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Casimir free energy at high temperatures: Grounded versus isolated conductors

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We evaluate the difference between the Casimir free energies corresponding to either grounded or isolated perfect conductors, at high temperatures. We show that a general and simple expression for that difference can be given, in terms of the electrostatic capacitance matrix for the system of conductors. For the case of close conductors, we provide approximate expressions for that difference, by evaluating the capacitance matrix using the proximity force approximation. Since the high-temperature limit for the Casimir free energy for a medium described by a frequency-dependent conductivity diverging at zero frequency coincides with that of an isolated conductor, our results may shed light on the corrections to the Casimir force in the presence of real materials.

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I. INTRODUCTION

Casimir forces and related phenomena constitute remarkable macroscopic manifestation of zero point or thermal fluctuations of the electromagnetic field. Different high precision experiments have been implemented in recent years in order to measure the Casimir force with ever increasing detail [1]. In spite of these efforts, the corresponding comparison between theory and experiment has not yet been, however, entirely satisfying [2]. This suggests that further theoretical and experimental developments may be required in order to tackle some of the long standing puzzles which arise in realistic descriptions of the forces and their detailed properties.

In this paper, we find a general expression for the difference, ΔF, between the high temperature free energies for two different cases, according to whether the conductors are: (a) grounded or (b) isolated. Albeit this is a question which has been partially addressed in previous works [3,4], we want to present a fuller answer here, allowing us to consider different concrete examples.

As we shall see, isolated perfect conductors can be used to describe, in the high temperature limit, real materials with a permittivity diverging in the zero-frequency limit. Therefore, ΔF may be used to account, in those cases, for real material corrections to the Casimir effect of grounded conductors at high temperatures. Thus, even though most experimental setups involve grounded conductors (in order to minimize spurious electrostatic effects), the question we address may be relevant to account for those corrections, apart from its conceptual interest.

II. FREE ENERGY FOR THE ELECTROMAGNETIC FIELD

Since our focus shall be on the high temperature limit of the free energy, we begin by deriving the expression for the free energy F for the quantum electromagnetic (EM) field at a finite temperature. It is a function of the inverse temperature $\beta = T^{-1}$ (in our conventions, Boltzmann's constant $k_B \equiv 1$). F may be written in terms of the partition function, $\mathcal{Z}$ as follows:

$$F = -\frac{1}{\beta}\log\left[\frac{\mathcal{Z}}{\mathcal{Z}_0}\right], \tag{1}$$

where the denominator, $\mathcal{Z}_0$, denotes the partition function for the free (i.e., in the absence of media) EM field. The effect of that denominator is to subtract the free energy of a free Bose gas of photons in the absence of the mirrors, which does not contribute to the force between them.

In the Matsubara formalism, a functional integral expression for the partition function $\mathcal{Z}$ can be constructed by integrating over field configurations depending on the spatial coordinates $\mathbf{x}$ and the imaginary time $x_0 \equiv \tau$. The fields are periodic, with period β, in the imaginary time. Denoting by $A = (A_\mu)$, ($\mu = 0, 1, 2, 3$) the 4-potential in Euclidean (imaginary time) spacetime, $\mathcal{Z}$ is given by:

$$\mathcal{Z} = \int [\mathcal{D}A] e^{-\mathcal{S}_{\rm inv}(A)} \tag{2}$$

where $\mathcal{S}_{\rm inv}(A)$ is the gauge-invariant action for A, while $[\mathcal{D}A]$ is used to denote the functional integration measure including gauge fixing.

In terms of the components of the field strength tensor $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, the form of the gauge-invariant action in the presence of real materials is

$$\mathcal{S}_{\rm inv}(A) = \int_0^\beta d\tau \int_0^\beta d\tau' \int d^3\mathbf{x} \bigg[\frac{1}{2} F_{0j}(\tau,\mathbf{x})\epsilon(\tau-\tau',\mathbf{x}) \times F_{0j}(\tau',\mathbf{x}) + \frac{1}{4} F_{ij}(\tau,\mathbf{x})\mu^{-1}(\tau-\tau',\mathbf{x}) F_{ij}(\tau,\mathbf{x})\bigg], \tag{3}$$

where indices from the middle of the Roman alphabet run over spatial indices (Einstein summation convention has been adopted), and $\epsilon(\tau-\tau',\mathbf{x})$ and $\mu(\tau-\tau',\mathbf{x})$ denote the Euclidean versions of the permittivity and permeability, respectively (μ^{-1} is the inverse integral kernel of μ, with respect to its time-like arguments). Space locality of those response functions has been assumed implicitly.

It is rather useful to adopt mixed Fourier transformations for the fields, as well as for the response functions:

$$A_\mu(\tau,\mathbf{x}) = \frac{1}{\beta}\sum_{n=-\infty}^{+\infty} \tilde{A}_\mu^{(n)}(\mathbf{x}) e^{i\omega_n\tau}$$
$$\epsilon(\tau-\tau',\mathbf{x}) = \frac{1}{\beta}\sum_{n=-\infty}^{+\infty} \tilde{\epsilon}^{(n)}(\mathbf{x}) e^{i\omega_n(\tau-\tau')}$$
$$\mu(\tau-\tau',\mathbf{x}) = \frac{1}{\beta}\sum_{n=-\infty}^{+\infty} \tilde{\mu}^{(n)}(\mathbf{x}) e^{i\omega_n(\tau-\tau')} \tag{4}$$

where $\omega_n \equiv \frac{2\pi n}{\beta}$ $(n \in \mathbb{Z})$ are the Matsubara frequencies. Note that, with the convention above for the definition of the Fourier expansion, $\tilde{A}_\mu^{(n)}$ is a dimensionless field.

The high temperature (classical) limit is dominated, for a Bose field, by the $n = 0$ Matsubara mode. In a previous work [5], we have shown that the zero mode free energy can be written as

$$F = F_s + F_v, \tag{5}$$

where F_s corresponds to the free energy of a scalar field in $2+1$ (Euclidean) dimensions

$$e^{-\beta F_s} = \int \mathcal{D}\tilde{A}_0^{(0)} e^{-\frac{1}{2\beta}\int d^3\mathbf{x}\,\tilde{\epsilon}^{(0)}(\mathbf{x})(\partial_j \tilde{A}_0^{(0)})^2}, \tag{6}$$

while F_v is the free energy of a vector field in $2+1$ dimensions

$$e^{-\beta F_v(\psi)} = \int \mathcal{D}\tilde{A}_j^{(0)} e^{-\frac{1}{\beta}\int d^3\mathbf{x}[\frac{1}{4\tilde{\mu}^{(0)}(\mathbf{x})}(\tilde{F}_{jk}^{(0)})^2 + \frac{1}{2}\Omega_0^2(\mathbf{x})(\tilde{A}_j^{(0)})^2]}. \tag{7}$$

Here, we have introduced the object:

$$\Omega_0^2(\mathbf{x}) \equiv \lim_{n\to 0}[\omega_n^2 \tilde{\epsilon}^{(n)}(\mathbf{x})] \tag{8}$$

(note that $\tilde{\epsilon}^{(0)}$, $\tilde{\mu}^{(0)}$ and Ω_0 are model-dependent).

Let us now discuss the limit of perfectly conducting materials, from the point of view of the scalar and vector contributions: regarding the field $\tilde{A}_0^{(0)}$, which behaves as a $2+1$ dimensional scalar, the infinite permittivity limit implies that its gradient inside the regions occupied by the material bodies vanishes identically. Therefore, the field is constant in those regions. On the other hand, if the conductors are grounded, those constants must vanish, so that the field itself is zero. Namely, the scalar field is subjected to Dirichlet boundary conditions, corresponding to the transverse magnetic (TM) EM mode. If the conductors are isolated, the field can take any value in each nonvacuum region. In this case, the functional integral should be performed over all possible configurations, including arbitrary (constant) values on the surfaces of the conducting bodies.

The vector zero mode, on the other hand, behaves as an EM field in $2+1$ dimensions. If Ω_0 tends to infinity, then the EM field will vanish identically on the regions filled up by media. It then satisfies perfect conductor boundary conditions. We have shown this to be equivalent to a real scalar field with Neumann conditions [6], corresponding to the transverse electric (TE) EM mode.

There is a well-known subtle point in the case of real materials, which manifests itself when considering two typical models for the permittivity, namely, the Drude or plasma models, where the permittivity diverges in the zero frequency limit. Therefore, in both cases the associated TM mode contribution is tantamount to that of a scalar field in the presence of an isolated perfect conductor. There is a difference, however, in the TE mode contribution for both models. Indeed, since Ω_0 vanishes in the Drude model, there is no TE contribution to the Casimir free energy, whilst the plasma model generates a nonvanishing TE mode. The latter coincides with that of a perfect conductor in the limit of a high plasma frequency.

III. GROUNDED VS ISOLATED FREE ENERGIES

In what follows we will consider in detail the scalar TM contribution, aiming to obtain the difference

$$\Delta F = F_s^{(g)} - F_s^{(i)} \tag{9}$$

between grounded and isolated perfect conductor boundary conditions. In view of the discussion at the end of the previous section, ΔF describes the difference between the scalar field term in the free energy of a system of grounded perfect conductors and that corresponding to the same geometry but involving materials which are described by Drude or plasma models.

In order to simplify the notation, we adopt a simpler notation for the only field we have to deal with henceforward, namely: $\tilde{A}_0^{(0)} \equiv \phi$ [we recall that ϕ is dimensionless, because of the definition for the Fourier transforms used in Eq. (4)]. Regarding the geometry, we assume that the system under consideration consists of N conductors, each one occupying a volume V_α enclosed by a surface S_α, with $\alpha = 1, 2, \ldots. N$.

An intermediate object that may be conveniently used as an ingredient to obtain both the grounded and isolated conductors partition functions, is a partition function where the (constant) value of ϕ on each surface S_α is fixed to a given but otherwise arbitrary value ϕ_α. The partition function for these particular boundary conditions is denoted by,

$$\mathcal{Z}[\{\phi_\alpha\}] = \int \mathcal{D}\phi e^{-\frac{1}{2\beta}\int d^3\mathbf{x}(\partial_j\phi)^2} \prod_{\alpha=1}^{N} \delta[\phi|_{S_\alpha} - \phi_\alpha]. \quad (10)$$

Thus, we may obtain the partition functions corresponding to grounded ($\mathcal{Z}^{(g)}$) and isolated ($\mathcal{Z}^{(i)}$) conductors as follows [3]:

$$\mathcal{Z}^{(g)} = \mathcal{Z}[\{\phi_\alpha\}]|_{\phi_\alpha=0} \quad (11)$$

and

$$\mathcal{Z}^{(i)} = \int_{-\infty}^{\infty} \left(\prod_{\alpha=1}^{N} d\phi_\alpha\right) \mathcal{Z}[\{\phi_\alpha\}]. \quad (12)$$

It is self-evident that Eq. (12) does not correspond to grounded conductors, since the values of the potentials at each surface are not fixed to zero; rather they have to be integrated out. One can show explicitly that the result of that integration corresponds to a situation in which the total charge of each conductor is zero, with vanishing charge fluctuations (there are of course ϕ_α fluctuations). We present a derivation of this property, within the context of our approach, in the Appendix (see also [7]).

In view of its relevance to both the grounded and isolated limits, let us then compute $\mathcal{Z}[\{\phi_\alpha\}]$. To that end, it is convenient to perform a shift (translation) in the integration variables: $\phi(\mathbf{x}) = \tilde{\phi}(\mathbf{x}) + \varphi(\mathbf{x})$, where $\tilde{\phi}$ is the (unique) solution of the classical electrostatic problem with prescribed boundary conditions for the potential on the conductors:

$$\nabla^2\tilde{\phi}(\mathbf{x}) = 0, \qquad \tilde{\phi}|_{S_\alpha} = \phi_\alpha, \quad (13)$$

and $\varphi(\mathbf{x})$ is a scalar field satisfying Dirichlet boundary conditions. It is rather straightforward to show that, after the shift, we have:

$$\begin{aligned}\mathcal{Z}[\{\phi_\alpha\}] &= e^{-\frac{1}{2\beta}\sum_{\gamma\delta} C_{\gamma\delta}\phi_\gamma\phi_\delta} \int \mathcal{D}\varphi e^{-\frac{1}{2\beta}\int d^3\mathbf{x}(\partial_j\varphi)^2} \prod_{\alpha=1}^{N} \delta[\varphi|_{S_\alpha}] \\ &= e^{-\frac{1}{2\beta}\sum_{\gamma\delta} C_{\gamma\delta}\phi_\gamma\phi_\delta} \mathcal{Z}^{(g)}, \quad (14)\end{aligned}$$

where the $C_{\gamma\delta}$ denote the capacitance coefficients of the system of conductors.

As discussed previously, $\mathcal{Z}^{(i)}$ is obtained by performing a Gaussian integral over the constant values of the potential on the conductors, obtaining

$$\Delta F = -\frac{1}{2\beta}\log[\det(\mathbb{C})/\beta^N], \quad (15)$$

where $\mathbb{C}$ is the capacitance matrix. This is the main result of this work. In what follows we will omit the factor β^N inside the logarithm, since it is irrelevant when computing the Casimir forces between conductors.

IV. EXAMPLES AND PFA APPROXIMATION

In this section we evaluate ΔF for some particular geometries, and analyze its behavior at long and short distances. The latter is elucidated by using an estimation of the capacitance matrix, obtained by using the proximity force approximation (PFA) [8].

A. Sphere-sphere geometry

Let us consider two facing spheres of equal radius a, separated by a distance d between centers. The elements of the capacitance matrix for this geometry are given by [9]

$$\begin{aligned}C_{11} = C_{22} &= a \sinh\psi \sum_{n=1}^{\infty} \mathrm{csch}((2n-1)\beta), \\ C_{12} = C_{21} &= -a \sinh\psi \sum_{n=1}^{\infty} \mathrm{csch}(2n\psi), \quad (16)\end{aligned}$$

where $\cosh\psi \equiv d/2a$. Inserting Eq. (16) into Eq. (15) one obtains an exact analytic expression for ΔF in this geometry.

When both spheres are very close, $d \to 2a$, we define $\xi = (d - 2a)/2a$ and take the limit $\xi \to 0$ in Eqs. (16). It can be shown that [10]

$$\begin{aligned}C_{11} = C_{22} &= a\left(-\frac{1}{4}\log\xi + \frac{\gamma}{2} + \frac{3}{4}\log 2 + \mathcal{O}(\xi\log\xi)\right), \\ C_{12} = C_{21} &= a\left(\frac{1}{4}\log\xi - \frac{\gamma}{2} + \frac{1}{4}\log 2 + \mathcal{O}(\xi\log\xi)\right), \quad (17)\end{aligned}$$

where γ denotes the Euler-Mascheroni constant. In this approximation:

$$\beta\Delta F \approx -\frac{1}{2}\log(-\log\xi). \quad (18)$$

We note that the last result coincides with the one obtained in [4], where the authors computed the high temperature Casimir free energies for the same geometry, considering Dirichlet and metallic boundary conditions, the latter described by a Drude model. It has also been shown there that, in the short distance limit, one has

$$\beta F_s^{(g)} \approx -\frac{\zeta(3)}{32\xi} + \frac{1}{48}\log\xi, \quad (19)$$

where the leading term is the usual PFA, while the next to leading order (NTLO) can be obtained using the derivative expansion approach [11,12]. We see that the difference ΔF is much smaller than the NTLO as $\xi \to 0$. Note, however, that due to the presence of the double logarithm this will only happen for exceedingly small values of ξ (and therefore the double logarithmic term becomes the main correction to the PFA for typical values of d and a).

Let us now consider the opposite limit, $d \gg a$ (large separation), where the capacitance coefficients have the expansions:

$$C_{11} = C_{22} = a\left(1 + \frac{a^2}{d^2} + \frac{2a^4}{d^4} + \cdots\right),$$
$$C_{12} = C_{21} = -a\left(\frac{a}{d} + \frac{a^3}{d^3} + \frac{3a^5}{d^5} + \cdots\right). \qquad (20)$$

Inserting this result into Eq. (15) we obtain:

$$\beta\Delta F \approx -\frac{1}{2}\left(\frac{a^2}{d^2} + \frac{5}{2}\frac{a^4}{d^4}\right). \qquad (21)$$

The free energy for grounded spheres has been obtained in Ref. [4]. Performing an expansion of their exact result in the large distance limit, we get $F_s^{(g)} \approx \Delta F$, and therefore $F_s^{(i)} = \mathcal{O}((\frac{a}{d})^6)$, which shows that the interaction between isolated spheres is dominated by the dipole-dipole interaction, as discussed in Ref. [3] for compact objects using a multipole expansion.

This example illustrates a general characteristic of the difference between the free energies for grounded and isolated objects. While at short distances both $F_s^{(g)}$ and $F_s^{(i)}$ have the same leading order behavior, at long distances the fluctuations of the charges (that occur for grounded conductors and not for isolated ones), radically change the nature of the leading interaction between conductors.

B. Sphere-plane geometry

We shall now consider a sphere of radius a, whose center is at a distance d from an infinite plane. The elements of the capacitance matrix can be obtained as a limiting case of a geometry involving two separated spheres with different radii a and b, in the limit $b \to \infty$ [9]. In this situation, $C_{22} \approx b$ while

$$C_{11} = -C_{12} = a \sinh\alpha \sum_{n=1}^{\infty} \mathrm{csch}(n\alpha), \qquad (22)$$

where $\cosh\alpha = d/a$. Therefore, the difference between free energies reads

$$\beta\Delta F = -\frac{1}{2}\log(C_{11}b - C_{11}^2) = -\frac{1}{2}\log C_{11} + \mathrm{const} \qquad (23)$$

where the constant, independent of d and diverging as $b \to \infty$, is irrelevant for the computation of the force between sphere and plane.

In the short distance limit $\alpha \to 0$, the sum that defines the coefficient C_{11} can be approximated by

$$\sum_{n=1}^{\infty} \mathrm{csch}(n\alpha) \approx \int_1^{\infty} dn\, \mathrm{csch}\,(n\alpha) = \frac{1}{\alpha}\log\left[\coth\left(\frac{\alpha}{2}\right)\right] \qquad (24)$$

and then

$$\beta\Delta F \approx -\frac{1}{2}\log\left[\frac{1}{2}\log\left(\frac{a}{d-a}\right)\right]. \qquad (25)$$

Once again, there is a double log term in the free energy for isolated objects. This behavior has been found numerically for the same geometry, when considering the classical limit of the Casimir interaction for Drude metallic boundary conditions [13].

C. General case: Two close conductors

In view of the examples above, the question presents itself about whether the appearance of double logarithms in ΔF at close distances is a general feature or not, i.e. if they always appear (independently of the geometry of the conductors involved). Exploration of other simple geometries shows that this is not the case. Indeed, the determinant of $\mathbb{C}$ does not show a logarithmic behavior at short distances for some elementary examples, like concentric cylinders or concentric spheres. One can show that this is also the case for eccentric cylinders or spheres, as well as for a cylinder in front of a plane.

In a general case, assuming that the geometry is such that one can use the PFA to estimate the capacitance matrix elements, the electrostatic energy between two conductors held at potentials ϕ_1 and ϕ_2 respectively can be approximated by [14]

$$U_{PFA} = \frac{1}{2}(\phi_1 - \phi_2)^2 \int d^2\mathbf{x}\, \frac{1}{d(\mathbf{x})} \equiv \frac{1}{2} C_{\mathrm{PFA}}(\phi_1 - \phi_2)^2, \qquad (26)$$

where $d(\mathbf{x})$ denotes the local distance between facing surface elements on both conductors. Note that, in this approximation, we have $C_{11} = C_{22} = -C_{12} = -C_{21} = C_{\mathrm{PFA}}$ and therefore $\det\mathbb{C}$ vanishes.

In general, including departures from the PFA result, we will have $C_{11} = C_{\mathrm{PFA}} + \Delta_{11}$, $C_{22} = C_{\mathrm{PFA}} + \Delta_{22}$, and $C_{12} = -C_{\mathrm{PFA}} + \Delta_{12}$, with $\Delta_{\alpha\beta}$ denoting the contributions coming from the subleading corrections. Therefore

$$\beta\Delta F \approx -\frac{1}{2}\log(C_{\mathrm{PFA}}) - \frac{1}{2}\log(\Delta_{11} + \Delta_{22} + 2\Delta_{12})$$
$$\approx -\frac{1}{2}\log(C_{\mathrm{PFA}}), \qquad (27)$$

where we have assumed that the contributions coming from the subleading corrections are much smaller than the

TABLE I. Capacitance matrix elements in the PFA for different geometries $C_{\rm PFA} = C_{11} = C_{22} = -C_{12}$. a and b denote the radii of spheres or cylinders, h is the distance between conducting surfaces, and L is the cylinders' length.

Geometry	$C_{\rm PFA}$
Sphere—sphere	$-\frac{a}{4}\log\frac{h}{a}$
Sphere—plane	$-\frac{a}{2}\log\frac{h}{a}$
Concentric spheres	$\frac{a^2}{h}$
Concentric cylinders	$\frac{La}{2h}$
Cylinder—plane	$\frac{L}{4\sqrt{2}}\sqrt{\frac{a}{h}}$
Eccentric cylinders	$\frac{L}{2}\sqrt{\frac{ab}{2(b-a)h}}$

leading PFA term [15]. From Eq. (27) we can derive the form of the short distance behavior, by using the corresponding expressions for $C_{\rm PFA}$. They are shown, for all the examples mentioned above, in Table I.

V. DISCUSSION

In this paper we have computed the difference between the high temperature Casimir free energies for a system of conductors, when these are either grounded or isolated. We have shown that difference comes from the TM Matsubara zero mode of the electromagnetic field, which can be described by a single scalar field. When the conductors are grounded, the scalar field satisfies Dirichlet boundary conditions. On the other hand, when the conductors are isolated, the scalar field may take any constant value on the surface of each conductor, and those constant values have to be integrated. Precisely because of that constant-potential integration, the difference ΔF becomes proportional to $\log\det\mathbb{C}$, where $\mathbb{C}$ denotes the (electrostatic) capacitance matrix of the system.

We have evaluated explicitly ΔF for particular geometries, and found a general expression for the case of two close conductors, using the PFA. Note that the use of the PFA for the approximate evaluation of ΔF could be convenient to derive, for example, the free energy for isolated conductors based on the knowledge of the result corresponding to grounded ones. The latter could be known by the use of any other method, not necessarily the PFA.

Essentially the same problem of evaluating the difference between the two free energies we have considered has been studied before [3], but an important caveat: in that reference, a multipole expansion is introduced at an early stage in the calculation. This is, indeed, adequate, in order to analyze the case of conductors when they are separated by long distances, but it cannot be used to write an expression of general validity. As we have shown here, such an expression may be written in terms of the determinant of the capacitance matrix of the system.

As shown in [3], in the long distance limit, the interaction between conductors changes drastically between the grounded and isolated case: the former is dominated by the monopole-monopole term, while, in the latter the leading interaction comes from the dipole-dipole term.

We have analyzed in some detail the behavior of ΔF in the opposite regime to the one of [3], namely, at short distances. This is the case that should be more relevant to Casimir effect calculations. In that context, we note that in previous works it has been shown that, at short distances, the corrections to PFA involve a double logarithm behavior in the free energy, for the particular cases of a sphere in front of a plane [13], and also for two spheres [4]. We have shown that those double logarithmic terms come from the logarithmic behavior of the capacitance coefficient C_{11}, and that their occurrence is not a general phenomenon, but in can nevertheless be predicted using an estimation of the capacitance matrix based on the PFA.

It is worthwhile to point out that the short distance corrections for isolated objects may formally be regarded as a next to NTLO correction to the PFA for grounded ones. Indeed, we have shown this to be the case for the concrete example of the sphere-sphere geometry. Note also that, in this context, when considering the corrections to the PFA calculation of the free energy for isolated objects, one gets contributions from two qualitatively different origins: on the one hand, one has the terms which arise from ΔF. On the other, we have the ones that proceed from the free energy for grounded conductors. The derivative expansion (DE) approach [11,12] has been used for the term which comes from the Dirichlet (grounded) term, and it gives of course the same result for either isolated and grounded conductors, since the difference between their free energies is in ΔF.

Finally, we note that there is another important difference between the contributions to the free energy of two isolated conductors coming from the two terms which may be identified as corresponding to grounded conductors and to ΔF. The interaction energy in the former, for many interesting cases, can be written as a functional of the (space dependent) vertical distance between the two surfaces. This functional becomes, in the limit of flat and parallel conductors, extensive in their area. This is the starting point of the DE [11], which for the Dirichlet case generates a correction depending on the distance function and its derivatives. The reason for this term to be extensive in the area, is that it proceeds from the contribution of field fluctuations, which form a continuum of degrees of freedom (to be integrated out), the number of which goes like the area of the surfaces times the differential volume in momentum space.

The ΔF term is, on the other hand, the result of evaluating the integral over just one (constant) mode: a single degree of freedom. Therefore there is no area factor in its contribution, even for flat parallel conductors. Even though the capacitance coefficients do depend on the areas, they appear inside a logarithm (in spite of the fact that the PFA may be correctly applied to calculate the capacitance coefficients). We see that the same cannot be done to evaluate, say, the ΔF term as the result of a single PFA (or even DE) calculation. Indeed, as shown in [11], the PFA is obtained as the "effective potential" for the corresponding functional. Namely, the ratio between the functional and the area, in the infinite area limit, for a constant distance between plates. And this ratio vanishes for ΔF.

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APPENDIX: PARTITION FUNCTION FOR ISOLATED CONDUCTORS

In this Appendix we prove that the partition function that includes an integration over the values of the surface potentials

$$\mathcal{Z}^{(i)} = \int_{-\infty}^{\infty}\left(\prod_{\alpha=1}^{N} d\phi_\alpha\right)\mathcal{Z}[\{\phi_\alpha\}] \tag{A1}$$

corresponds to an isolated conductor with fixed vanishing total charge. To this end, we introduce a generating functional for the mean values $\langle Q_{\alpha_1}^{n_1} Q_{\alpha_2}^{n_2} \ldots \rangle$:

$$\mathcal{Z}[\{\mu_\alpha\}] = \int_{-\infty}^{\infty}\left(\prod_{\alpha=1}^{N} d\phi_\alpha\right)\int \mathcal{D}\phi e^{-\frac{1}{2\beta}\int d^3\mathbf{x}(\partial_j\phi)^2} \times \prod_{\alpha=1}^{N}\delta[\phi|_{S_\alpha} - \phi_\alpha] e^{-\mu_\alpha Q_\alpha} \tag{A2}$$

and prove that it does not depend on μ_α. The charge on each conductor reads

$$Q_\alpha = -\int_{S_\alpha} \vec{\nabla}\phi \cdot \vec{dS}. \tag{A3}$$

In order to compute the functional integral in $\mathcal{Z}[\{\mu_\alpha\}]$, we proceed as before and perform a shift in the integration variables $\phi = \tilde{\phi} + \varphi$ [see Eq. (13)]. In terms of the new integration variable, φ, the charge is given by

$$Q_\alpha = \tilde{Q}_\alpha - \int_{S_\alpha} \vec{\nabla}\varphi \cdot \vec{dS}, \tag{A4}$$

where $\tilde{Q}_\alpha$ is the charge associated to the classical field $\tilde{\phi}$. We obtain:

$$\mathcal{Z}[\{\mu_\alpha\}] = \int_{-\infty}^{\infty}\left(\prod_{\alpha=1}^{N} d\phi_\alpha\right) e^{-\frac{1}{2\beta}\sum_{\gamma\delta} C_{\gamma\delta}\phi_\gamma\phi_\delta} \times e^{-\sum_{\gamma\delta}\mu_\gamma C_{\gamma\delta}\phi_\delta} \int \mathcal{D}\varphi e^{-\frac{1}{2\beta}\int d^3\mathbf{x}(\partial_j\varphi)^2} \times e^{\sum_\alpha \mu_\alpha \int_{S_\alpha}\vec{\nabla}\varphi\cdot\vec{dS}} \prod_{\alpha=1}^{N}\delta[\varphi|_{S_\alpha}], \tag{A5}$$

which are two independent integrals. The first one [upper line in Eq. (A5)] is an ordinary Gaussian integral. The second one [lower line in Eq. (A5)] is a functional integral for a free scalar field satisfying Dirichlet boundary conditions, in the presence of a source J defined by

$$\sum_\alpha \mu_\alpha \int_{S_\alpha} \vec{\nabla}\varphi \cdot \vec{dS} \equiv \int d^3\mathbf{x} J\varphi, \tag{A6}$$

so

$$J = -\sum_\alpha \mu_\alpha \int_{S_\alpha} \vec{dS}_\alpha \cdot \vec{\nabla}\delta(\mathbf{x} - \mathbf{x}_{S_\alpha}), \tag{A7}$$

where $\mathbf{x}_{S_\alpha}$ denotes points on the surface S_α. Therefore

$$\mathcal{Z}[\{\mu_\alpha\}] = e^{\frac{\beta}{2}\sum_{\gamma\delta} C_{\gamma\delta}\mu_\gamma\mu_\delta} e^{\frac{\beta}{2}\int d^3\mathbf{x}\int d^3\mathbf{y} J(\mathbf{x}) G(\mathbf{x},\mathbf{y}) J(\mathbf{y})}, \tag{A8}$$

where G is the Green's function of the electrostatic problem

$$\nabla^2 G(\mathbf{x},\mathbf{y}) = -\delta(\mathbf{x}-\mathbf{y}) G|_{S_\alpha} = 0, \tag{A9}$$

and we omitted an overall constant that is independent of μ_α.

Using the explicit expression for the current J, and after integration by parts we obtain

$$\int d^3\mathbf{x}\int d^3\mathbf{y} J(\mathbf{x}) G(\mathbf{x},\mathbf{y}) J(\mathbf{y}) = \sum_{\alpha\beta}\mu_\alpha\mu_\beta \int dS_\alpha \int dS_\beta \partial_{n_\alpha}\partial_{n_\beta} G, \tag{A10}$$

where we recognize the (not so well known) formal expression of the coefficients of capacitance in terms of the Green's function [16]

$$C_{\gamma\delta} = -\int dS_\gamma \int dS_\delta \partial_{n_\gamma}\partial_{n_\delta} G. \tag{A11}$$

Combining Eqs. (A8)–(A11) we see that $\mathcal{Z}[\{\mu_\alpha\}]$ does not depend on μ_α. Therefore, all the mean values $\langle Q_{\alpha_1}^{n_1} Q_{\alpha_2}^{n_2} \ldots \rangle$ vanish.

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