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Lyapunov decay in quantum irreversibility

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The Loschmidt echo—also known as fidelity—is a very useful tool to study irreversibility in quantum mechanics due to perturbations or imperfections. Many different regimes, as a function of time and strength of the perturbation, have been identified. For chaotic systems, there is a range of perturbation strengths where the decay of the Loschmidt echo is perturbation independent, and given by the classical Lyapunov exponent. But observation of the Lyapunov decay depends strongly on the type of initial state upon which an average is done. This dependence can be removed by averaging the fidelity over the Haar measure, and the Lyapunov regime is recovered, as it was shown for quantum maps. In this work, we introduce an analogous quantity for systems with infinite dimensional Hilbert space, in particular the quantum stadium billiard, and we show clearly the universality of the Lyapunov regime.

1. Introduction

Understanding the emergence of irreversibility from the basic laws of physics has been a longstanding problem. Although, since the eighteenth century it is known that the second law describes the arrow of time, its microscopic foundation has been matter of debate until these days [1,2]. The main problem is that classical mechanics is time-symmetric and cannot explain the emergence of the thermodynamic arrow of time. This contradiction has been apparently resolved with the understanding of chaos. The sensitivity to initial conditions of chaotic systems, along with the notions of mixing and coarse graining, has been the main argument to explain irreversibility in classical systems [3].

In quantum mechanics, the situation is more involved. Owing to the linearity of the Schrödinger equation there is no sensitivity to initial conditions, and therefore the origin of irreversibility in quantum mechanics lies elsewhere. For this reason, an alternative idea was proposed by Peres [4]. He suggested that quantum mechanics is sensitive to perturbations in the evolution rather than to the initial conditions. A suitable dynamical quantity to study such a behaviour was coined fidelity or Loschmidt echo (LE), which is defined as

$$M_{\psi}(t) = |\langle \Psi | U^{\dagger}_{\xi+\delta\xi}(t) U_{\xi}(t) | \Psi \rangle|^2, \tag{1.1}$$

where $U_{\xi}(t)$ is an evolution operator and, $U_{\xi+\delta\xi}(t)$ is a corresponding perturbed one, $|\psi\rangle$ an initial state, and the parameter $\delta\xi$ characterizes the strength of the perturbation. Thus, equation (1.1) can be interpreted in two different ways. On the one hand, it is the overlap between an initial state $|\psi\rangle$ evolved forward up to time t with the evolution operator $U_{\xi}(t)$, and the same state evolved backward in time with a perturbed evolution operator $U_{\xi+\delta\xi}(t)$. On the other hand, it can also be interpreted as the overlap at time t of the same state evolved forward in time with slightly different Hamiltonians. While the first interpretation gives the idea of irreversibility, the second is related to the sensitivity to perturbations of quantum evolutions.

The LE has been intensively studied in the first decade of this century [5–7], and several time and perturbation regimes were clearly identified using different techniques like random matrix theory, semi-classical and numerical simulations [7]. The progress in experimental techniques has permitted to study the LE in various different settings like NMR [8,9], microwave billiards [10,11], elastic waves [12] and cold atoms [13,14]. The most relevant result in connection with chaos and irreversibility is that the LE has a regime where the decay becomes independent of the perturbation strength and it is given by the Lyapunov exponent [15], a classical measure of the divergence of neighbouring trajectories [3]. The Lyapunov regime has been observed in several systems [16–23]. In these works, a crucial feature is that the initial states need to be coherent (Gaussian) wave functions [24]. Besides, an average on initial condition or perturbations is required.

The dependence of the LE with the type of initial state can be removed by considering an average over initial states according to the Haar measure for finite dimensional systems [25]

$$\int \mathrm{d}|\Psi\rangle M_{\psi}(t) = \frac{1}{d(d+1)} \left[d + |\langle U_{\xi}(t), U_{\xi+\delta\xi}(t)\rangle_{\mathrm{HS}}|^2 \right], \tag{1.2}$$

where $\langle U_{\xi}(t), U_{\xi+\delta\xi}(t)\rangle_{\mathrm{HS}} \equiv \mathrm{tr}[U^{\dagger}_{\xi}(t) U_{\xi+\delta\xi}(t)]$ is the Hilbert–Schmidt product between the operators, and d is the dimension of the Hilbert space. Thus, the average fidelity amplitude

$$|f(t)| = |\mathrm{tr}[U^{\dagger}_{\xi}(t) U_{\xi+\delta\xi}(t)]|, \tag{1.3}$$

which is directly related to the LE and is a state-independent quantity. This quantity was studied in detail for quantum maps [26]. Analytical results were obtained using a semi-classical theory known as dephasing representation (DR) [27–29]. It is shown that $|f(t)|$ has two clear decay regimes. For short times, the decay rate depends on the perturbation and it is predicted considering random dynamics. This corresponds to the limit of infinite Lyapunov exponent. If the strength of the perturbation is small enough, this regime lasts up to the saturation point. The other regime, was obtained considering that the perturbation is completely random. That is, after each step of the map, the perturbation contributes with a random phase to each trajectory. In that case, using the DR and transfer matrix theory it is shown that the asymptotic decay rate of $|f(t)|$ is controlled by the largest classical Lyapunov exponent λ. Numerical tests of the analytical predictions were given for the quantum baker and a family of perturbed cat maps (see §2 below).

In this work, we go one step further by studying $|f(t)|$ in a realistic system. We consider a particle inside a stadium billiard that is perturbed by a smooth potential consisting of a number of Gaussians randomly distributed inside the cavity. A two-dimensional billiard has an infinite-dimensional Hilbert space. For this reason, instead of computing equation (1.3) for a complete set, we consider an initial state defined as an incoherent sum of all the energy projectors from the ground state up to a given high energy level. We have numerically computed this quantity that we call $|f_\Omega(t)|$ for the quantum stadium billiard and show that it has similar behaviour to that observed for quantum maps. For short times, $|f_\Omega(t)|$ has a decay that depends on the perturbation strength. But, after a crossover and for sufficiently large perturbation strength, we can clearly see that $|f_\Omega(t)|$ decays exponentially with a decay rate given by the Lyapunov exponent of the classical billiard. In order to confirm these results, we have also computed $|f_\Omega(t)|$ using the DR. We also show that the DR describes very well the quantum behaviour and that the Lyapunov regime is also clearly observed in this approximation.

The rest of the paper is organized as follows. In §2, we summarize the results obtained in [26] in quantum maps. We show that the DR works very nicely to describe the quantum behaviour and we show the different decay regimes of $|f(t)|$. In §3, we show the $|f_\Omega(t)|$ for the stadium billiard. Final remarks and outlook are given in §4.

2. Quantum maps

For this work to be self-contained, in this section, we briefly review the results previously obtained in [26] for quantum maps on a two-dimensional phase space with periodic boundary conditions (2-torus). These maps have the essential ingredients of chaotic systems and are simple to treat numerically and sometimes even analytically. The torus geometry implies that, upon quantization, position q and momentum p are discrete and related by the discrete Fourier transform. The Hilbert space then has finite dimension N, and the semi-classical limit is given by $N \to \infty$. An efficient Planck constant can be thus defined as $\hbar = 1/(2\pi N)$.

As a tool we use the DR [27–29] which avoids some of the drawbacks of other semi-classical methods. The fidelity in the DR can be written as

$$f_{\mathrm{DR}}(t) = \int \mathrm{d}q\,\mathrm{d}p W_\rho(q,p) \exp\left(-\frac{\mathrm{i}\Delta S_{\delta\xi}(q,p,t)}{\hbar}\right), \tag{2.1}$$

where $W(q,p)$ is the Wigner function of the initial state ρ, and

$$\Delta S_{\delta\xi}(q,p,t) = -\delta\xi \int_0^t \mathrm{d}\tau\, V(q(\tau),p(\tau)) \tag{2.2}$$

is the action difference evaluated along the unperturbed classical trajectory. If ρ is a maximally mixed state then equation (2.1) is an average over a complete set (and becomes basis independent),

$$f_{\mathrm{DR}}(t) = \frac{1}{\mathcal{V}} \int \mathrm{d}q\,\mathrm{d}p \exp\left(-\frac{\mathrm{i}\Delta S_{\delta\xi}(q,p,t)}{\hbar}\right), \tag{2.3}$$

this quantity is the semi-classical expression for the fidelity given in equation (1.3). Here, for simplicity, we write phase space variables q and p, and their differentials, as one dimensional. But equation (2.3) holds for arbitrary dimensions. For maps time is discrete so, for the reminder of this section, we define $t := n$, with n integer.

We use the DR to study the decay of the fidelity in the chaotic regime as follows. First, we suppose that the system is very strongly chaotic, $\lambda \to \infty$. This is essentially equivalent to assuming that the evolution is random without any correlation. In order to compute $f(n)$, we partition the phase space into N_c cells and consider that the probability of jumping from one cell to the other is uniform. It can then be shown that

$$f_{\mathrm{DR}}(n) = \frac{1}{N^n}\sum_{j_1}\cdots\sum_{j_n}\exp\left[-\frac{\mathrm{i}(\Delta S_{\delta\xi,j_1}+\cdots+\Delta S_{\delta\xi,j_n})}{\hbar}\right]$$
$$= \left[\frac{1}{N^n}\sum_{j}\exp\left(-\frac{\mathrm{i}\Delta S_{\delta\xi,j}}{\hbar}\right)\right]^n, \tag{2.4}$$

where $\Delta S_{\delta\xi,j_k}$ is the action difference on the cell j at time k. Taking the limit $N_c \to \infty$, we get

$$f_{\mathrm{DR}}(n) = \left[\int\!\!\!\int \mathrm{d}q\,\mathrm{d}p\,\exp(-\mathrm{i}\Delta S_{\delta\xi}(q,p))\hbar\right]^n. \tag{2.5}$$

The absolute value of $f_{\mathrm{DR}}(n)$ can then be written as

$$|f_{\mathrm{DR}}(n)| = \exp(-\Gamma n), \tag{2.6}$$

where

$$\Gamma = -\log\left|\int\!\!\!\int \exp\left[-\frac{\mathrm{i}\Delta S_{\delta\xi}(q,p)}{\hbar}\right]\mathrm{d}q\,\mathrm{d}p\right|. \tag{2.7}$$

Then, if the dynamics is completely random, which is approximately the case for strongly chaotic systems, then the fidelity decays exponentially with a rate Γ. As we shall see, this decay also explains the short time behaviour regardless of λ because for short times the dynamics can always be supposed to be uncorrelated.

To unveil the intermediate time regime, we consider the limit of random perturbation. In [26], using the DR it is shown that, for the baker map with a random perturbation, the fidelity can be written as a sum of products of transfer matrices

$$f_{\mathrm{DR}}(n) = \frac{1}{2^{n/2+L-1}}\sum_{k0,\ldots,k_n} M_{k_0,k_1}\ldots M_{k_{n-1},k_n}, \tag{2.8}$$

where

$$k_i = 2^{(L-1)} \times \cdot\mu_i\ldots\mu_{L+i-2}, \tag{2.9}$$

the digits $\mu_i = 0, 1$ define position and momentum

$$q = \sum_{j=0}^{\infty}\frac{\mu_j}{2^j+1} \overset{\text{def}}{=} \cdot\mu_0\mu_1\ldots \tag{2.10}$$

and

$$p = \sum_{j=0}^{\infty}\frac{\mu_{-j}}{2^j+1} \overset{\text{def}}{=} \cdot\mu_{-1}\mu_{-2}\ldots \tag{2.11}$$

in symbolic dynamics (see e.g. [30]). A point in phase space is then $(q,p) = \ldots\mu_{-2}\mu_{-1}\cdot\mu_0\mu_1\ldots$, and one step of the map consists in shifting the point to the right. The letter L in the previous equations indicates a truncation size of the symbolic dynamics expansion. Defining the unit norm vector $|1\rangle = 2^{-(L-1)/2}(1,1,\ldots,1)$, equation (2.8) can be written in compact form as

$$f_{\mathrm{DR}}(n) = 2^{-n/2}\langle 1|M^n|1\rangle. \tag{2.12}$$

The properties of the fidelity are then determined by the spectrum of the finite matrix M. In particular, the asymptotic decay is ruled by the largest eigenvalue (in modulus) of M. Considering the special structure of the transfer matrices for the bakers map, it was shown that

$$|f_{\mathrm{DR}}(n)|^2 \approx 2^n = e^{\lambda_B n}, \tag{2.13}$$

where $\lambda_B = \ln 2$ is the largest Lyapunov exponent of the baker map. This analytical result was further extended to more general types of maps [26].

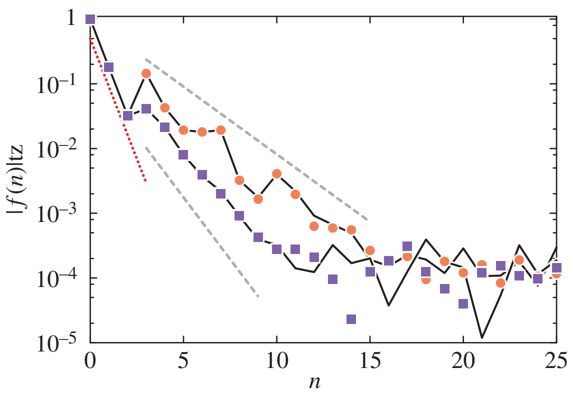

Figure 1. Fidelity as a function of discrete time n using straight quantum calculation (points) and DR (black solid line) for the perturbed cat map, with $a=1$ (circles), corresponding to $\lambda \approx 0.962$ and $a=2$ (squares) corresponding to $\lambda \approx 1.76$. Here $N=10^4$ and $\xi = 0.025$ and $\delta\xi/\hbar = 2$. The dashed (grey) lines correspond to the Lyapunov regime, $|f(n)| \sim \exp(-\lambda n/2)$. Q2 The dotted (red) line marks the short time decay, $|f(n)| \sim \exp(-\Gamma n)$, with Γ obtained from equation (2.7). (Online version in colour.)

We show numerically these two regimes for a family of perturbed cat maps [31]

$$\left.\begin{aligned} \bar{p} &= p - a\,q + \xi f(q) \\ \text{and} \qquad \bar{q} &= q - b\bar{p} + \tilde{\xi} h(\bar{p}) \end{aligned}\right\} \quad (\text{mod } 1). \tag{2.14}$$

For simplicity, let $\tilde{\xi} = \xi$. For a, b integer, these maps are uniformly hyperbolic and for small enough K, the Lyapunov exponent is approximately given by

$$\lambda \approx \log \frac{1 + ab + \sqrt{ab(4+ab)}}{2}. \tag{2.15}$$

For simplicity from now on we take $a=b$. These maps can be written in the general form

$$\left.\begin{aligned} \bar{p} &= p - \frac{\mathrm{d}V_\xi(q)}{\mathrm{d}q} \\ \text{and} \qquad \bar{q} &= q + \frac{\mathrm{d}T_\xi(\bar{p})}{\mathrm{d}\bar{p}} \end{aligned}\right\} \quad (\text{mod } 1) \tag{2.16}$$

and can be simply quantized as a product of two operators

$$U_\xi = \mathrm{e}^{-\mathrm{i}2\pi N T_\xi(\hat{p})}\,\mathrm{e}^{-\mathrm{i}2\pi N V_\xi(\hat{q})}. \tag{2.17}$$

Many well-known quantizations of classical maps can be expressed in this way, e.g. the kicked Harper map [32] and the Chirikov standard map [33]. For the numerical examples, we consider

$$f(q) = 2\pi[\cos(2\pi q) - \cos(4\pi q)] \tag{2.18}$$

and

$$h(\bar{p}) = 0 \tag{2.19}$$

as the perturbing 'forces' of equation (2.14).

In figure 1, we show two things. On the one hand, the almost prefect agreement of the DR Q2 calculation of $|f(t)|$ against the straightforward quantum result. On the other hand, it is shown that the two different exponential regimes can be distinguished. For the sake of clarity, we show results for $a=1$ and 2 which correspond to $\lambda \approx 0.962$ and 1.72, respectively.

In figure 2, we show a detailed example illustrating how different the two regimes can be. There, five examples of $|f(t)|$ for different values of δK are displayed. It can be clearly observed that the initial decay rate is given by Γ. In the inset, we show Γ as a function of the perturbation and the points mark the decay rate value indicated by the dashed (red lines) in the main panel.

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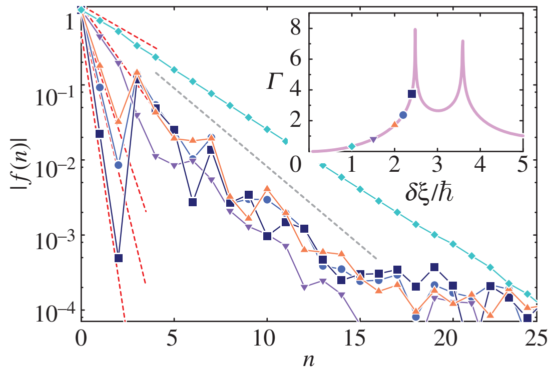

Q2 **Figure 2.** Quantum calculation of the fidelity for the perturbed cat map with $a = 1$ and different values of the perturbation, $\delta\xi/\hbar = 1$ (diamonds), 1.5 (inverted triangles), 2 (triangles), 2.2 (circles) and 2.4 (squares). The dashed red lines show the small-time behaviour $|f(t)| \sim \exp(-\Gamma n)$. The values of Γ obtained from equation (2.7) are shown in the inset. (Online version in colour.)

After this short-time decay there is a revival and then the fidelity again decays exponentially with a rate given by λ, except in the case where $\Gamma \ll \lambda$.

From the evidence of figures 1 and 2, a behaviour like

$$|f(t)| \sim \exp(-\Gamma t) + A \exp\left(-\frac{\lambda t}{2}\right) \tag{2.20}$$

can be hinted. The decay given by the rate Γ is explained by an initial lack of correlations. If the dynamics is strongly chaotic, then this is the decay that dominates throughout the evolution. This can be simulated by random evolution. In other cases, there is a crossover from to the perturbation independent Lyapunov regime. In [26], a random perturbation model was used to demonstrate this crossover, and also the crossover time could be inferred.

3. Stadium billiard

In the previous sections, we show that $|f(t)|$ is a suitable quantity to characterize quantum irreversibility in d-dimensional systems. It does not depend on the initial conditions because it is the trace of the echo operator $U^{\dagger}_{\xi+\delta\xi}(t)U_{\xi}(t)$. Moreover, in the case of abstract maps, the DR can be used to show analytically that there is a Lyapunov regime that does not depend on the type of initial states, contrary to what happens in the case of the LE. Now we will study the behaviour of a similar quantity in a realistic system, a particle inside a billiard. In this system, the Hilbert space is infinite-dimensional and it is not possible to compute the trace of the echo operator. For this reason, we consider

$$|f_{\Omega}(t)| = |\mathrm{tr}[U^{\dagger}_{\xi+\delta\xi}(t)U_{\xi}(t)\rho_{\Omega}]|, \tag{3.1}$$

where the initial density function $\rho_{\Omega}(m) = m^{-1}\sum_{i=0}^{m}|E_i(\xi)\rangle\langle E_i(\xi)|$, a microcanonical state located in an energy window that start in the ground state up to the mth excited state. We note that $|f_{\Omega}(t)|$ is related to a well-known quantity in non-equilibrium statistical mechanics: the probability of doing work W. This can be seen by considering a system with Hamiltonian $H(\xi)$ that is in an initial equilibrium state ρ. At $t = 0$, the energy is measured and a quench $H(\xi) \to H(\xi + \delta\xi)$ is done. Then, the system evolves a time t and another energy measurement is done. If $E_i(\xi)$ and $E_j(\xi + \delta\xi)$ are the results of the measurements, the work done on the system is $W = E_j(\xi + \delta\xi) - E_i(\xi)$. Then, it is easy to show that the probability of work $P(W)$ is the Fourier transform of

$$f_{\beta}(t) = \mathrm{tr}[U^{\dagger}_{\xi+\delta\xi}(t)U_{\xi}(t)\rho], \tag{3.2}$$

here we consider the absolute value of this quantity equation (3.1).

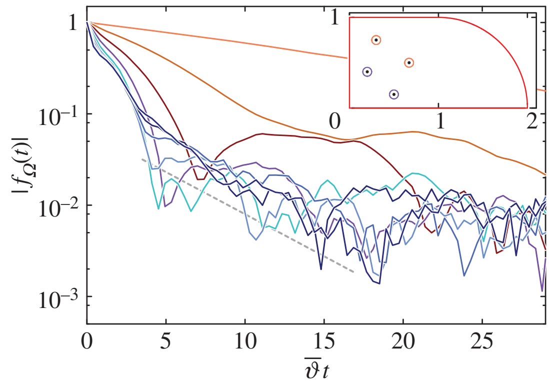

Figure 3. Fidelity for the stadium billiard with initial state $\rho_\Omega(m = 500)$ and a basis of 3134 states, as a function of the rescaled time $\bar{\upsilon}t$. $\delta\xi = 10, 30, 50, 70, 80, 90$. In the inset, we show the position of the four Gaussians and diameter of the circles is $\sigma =$ 0.1 (blue indicate positive Gaussian and red negative). The dashed grey line is $\exp(-\lambda_1 t)$. (Online version in colour.) Q2

We have studied $|f_\Omega(t)|$ for a particle in the desymmetrized stadium billiard with radius $r = 1$ and straight line of length $l = 1$ (see inset of figure 3). The perturbation is a smooth potential consisting in a series of four Gaussians, Q3

$$V(x, y, \delta\xi) = \delta\xi \sum_{i=1}^{4} \text{sign}_i \exp\left(-\frac{[(x - x_i)^2 - (y - y_i)^2]}{(2\sigma^2)}\right] \qquad (3.3)$$

with $\delta\xi$ the perturbation strength, $\sigma = 0.1$ their widths $(x_1, y_1) = (0.2, 0.4)$, $(x_2, y_2) = (0.67, 0.5)$, $(x_3, y_3) = (0.5, 0.15)$ and $(x_4, y_4) = (0.3, 0.75)$, the positions of the centres and $\text{sign}_1 = \text{sign}_3 = 1$, $\text{sign}_2 = \text{sign}_4 = -1$. The eigenstates of the unperturbed stadium $(\xi = 0)$ are obtained using the scaling method [34]. The states of the perturbed system are obtained by diagonalizing the perturbed Hamiltonian in the basis of the stadium billiard. We have used several number of unperturbed states to check the convergence of the results. We point out that the perturbations considered in this work and in [26] affect the whole phase space. There has been some theoretical [35–38] and experimental efforts to study the effect of local perturbations. In theses works, they either find a crossover from a Fermi golden rule regime to an exponential regime with rate given by the so-called escape rate (given by a representative size of the perturbation) [35,38], or an algebraic decay regime [37]. But, even though they also consider the average fidelity amplitude, they do not find a Lyapunov regime. Therefore, in this work we only consider global perturbations.

In figure 3, we show $|f_\Omega(t)|$ for the stadium billiard. The unperturbed evolution is given by the free dynamics inside the cavity $(\xi = 0)$ and the perturbed one is with the potential of equation (3.3). Results for several perturbation strengths $\delta\xi$ are shown. The initial microcanonical state corresponds to the first 500 eigenstates of the unperturbed system. The calculations were done using the first 3135 eigenstates of the stadium billiard. The convergence of the results were tested using a bigger basis of up to 5600 states. For a smaller basis of 1300 states, the results are also well behaved.

Let us first analyse the small-time behaviour. For small $\delta\xi \lesssim 25$, $|f_\Omega(t)|$ decays exponentially $\sim \exp(\Gamma t)$. In figure 4 (top), we show Γ as a function of $\delta\xi$. As expected, we can clearly see that $\gamma \sim \delta\xi^2$. Such behaviour is referred to as the Fermi golden rule regime [16]. For $\delta\xi \gtrsim 25$, the short-time decay $|f_\Omega(t)|$ is approximately a Gaussian function $\sim \exp[-t^2/(\tau^2)]$ (figure 3). In figure 4 (bottom), we show $1/\tau$ as a function of $\delta\xi$. We can see that after a transient, there is a region where $1/\tau$ does not depend on the perturbation strength $\delta\xi$.

The behaviour of the fidelity $|f_\Omega(t)|$ can be related to the spread of the initial state in the perturbed basis. The natural quantity to study this type of localization properties is the inverse

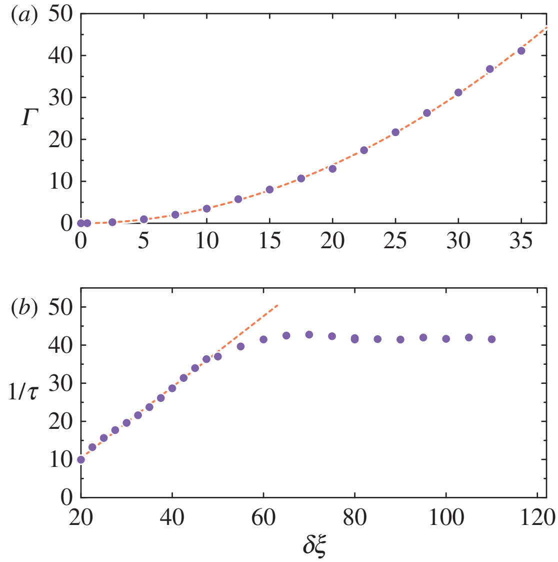

Figure 4. (*a*) Decay rate Γ for the short-time decay in the perturbative regime, $|f_\Omega(t)| \sim \exp[-\Gamma t]$. As expected from the Fremi golden rule Γ has approximately quadratic dependence with $\delta\chi$ (the dashed red line is a fit where $\Gamma \propto \delta\chi^{1.969\pm0.034}$). (*b*) Characteristic time $1/\tau$ for the Gaussian decay $|f_\Omega(t)| \sim \exp[-(t/\tau)^2]$, for small t. The slope of the linear fit is 0.93 ± 0.013, so $1/\tau \approx \delta\chi$ until saturation. Initial state is $\rho_\Omega(n = 800)$ and the basis used has 3134 states, the width of the Gaussian perturbation is $\sigma = 0.1$. (Online version in colour.)

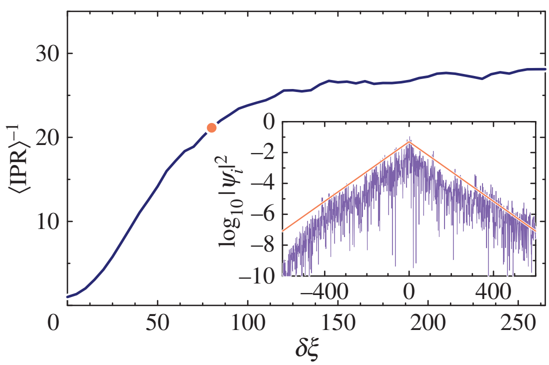

Figure 5. Inverse of the average IPR for the first 800 perturbed states as a function of $\delta\xi$. Inset: $||E_j\rangle_i|^2 = |\langle E_j(\xi + \delta\xi)|E_i(\xi)\rangle|^2$, for the state $j = 600$ for $\delta\xi = 80$. (Online version in colour.)

participation ratio (IPR) [39–45]. The IPR of a perturbed eigenstate $|j(\xi + \delta\xi)\rangle$ in the unperturbed basis $|E_i(\xi)\rangle$ is

$$\mathrm{IPR}(|E_i\rangle) = \left(\sum_m |\langle E_m(\xi)|E_i(\xi + \delta\xi)\rangle|^4\right) \tag{3.4}$$

(throughout this section $\xi = 0$). The inverse of this quantity—also called the participation number—gives an estimation of the number of unperturbed states contributing to a given perturbed state. In figure 5, we show the inverse of the IPR averaged over the first 800 states, as a function of $\delta\xi$. We can see that it has an approximately quadratic growth up to $\delta\xi = 20$. After that the inverse of the IPR grows linearly in the interval $20 \lesssim \delta\xi \lesssim 60$. Finally the growth rate tends to a saturation at value which is much smaller than the basis size. This shows that

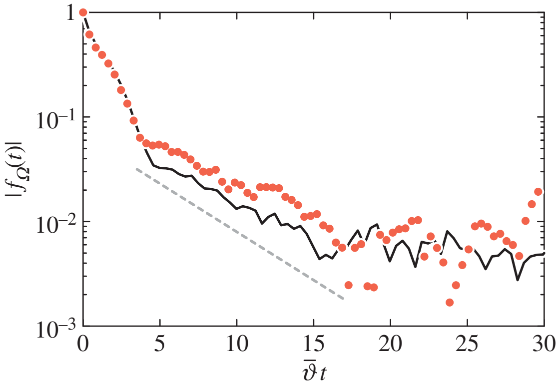

Figure 6. $|f_\Omega(t)|$ for the stadium billiard with perturbation strength $\delta\xi = 70$ and width $\sigma = 0.1$. The (red) circles correspond to the quantum evolution, with initial state $\rho_\Omega(m = 500)$ and a basis of 3134 states. The solid line corresponds to DR calculation using 189 991 initial conditions with energies chosen in the range corresponding to the first 500 quantum energies. The dashed grey line corresponds to the Lyapunov decay. (Online version in colour.) Q2

the perturbed states remain localized in energy. As an example, in the inset of figure 5, we show $|\psi_i|^2 = \langle i(\xi + \delta\xi)|i(\xi)\rangle|^2$ for the state corresponding to the level 600 for $\delta\xi = 80$ (marked by a red dot on the main panel). The exponential decay of the tails is a manifestation of its localization. This localization is responsible for the plateau of $1/\tau$ shown in figure 4 (bottom). We remark that the three perturbation regimes of the short-time decay of $f_\Omega(t)$ (figures 3 and 4) are manifested in the IPR behaviour (figure 5).

After the Gaussian decay shown in figure 3, we can see a second exponential decay with a rate given by the classical Lyapunov exponent of the stadium billiard. The Lyapunov exponent is $\lambda = \lambda_1\bar{v}$, where $\lambda_1 = 0.43$ corresponds to $l = r = 1$ [46,47]. Here $\bar{v} = 2\bar{k}$ is the average velocity computed from the eigenenergies $E_i = k_i^2$ (k_i being the wavenumber of the eigenstate $|E_i\rangle$) in the energy window Ω considered. Evidently, the fidelity computed using an initial state ρ_Ω and the fidelity obtained from the Haar measure for the quantum maps share the same decay behaviour. A short-time decay which depends on the characteristics and strength of the perturbation, followed by a Lyapunov regime depending on a classical feature.

We also compute $|f_\Omega(t)|$ using the semi-classical DR. This is a simple task due to the fact that unperturbed trajectories are geometrically obtained in the billiard and the perturbation only gives a phase as dictated by equation (2.1). To take into account the initial $\rho_\Omega(m)$ state, we compute the semi-classical $|f_\Omega(t)|$ using that the initial conditions are uniformly random inside the billiard, the same was assumed for the direction of the initial momentum. The modulus squared of the momenta are distributed as the eigenenergies of the unperturbed system. In figure 6, we show $|f_\Omega(t)|$ computed using the DR and the quantum results. We can see that the semi-classical approximation provides an accurate fitting of the quantum results. Moreover, the Lyapunov decay is clearly observed in the DR approximation.

4. Conclusion

From the outset, the LE emerged as a viable quantity to characterize instability and irreversibility in quantum systems. A large amount of work was dedicated to describe the different regimes depending on the perturbation strength, but it received a real important boost when the Lyapunov regime was first described linking classical an quantum chaotic behaviour. However, although it was shown to exist in many different systems, all the semi-classical and numerical calculations showed that the Lyapunov regime could only be observed if the initial states considered were ‘classically meaningful’ [6] (typically coherent states). We found a solution to this problem considering the average fidelity amplitude, a basis independent quantity, which is closely related to the LE if one considers an average over the Haar measure (equations (1.2) and (1.3)). Indeed, a recent work [26], briefly reviewed in §2, shows analytically and numerically that for quantum maps on the torus the average fidelity amplitude decays as a double exponential, where the first decay rate depends on the strength and type of perturbation, whereas the second decay rate is given by the classical Lyapunov regime.

But, although quantum maps have some generic properties of quantum chaos, they are not very generic systems themselves. So the challenge in this work was to take the analysis one step further and study the average fidelity amplitude in a more realistic system, paradigmatic of quantum chaos, like the stadium billiard. To overcome the problem of the infinite-dimensional Hilbert space, where averaging over the Haar measure is unfeasible, we introduced an energy cut-off and considered the system to be initially in a state that has an equiprobable distribution over some energy window, in the same spirit of a microcanonical ensemble. In this way, we were able to recover the same behaviour of the fidelity amplitude as the one shown for quantum maps, in particular, we could clearly observe the Lyapunov regime. Therefore, we have made a step forward towards the settlement of this longstanding problem: we showed an example of a realistic system where the Lyapunov regime is observed, independent of the type of initial state, if the appropriate quantity—the fidelity amplitude—is considered.

Additionally, we have shown that these regimes were also manifested in the behaviour of the IPR, which is a very relevant quantity in the study of localization and quantum chaos at the level of the structure of eigenstates. Finally, for completeness, we studied the dynamics of the fidelity amplitude using the semi-classical DR approximation and showed a good agreement with the quantum results. This suggests that as fidelity is also related to the Fourier transform of the work probability distribution after a quench [48], further insight into this issue can be obtained by considering tools such as the semi-classical DR approximation.

Data accessibility. All the programmes used to generate the data shown in the manuscript can be found in the following public link: https://goo.gl/eXZC3l.

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