}$. Let

$$W_k := \prod_{i=1}^{k} E_i, \qquad \overline{W_m} = \bigcup_{k=1}^{m} W_k.$$

Let us consider the points $x_j, y_j \in E'_j$, $x_j \neq y_j$, as we pointed out earlier, if $x \in B_{N'_j,\varepsilon'_j}\left(g, x_j\right)$, $y \in B_{N'_j,\varepsilon'_j}\left(g, y_j\right)$ then $d\left(T^l(x), T^l(y)\right) > 2\varepsilon$, for any $l = 0, ..., N_j - 1$, and with $\varepsilon < \overline{\varepsilon}/4$. Now for any $x \in B_{M_r,\varepsilon}(z) \cap G$ there is a uniquely determined $z = z(x) \in W_r$.

We have, as a particular case of the earlier results, that

$$h_{top}(G_{\nu,\varphi}) = h_\nu(T), \text{ for any } \nu \in \mathcal{M}(X).$$

Recall that $x \in K_{y,\varphi}$ and $V_\varphi(x) \subset A(y) := \{\mu : bar(\mu) = y\}$ are equivalent statements. Let $\nu \in \mathcal{M}(X)$ with $\varphi_*(\nu) \in A(y) \cap \mathcal{M}_2(Y)$, therefore $G_{\nu,\varphi} \subset G_{A(y),\varphi}$ and

$$h_{top}(G_{A(y),\varphi}) \geq h_{top}(G_{\nu,\varphi}) \geq h_\nu(T) \text{ for any } \nu \in \mathcal{M}(X) \text{ with} \varphi_*(\nu) \in A(y) \cap \mathcal{M}_2(Y).$$

For the other inequality, we have

$$G_{A(y),\varphi} \subset g_{A(y),\varphi},$$

and so

$$h_{top}(G_{A(y),\varphi}) \leq h_{top}(g_{A(y),\varphi}) \leq \sup\{h_\nu(T) : \varphi_*(\nu) \in A(y) \cap \mathcal{M}_2(Y)\},$$

(by Proposition 2). □

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