Journal of Interdisciplinary Mathematics

ISSN: 0972-0502 (Print) 2169-012X (Online) Journal homepage: http://www.tandfonline.com/loi/tjim20

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To cite this article: María B. Pintarelli & Fernando Vericat (2016) On the numerical solution of the linear and nonlinear Poisson equations seen as bi-dimensional inverse moment problems, Journal of Interdisciplinary Mathematics, 19:5-6, 927-944, DOI: 10.1080/09720502.2014.916845

To link to this article: http://dx.doi.org/10.1080/09720502.2014.916845

Published online: 27 Dec 2016.

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Journal of Interdisciplinary Mathematics
Vol. 19 (2016), No. 5&6, pp. 927–944
DOI : 10.1080/09720502.2014.916845

On the numerical solution of the linear and nonlinear Poisson equations seen as bi-dimensional inverse moment problems

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Abstract

The numerical solution of the bi-dimensional nonlinear Poisson equations under Cauchy boundary conditions is considered. Using Green identity we show that this problem is equivalent to solve a bi-dimensional Fredholm integral equation of the first kind which can in turn be handled as a bi-dimensional generalized inverse moment problem. In the particular linear case the Helmholtz PDE is recovered and, within our scheme, the problem reduces to a bi-dimensional Hausdorff moment problem. In all the cases we find approximated solutions for the associated finite moment problems and bounds for the corresponding errors.

Keywords: *Nonlinear Poisson equation, Fredholm integral equations, Hausdorff moment problem, generalized moment problem*

Mathematics Subject Classification: *44A60; 65J22*

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1. Introduction

In this article we consider the bi-dimensional nonlinear Poisson equation under Cauchy boundary conditions on a rectangle. We write the equation in the general form

$$u_{xx}(x,y)+u_{yy}(x,y)=\alpha g\left[u(x,y)\right]+\beta f(x,y), \tag{1.1}$$

where we assume that the unknown function $u(x,y)$ is defined in the rectangle $D\equiv[a_1,b_1]\times[a_2,b_2]\subset\mathbb{R}^2$ over the reals $\mathbb{R}$, α and β are real numbers and g and f are functions: $g:\mathbb{R}\to\mathbb{R}$; $f:\ D\to\mathbb{R}$. According to the values taken by the constants α and β and the functions g and f, Eq.(1.1) yields particular equations which can be interpreted in terms of phenomena that are of interest to diverse branches of Science and Technology. Physically, the most relevant particular case is, perhaps, for $\alpha=-\frac{4\pi e}{\varepsilon}$; $\beta=0$; $g[\bullet]:=\sum_{s=1}^{m}c_sZ_s\exp\left[-\frac{eZ_s}{k_bT}\bullet\right]$ and $u(x,y)$ representing the electrostatic potential for a system of m species of point particles at number concentrations c_s, constrained to move on a plane with dielectric constant ε. Species s ($s=1,2,\cdots,m$) are assumed to carry a charge eZ_s ($e=$ electron charge) such that electroneutrality $\sum_{s=1}^{m}c_sZ_s=0$ is verified. In this case Eq.(1.1) is named nonlinear Poisson-Boltzmann[1]. By linearization of $g[\bullet]$ the linear version of the Poisson-Boltzmann equation is obtained in the form $\nabla^2u(x,y)=\kappa^2u(x,y)$ with $\kappa^2=\frac{4\pi e^2}{\varepsilon k_BT}\sum_{s=1}^{m}c_sZ_s^2$ the square of the so called Debye-Hückel inverse length κ. Here T denotes the system temperature and k_B is the Boltzmann constant.

In general, the linear version of Eq.(1.1):

$$u_{xx}(x,y)+u_{yy}(x,y)=\alpha u(x,y)+\beta f(x,y), \tag{1.2}$$

is called the Helmholtz equation. In electrostatics Eq.(1.2), with $\alpha=0$, $\beta=1$ and $f(x,y)$ giving the charge distribution in D, yields the specifically called Poisson equation which can be derived from the Gauss law. If still we consider the absence of charge distribution ($f(x,y)\equiv0$) the Laplace equation remains. If we take $\alpha=\kappa^2$; $\beta=0$, we obviously recover the linear Poisson-Boltzmann equation above mentioned. The Helmholtz equation appears also in many other branches of knowledge. With $\beta=0$ it often arises in the study of physical problems involving partial differential equations (PDEs) in both space (x,y) and time t and represents the time-independent form that results from applying the method of variables separation to the original equation[2], α being the separation constant. A typical example is the wave equation

$$u_{xx}(x,y,t)+u_{yy}(x,y,t)=\frac{1}{c^2}u_{tt}(x,y,t)$$

that appears in areas of physics such as the study of electromagnetic radiation, seismology, acoustics, etc. and so the associated Helmholtz equation may be related to these problems.

The non linear Poisson Eq.(1.1) as well as its linear version Eq.(1.2) have been numerically solved for several values of their parameters and expressions for the functions g and f and also for different boundary conditions using diverse methods. For example, Atkinson and Hansen[3] used Garlekin method to solve the nonlinear Poisson equation on the unit disk with zero Dirichlet boundary conditions. In general the techniques to solve Eqs. 1.1 and 1.2 include finite difference[4], finite elements[5], boundary integral methods (BIM)[6]-[8], etc.

In this paper we consider Eqs.(1.1-1.2) in their general form under Cauchy conditions on the boundary ∂D:

$$u(x=a_1,y)=\varphi_1(y);\, u_x(x=a_1,y)=\varphi_2(y) \tag{1.3 a}$$

$$u(x=b_1,y)=\varphi_3(y);\, u_x(x=b_1,y)=\varphi_4(y) \tag{1.3 b}$$

$$u(x,y=a_2)=\varphi_5(x);\, u_y(x,y=a_2)=\varphi_6(x) \tag{1.3 c}$$

$$u(x,y=b_2)=\varphi_7(x);\, u_y(x,y=b_2)=\varphi_8(x), \tag{1.3 d}$$

and numerically solve them by transforming the involved problem into a bi-dimensional inverse moment one. This approach was already suggested by Ang et. al.[9] in relation with the heat conduction equation and we have applied it to the nonlinear Klein-Gordon equation[10].

The work is organized as follows. In principle, we consider separately the linear and the nonlinear equations. Next section is devoted to the first one. There we transform Eq.(1.2) into an integral equation by using Green identity. The resulting integral equation is considered as a bi-dimensional Hausdorff moment problem which is regularized by solving a related finite problem as we did in reference [11] and also discuss in Appendix A.

In section III the nonlinear Poisson equation is considered. Now we view the resulting integral equation, obtained from the application of Green identity, as a bi-dimensional generalized moment problem of the type we have discussed in reference [12] for just one dimension and that we extend to involve two dimension integrals in[10] (see also Appendix B). In each of both cases we validate the corresponding numerical procedure by applying it to solve an equation that we specifically build from a given function in such a way that it be the exact solution.

2. Linear Poisson equation (Helmholtz equation)

2.1 *Green identity*

This section will be devoted to the linear Poisson equation (1.2) in the domain $D \equiv [0, M] \times [0, M]$ with $M \to \infty$. According to our program we firstly transform the PDE into an integral equation using Green identity. To this let take the auxiliary function

$$h(x, y; r, s) = e^{-xr} e^{-ys} \tag{2.1}$$

that verifies

$$h_{rr}(x, y; r, s) + h_{ss}(x, y; r, s) = (x^2 + y^2) h(x, y; r, s). \tag{2.2}$$

In the region $D = [0, M] \times [0, M]$ we apply the planar Green identity

$$\iint_D u \nabla^2 h dA + \iint_D (\nabla u . \nabla h) dA = \oint_{\partial D} u \nabla h . \breve{n} d\ell \tag{2.3}$$

where ∂D is the contour of the region D.

Replacing here expression 2.1 for h and using Eq.(2.2) together with Eq.(1.2) we obtain

$$\int_0^M \int_0^M \left[(x^2 + y^2) h(x, y; r, s) u(r, s) - \alpha u(r, s) h(x, y; r, s) \right] drds = \phi(x, y) \tag{2.4}$$

where

$$\begin{aligned}
&\phi(x,y) \\
&\equiv \int_0^M h(x,y;r,M)\left[u_s(r,M)+yu(r,M)\right]dr-\int_0^M h(x,y;M,s)\left[u_r(M,s)+xu(M,s)\right]ds \\
&-\int_0^M h(x,y;r,0)\left[u_s(r,0)+yu(r,0)\right]dr+\int_0^M h(x,y;0,s)\left[u_r(0,s)+xu(0,s)\right]ds \\
&+\beta\int_0^M\int_0^M h(x,y;r,s)f(r,s)\,drds.
\end{aligned} \tag{2.5}$$

2.2 *Hausdorff moment problem*

We assume that

$$\begin{aligned}
&\lim_{M\to\infty}\int_0^M h(x,y;M,s)\left[u_r(M,s)+xu(M,s)\right]ds \quad \to 0 \\
&\lim_{M\to\infty}\int_0^M h(x,y;r,M)\left[u_s(r,M)+yu(r,M)\right]dr \quad \to 0.
\end{aligned}$$

Then, taking the limit $M\to\infty$ in Eq.(2.5), we have

$$\int_0^\infty\int_0^\infty\left(x^2+y^2-\alpha\right)h(x,y;r,s)u(r,s)\,drds=\phi^*(x,y). \tag{2.6}$$

with

$$\begin{aligned}
&\phi^*(x,y) \\
&\equiv -\int_0^\infty h(x,y;r,0)\left[u_s(r,0)+yu(r,0)\right]dr+\int_0^\infty h(x,y;0,s)\left[u_r(0,s)+xu(0,s)\right]ds \\
&+\beta\int_0^\infty\int_0^\infty h(x,y;r,s)f(r,s)\,drds.
\end{aligned} \tag{2.7}$$

Setting $x=m$ and $y=n$ $\left(m,n\in\mathbb{N}\right)$ we obtain

$$\int_0^\infty\int_0^\infty e^{-mr}e^{-ns}u(r,s)\,drds=\mu_{mn},\quad m,n=0,1,2,\cdots \tag{2.8}$$

where

$$\mu_{mn} = \frac{\phi^*(m,n)}{(m^2+n^2-\alpha)}. \tag{2.9}$$

This can be viewed as a bi-dimensional generalized Stieltjes moment problem. By changing variables $(r,s)\to(z_1,z_2)$ where $z_1 = e^{-r}$; $z_2 = e^{-s}$, we have a Hausdorff moment problem given by

$$\int_0^1\int_0^1 z_1^m z_2^n w(z_1,z_2)\,dz_1dz_2 = \tilde{\mu}_{mn},\ \ m,n = 0,1,2,\cdots \tag{2.10}$$

where $\tilde{\mu}_{mn} \equiv \phi^*(m+1+\alpha_1, n+1+\alpha_2)/\left[(m+1+\alpha_1)^2+(n+1+\alpha_2)^2-\alpha\right]$ and

$$w(z_1,z_2) = z_1^{\alpha_1} z_2^{\alpha_2} u(-\ln z_1, -\ln z_2).$$

Here, α_1 and α_2 are conveniently chosen numbers so that the moments $\tilde{\mu}_{mn}$ be well defined. Eqs. (2.9) and (2.10) represent a bi-dimensional Hausdorff moment problem for $w(z_1,z_2)$. We have studied this problem in reference [11]. There we first consider the relative finite moment problem, say Eq. 2.9 but with $m,n = 0,1,2,\cdots,N$; $(N\in\mathbb{N})$ whose solution is expanded

$$w(z_1,z_2) = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\lambda_{ij}P_{ij}(z_1,z_2)$$

where $P_{ij}(z_1,z_2) = P_i(z_1)P_j(z_2)$ with $P_i(z)$ $(i = 0,1,2,\cdots)$ the Legendre polynomials defined in [0, 1 and the coefficients λ_{ij} are

$$\lambda_{ij} = \int_0^1\int_0^1 w(z_1,z_2)P_{ij}(z_1,z_2)\,dz_1dz_2 \qquad (i,j = 0,1,2,\cdots).$$

Then we estimate $w(z_1,z_2)$ by truncating the expansion:

$$w(z_1,z_2) \approx w_N(z_1,z_2) = \sum_{i=0}^{N}\sum_{j=0}^{N}\lambda_{ij}P_{ij}(z_1,z_2) \tag{2.11}$$

where the coefficients λ_{ij} are explicitly given by

$$\lambda_{ij} = \sum_{k_1=0}^{i}\sum_{k_2=0}^{j} c_{ik_1}c_{jk_2}\mu_{k_1k_2} \quad (i,j = 0,1,2,\cdots,N) \tag{2.12}$$

with

$$c_{ik} = \sqrt{2i+1}(-1)^k\binom{i}{k}\binom{i+k}{k} \tag{2.13}$$

In order that this method of truncated expansion[13] be valid we require[10] that $\phi^*(x,y) \in L^2[[0,\infty)\times[0,\infty)]$ and

$$\int_0^\infty\int_0^\infty \left[x u_x^2(x,y) + y u_y^2(x,y) \right] e^{(x+y)} dxdy < \infty. \quad (2.14)$$

In the Appendix A can be viewed the proof of the following theorem, which adapts some of the results of reference [10] to the present context:

Theorem 1: *Calling* $u_N(x,y) = e^{\alpha_1 x} e^{\alpha_2 y} w_N(e^{-x}, e^{-y})$, *if* $u(x,y)$ *verifies*

$$\left\| u(x,y) e^x \right\|_w \le E_1$$

$$\left\| u_x(x,y) e^x \right\|_w \le E_2$$

$$\left\| u(x,y) e^y \right\|_w \le E_3$$

$$\left\| u_y(x,y) e^y \right\|_w \le E_4,$$

where $E_1, E_2, E_3,\ E_4$ are positive constants and the norm $\left\| f(x,y) \right\|_w^2$ is defined as

$$\left\| f(x,y) \right\|_w^2 \equiv \int_0^\infty\int_0^\infty \left| f(x,y) \right|^2 e^{-(1+2\alpha_1)x} e^{-(1+2\alpha_2)y} dxdy, \quad (2.15)$$

then

$$\left\| u_N(x,y) - u(x,y) \right\|_w^2 \le \frac{1}{4(N+1)^2} \left(\alpha_1^2 E_1^2 + E_2^2 + \alpha_2^2 E_3^2 + E_4^2 \right). \quad (2.16)$$

Moreover if the moments $\tilde{\mu}_{mn} \equiv \phi(m+1+\alpha_1, n+1+\alpha_2)$ have error such that $Tr(\mu\mu^T) = \sum_{m=1}^N \sum_{n=1}^N \mu_{mn} \le \varepsilon^2$, then

$$\left\| u_N(x,t) - u(x,t) \right\|_w^2 \le \frac{1}{4(N+1)^2} \left(\alpha_1^2 E_1^2 + E_2^2 + \alpha_2^2 E_3^2 + E_4^2 \right) + \varepsilon^2 c^2, \quad (2.17)$$

with $c = (2N+1)(N+1)^2\, 2^{6N} \dfrac{2^8}{2^6 - 1}$.

Example 1: As an example of application of the procedure outlined, we numerically solve the linear Poisson equation

$$u_{xx}(x,y)+u_{yy}(x,y)=u(x,y)-2\left[2x-y(x-2)\right]e^{-(x+y)}$$

with the boundary conditions:

$$u(x=0,y)=0;\, u_x(x=0,y)=ye^{-y}$$

$$u(x,y=0)=0;\, u_y(x,y=0)=xe^{-x}$$

and compare our result with the exact solution

$$u(x,y)=xye^{-(x+y)}.$$

Here we use $N=5$ moments and obtain $\left\|u_5(x,t)-u(x,t)\right\|_w=0.000556895$. In Fig.1 the two surfaces are shown.

3. Nonlinear Poisson equation

In this Section we consider the nonlinear Poisson equation defined in general by Eq.(1.1) with the boundary conditions 1.3.

3.1 *Green identity*

In this case we use as the auxiliary function

$$h(x,t;r,s)=e^{-x(r+1)}\cos[ys] \tag{3.1}$$

that verifies

$$h_{rr}(x,y;r,s)+h_{ss}(x,y;r,s)=\left(x^2-y^2\right)h(x,y;r,s). \tag{3.2}$$

If we take here $x=y$ then we have

$$h_{rr}(y,y;r,s)+h_{ss}(y,y;r,s)=0.$$

In the region $D\equiv[a_1,b_1]\times[a_2,b_2]$ we apply the planar Green identity (2.3). Replacing there the expression 3.1 for h and using Eq.(3.2) together with Eqs.(1.1) and (1.3d) we obtain

$$\alpha\int_{a_1}^{b_1}\int_{a_2}^{b_2} g\left[u(r,s)\right]h(y,y;r,s)\,drds=\phi(y) \tag{3.4}$$

where

$$\begin{aligned}\phi(y) \equiv &-\int_{a_2}^{b_2}\left[h(y,y;b_1,s)u_r(b_1,s)-h(y,y;a_1,s)u_r(a_1,s)\right]ds\\ &+\int_{a_1}^{b_1}\left[h(y,y;r,b_2)u_s(r,b_2)-h(y,y;r,a_2)u_s(r,a_2)\right]dr\\ &+\int_{a_1}^{b_1}\left[h_s(y,y;r,b_2)u(r,b_2)-h_s(y,y;r,a_2)u(r,a_2)\right]dr\\ &-\int_{a_2}^{b_2}\left[h_r(y,y;b_1,s)u(b_1,s)-h_r(y,y;a_1,s)u(a_1,s)\right]ds\\ &+\beta\int_{a_1}^{b_1}\int_{a_2}^{b_2} h(y,y;r,s)u(r,s)\,drds.\end{aligned} \tag{3.5}$$

3.2 *Generalized moment problem*

Defining the kernel

$$K(r,s;y)\equiv h(y,y;r,s)=e^{-yr}\cos[ys] \tag{3.6}$$

we have

$$\int_{a_1}^{b_1}\int_{a_2}^{b_2} g(u(r,s))K(r,s;y)\,drds=-\frac{1}{\alpha}\phi(y) \tag{3.7}$$

By using a basis $\{\Psi_m(y)\}_{m=0}^{\infty}$ we transform this bi-dimensional Fredholm integral equation of the first kind into a bi-dimensional generalized moment problem of the type we study in reference [12]:

$$\int_{a_1}^{b_1}\int_{a_2}^{b_2} g(u(r,s))K_m(r,s)\,drds=\mu_m \qquad (m=0,1,2,\cdots) \tag{3.8}$$

where

$$K_m(r,s)=\int_{a_2}^{b_2} K(r,s;y)\Psi_m(y)\,dy \tag{3.9}$$

and the moments μ_m are

$$\mu_m=-\frac{1}{\alpha}\int_{a_2}^{b_2}\phi(y)\Psi_m(y)\,dy. \tag{3.10}$$

If the functions $\{K_m(r,s)\}_m$ are linearly independent the problem of generalized moments defined by Eqs. (3.8)-(3.10) can be solved as we do in [12]: finding the solution $\chi(r,s)=g(u(r,s))$ to the corresponding finite problem, say with $m=0,1,2,\cdots,N$ ($N\in\mathbb{N}$). Thus, if $g(u)$ has continuous inverse, then $g^{-1}\left[\chi(r,s)\right]$ will be a estimation of $u(r,s)$.

Let consider the basis $\{\psi_i(r,s)\}_{i=0}^{\infty}$ obtained by applying the Gram-Schmidt orthonormalisation process on $\{K_m(r,s)\}_{m=0}^{N}$ and then adding to the resulting set the necessary functions until an orthonormal basis is achieved. Thus

$$\left\langle \psi_i(r,s)\,|\,\psi_j(r,s)\right\rangle=\int_{a_1}^{b_1}\int_{a_2}^{b_2}\psi_i(r,s)\psi_j(r,s)\,drds=\delta_{ij} \qquad (i,j=0,1,2,...)$$

and the solution $\chi(r,s)$ can be expanded:

$$\chi(r,s)=\sum_{i=0}^{\infty}\lambda_i\psi_i(r,s),$$

but we approximate it by truncating the expansion[11]:

$$\chi(r,s)\approx\chi_N(r,s) \tag{3.11}$$

where

$$\lambda_i=\sum_{j=0}^{i}C_{ij}\mu_j \quad (i=0,1,2,\cdots,N) \tag{3.12}$$

with the coefficients C_{ij} verifying the linear system

$$C_{ij} = \left(\sum_{k=j}^{i-1} (-1) \frac{\langle K_i(r,s) \mid \psi_k(r,s) \rangle}{\left\| \psi_k(r,s) \right\|^2} C_{kj} \right) . \left\| \psi_i(r,s) \right\|^{-1} \quad \left(1 < i \leq N; 1 \leq j < i \right). \tag{3.13}$$

The diagonal terms are $C_{ii} = \left\| \psi_i(x) \right\|^{-1} \quad (i = 0,1,...,N)$ and $\langle u(r,s) \mid v(r,s) \rangle$ denotes the inner product in the Hilbert space.

In the Appendix B we extend to the bi-dimensional case the arguments used in reference [12] to demonstrate the

Theorem 2: *Let the set of real numbers $\{\mu_k\}_{k=0}^{N}$ and let ε and E be two positive numbers such that*

$$\sum_{k=0}^{N} \left| \int_{a_1}^{b_1} \int_{a_2}^{b_2} K_k(r,s) \chi(r,s) \, drds - \mu_k \right|^2 \leq \varepsilon^2 \tag{3.14}$$

and

$$\int_{a_1}^{b_1} \int_{a_2}^{b_2} \left[\left(b_1 - \alpha_1 \right)^2 \chi_r^2 + \left(b_2 - \alpha_2 \right)^2 \chi_s^2 \right] drds \leq E^2, \tag{3.15}$$

then

$$\int_{a_1}^{b_1} \int_{a_2}^{b_2} \left| \chi(r,s) \right|^2 drds \leq \min_n \left\{ \left\| \mathbf{C} \ \mathbf{C} \right\|^2 \varepsilon^2 + \frac{1}{2(n+1)} E^2; \qquad n = 0,1,...N \right\} \tag{3.16}$$

where $\mathbf{C}$ is the lower triangular matrix with elements $C_{ij} \left(1 < i \leq N; 1 \leq j < i \right)$ (Eq.3.13) and $\mathbf{C}^{\dagger}$ its transpose. Moreover the truncated solution $\chi_N(r,s)$ given by Eq.(3.11) verifies

$$\int_{a_1}^{b_1} \int_{a_2}^{b_2} \left| \chi_N(r,s) - \chi(r,s) \right|^2 drds \leq \left\| \mathbf{C} \ \mathbf{C} \right\|^2 \varepsilon^2 + \frac{1}{2(N+1)} E^2. \tag{3.17}$$

If $g^{-1}(x)$ is Lipschitz in $\mathbb{R}^2$, say if $\left\| g^{-1}(x) - g^{-1}(x') \right\| \leq \lambda \left\| x - x' \right\|$ for some λ and $\forall x, x' \in \mathbb{R}^2$ then, according to the previous theorem, is

$$\int_{a_1}^{b_1}\int_{a_2}^{b_2} \left|u_N(r,s)-u(r,s)\right|^2 drds \le \lambda\left\{\left\|\mathbf{C}\ \mathbf{C}\right\|^2 \varepsilon^2 + \frac{1}{2(N+1)}E^2\right\}.$$

Example 2: We apply the method to the nonlinear Poisson equation

$$u_{xx}(x,y)+u_{yy}(x,y)=e^{u(x,y)}$$

in the domain $D=[0,1]\times[0,1]$ and boundary conditions given by

$$u(x=0,y)=\ln\frac{4}{(1+y)^2};\ u_x(x=0,y)=-\frac{2}{1+y}$$

$$u(x=1,y)=\ln\frac{4}{(2+y)^2};\ u_x(x=1,y)=-\frac{2}{2+y}$$

$$u(x,y=0)=\ln\frac{4}{(1+x)^2};\ u_y(x,y=0)=-\frac{2}{2+x}$$

$$u(x,y=1)=\ln\frac{4}{(2+x)^2};\ u_y(x,y=1)=-\frac{2}{2+x},$$

In Fig. 2 we compare our numerical solution, obtained with $N=4$ moments, with the exact solution

$$u(x,y)=\ln\frac{4}{(1+x+y)^2}.$$

In this case the error is $\left\|u_4(x,t)-u(x,t)\right\|_{L^2}=0.137322$.

Support of this work by Consejo Nacional de Investigaciones Cientficas y Técnicas (PIP 112-200801-01192), Universidad Nacional de La Plata (Grant 11/I153) and Agencia Nacional de Promoción Cientfica y Tecnológica of Argentina (PICT 2007-00908) is greatly appreciated. F.V. is a member of CONICET.

Appendix A: Proof of Theorem 1

Taking into account the definitions of $u_N(x,t)$ and of the norm $\|\bullet\|_w^2$ we have

$$\|u_N(x,t)-u(x,t)\|_w^2 = \|w_N(z_1,z_2)-w(z_1,z_2)\|^2$$

where $\|f(x,y)\|^2 \equiv \int_0^1\int_0^1 |f(x,y)|^2\, dxdt$. But it is proved that

$$\|w_N(z_1,z_2)-w(z_1,z_2)\|^2 \le \frac{1}{4(N+1)^2}(I_1+I_2)$$

with

$$I_\gamma \equiv \left\|w_{z_\gamma}(z_1,z_2)\right\|^2 = \int_0^1\int_0^1 w_{z_\gamma}(z_1,z_2)^2\, dz_1dz_2 \qquad (\gamma=1,2)$$

Derivating $w(z_1,z_2)$ in Eq.(2.9) with respect to z_γ $(\gamma=1,2)$ and effecting the double integral we obtain for I_γ the expression

$$\begin{aligned} I_\gamma &= \int_0^\infty\int_0^\infty \alpha_\gamma^2 u(x,y)^2\, e^{-2\left(\alpha_1+\frac{(-1)^\gamma}{2}\right)x}\, e^{-2\left(\alpha_2-\frac{(-1)^\gamma}{2}\right)y}\, dxdy \\ &+\int_0^\infty\int_0^\infty \left[u_x(x,y)^{2-\gamma}+u_y(x,y)^{\gamma-1}\right]^2 e^{-2\left(\alpha_1+\frac{(-1)^\gamma}{2}\right)x}\, e^{-2\left(\alpha_2-\frac{(-1)^\gamma}{2}\right)y}\, dxdy. \end{aligned}$$

Besides, if noise is considered such that $Tr\left(\mu\mu^T\right)\le\varepsilon^2$, since in this case is[10]

$$\begin{aligned} \|u_N(x,t)-u(x,t)\|_w^2 &= \|w_N(z_1,z_2)-w(z_1,z_2)\|^2 \\ &\le \frac{1}{4(N+1)^2}(I_1+I_2)+c^2Tr\left(\mu\mu^T\right) \\ &\le \frac{1}{4(N+1)^2}\left(\alpha_1^2E_1^2+E_2^2+\alpha_2^2E_3^2+E_4^2\right), \end{aligned}$$

with $c=(2N+1)(N+1)^2\, 2^{6N}\dfrac{2^8}{2^6-1}$, then Eq.(2.17) is recovered. □

Appendix B: Proof of Theorem 2

We closely follow the demonstration given in reference [12] for the one-dimensional moment problem which in turn is based in Talenti work[13] for the Hausdorff problem. Here we just introduce the necessary modifications for the general bi-dimensional problem.

Without lost of generality we take $\mu_k = 0 \quad (k = 0,1,2,\cdots,N)$ in Eq.(3.14). Let write $\psi(r,s)$ in the form

$$\chi(r,s) = h_N(r,s) + t_N(r,s)$$

where $h_N(r,s)$ is the orthogonal projection of $\chi(r,s)$ on the linear space generated by the set $\{K_m(r,s)\}_{m=0}^{N}$ and $t_N(r,s) = \chi(r,s) - h_N(r,s)$ the orthogonal projection of $\chi(r,s)$ onto the orthogonal complement. The functions $h_N(r,s)$ and $t_N(r,s)$ can be expanded in the basis $\{\psi_i(r,s)\}_{i=0}^{\infty}$:

$$h_N(r,s) = \sum_{i=0}^{N} \lambda_i \psi_i(r,s); \qquad t_N(r,s) = \sum_{i=N+1}^{\infty} \lambda_i \psi_i(r,s)$$

with

$$\lambda_i = \int_{a_1}^{b_1}\int_{a_2}^{b_2} \psi_i(r,s)\chi(r,s)\,drds \qquad (i = 1,2,\cdots)$$

The relation between the coefficients λ_i and the moments

$$\mu_i = \int_{a_1}^{b_1}\int_{a_2}^{b_2} K_i(r,s)\chi(r,s)\,drds \qquad (i = 1,2,\cdots)$$

reads

$$\lambda_i = \sum_{j=0}^{i} C_{ij}\mu_j \quad (i = 0,1,2,\cdots)$$

where the matrix components C_{ij} are given by Eq.(3.13) in the text. Thus, we have

$$\sum_{j=0}^{i} C_{ij} g_j(r,s) = \psi_i(r,s) \qquad (i = 0,1,2,\cdots.)$$

or, in matricial form, $\lambda = C\mu$ where

$$\lambda = \begin{bmatrix} \lambda_0 \\ \lambda_1 \\ \vdots \\ \lambda_N \end{bmatrix} \qquad \mu = \begin{bmatrix} \mu_0 \\ \mu_1 \\ \vdots \\ \mu_N \end{bmatrix}$$

and

$$\mathbf{C} = \begin{bmatrix} C_{00} & & & \\ C_{10} & C_{11} & & \\ \vdots & \vdots & & \\ C_{N0} & C_{N1} & \cdots & C_{NN} \end{bmatrix}$$

Taking into account previous equations, the orthonormalisation condition of the set $\{\varphi_i(r,s)\}_{i=0}^{\infty}$ and the condition given by Eq.(3.14), we have

$$\int_{a_1}^{b_1}\int_{a_2}^{b_2} |h_N(r,s)|^2 \, drds = \langle \lambda | \lambda \rangle = \langle \mathbf{C}\ \mathbf{C}\mu | \mu \rangle \leq \|\mathbf{C}\ \mathbf{C}\| . \|\mu\|^2 = \|\mathbf{C}\ \mathbf{C}\|^2 \, \varepsilon^2 .$$

In order to estimate the norm of $t_N(r,s)$, we observe that each element of the orthonormal set $\{\psi_i(r,s)\}_{i=0}^{\infty}$ can in turn be expanded in terms of the elements of another orthonormal basis, in particular the set $\{P_{kl}(r,s)\}_0^{\infty}$, with $P_{kl}(r,s) = e^{-r}L_k(r)e^{-s}L_l(s)$ where $L_k(r)$ denotes Laguerre polynomials:

$$\psi_i(r,s) = \sum_{k=0}^{\infty}\sum_{l=0}^{\infty} \gamma_{i;kl} P_{kl}(r,s)$$

Then, calling $\lambda_{kl} = \sum_{i=N+1}^{\infty} \lambda_i \gamma_{i;kl}$, it follows

$$\int_{a_1}^{b_1}\int_{a_2}^{b_2} |t_N(r,s)|^2 \, drds \leq \sum_{k=0}^{\infty}\sum_{l=0}^{\infty} \frac{(k+1)}{N+1} \lambda_{kl}^2$$

and also

$$\int_{a_1}^{b_1}\int_{a_2}^{b_2} \left|t_N(r,s)\right|^2 drds \leq \sum_{k=0}^{\infty}\sum_{l=0}^{\infty} \frac{(l+1)}{N+1}\lambda_{kl}^2.$$

Therefore,

$$\int_{a_1}^{b_1}\int_{a_2}^{b_2} \left|t_N(r,s)\right|^2 drds \leq \frac{1}{2(N+1)}\int_{a_1}^{b_1}\int_{a_2}^{b_2} \left(r\chi_r^2 + s\chi_s^2\right)e^r e^s drds.$$

By adding these expressions for the two norms $\left\|h_N(r,s)\right\|^2$ and $\left\|t_N(r,s)\right\|^2$ the result 3.16 in the text is obtained. In a similar way is proved the inequality 3.17.

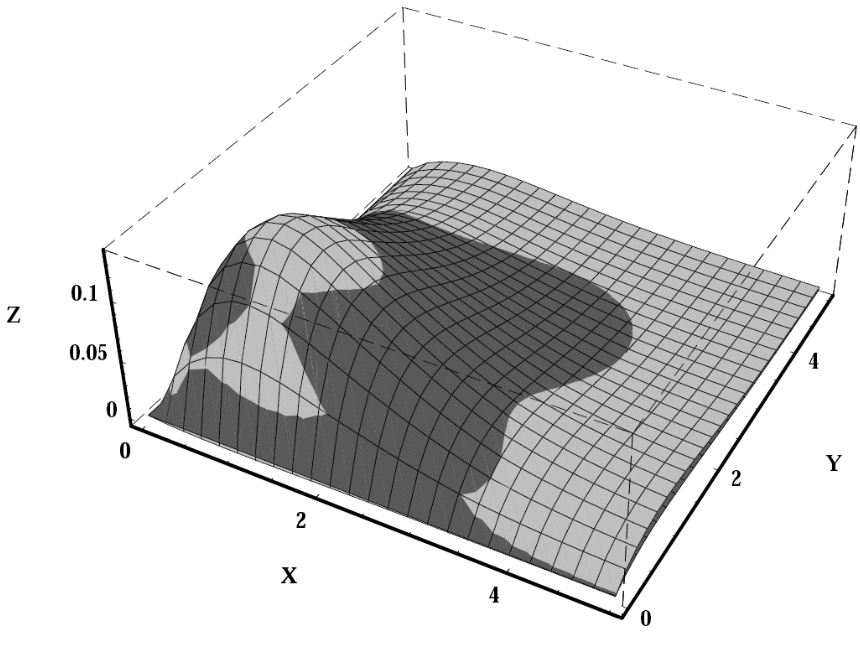

Fig. 1

Comparison of the exact solution $u(x, y)$ (dark grey) with the estimate $u_5(x, y)$ (light grey) for example 1. The function $u_5(x, y)$ is a truncated solution to the bi-dimensional finite Hausdorff moment problem obtained by transforming the linear Poisson equation using Green identity.

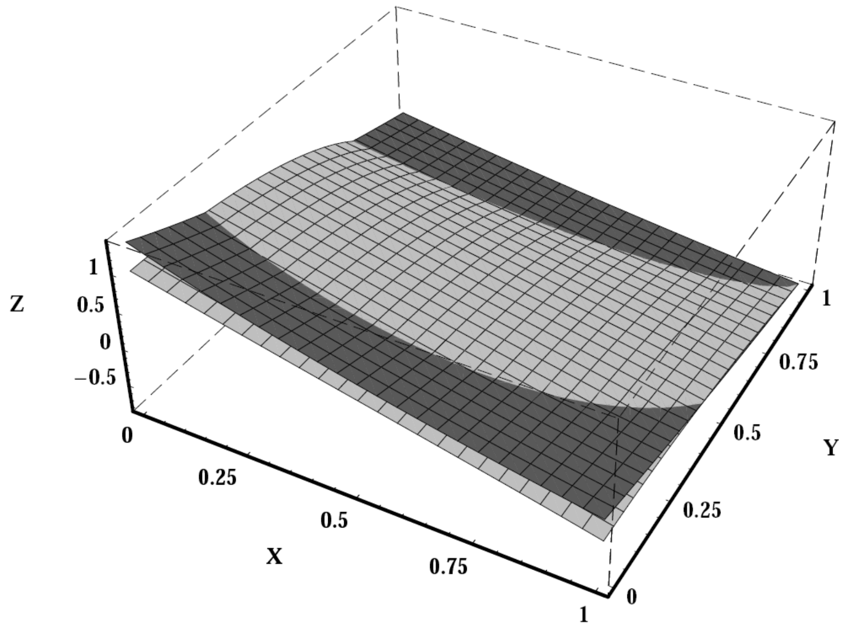

Fig. 2

Comparison of the exact solution *u(x, y)* (dark grey) with the estimate $u_4(x, y)$ (light grey) for example 2. The function $u_4(x, y)$ is a truncated solution to the bi-dimensional finite generalized moment problem obtained by transforming the nonlinear Poisson equation using Green identity.

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Received July, 2013