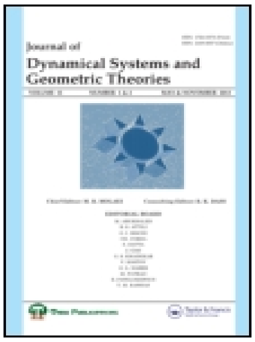

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SATURATEDNESS OF DYNAMICAL SYSTEMS UNDER THE ALMOST SPECIFICATION PROPERTY

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Abstract. A dynamical system is saturated when for any invariant measure μ, the topological entropy of the set of the $\mu-$generic points equals the measure-theoretic entropy of the system. This fact was confirmed by Fan, Liao and Peyrière for systems with specification. In a recent article we extended this result under the condition of non-uniform specification. In this work we consider another weaker condition than specification called almost specification property. This concept was introduced by Thompson as a modification of the almost property product by Pfister and Sullivan. We prove herein the saturatedness of systems under the Thompson condition. The saturatedness is a key point to establish a variational principle for $V-$statistics, as was developed by Fan, Schmeling and Wu.

AMS Classification: 37C45, 37B40

Keywords: Almost specification property; $V-$statistics

1. Introduction

The study of multiple ergodic averages was motivated in part by its relationship with combinatorial number theory. In particular by mean of multiergodic averages can be proved using arguments from Ergodic Theory and Dynamical Systems a version of the result of Szemeredi about the existence arithmetic progressions of arbitrary length. This was made by Furstenberg[7]. Another interesting motivation is the multifractal analysis of $V-$statistics, let us consider a topological dynamical system (X, f), with X a compact metric space and f a continuous map. Let $X^r = X \times ... \times X$ be the product of $r-$copies of X with $r \geq 1$, if $\Phi : X^r \to \mathbf{R}$ is a continuous map, then let

$$V_\Phi(n, x) = \frac{1}{n^r} \sum_{1 \leq i_1, \ldots, i_r \leq n} \Phi\left(f^{i_1}(x), ..., f^{i_r}(x)\right). \tag{1}$$

These averages are called the $V-$*statistics of order* r *with kernel* Φ. The multifractal decomposition for the spectra of $V-$statistics is

$$E_\Phi(\alpha) = \left\{x : \lim_{n\to\infty} V_\Phi(n, x) = \alpha\right\}.$$

Fan, Schemeling and Wu[5] have obtained the following variational principle for dynamical systems with the specification property.:

$$h_{top}(E_\Phi(\alpha)) = \sup\left\{h_\mu(f) : \int \Phi d\mu^{\otimes r} = \alpha\right\}, \tag{2}$$

where h_{top} is the topological entropy for non-compact nor invariant sets and $h_\mu(f)$ is the measure-theoretic entropy of μ. Here $\mu^{\otimes r}$ means $\mu \times ... \times \mu$, $r-$times. This generalizes the variational principle established by Takens and Verbitski for $r = 1$[12]. It is also interesting the study of the *irregular part* of the spectrum, or *historic set*, i.e. the set of points x for which $\lim_{n\to\infty} V_\Phi(n, x)$ does not exist. The denomination of historic corresponds to Ruelle, and is due to that these points may be interpreted as the changes in the ¨epochs¨ of the system. We have proved[8] that for topological dynamical systems satisfying the specification property, if the irregular part of the spectrum of multiple ergodic averages, or $V-$statistics is non-empty then it has the same topological entropy of the whole space X. In a recently submitted article[9] we considered a weak form of specification known as *non-uniform specification* condition, notion introduced by Varandas[14], and we proved that the result of [8] can be extended to systems with this property.

A key point to establish the variational principle for $V-$statistics as well the full entropy of the irregular set is the *saturatedness.* as seen in [5] and [8]. A dynamical system is called *saturated* when the topological entropy of the set of the $\mu-$generic points equals the measure-theoretic entropy of the system. for any invariant measure μ. Therefore, our objective is to establish saturatedness under weaker conditions than specification in order to extend the variational principle and the full entropy of the irregular part for systems under these conditions. In this work we consider systems satisfying an awakened version of specification called the *almost specification property*, which was introduced by Thompson[13] inspired in the $g-$*almost product property* of Pfister and Sullivan[10]. Thompson proved the full entropy of the irregular part of the Birkhoff averages spectrum for systems with the almost specification property. He applied this result to the case of $\beta-$shifts, which are systems having the almost specification property but the set of values of β such that the corresponding $\beta-$shift has the specification property has zero Lebesgue measure[4],[11]. The demonstration of Thompson is not based on saturatedness. Once proved that systems having the almost specification property are saturated, going along the lines of [8],[9] can be extended the result of Thompson to $V-$statistics or equivalently the result of [8],[9] to systems with the almost specification property.

The result to be proved is

Theorem: Let (X, f) be a dynamical system with the almost specification property. Let μ be a probability, $f-$invariant measure on X. If $G(\mu)$ denotes the set of $\mu-$generic points then

$$h_{top}(G(\mu)) = h_{\mu}(f), \qquad (3)$$

where h_{top} is the topological entropy for non-compact nor invariant sets and $h_{\mu}(f)$ is the measure-theoretic entropy of μ.

The inequality

$$h_{top}(G(\mu)) \leq h_{\mu}(f),$$

holds for any measure μ [3]. In [6] was proved that opposite inequality holds for dynamical systems with specification.

2. Preliminaries

Let $f : X \to X$, with X a compact metric space, and f be a continuous map. If $n \geq 1$, then the dynamical metric, or Bowen metric, is $d_n(x,y) = \max\left\{d\left(f^i(x), f^i(y)\right) : i = 0,1,...,n-1\right\}$. We denote by $B_{n,\varepsilon}(x)$ the ball of center x and radius ε in the metric d_n. By $\mathcal{M}(X)$ we denote the space of probability measures on X, and by $\mathcal{M}_{inv}(X,f)$ the space of $f-$invariant measures on X. The space $\mathcal{M}(X)$ is endowed the weak $*-$ topology, and if X is compact then $\mathcal{M}(X)$ is compact in the weak topology.

Let us recall the Bowen definition of topological entropy of non-compact nor invariant sets. Let $Z \subset X$ and let $\mathcal{C}(n,\varepsilon,Z)$ be the collection of finite or countable coverings of the set Z by balls $B_{m,\varepsilon}(x)$ with $m \geq n$. Let

$$M(Z,s,n,\varepsilon) = \inf_{\mathcal{B}\in\mathcal{C}(n,\varepsilon,Z)} \sum_{B_{m,\varepsilon}(x)\in\mathcal{B}} \exp(-sm),$$

and set

$$M(Z,s,\varepsilon) = \lim_{n\to\infty} M(Z,s,n,\varepsilon).$$

There is an unique number $\overline{s}$ such that $M(Z,s,\varepsilon)$ jumps from $+\infty$ to 0. Let

$$h_{top}(Z,\varepsilon) = \overline{s} = \sup\{s : M(Z,s,\varepsilon) = +\infty\} = \inf\{s : M(Z,s,\varepsilon) = 0\},$$

and

$$h_{top}(Z) = \lim_{\varepsilon\to 0} h_{top}(Z,\varepsilon). \tag{4}$$

The number $h_{top}(Z,\varepsilon)$ is the *topological entropy* of Z.

Theorem (Distribution mass principle)[12]: Let $f : X \to X$ be a continuous map, let $Z \subset X$. Let us assume that there are a $\varepsilon > 0$, $s > 0$ such that can be found a sequence of probability measures$\{m_k\}$, a constant $K > 0$ and a natural N satisfying

$$\limsup_{n\to\infty} m_k(B_{n,\varepsilon}(x)) \leq K\exp(-ns),$$

for any ball $B_{n,\varepsilon}(x)$ with $B_{n,\varepsilon}(x) \cap Z \neq \varnothing$ for any $n \geq N$. If it also assumed that one $*-$limit m of the sequence $\{\mu_k\}$ verifies $m(Z) > 0$ then $h_{top}(Z,\varepsilon) > s$.

Definition: Let $\varepsilon_0 > 0$, a function $g : \mathbf{N}\times (0,\varepsilon_0) \to \mathbf{N}$ is called a *mistake function* if for any $\varepsilon \in (0,\varepsilon_0)$ and for any $n \in \mathbf{N}$ holds $g(n,\varepsilon) \leq g(n+1,\varepsilon)$ and $\dfrac{g(n,\varepsilon)}{n} \to 0$, as $n \to \infty$. For $\varepsilon > \varepsilon_0$ and for a given mistake function g is defined $g(n,\varepsilon) . = g(n,\varepsilon_0)$.

This class of mistake functions, introduced by Thompson in [13], is slightly more general than the class of blow-up functions by Pfister and Sullivan[10] to define the $g-$almost product property. This last map does not depend on ε.The function $g(n,\varepsilon) = \dfrac{\log n}{\varepsilon}$ is a mistake function but it does not fall in the class of Pfister and Sullivan.

For $m, n \in \mathbf{N}$, $m < n$, let

$$I(n,m) := \{\Lambda \subset \{0,1,...,n-1\} : card\Lambda \geq n-m\}.$$

Let g be a mistake function and $\varepsilon > 0$, with $g(n,\varepsilon) < n$ for enough large n, set

$$I(g,n,\varepsilon) := \{\Lambda \subset \{0,1,...,n-1\} : card\Lambda \geq n - g(n,\varepsilon)\}.$$

If $\Lambda \subset \{0,1,...,n-1\}$ then is introduced the metric

$$d_\Lambda(x,y) = \max\left\{d\left(f^i(x), f^i(y)\right) : i \in \Lambda\right\},$$

and the ball

$$B_{\Lambda,\varepsilon}(x) = \{y : d_\Lambda(x,y) < \varepsilon\}.$$

The ball $B_{n,\varepsilon}(g,x)$ is defined by

$$B_{n,\varepsilon}(g,x) = \{y \in B_{\Lambda,\varepsilon}(x),\ \text{for some } \Lambda\ \in I(g,n,\varepsilon)\},$$

or equivalently

$$B_{n,\varepsilon}(g,x) = \bigcup_{\Lambda\ \in I(g,n,\varepsilon)} B_{\Lambda,\varepsilon}(x).$$

Definition: A dynamical system (X,f) has the *almost specification property* if there exists a mistake function g such that for any $\varepsilon_1,...,\varepsilon_k > 0$ there are numbers $N_1,...,N_k$, $N_i = N_i(g,\varepsilon_i)$, $i = 1,2,...,k$, such that for any points $x_1,...,x_k \in X$ and integers $n_i \geq N_i$

$$\bigcap_{j=1}^{k} f^{-\sum_{i=1}^{j-1} n_i}\left(B_{n_j,\varepsilon_j}(g,x_j)\right) \neq \varnothing. \tag{5}$$

The function g indicates how many mistakes are allowed to shadow an orbit in the almost specification property. Since the class of mistake functions is larger then the blow-up functions, the almost specification property is more general than the $g-$almost product property.

A dynamical system (X,f) has the *specification property* if: for any $\varepsilon>0$ there is an integer $M(\varepsilon)$ such that for any collection of intervals $I_j=[a_j,b_j]\subset\mathbb{Z}^+$, $j=0,\cdots,k-1$ such that $a_j-b_{j-1}\geq M(\varepsilon)$, and for any $x_0,\cdots,x_{k-1}\in X$ there is a $x\in X$ such that

$$d\left(f^{a_j+\ell}(x),f^{\ell}(x_j)\right)<\varepsilon,\quad \text{for } \ 0\leq\ell\leq b_j-a_j,\quad j=0,1,2.,,,k-1.$$

Pfister and Sullivan proved[10] that specification implies $g-$almost product property, which in turn implies almost specification property. To see directly that specification implies the Thompson condition, set $g(n,\varepsilon):=M(\varepsilon)$ for any $n\geq M(\varepsilon)$ and $N=N(g,\varepsilon)+1$, ε can be replaced by $\varepsilon_1,...,\varepsilon_k$ using the trick of [10].

Definition: A set $E\subset Z$ is $(n,m,\varepsilon)-$*separated* for Z if for any $\Lambda\in I(n,m)$ and for any $x,y\in Z$.holds $d_\Lambda(x,y)>\varepsilon$.A set $E\subset Z$ is $(g,n,\varepsilon)-$*separated* for Z if it is $(n,g(n,\varepsilon),\varepsilon)-$separated for Z.

Let $s_n(g,\varepsilon,Z)=\max\{cardE:\ E\subset Z \text{ and } E \text{ is } (g,n,\varepsilon)-\text{separated for } Z\}.$

The following result was obtained by Thompson[13] as a modification to the Katok formula for the entropy.

Theorem: Let $f:X\to X$ be continuous with X be a compact metric space. Let $\mu\in\mathcal{M}_{inv}(X,f)$ ergodic, for any $\gamma\in(0,1)$ and for any mistake function g is valid

$$h_\mu(f)=\lim_{\varepsilon\to0}\limsup_{n\to\infty}\left(\inf\{s_n(g,\varepsilon,Z):Z\subset X,\ \mu(Z)>1-\gamma\}\right). \tag{6}$$

The so called *empirical measures* on X associated to the dynamical system (X,f) are

$$\mathcal{E}_n(x)=\frac{1}{n}\sum_{i=0}^{n-1}\delta_{f^i(x)}.$$

Here δ is the point mass measure. We denote the weak limit of the sequence$\{\mathcal{E}_n(x)\}$ by $V(x)$. Since X is compact, $V(x)\neq\varnothing$. If μ is a measure on X then a point $x\in X$ is $\mu-$generic if $V(x)=\{\mu\}$, by $G(\mu)$ is denoted the set of $\mu-$generic points.

Following [6] the set of generic points can be characterized in the following way. Let $\{p_i\}$ be a sequence of numbers with $\sum_{i=1}^{\infty} p_i = 1$ and let $\{s_i\}$ be a sequence in ℓ^∞. The sequence $\{s_i = s_{n,i}\}_i$ converges to $\alpha = (\alpha_i) \in \ell^\infty$ in the weak $*-$ topology if and only if $\lim_{n\to\infty} |s_{n,i} - \alpha_i| = 0$. Let $\{\varphi_1, \varphi_2, ...\}$ a dense subset in unit ball of $C/X)$, for a fixed $\mu \in \mathcal{M}_{inv}(X,f)$, let $\alpha = (\alpha_1, \alpha_2, ...)$, with $\alpha_i = \int \varphi_i d\mu$- Thus

$$G(\mu) = \left\{ x : \lim_{n\to\infty} \sum_{i=1}^{\infty} p_i \left| \frac{1}{n} S_n(\varphi_i(x)) - \alpha_i \right| = 0 \right\},$$

where $S_n(\varphi_i(x)) = \sum_{k=0}^{n-1} \varphi_i\left(f^k(x)\right)$.

Lemma 2.1 ([15],[14]): For any $\mu \in \mathcal{M}_{inv}(X,f)$, $0 < \delta < 1$, $0 < \gamma < 1$, there is a measure ν such that $\nu = \sum_{j=1}^{k} \lambda_j \nu_j$, where each ν_j is ergodic and $\sum_{j=1}^{k} \lambda_j = 1$, and such that

i) $h_\nu(f) \geq h_\mu(f) - \gamma$.

ii) $\sum_{i=1}^{\infty} p_i \left| \int \varphi_i d\mu - \int \varphi_i d\nu \right| < \delta$, where $\{\varphi_i\}$ and $\{p_i\}$ are sequences like above.

Let $N \geq 1$ and

$$Y_j(N) = \left\{ x : \sum_{i=1}^{\infty} p_i \left| \frac{1}{n} S_n(\varphi_i(x)) - \int \varphi_i d\nu_j \right| < \delta, \text{ for } n > N \right\},$$

where $S_n(\varphi_i(x)) = \sum_{k=0}^{n-1} \varphi_i\left(f^k(x)\right)$. By the Birkhoff ergodic theorem and the Egorov theorem we have that

$$\lim_{n\to\infty} \sum_{i=1}^{\infty} p_i \left| \frac{1}{n} S_n(\varphi_i(x)) - \int \varphi_i d\nu_k \right| = 0, \ \nu_k - a.e,$$

and $\nu_j(Y_j(N)) > 1 - \gamma$, for sufficiently large N.

Let $\alpha = (\alpha_1, \alpha_2, ...) \in \ell^\infty$ and $\Theta = \{\varphi_1, \varphi_2, ...\}$ be a dense subset in unit ball of $C(X)$. If $\delta > 0$, $n \geq 1$, then set

$$X_\Theta(\alpha, \delta, n) = \left\{ x : \sum_{i=1}^{\infty} p_i \left| S_n(\varphi_i(x)) - \alpha_i \right| < \delta, \ \alpha = (\alpha_i) \in \ell^\infty \right\} \quad (7)$$

$$\Lambda_\Theta(\alpha) := \lim_{\varepsilon\to 0} \lim_{\delta\to 0} \lim_{n\to\infty} \sup \frac{1}{n} \log N(\alpha, \delta, \epsilon, n),$$

where $N(g,\alpha,\delta,\epsilon,n)$ is the minimal number of balls $B_{n,\varepsilon}(g,x)$ needed to cover the set $X_{\Theta}(\alpha,\delta,n)$.

3. Proof of the theorem

We begin with the construction of a fractal set F, for this is followed that made in [13], which is in part inspired in [12]. Let $\alpha=(\alpha_1,\alpha_2,...)\in\ell^{\infty}$ and $\Theta=\{\varphi_1,\varphi_2,...\}$ be a dense subset in unit ball of $C(X)$.Let us introduce a sequence of positive integers $\{n_k\}$ and an increasing sequence of integers $\{N_k\}$ with $N_0=0$ and $N_k\to\infty$ and such that $\frac{n_{k+1}}{N_k}\to 0$ and $\frac{\sum_{i=1}^{k} n_{iN_i}}{N_{k+1}}\to 0$ as $k\to\infty$.

Let $\{\delta_k\}$ be a sequence of real numbers with $\delta_k\to 0$, as $k\to\infty$ and let $\{S_k\}$ be a family of finite subsets of $X_{\Theta}(\alpha,\delta_k,n_k)$ with $\alpha=(\alpha_1,\alpha_2,...)$ and $\alpha_i=\int\varphi_i d\mu$. If g is a mistake function then define maps $h_k(n,\varepsilon):=2g\left(n,\varepsilon/2^k\right)$ and assume that each S_k is $(h_k,n_k,5\varepsilon)$ −separated. Let us consider points

$\left(z_1=\left(x_1^1,x_2^1,...,x_{N_1}^1\right),z_2=\left(x_1^2,x_2^2,...,x_{N_2}^2\right),...,z_k=\left(x_1^k,x_2^k,...,x_{N_k}^k\right)\right)\in S_1^{N_1}\times S_2^{N_2}\times...\times S_k^{N_k}$. By the almost specification property there exists a point $z=z(z_1,z_2,...,z_k)$ such that

$$f^{t_i+(j-1)n_i}(z)\in B_{n_i,\varepsilon_{/2^i}}\left(g,x_j^i\right),$$

for any $i=1,...,k$, $j=1,...,N_k$ and where $t_i=\sum_{l=0}^{i-1} n_{lN_l}$. Let

$$C(z_1,z_2,...,z_k):=\bigcap_{i=1}^{k}\bigcap_{j=1}^{N_k} f^{-t_i-(j-1)n_i}\left(B_{n_i,\varepsilon_{/2^i}}\left(g,x_j^i\right)\right)\neq\varnothing \tag{8}$$

Then, let us define sets

$$F_k=\left\{\overline{C(z_1,z_2,...,z_k)}:(z_1,z_2,...,z_k)\in S_1^{N_1}\times S_2^{N_2}\times...\times S_k^{N_k}\right\},$$

and let

$$F:=\bigcap_{k\geq 1}F_k.$$

Let $t_k = \sum_{i=0}^{k} n_{iN_i}$, for each $n \in \mathbf{N}$ let $j \in \{0, 1, ..., N_{k+1} - 1\}$ be the unique number such that

$$t_k + jn_{k+1} \leq n < t_k + (j+1)\, n_{k+1}.$$

Let us recall that for any $(z_1, z_2, ..., z_k) \in S_1^{N_1} \times S_2^{N_2} \times ... \times S_k^{N_k}$ there is a $z \in C(z_1, z_2, ..., z_k)$, now let

$$L_k = \{z = z(z_1, z_2, ..., z_k) \in C(z_1, z_2, ..., z_k)\}.$$

Lemma 3.1[13]**:** If $(z_1, z_2, ..., z_k) \neq (w_1, w_2, ..., w_k)$ then $z = (z_1, z_2, ..., z_k) \neq w = w(w_1, w_2, ..., w_k)$. Consequently $cardL_k = M_1^{N_1} ... M_k^{N_k}$, $M_k = cardS_k$.

With this above result can be defined a sequence of measures concentrated on F_k by

$$m_k = \frac{1}{A_k} \nu_k,$$

with $\nu_k = \sum_{x \in L_k} \delta_x$ and $A_k = cardL_k$. Let $\mathcal{B} = B_{n,\varepsilon/2}(x)$ such that $\mathcal{B} \cap F \neq \varnothing$, it holds[13]

$$m_{k+p}(\mathcal{B}) \leq \frac{M_{k+1}^{N_{k+1}-j}}{M_1^{N_1} ... M_k^{N_k} M_{k+1}^{N_{k+1}}} = \frac{1}{cardL_k \times M_{k+1}^{j}},$$

for any $p \geq 1$.Let m be the w^*−limit of the sequence$\{\mu_k\}$, the measure m is concentrated on F, and by the distribution mass principle we have

$$h_{top}(F) \geq \liminf_{n \to \infty} \frac{1}{n} \left(\sum_{i=1}^{k} N_i \log M_i + j \log M_{k+1} \right).$$

The sets S_k, used in the construction of the set F, may be chosen such that

$$\begin{aligned} M_k = cardS_k &\geq \exp\left[n_k\left(\Lambda_\Theta(\alpha) - \gamma\right)\right] \geq M_1^{N_1} ... M_k^{N_k} M_{k+1}^{j} \\ &\geq \exp\left[\Lambda_\Theta(\alpha) - \gamma\right] \left(\sum_{i=1}^{k} n_i N_i + jn_{k+1} \right) \geq \exp\left[n\left(\Lambda_\Theta(\alpha) - \gamma\right)\right]. \end{aligned}$$

So that

$$h_{top}(F) \geq \liminf_{n \to \infty} \frac{1}{n} \left(\sum_{i=1}^{k} N_i \log M_i + j \log M_{k+1} \right) \geq \Lambda_\Theta(\alpha) - \gamma.$$

Proposition 3.2: The fractal F is contained in the set of generic points $G(\mu)$, for any $\mu \in \mathcal{M}_{inv}(X, f)$.

Proof: Let $\Theta=\{\varphi_1,\varphi_2,...\}$ and recall that

$$G(\mu)=\left\{x:\lim_{n\to\infty}\sum_{i=1}^{\infty}p_i\left|\frac{1}{n}S_n(\varphi_i(x))-\int\varphi_i d\mu\right|=0\right\},$$

where $\{p_i\}$ is a sequence of numbers with $\sum_{i=1}^{\infty}p_i=1$. Let $z\in F$, $a_j=(j-1)n_k$ and $z_{k\cdot}=f^{t_k-1}(z)$, if $var(\varphi_i,\varepsilon)=\sup\{|\varphi_i(x)-\varphi_i(y)|:d(x,y)<\varepsilon\}$, then

$$\left|S_{n_k}(\varphi_i(x_j^s))-S_{n_k}(\varphi_i(f^{a_j}(z_k)))\right|\le$$
$$n_k var(\varphi_i,\varepsilon)+g\left(n_k,\varepsilon/2^k\right)\|\varphi_i\|_0,\ s=1,...,k.$$

Therefore

(9)
$$\begin{aligned}&\left|S_{n_k}(\varphi_m(x_j^s))-\alpha_i n_k\right|\\&\le\left|S_{n_k}(\varphi_m(x_j^s))-S_{n_k}(\varphi_i(f^{a_j}(z_k)))\right|+\left|S_{n_k}(\varphi_i(f^{a_j}(z_k)))-\alpha_i n_k\right|\\&\le n_k var(\varphi_m,\varepsilon)+g\left(n_k,\varepsilon/2^k\right)\|\varphi_m\|_0+n_k\delta_k,\end{aligned}$$

with $\alpha_i=\int\varphi_i d\mu$.

Let n be the unique number such that $t_k\le n<t_k+1$ and recall that $j\in\{0,1,...,N_{k+1}-1\}$ is the unique number such that

$$t_k+jn_{k+1}\le n<t_k+(j+1)n_{k+1}.$$

Let us consider a partition of the interval $[0,n-1]$ into the subintervals $[0,t_k-1]$ and $I_1^\ell=[t_k+(\ell-1)n_{k+1},t_k+\ell n_{k+1}]$, $I_2^\ell=[t_k+jn_{k+1},n-1]$, $\ell=1,2,...,j$. Thus we have

$$S_n(\varphi_i(z))=S_{t_k}(\varphi_i(z))+\sum_{t=a_\ell}^{a_\ell+n_{k+1}-1}\varphi_i\left(f^t(z)\right)+\sum_{t=t_k+\ell n_{k+1}}^{n-1}\varphi_i\left(f^t(z)\right).$$

Let us begin with the estimation in $[0,t_k-1]$, we have that $\frac{1}{t_k}S_{t_k}(\varphi_i(z))$ asymptotically behaves like $\frac{1}{n_{kN_k}}S_{n_{kN_k}}(\varphi_i(z_k))$, with $z_{k\cdot}=f^{t_k-1}(z)$, $z\in F$. Thus

$$S_{t_k}(\varphi_i(z))=S_{t_k-1}(\varphi_i(z))+S_{n_{kN_k}}(\varphi_i(z_k)),$$

and

$$S_{n_{kN_k}}(\varphi_i(z_k))=\sum_{j=1}^{N_k}S_{n_j}\left(\varphi_i\left(f^{t_j-1}(z_k)\right)\right).$$

Let $x_j^k \in S_k$, so

(10)
$$S_{n_kN_k}\left(\ \varphi_i\left(z_k\right)\right)-\sum_{j=1}^{N_k}S_{n_j}\left(\ \varphi_i\left(x_j^k\right)\right)\right) \leq n_kN_k var\left(\ \varphi_i,\varepsilon/2^k\right)++g\left(n_k,\varepsilon/2^k\right)\parallel\ \varphi_i\|_0\,.$$

Then

$$(11)\ \ S_{n_kN_k}\left(\varphi_i\left(z_k\right)\right) \leq \left[\int\varphi_i d\mu+\delta_k\right]n_k+n_{kN_k}var\left(\varphi_i,\varepsilon/2^k\right)+g\left(n_k,\varepsilon/2^k\right)\|\varphi_i\|_0\,,$$

so that

$$\left|\frac{1}{n_{kN_k}}S_{n_kN_k}\left(\varphi_m\left(z_k\right)\right)-\int\varphi_m d\mu\right|\to 0\ \text{ as } k\to\infty,$$

and, since $\dfrac{n_{kN_k}}{t_k}\to 1$ as $k\to\infty$, we get

$$\left|\frac{1}{n_{kN_k}}S_{n_kN_k}\left(\varphi_i\left(z_k\right)\right)-\frac{1}{t_k}S_{t_k}\left(\varphi_i\left(z\right)\right)\right|\to 0\ \text{ as } k\to\infty.$$

Therefore

$$\lim_{k\to\infty}\sum_{i=1}^{\infty}p_i\left|\ \frac{1}{t_k}S_{t_k}\left(\varphi_m\left(z\right)\right)-\int\varphi_{io}d\mu\ \right|=0,$$

for $z\in F$ and on the interval $[0,t_k-1]$. Now we estimate in each interval I_1^ℓ, $\ell=1,2,...,j$, let $x_\ell^{k+1}\in S_{k+1}$ and $f^{t_k+(\ell-1)n_{k+1}}\left(z\right)\in B_{n_{k+1},\varepsilon/2k+1}\left(g,x_\ell^{k+1}\right)$, thus

$$\left|\sum_{t=a_\ell}^{a_\ell+n_{k+1}-1}\varphi_i\left(f^t\left(z\right)\right)-n_{k+1}\int\varphi_i d\mu\right|\leq\left|\sum_{t=a_\ell}^{a_\ell+n_{k+1}-1}\varphi_i\left(f^t\left(z\right)\right)-S_{n_{k+1}}\left(\varphi_i\left(x_\ell^{k+1}\right)\right)\right|$$

$$+\left|S_{n_{k+1}}\left(\varphi_i\left(x_\ell^{k+1}\right)\right)-n_{k+1}\int\varphi_i d\mu\right|\leq card\Lambda_{n_{k+1}}var\left(\ \varphi_i,\varepsilon/2^{k+1}\right)$$

$$+g\left(n_{k+1},\varepsilon/2^{k+1}\right)\|\varphi_i\|_0+n_{k+1}\delta_{k+1}.$$

with $\Lambda_{n_{k+1}}\subset\{0,1,2,...,n_{k+1}\}$.

$$\lim_{k\to\infty}\sum_{i=1}^{\infty}p_i\left|\frac{1}{n_{k+1}}\sum_{t=a_\ell}^{a_\ell+n_{k+1}-1}\varphi_i\left(f^t\left(z\right)\right)-\int\varphi_i d\mu\right|=0,$$

for $z\in F$ and on the intervals I_1^ℓ.

Finally we do the estimations on the intervals I_2^ℓ, $\ell = 1, 2, ..., j$. We have

$$\left| \sum_{t=t_k+jn_{k+1}}^{n-1} \varphi_i \left(f^t (z) \right) - (n - t_k + jn_{k+1}) \int \varphi_i d\mu \right| \leq$$

$$2 \left((n - \ t_k + jn_{k+1}) \left\| \varphi_i \right\|_0 \right) \leq 2n_{k+1} \left\| \varphi_i \right\|_0 \tag{12}$$

Then, since $\ t_k > N_k$ and $n > \ t_k + jn_{k+1}$,

$$\begin{aligned}
&\left| S_n \left(\varphi_m (z) \right) - n \int \varphi_i d\mu \right| \\
&\leq S_{t_k} \left(\varphi_m (z) \right) + jcard\Lambda_{n_{k+1}} var \left(\ \varphi_i, \varepsilon/2^{k+1} \right) + \\
&\qquad \left\| \varphi_i \right\|_0 g \left(n_{k+1}, \varepsilon/2^{k+1} \right) + jn_{k+1}\delta_{k+1} + 2n_{k+1} \left\| \varphi_i \right\|_0 \\
&\leq \frac{1}{t_k} S_{t_k} \left(\varphi_i (z) \right) + \delta_{k+1} + 2\frac{n_{k+1}}{N_k} \left\| \varphi_i \right\|_0 + \\
&\qquad var \left(\ \varphi_i, \varepsilon/2^{k+1} \right) + \frac{g \left(n_{k+1}, \varepsilon/2^{k+1} \right)}{n_{k+1}} \left\| \varphi_i \right\|_0,
\end{aligned}$$

with $\Lambda_{n_k} \subset \{0, 1, ..., n_k - 1\}$, so that

$$\lim_{k\to\infty} \sum_{i=1}^{\infty} p_i \left| \frac{1}{n} \sum_{t=0}^{n-1} \varphi_i \left(f^t (z) \right) - \int \varphi_i d\mu \right| = 0 \tag{13}$$

Therefore $z \in G(\mu)$ $\square$

Next we prove

Proposition 3.3: It holds $\Lambda_\Theta (\alpha) \geq h_\mu (f)$, where recall that

$$\Lambda_\Theta (\alpha) := \lim_{\varepsilon\to 0} \lim_{\delta\to 0} \lim_{n\to\infty} \sup \frac{1}{n} \log N (\alpha, \delta, \epsilon, n),$$

with $N(g, \alpha, \delta, \epsilon, n)$ the minimal number of balls $B_{n,\varepsilon}(g, x)$ needed to cover the set $X_\Theta (\alpha, \delta, n) \ = \left\{ x : \sum_{i=1}^{\infty} p_i \left| S_n (\varphi_i (x)) - \ \alpha_i \right| < \delta,\ \alpha_i = \int \varphi_i d\mu \right\}$.

Proof: Let $N \geq 1, \delta > 0$, and

$$Y_j (N) = \left\{ x : \sum_{i=1}^{\infty} p_i \left| \frac{1}{n} S_n (\varphi_i (x)) - \int \varphi_i d\nu_j \right| < \delta, \text{ for } n > N \right\},$$

and $\nu = \sum_{j=1}^{k} \lambda_j \nu_j$, ν_j ergodic given by the lemma 2.1, which satisfy $\nu_j (Y_j (N)) > 1 - \gamma$, for sufficiently large N, $0 < \gamma < 1$. Recall that by the modified Katok entropy formula we have

$$h_{\nu_j}(f) \geq \lim_{\varepsilon\to 0}\left\{\limsup_{n\to\infty} S_n(g,\varepsilon,Y_j(N),\nu_j,\gamma)\right\},$$

where $S_n(g,\varepsilon,Z,\mu,\gamma)=\inf\{s_n(g,\varepsilon,Z): Z\subset X,\ \mu(Z)>1-\gamma\}$, and $s_n(g,\varepsilon,Z)=\max\{cardE:\ E\subset Z \text{ and } E \text{ is } (g,n,\varepsilon)-\text{separated for } Z\}$. Thus there is a $N_j=N_j(\nu_j,4\varepsilon,\gamma)$ such that

$$S_n(2g,4\varepsilon,Y_j(N),\nu_j,\gamma)\geq \exp\left[n\left(h_{\nu_j}(f)-\gamma\right)\right], \tag{14}$$

therefore if E_j is a $(2g,n,4\varepsilon)$ −separated set for $Y_j(N)$ then

$$cardE_j \geq \exp\left[n\left(h_{\nu_j}(f)-\gamma\right)\right],\ \text{for } n\geq N_j,\ j=1,...,k.$$

Like in the construction of the set F, for any $(x_1,...,x_k)\in E_1\times...\times E_k$, we can choose a point $z=z(x_1,...,x_k)\in C(x_1,x_2,...,x_k)$. The next auxiliary lemma is similar to lemma 3.1 but for the sets $E_1,...,E_k$

Lemma 3.4: If $(x_1,...,x_k)$, $(x_1,...,x_k)\in E_1\times...\times E_k$ distinct $k-$uples the points $z_1=z_1\ (x_1,...,x_k)$ and $z_2=z_2(x_1,...,x_k)$ are distinct

Proof: Let r be the coordinate for which $x_r\neq x_r'$ and let $\Lambda_1,\Lambda_{2,}\in I(g,n_r,\varepsilon/2^r)$, with $n_r=[n\lambda_r]$, so

$d_{\Lambda_{1,}}(f^a(z_1),x_r)<\varepsilon/2^j$ and $d_{\Lambda_2}(f^a(z_2),\ x_r')<\varepsilon/2^j$. Let $\Lambda=\Lambda_1\cap\Lambda_2\in I(g,n_r,\varepsilon/2^r)$, thus

$4\varepsilon<d_\Lambda(x_r',x_r)\leq d_{\Lambda_{1,}}(x_r,f^a(z_1)\ )+d_\Lambda(f^a(z_1),\ f^a(z_2))+d_{\Lambda_2}(f^a(z_2),\ x_r')\leq \varepsilon/2^{r-1}+d_\Lambda(f^a(z_1),\ f^a(z_2))$, and then

$d_\Lambda(f^a(z_1),\ f^a(z_2))>4\varepsilon-\ \varepsilon/2^{r-1}>3\varepsilon$ □

Let $\overline{n}=\sum_{\ell=1}^{k}n_\ell=\sum_{\ell=1}^{k}[n\lambda_\ell]$, $\sum_{\ell=1}^{k}\lambda_\ell=1$,we must prove that any point $z=z\ (x_1,...,x_k)$ belongs to $X_\Theta(\alpha,5\delta,\overline{n})$, for n enough large, in this case we would have

$$N(2g,\alpha,5\delta,\varepsilon,\overline{n})\geq cardE_1...cardE_k\geq \exp\left[\sum_{\ell=1}^{k}[n\lambda_\ell]\left(h_{\nu_j}(f)-\gamma\right)\right],$$

since $\dfrac{[n\lambda_\ell]}{n}\to\lambda_\ell$ as $n\to\infty$ and $\sum_{\ell=1}^{k}\lambda_\ell=1$ we have that $\sum_{\ell=1}^{k}\dfrac{[n\lambda_\ell]}{n}\to 1$ as $n\to\infty$. Hence

$$\limsup_{n\to\infty}\frac{1}{n}\ \log N(2g,\alpha,5\delta,\varepsilon,\overline{n})$$

$$\geq \sum_{\ell=1}^{k} h_{\nu_j}(f) - \gamma = h_\nu(f) - \gamma \geq h_\mu(f) - \gamma,$$

therefore

$$\Lambda_\Theta(\alpha) \geq h_\mu(f),$$

this proved the proposition 3.3, modulo $z \in X_\Theta(\alpha, 5\delta, \overline{n})$ □

To see that, for n enough large, $z \in X_\Theta(\alpha, 5\delta, \overline{n})$, let

$$\frac{1}{\overline{n}} S_{\overline{n}}(\varphi_i(z)) = \frac{1}{\overline{n}} \sum_{\ell=1}^{k} \sum_{t=0}^{n_\ell - 1} \varphi_i\left(f^{n_\ell + t}(z)\right)$$

$$= \frac{1}{\sum_{\ell=1}^{k} [n\lambda_\ell]} \sum_{\ell=1}^{k} [n\lambda_\ell] S_{[n\lambda_\ell]}(\varphi_i((f^{n_\ell}(z))),$$

thus we have

$$\left|\frac{1}{\overline{n}} S_{\overline{n}}(\varphi_i(z)) - \int \varphi_i d\mu\right|$$

$$\leq \sum_{\ell=1}^{k} \frac{1}{\overline{n}} [n\lambda_\ell] \left|S_{[n\lambda_\ell]}(\varphi_i((f^{n_\ell}(z))) - S_{[n\lambda_\ell]}(\varphi_i(x_l))\right| +$$

$$\sum_{\ell=1}^{k} \frac{1}{\overline{n}} [n\lambda_\ell] \left|S_{[n\lambda_\ell]}(\varphi_i(x_l)) - \int \varphi_i d\nu_l\right| +$$

$$\sum_{\ell=1}^{k} \left|\frac{1}{\overline{n}} [n\lambda_\ell] - \lambda_\ell\right| \left|\int \varphi_i d\nu_l\right| + \left|\int \varphi_i d\nu_l - \int \varphi_i d\mu\right|,$$

since $[n\lambda_\ell] \leq n\lambda_\ell$, and by the lemma 2.1, we get

$$\sum_{i=1}^{\infty} p_i \left|\frac{1}{\overline{n}} S_{\overline{n}}(\varphi_i(z)) - \int \varphi_i d\mu\right|$$

$$\leq \sum_{i=1}^{\infty} p_i \left(\frac{card\Lambda_{n_l}}{n_l} var\left(\varphi_i, \varepsilon/2^l\right) + \frac{g\left(n_l, \varepsilon/2^l\right)}{n_l} \|\varphi_i\|_0 + \delta + \delta + \delta\right)$$

$$\leq \sum_{i=1}^{\infty} p_i (2\delta + \delta + \delta + \delta) = 5\delta, \qquad \text{for } n_l > N$$

and therefore $z \in X_\Theta(\alpha, 5\delta, \overline{n})$.

Now, let us recap, we have obtained

$-F \subset G(\mu)$

$$-h_{top}(F) \geq \liminf_{n\to\infty} \frac{1}{n}\left(\sum_{i=1}^{k} N_i \log M_i + j \log M_{k+1}\right) \geq \Lambda_{\Theta}(\alpha)$$
$$-\ \Lambda_{\Theta}(\alpha) \geq h_{\mu}(f).$$ Therefore we get

$$h_{top}(G(\mu)) \geq h_{\mu}(f).$$

As it is known the opposite inequality was proved by Bowen, and the theorem is proved. □

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