EIGENFREQUENCIES OF GENERALLY RESTRAINED TIMOSHENKO BEAMS WITH AN INTERNAL HINGE

Maria Virginia Quintana[1], Javier Leandro Raffo[2]

ABSTRACT

This paper deals with the free transverse vibration of a Timoshenko beam with an arbitrarily located internal elastic hinge. The ends of the beam were elastically restrained against both rotation and translation. The results are obtained with the exact solution and with an analytical method that consists of a combination of the Ritz and the Lagrange multiplier methods. Both were used to determine free vibration characteristics of the employed beam. In the combined method, trial functions denoting the transverse deflections and the normal rotations of the cross section of the beam are expressed in polynomial forms. In order to verify the accuracy of the developed mathematical model, cases available in the literature have been considered. Results have also been presented for different end and restraint conditions (in the intermediate point) of the beam. In addition, a comparison with a crack model is also provided.

Keywords: vibrations; Timoshenko beams; elastically restrained; Lagrange multiplier; Ritz method.

1. INTRODUCTION

Timoshenko proposed a beam theory which includes the effects of shear distortion and rotatory inertia to the Euler-Bernoulli model [1, 2]. Afterwards there has been a considerable interest in developing techniques for the solutions of equations according to the Timoshenko theory. The problem of free vibration of Timoshenko beams with classical end conditions has been extensively treated and numerous studies have been devoted to it. The initial studies have been described in reference [3]. Although it is not possible to present a full account of all these studies here due to the enormous volume of information some important references have been cited in the presented paper. The problem of elastic end restraints has also received considerable attention. The problem of free vibration of Timoshenko beams with elastically supported ends by using a finite element model (FEM), which satisfies all the geometric and natural boundary conditions, has been studied by Abbas [4]. The natural frequencies and the critical buckling load coefficients for a multi-span Timoshenko beam elastically supported have been investigated by Farghaly [5]. Kocaturk and Simsek [6, 7] analysed free vibration problem of Timoshenko beams (having classical and elastically supported ends) by using the Lagrange equations with the trial functions expressed in the power

[1]Doctoral Degree Researcher, Chemical Industry Research Institute (INIQUI), Facultad de Ingenieria, Universidad Nacional de Salta, Argentina, Ph. +54 387 4255379, Email vquintan@unsa.edu.ar.
[2]Professor, Computational Mechanics Group, Facultad Regional Delta, Universidad Tecnológica Nacional, Argentina, Ph. +54 3489 420400, Fax. +54 3489 420400, Email jraffo@frd.utn.edu.ar.

Manuscript received on 18th March 2014 reviewed and accepted on 22nd June 2014 as per publication policies of *NED University Journal of Research*. Pertinent discussion including authors' closure will be published in June 2015 issue of the Journal if the discussion is received by 30th November 2014.


María Virginia Quintana is a Professor in the Civil Engineering School at Universidad Nacional de Salta and Universidad Católica de Salta, Salta, Argentina. She received her Bachelors in Civil Engineering from Universidad Nacional de Salta, Salta, Argentina in 2005 where she is currently enrolled as a doctoral student. Her research interests include applied mathematics, computational mechanics, mechanical vibration and composite materials.


Javier Leandro Raffo is an Associate Professor in the Department of Mechanical Engineering at Universidad Tecnologica Nacional, Buenos Aires, Argentine. His research interests include applied mathematics, computational mechanics and experimental mechanics.

series form. The free vibration of multi-span Timoshenko beams by the Rayleigh-Ritz method using static Timoshenko beam functions has been treated by Zhou [8]. Grossi and Aranda [9] applied the Ritz method in the variational formulation of Timoshenko beams with elastically restrained ends. A full development and analysis of four theories (including the Timoshenko model) for the transversely vibrating uniform beam have been presented in reference [10].

A review of the existing literature reveals that there is a limited amount of information for the vibration of beams with internal hinges. The forced vibrations of two beams joined with a non-linear rotational joint have been analysed by Ewing and Mirsafian [11]. The studies on the fundamental frequency of a beam with an internal hinge and subjected to an axial force were presented in reference [12]. Chang et al. [13] investigated the dynamic response of a beam with an internal hinge, subjected to a random moving oscillator. The free transverse vibration of a non-homogeneous tapered beam subjected to general axial forces, with arbitrarily located internal hinge and elastics supports and ends elastically restrained against rotation and translation have been analysed by Grossi and Quintana [14]. All these studies refer to beams treated with the Euler-Bernoulli theory.

The problem of vibration of Timoshenko beams with internal hinges (out of the context of cracks) has not been treated with the exception of Lee et al. [15] who considered a Timoshenko beam with a free hinge by determining the exact vibration frequencies. Free vibration determination of Timoshenko beams with cracks using a method of detection of location of cracks in beams based on frequency measurements have been presented in reference [16]. In order to compare the results with those used in a crack model, a comparison with the model used in Khaji et al. [17] was presented by Lele and Maiti [16]. The cracked section of the Timoshenko beam was modeled as local flexibility that was assumed to be a rotational spring. This model was first proposed by Ostachowicz and Krawczuk [18] using a theory which is based on the stress intensity factor developed by Haisty and Springer [19]. Later, Narkis [20] compared the results of this model with three different authors and a FE analysis (FEA) model. Khaji et al. [17] and Narkis [20] used the model to solve the inverse problem of identifying crack locations and crack depths from frequency data first obtained from a FEA model. The comparison of these works indicated that the proposed crack model performed well.

The aim of the presented paper is to investigate the natural frequencies and mode shapes of Timoshenko beams with several complicating effects such as an internal hinge elastically restrained against both rotation and translation, and with restrained ends. Several cases have been solved with the exact solution and with a method that employs a combination of the Ritz and the Lagrange multipliers methods in conjunction with sets of simple polynomials as trial functions. In order to asses the accuracy of the developed mathematical models, few cases available in the literature have been considered and comparisons of the numerical results were made. The algorithms developed can be applied to a wide range of elastic restraint conditions. A large number of problems have been solved; however, the results of only few cases have been presented here in view of space limitations. Since the presence of intermediate elastic restraints and a hinge allow the simulation of a crack model, a comparison with the results of Khaji et al. [17] has also been included in the presented paper.

2. THEORY AND FORMULATION

Consider a uniform Timoshenko beam of length (l) with two spans which are connected an intermediate point ($x = c$) by an hinge elastically restrained against rotation and translation, and with specified boundary conditions as shown in **Figure 1**.

According to Timoshenko beam theory, two independent variables (transverse deflection [$w = w(x, t)$] and normal rotational angle [$\phi = \phi(x, t)$] due to bending) are used to describe the deformation of beam.

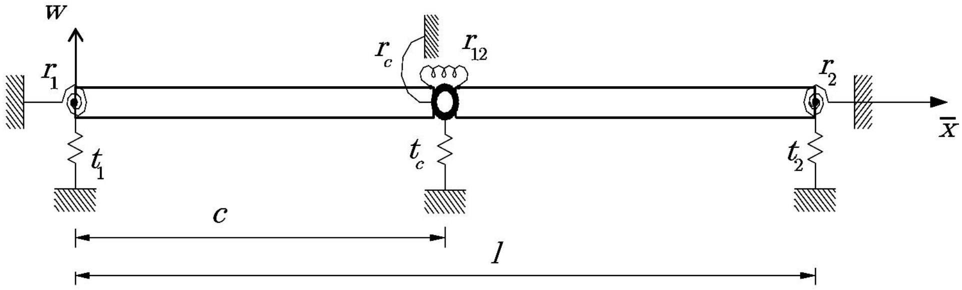


Figure 1. Beam model description.

2.1 Exact Solution using General Solution

The coupled equation of motion, and its associated boundary and transition conditions have been rigorously derived using Hamilton's principle [21] and are given by Eqs. (1) to (10).

$$
\begin{aligned}
&-\frac{\partial}{\partial t}\left(\rho_1(\bar{x})A_1(\bar{x})\frac{\partial w}{\partial t}(\bar{x},t)\right) \\
&\qquad +\frac{\partial}{\partial \bar{x}}\left[\kappa_1 G_1(\bar{x})A_1(\bar{x})\left(\frac{\partial w}{\partial \bar{x}}(\bar{x},t)-\phi(\bar{x},t)\right)\right]+q(\bar{x},t)=0 \\
&-\frac{\partial}{\partial t}\left(\rho_1(\bar{x})I_1(\bar{x})\frac{\partial \phi}{\partial t}(\bar{x},t)\right)+\frac{\partial}{\partial \bar{x}}\left(E_1(\bar{x})I_1(\bar{x})\frac{\partial \phi}{\partial \bar{x}}(\bar{x},t)\right) \\
&\qquad +\kappa_1 G_1(\bar{x})A_1(\bar{x})\left(\frac{\partial w}{\partial \bar{x}}(\bar{x},t)-\phi(\bar{x},t)\right)=0, \quad \forall \bar{x}\in(0,c),\ t\geq 0
\end{aligned}
\tag{1}
$$

$$
\begin{aligned}
&-\frac{\partial}{\partial t}\left(\rho_2(\bar{x})A_2(\bar{x})\frac{\partial w}{\partial t}(\bar{x},t)\right) \\
&\qquad +\frac{\partial}{\partial \bar{x}}\left[\kappa_2 G_2(\bar{x})A_2(\bar{x})\left(\frac{\partial w}{\partial \bar{x}}(\bar{x},t)-\phi(\bar{x},t)\right)\right]+q(\bar{x},t)=0 \\
&-\frac{\partial}{\partial t}\left(\rho_2(\bar{x})I_2(\bar{x})\frac{\partial \phi}{\partial t}(\bar{x},t)\right)+\frac{\partial}{\partial \bar{x}}\left(E_2(\bar{x})I_2(\bar{x})\frac{\partial \phi}{\partial \bar{x}}(\bar{x},t)\right) \\
&\qquad +\kappa_2 G_2(\bar{x})A_2(\bar{x})\left(\frac{\partial w}{\partial \bar{x}}(\bar{x},t)-\phi(\bar{x},t)\right)=0, \quad \forall \bar{x}\in(c,l);\ t\geq 0
\end{aligned}
\tag{2}
$$

$$t_1 w(0^+,t)=\kappa_1 G_1(0^+)A_1(0^+)\left(\frac{\partial}{\partial x}w(0^+,t)-\phi(0^+,t)\right) \tag{3}$$

$$r_1\phi(0^+,t)=E_1(0^+)I_1(0^+)\frac{\partial \phi}{\partial x}(0^+,t) \tag{4}$$

$$
\begin{aligned}
t_c w(c^-,t)=&-\kappa_1 G_1(c^-)A_1(c^-)\left(\frac{\partial w}{\partial x}(c^-,t)-\phi(c^-,t)\right) \\
&+\kappa_2 G_2(c^+)A_2(c^+)\left(\frac{\partial w}{\partial x}(c^+,t)-\phi(c^+,t)\right)
\end{aligned}
\tag{5}
$$

$$r_c\phi(c^-,t)=E_1(c^+)I_1(c^+)\frac{\partial \phi}{\partial x}(c^+,t)-E_1(c^-)I_1(c^-)\frac{\partial \phi}{\partial x}(c^-,t) \tag{6}$$

$$w(c^+,t)=w(c^-,t) \tag{7}$$

$$r_{12}\left(\phi(c^+,t)-\phi(c^-,t)\right)=E_2(c^+)I_2(c^+)\frac{\partial \phi}{\partial x}(c^+,t) \tag{8}$$

$$t_2 w(l^-,t)=-\kappa_2 G_2(l^-)A_2(l^-)\left(\frac{\partial w}{\partial x}(l^-,t)-\phi(l^-,t)\right) \tag{9}$$

$$r_2\phi(l^-,t)=-E_2(l^-)I_2(l^-)\frac{\partial \phi}{\partial x}(l^-,t) \tag{10}$$

where r_l is the mass per unit length; A_l is the area of cross-section; k_l is the shear correction factor; G_l is the transverse shear modulus; E_l is the Young's modulus and I_l is the moment of inertia. The subscript i denotes the ith span of the beam. The rotational restraints are characterized by the spring constants r_1, r_2, r_3 and r_{12} and the translational restraints by the spring constants t_1, t_2, and tc. The set of all the transition conditions of the problem are given by Eqs. (5)-(8).

Using the well-known method of separation of variable, Eqs. (11) and (12) may be taken as solution of Eqs. (1) and Eqs. (2), respectively.

$$w_i\left(\bar{x},t\right)=\sum_{n=1}^{\infty}\bar{W}_i\left(\bar{x}\right)\cos\omega t,\quad i=1,2,... \tag{11}$$

$$\phi_i\left(\bar{x},t\right)=\sum_{n=1}^{\infty}\bar{\Phi}_i\left(\bar{x}\right)\cos\omega t,\quad i=1,2,... \tag{12}$$

where $\bar{w}_i=\bar{w}_{i,n}$ and $\bar{\Phi}_i=\bar{\Phi}_{i,n}$ are the corresponding nth mode of natural vibration and w is the radian frequency. Introducing the change of variable $X=\bar{X}/l$, $\Phi_i(X)=\bar{\Phi}_i(\bar{X})$ and $W_i(X)=\bar{W}_i(\bar{X})/l$ into Eqs. (1)-(10), the functions $W_i(X)$ and $\Phi_i(X)$ are given by

$$W_1(x)=C_1\cosh(\beta_{1,1}x)+C_2\sinh(\beta_{1,1}x)+C_3\cos(\beta_{1,2}x)+C_4\sin(\beta_{1,2}x) \tag{13}$$

$$W_2(x)=C_5\cosh(\beta_{2,1}x)+C_6\sinh(\beta_{2,1}x)+C_7\cos(\beta_{2,2}x)+C_8\sin(\beta_{2,2}x) \tag{14}$$

$$\begin{aligned}\Phi_1(x)=&C_1m_{1,1}\sinh(\beta_{1,1}x)+C_2m_{1,1}\cosh(\beta_{1,1}x)\\&+C_3m_{1,2}\sin(\beta_{1,2}x)-C_4m_{1,2}\cos(\beta_{1,2}x)\end{aligned} \tag{15}$$

$$\begin{aligned}\Phi_2(x)=&C_5m_{2,1}\sinh(\beta_{2,1}x)+C_6m_{2,1}\cosh(\beta_{2,1}x)\\&+C_7m_{2,2}\sin(\beta_{2,2}x)-C_8m_{2,2}\cos(\beta_{2,2}x)\end{aligned} \tag{16}$$

The set of dimensionless parameters can be defined by Eqs. (17) to (23).

$$\beta_{i,1}=\sqrt{\sqrt{a_i^2-b_i}-a_i} \tag{17}$$

$$\beta_{i,2}=\sqrt{\sqrt{a_i^2-b_i}+a_i} \tag{18}$$

$$a_i=\frac{\alpha_i(r_i+s_i)}{2} \tag{19}$$

$$b_i=\alpha_i\left(\alpha_i r_i s_i-1\right) \tag{20}$$

$$m_{i,j}=\frac{\alpha_i s_i+\beta_{i,j}^2}{\beta_{i,j}},\ j=1,2,... \tag{21}$$

$$r_i=\frac{I_i}{A_i l^2} \tag{22}$$

$$s_i=\frac{E_i}{\kappa_i G_i}r_i,\quad i=1,2 \tag{23}$$

where $\alpha_1=\alpha$, $\alpha=\frac{\rho_1A_1}{E_1I_1}l^4\omega^2$ is the frequency parameter and $\alpha_2=\alpha\frac{\rho_2A_2}{\rho_1A_1}\frac{E_1I_1}{E_2I_2}$.
Substitution of Eqs. (13) to (16) into Eqs. (1) and (2) and, subsequently, in the boundary and transition conditions given by Eqs. (3) to (10) a set of eight homogeneous equations in the constants C_l are obtained. Since the system is homogeneous for existence of a non-trivial solution the determinant of coefficients must be equal to zero.

2.2 Combination of Ritz and Lagrange Multiplier Methods

The Ritz method has been proposed to obtain a solution when the geometrical or mechanical properties of the beam are variable at the *ith* span. When the Ritz method is applied to a structure which is constructed by joining several components together, the transition conditions require a continuity of displacement between all the joints of the structural components. These transition conditions give rise to several problems in the rational choice of the coordinate functions. Fortunately it is not necessary to subject the coordinate functions to the natural boundary conditions [22, 23]. This is particularly true in the case of a beam with an internal hinge. For this reason, only the essential transition condition in the hinge is taken into account with the Lagrange multipliers [21-23].

The elastic strain energy due to the beam and the elastic restraints at any instant *t* is given by Eq. (24).

$$U=\frac{1}{2}\int_0^l\left\{EI\left(\frac{\partial\phi(\bar{x},t)}{\partial\bar{x}}\right)^2+kGA\left(\frac{\partial w(\bar{x},t)}{\partial\bar{x}}-\phi(\bar{x},t)\right)^2\right\}d\bar{x}+\frac{1}{2}\Big[t_1w^2(0,t)$$
$$+r_{12}\left(\phi\left(c^+,t\right)-\phi\left(c^-,t\right)\right)^2+r_1\phi^2(0,t)+t_cw^2(c,t)+r_c\phi^2(c^-,t)+$$
$$t_2w^2(l,t)+r_2\phi^2(l,t)\Big] \tag{24}$$

The kinetic energy of the beam at instant t is given by Eq. (25).

$$T=\frac{1}{2}\int_0^l\left\{\rho A\left(\frac{\partial w(\bar{x},t)}{\partial t}\right)^2+\rho I\left(\frac{\partial\phi(\bar{x},t)}{\partial t}\right)^2\right\}d\bar{x} \tag{25}$$

When the beam is in free vibration, transverse deflection and normal rotation can be expressed as

$$w(\bar{x},t)=\bar{W}(\bar{x})\sin(\omega t),\quad \phi(\bar{x},t)=\bar{\Phi}(\bar{x})\sin(\omega t) \tag{26}$$

where $\bar{W}(\bar{x})$ and $\bar{\Phi}(\bar{x})$ are the amplitude of deflection and rotation of the beam, respectively.

By introducing non-dimensional parameters [Eq. (27)], the Lagrangian functional (L_o) of the problem can be written as Eq. (28).

$$x=\frac{\bar{x}}{l},W=\frac{\bar{W}}{l},\Phi=\bar{\Phi} \tag{27}$$

$$L_0=U-T=\frac{1}{2}\int_0^1\left\{\left(\frac{d\Phi}{dx}\right)^2+\gamma\left(\frac{l}{r}\right)^2\left(\frac{dW}{dx}-\Phi\right)^2\right\}dx+\frac{1}{2}\Big[T_1W^2(0)+R_1\Phi^2(0)+$$
$$+R_{12}\left(\Phi^2(c_l^+)-\Phi^2(c_l^-)\right)+T_cW^2(c_l)+R_c\Phi^2(c_l)+$$
$$+T_2W^2(1)+R_2\Phi^2(1)\Big]-\frac{1}{2}\Omega^2\int_0^1\left[\left(\frac{r}{l}\right)^2\Phi^2+W^2\right]dx \tag{28}$$

where

$$\gamma=\frac{kG}{E};\ r=\sqrt{\frac{I}{A}},\ \Omega=\sqrt{\alpha}=\omega l^2\sqrt{\frac{\rho A}{EI}},\ c_l=\frac{c}{l}$$

$$T_i=\frac{t_il^3}{EI},\ R_i=\frac{r_il}{EI},\ i=1,2,\ \text{and}\ R_c=\frac{r_cl}{EI},R_{12}=\frac{r_{12}l}{EI},\ T_c=\frac{t_cl^3}{EI}$$

Since it is difficult to obtain a simple and adequate deflection function which can be applied to the entire beam and also to show the continuity of displacement and the discontinuities of the slope crossing the hinge, the minimization of the functional given by Eq. (28) is achieved using subsidiary conditions. Therefore, it can be assumed that $W(x)$ and $F(x)$ are given by Eq. (29).

$$W(x)=\begin{cases}W_1(x)\forall x\in\left[0,c_l\right]\\W_2(x)\forall x\in\left[c_l,1\right]\end{cases}$$
$$\Phi(x)=\begin{cases}\Phi_1(x)\forall x\in\left[0,c_l\right]\\\Phi_2(x)\forall x\in\left[c_l,1\right]\end{cases} \tag{29}$$

Considering the compatibility requirement on the intermediate elastically restrained point, the relationships between two adjacent spans can be expressed as

$$W_1(c_l) - W_2(c_l) = 0 \tag{30}$$

The problem can now be presented as one of extremizing the given functional in Eq. (28)subjected to the following constraint

$$H = W_1(c_l) - W_2(c_l) \tag{31}$$

This constraint may be incorporated into the energy functional given by Eq. (28) by using the Lagrange multiplier method [23] as

$$L_L = L_0 + \lambda H \tag{32}$$

where L_L is called the Lagrangian functional; and $\lambda \in$ ¡ is a time independent Lagrange multiplier.

The transverse deflection and the normal rotation can be represented by a set of characteristic polynomials $p_{ki}(X)$ and $q_{kj}(X)$ as

$$\Phi_k = \sum_{i=1}^{M} a_{ki} p_{ki}\left(x\right), \quad k = 1,2,... \tag{33}$$

$$W_k = \sum_{j=1}^{N} b_{kj} q_{kj}\left(x\right), \quad k = 1,2,... \tag{34}$$

where both a_{ki} and b_{kj} are unknown coefficients to be determined and $p_{ki}(X)$, $q_{kj}(X)$ are the trial functions. It is sufficient that they satisfy the geometric boundary conditions of the beam since (as the number of trial functions approaches infinity) the natural boundary conditions will be exactly satisfied [22]. The first member of the set $p_{11}(X)$ is obtained as the simplest polynomial that satisfies at least the geometric boundary condition of the first span.

In the case of beam involving free edges or ends elastically restrained against rotation and translation simpler starting member of zero order are used.

The higher members of the set $\{p_1\}$ are obtained as

$$p_{1i} = p_{11} x^{i-1}, \quad i = 2,3,..,N \tag{35}$$

The polynomials set $\{p_2\}$ and $\{q_k\}$ are also generated using the same procedure. Thus

$$p_{ki} = p_{k1} x^{j-1}, \quad i = 2,3,..,N \tag{36}$$

$$q_{kj} = q_{k1} x^{j-1}, \quad j = 2,3,..,M, \quad k = 1,2 \tag{37}$$

In the presented paper, beams having a variety of boundary conditions including elastic restrained ends have been considered; the starting functions used are given in Appendix.

Substituting Eq. (33) and (34) into Eq. (32) and minimizing with respect to the unknown coefficients a_{ki}, b_{kj} and the Lagrange multiplier (l) one obtains

$$\frac{\partial L_L}{\partial a_{ki}} = 0, \quad i = 1,2,...,M, \quad k = 1,2 \tag{38}$$

$$\frac{\partial L_L}{\partial b_{kj}} = 0, \quad j = 1,2,...,N, \quad k = 1,2 \tag{39}$$

$$\frac{\partial L_L}{\partial \lambda} = 0 \tag{40}$$

By using Eq. (35) to (37), the simultaneous set of linear algebraic equations is obtained which can be expressed in the following matrix form as given by Eq. (41).

$$\left(\left[K\right] - \Omega^2\left[M\right]\right)\left\{\overline{c}\right\} = \left\{0\right\} \tag{41}$$

$$
[K] = \begin{bmatrix} [K^{(aa,1)}] & [K^{(ab,1)}] & [K^{(aa,12)}] & [0] & [0] \\ & [K^{(bb,1)}] & [0] & [0] & [L^{(1,\lambda)}] \\ & & [K^{(aa,2)}] & [K^{(ab,2)}] & [0] \\ & \text{symm} & & [K^{(bb,2)}] & [L^{(2,\lambda)}] \\ & & & & [0] \end{bmatrix} \tag{42}
$$

$$
[M] = \begin{bmatrix} [M^{(a,1)}] & [0] & [0] & [0] & [0] \\ & [M^{(b,1)}] & [0] & [0] & [0] \\ & & [M^{(a,2)}] & [0] & [0] \\ & & & [M^{(b,2)}] & [0] \\ & \text{symm} & & & [0] \end{bmatrix} \tag{43}
$$

$$
\{\bar{c}\} = \left\{\{a_1\},\{b_1\},\{a_2\},\{b_2\},\{\lambda\}\right\}^T \tag{44}
$$

$$
\begin{aligned} \{a_k\} &= \{a_{k1}, a_{k2}, \ldots, a_{kM}\}^T \\ \{b_k\} &= \{b_{k1}, b_{k2}, \ldots, b_{kN}\}^T, \quad k = 1,2. \end{aligned} \tag{45}
$$

The expressions for the various elements of the stiffness matrix $[K]$ and the mass matrix $[M]$ are given as

$$
\begin{aligned} K_{ij}^{(aa,1)} = &\int_0^{c_l} \left[\frac{dp_{1i}(x)}{dx}\frac{dp_{1j}(x)}{dx} + \gamma\left(\frac{l}{r}\right)^2 p_{1i}(x)p_{1j}(x)\right]dx + R_1 p_{1i}(0)p_{1j}(0) + \\ &+ R_c p_{1i}(c_l)p_{1j}(c_l) + R_{12} p_{1i}(c_l)p_{1j}(c_l) \end{aligned} \tag{46}
$$

$$
K_{ij}^{(ab,1)} = -\int_0^{c_l} \gamma\left(\frac{l}{r}\right)^2 p_{1i}\frac{dq_{1j}}{dx}dx \tag{47}
$$

$$
K_{ij}^{(aa,3)} = -R_{12} p_{1i}(c_l)p_{2j}(c_l) \tag{48}
$$

$$
K_{nm}^{(bb,1)} = \int_0^{c_l} \gamma\left(\frac{l}{r}\right)^2 \frac{dq_{1n}(x)}{dx}\frac{dq_{1m}(x)}{dx}dx + T_1 q_{1n}(0)q_{1m}(0) + T_c q_{1n}(c_l)q_{1m}(c_l) \tag{49}
$$

$$
L_{1n}^{(1,\lambda)} = q_{1n}(c_l) \tag{50}
$$

$$
K_{ij}^{(aa,2)} = \int_{c_l}^1 \left[\frac{dp_{2i}(x)}{dx}\frac{dp_{2j}(x)}{dx} + \gamma\left(\frac{l}{r}\right)^2 p_{2i}(x)p_{2j}(x)\right]dx + R_2 p_{2i}(1)p_{2j}(1) + R_{12} p_{2i}(c_l)p_{2j}(c_l) \tag{51}
$$

$$
K_{in}^{(ab,2)} = -\int_{c_l}^1 \gamma\left(\frac{l}{r}\right)^2 p_{2i}\frac{dq_{2n}}{dx}dx \tag{52}
$$

$$
K_{nm}^{(bb,2)} = \int_{c_l}^1 \gamma\left(\frac{l}{r}\right)^2 \frac{dq_{2n}(x)}{dx}\frac{dq_{2m}(x)}{dx}dx + T_2 q_{2n}(1)q_{2m}(1) \tag{53}
$$

$$
L_{1n}^{(2,\lambda)} = -q_{2n}(c_l) \tag{54}
$$

$$
M_{ij}^{(a,1)} = \int_0^{c_l} \left(\frac{r}{l}\right)^2 p_{1i}(x)p_{1j}(x)dx \tag{55}
$$

$$M_{nm}^{(b,1)} = \int_0^{c_1} q_{1n}(x) q_{1m}(x) dx \tag{56}$$

$$M_{ij}^{(a,2)} = \int_{c_1}^{1} \left(\frac{r}{l}\right)^2 p_{2i}(x) p_{2j}(x) dx \tag{57}$$

$$M_{nm}^{(b,2)} = \int_{c_1}^{1} q_{2n}(x) q_{2m}(x) dx \tag{58}$$

with $i, j = 1,2,\ldots, M$, $n, m = 1,2,\ldots, N$

The eigenvalues Ω^2 were found from the condition that the determinant of the system of equations given by Eq. (41) must vanish.

3. CONVERGENCE AND COMPARISON STUDY

The computations for this study were performed by using computer programme entitled Maple [24]. This subroutine computes in exact way the definite integral over the straight line from x_o to x_1. The eigenvalues are computed by the QR decomposition method. The matrix is first balanced and transformed into upper Hessenberg form. Thereafter, the eigenvalues were computed. Throughout the presented analysis, the beams were modelled with shear correction factor $k = 5/6$ and Poisson's ratio $\mu = 0.3$.

The first five dimensionless natural frequencies $\Omega = \sqrt{\alpha}$, $\alpha = (\rho_1 A_1/E_1 I_1) l^4 \omega^2$ for a cantilever stepped beam (**Figure 2**) are given in **Table 1**. These were determined using the aforementioned method. The exact solutions were obtained for $h_2/h_1 = 0.8$ and $c_1 = 2/3$. The results were compared with those obtained by Rossi et al. [25] and Tong and Tabarrok [26].

For the rest of the presented work, the beam was considered to be of the same material properties and cross-section.

Exact results and a convergence study of the Ritz and Lagrange multiplier method (R&LM) of the first six values of the dimensionless frequency parameter W of a simply supported (S-S) and a clamped-clamped (C-C) beam with an intermediate support located at $c_1 = 0.4$ for $\sqrt{12}\ r/l = 0.1$ are presented in **Table 2**.

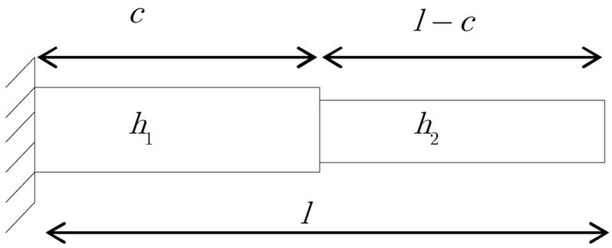


Figure 2. Cantilever stepped beam.

Table 1. Comparative study of first five exact values of frequency parameter $\Omega = \omega l^2 \sqrt{\rho_1 A_1 / E_1 I_1}$ for a cantilever stepped beam with $h_2 / h_1 = 0.8$ and $c_1 = 2/3$.

r_1	Reference	Ω_1	Ω_2	Ω_3	Ω_4	Ω_5
0.0133	Rossi et al. [25]	3.82	21.35	55.04	107.5	173.62
	Tong and Tabarrok [26]	3.8219	21.3540	55.0408	107.4993	173.6322
	Exact solution	3.8243	21.3559	55.0510	107.5298	173.6753
0.0267	Rossi et al. [25]	3.80	20.72	51.68	96.39	148.97
	Tong and Tabarrok [26]	3.8034	20.7283	51.6851	96.3918	148.9651
	Exact solution	3.8047	20.7275	51.6754	96.3656	148.9066
0.04	Rossi et al. [25]	3.77	19.80	47.35	84.14	125.06
	Tong and Tabarrok [26]	3.7716	19.8036	47.3540	84.1399	125.0681
	Exact solution	3.7730	19.8047	47.3531	84.1407	125.0650

Table 2. Convergence study of first six values of frequency parameter W of a two-span Timoshenko beam ($T_c \rightarrow \infty$ and $R_{12} \rightarrow \infty$) located at $c_1 = 0.4$ for $\sqrt{12}$ $r/1$ 0.1

Boundary condition	N=M	W_1	W_2	W_3	W_4	W_5	W_6
SS-SS	3	35.9280	78.4428	139.1719	250.1963	679.3662	690.4346
	4	31.3524	67.3305	129.6826	226.1855	268.0760	429.2906
	5	31.3505	67.0022	104.4706	194.6074	255.8360	396.4166
	6	31.3372	66.9616	104.3777	186.7849	205.4627	349.6778
	7	31.3371	66.9553	103.9240	186.5327	204.6538	300.6535
	8	31.3371	66.9552	103.9223	185.3377	203.2441	300.4937
	9	31.3371	66.9552	103.9196	185.3368	203.2227	293.0330
	10	31.3371	66.9552	103.9196	185.3184	203.1968	293.0235
	11	31.3371	66.9552	103.9196	185.3184	203.1966	292.7686
	12	31.3371	66.9552	103.9196	185.3183	203.1965	292.7684
	13	31.3371	66.9552	103.9196	185.3183	203.1965	292.7653
	14	31.3371	66.9552	103.9196	185.3183	203.1965	292.7653
	Exact	31.3371	66.9552	103.9196	185.3183	203.1965	292.7652
	Zhou [8]	31.3365	66.9549	103.9197	185.3192	203.2250	292.8411
C-C	3	53.0692	103.8763	174.1909	276.5284	690.3621	721.8255
	4	44.9647	90.0285	141.7985	236.7698	290.3811	448.5260
	5	44.9174	89.4652	121.2447	214.0275	251.5873	403.2562
	6	44.8972	89.3835	120.6392	204.6194	222.1686	361.0645
	7	44.8970	89.3755	120.3072	203.0319	220.9901	313.1963
	8	44.8970	89.3751	120.3001	202.1035	220.3835	308.1262
	9	44.8970	89.3751	120.2983	202.0641	220.3560	304.0140
	10	44.8970	89.3751	120.2982	202.0523	220.3466	303.7682
	11	44.8970	89.3751	120.2982	202.0520	220.3463	303.6562
	12	44.8970	89.3751	120.2982	202.0519	220.3463	303.6523
	13	44.8970	89.3751	120.2982	202.0519	220.3463	303.6512
	14	44.8970	89.3751	120.2982	202.0519	220.3463	303.6512
	Exact	44.8970	89.3751	120.2982	202.0519	220.3463	303.6512
	Zhou [8]	44.8967	89.3762	120.3014	202.0662	220.4037	303.7840

The convergence of the mentioned eigenvalues was studied by gradually increasing the number of the trial functions. A comparison of values with those of Zhou [8] is also included in **Table 2**. The results indicate that $N = M = 11$ is adequate to reach stable convergence for all cases in R&LM. Exact values and the R&LM results show an excellent agreement with those of Zhou [8].

A comparison with the crack model used by Khaji et al. [17] is presented in **Table 3**. The cracked section of the Timoshenko beam was modelled as local flexibility that was assumed to be a rotational spring.

The discontinuity in the slope of beam was modelled as

$$\left(\frac{\partial W_2}{\partial x} - \frac{\partial W_1}{\partial x}\right)\Bigg|_{X=c_1} = \theta \cdot \frac{\partial \Phi_2}{\partial x}\Bigg|_{X=c_1} \tag{59}$$

Table 3. Comparative study of first value of frequency parameter W which correspond to crack model [17] with the R&LM method and exact results varying R_{12} values as a function of η, with r/l = 0.25

Boundary			W_1		
condition	η	R_{12}	Khaji et al. [17]	R&LM, N=M=7	Exact
S-S	0.20	9.6689	8.2760	8.2733	8.27327
	0.35	2.9396	7.1126	7.1102	7.11021
	0.50	1.2380	5.7693	5.7674	5.76738
	0.70	0.5185	4.2726	4.2711	4.27108
S-C	0.20	9.6689	12.0286	12.0246	12.0246
	0.35	2.9396	11.1045	11.1007	11.1007
	0.50	1.2380	10.1282	10.1248	10.1248
	0.70	0.5185	9.1955	9.1919	9.19187
C-C	0.20	9.6689	15.8527	15.8474	15.8474
	0.35	2.9396	14.9282	14.9231	14.9231
	0.50	1.2380	13.9791	13.9743	13.9743
	0.70	0.5185	13.1041	13.0997	13.0997

where $\theta = 6\pi\eta^2 f(\eta)(h/l)$ is the non-dimensional crack sectional flexibility and depends on the extension of the crack, $\eta = a/h$ is the crack depth ratio where a is the crack depth and h is the beam depth of a rectangular section.

Assuming a one side open crack

$$f(\eta) = 0.6384 - 1.035\eta + 3.7201\eta^2 - 5.1773\eta^3 + 7.553\eta^4 - 7.332\eta^5 + 2.4909\eta^6 \tag{60}$$

To perform a comparison between modal frequency results from this work and the ones obtained by Khaji et al. [17], the relationship between the non-dimensional hinge rigidity and the non-dimensional crack sectional flexibility is

$$R_{12} = \frac{1}{\theta}. \tag{61}$$

Table 3 compares the fundamental frequency parameter Ω for S-S, S-C and C-C beams with η equal to 0.20, 0.35, 0.50 and 0.70, and $c_1 = 0.5$ for $r/l = 0.25$. The values were obtained with both R&LM method using $N = M = 7$ and the exact values.

4. NUMERICAL EXAMPLES

In order to investigate the influence of the internal elastically restrained hinge on the free vibration characteristics of Timoshenko beams, numerical results were computed by using the proposed methods. A large number of problems were solved. Since the number of cases are extremely large, the results for only a few cases are presented. All calculations have been performed taking $N = M = 7$, $k = 5/6$ and $\mu = 0.3$ unless otherwise specified.

Table 4 provides values of the fundamental frequency parameter W of a Timoshenko beam for different values of R_{12} with $T_o = R_o = 0$ located at $c_1 = 0.1$, 0.3 and 0.5 for $\sqrt{12}\ r/l$ 0.001, 0.01 and 0.1. The results correspond to S-S, C-C, F-F, C-F, C-S and S-F boundary conditions.

Tables 5 and **6** present results of the R&LM method and the exact solution, respectively, of the fundamental frequency parameter W of a uniform Timoshenko S-S, C-C and F-F beam with $\sqrt{12}$ r/l 0.1, 0.3 and 0.6 (obtained with R&LM method) and the exact solution for $R_c = 0$ and different values of T_c and R_{12} located at $c_1 = 0.5$. Modal shapes shown in **Tables 5** and **6** correspond to $\sqrt{12}$ r/l 0.1 and $R_{12} = 0$.

Table 4. Values of fundamental frequency parameter Ω of a Timoshenko beam for different boundary conditions and different values of $\sqrt{12}\ r/l$, with $T_c = R_c = 0$ and different values of R_{12} located at c_1 = 0.1, 0.3 and 0.5

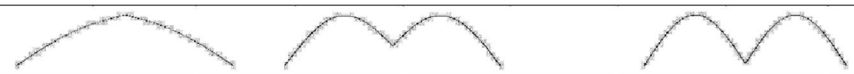
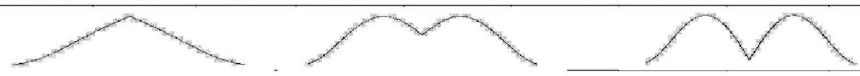
Boundary condition	R_{12}	$\sqrt{12}\ r/l$								
		0.001	0.01	0.1	0.001	0.01	0.1	0.001	0.01	0.1
			$c_1 = 0.1$			$c_1 = 0.3$			$c_1 = 0.5$	
S-S	0	17.8621	17.8545	17.1396	26.3351	26.3169	25.2004	39.4761	39.4510	37.0962
	1	8.9962	8.9948	8.8644	6.3947	6.3941	6.3317	5.6796	5.6791	5.6308
	10	9.7760	9.7744	9.6176	9.2746	9.2732	9.1353	9.0078	9.0065	8.8779
	100	9.8602	9.8585	9.6984	9.8055	9.8039	9.6460	9.7723	9.7707	9.6141
	1000	9.8686	9.8670	9.7066	9.8631	9.8615	9.7013	9.8597	9.8581	9.6980
C-C	0	18.9073	18.8972	17.9663	20.0982	20.0865	19.0201	14.0640	14.0596	13.6391
	1	19.6422	19.6312	18.6246	21.0117	20.9987	19.8173	16.8748	16.8678	16.2113
	10	21.4330	21.4194	20.1814	22.0828	22.0680	20.7290	20.9977	20.9849	19.8188
	100	22.2488	22.2337	20.1814	22.3405	22.3252	20.9450	22.2111	22.1960	20.8377
	1000	22.3604	22.3450	20.9616	22.3698	22.3545	20.9695	22.3566	22.3413	20.9586
F-F	0	26.3124	26.2932	24.7237	39.7090	39.6812	37.2193	61.6725	61.6083	56.2079
	1	21.9475	21.9392	21.1615	14.4029	14.3985	13.9808	11.8182	11.8154	11.5448
	10	22.3331	22.3250	21.5673	21.0996	21.0922	20.4056	19.9794	19.9729	19.3621
	100	22.3694	22.3612	21.6045	22.2396	22.2316	21.4827	22.0961	22.0882	21.3498
	1000	22.3730	22.3649	21.6082	22.3598	22.3517	21.5959	22.3450	22.3370	21.5822
C-F	0	18.8924	18.8853	18.2209	19.1291	19.1186	18.1464	9.8696	9.8665	9.5771
	1	19.5378	19.5299	18.7885	20.2577	20.2466	19.2208	14.2254	14.2202	13.7321
	10	21.1604	21.1499	20.1790	21.6476	21.6357	20.5404	20.1497	20.1398	19.2178
	100	21.9180	21.9061	20.8108	21.9907	21.9786	20.8656	21.8141	21.8023	20.7111
	1000	22.0224	22.0103	20.8970	22.0300	22.0179	20.9027	22.0120	21.9999	20.8870
C-S	0	12.1297	12.1268	11.8480	15.0959	15.0897	14.5074	9.0711	9.0688	8.8515
	1	12.8700	12.8666	12.5364	15.2344	15.2283	14.6492	11.4895	11.4862	11.1740
	10	14.5730	14.5679	14.0854	15.3805	15.3744	14.7979	14.5168	14.5114	14.0093
	100	15.3080	15.3020	14.7389	15.4139	15.4078	14.8318	15.3145	15.3084	14.7414
	1000	15.4068	15.4007	14.8261	15.4177	15.4116	14.8356	15.4076	15.4015	14.8264
S-F	0	25.9582	25.9442	24.6665	38.5079	38.4833	36.2888	46.0557	46.0191	42.8134
	1	13.2815	13.2785	12.9936	8.9482	8.9468	8.8109	8.6977	8.6962	8.5557
	10	15.1810	15.1771	14.8074	14.1399	14.1366	13.8198	14.0154	14.0121	13.6969
	100	15.3943	15.3903	15.0088	15.2763	15.2724	14.8973	15.2596	15.2557	14.8810
	1000	15.4158	15.4118	15.0290	15.4038	15.3998	15.0177	15.4021	15.3981	15.0161

Tables 7 and **8** present results of the R&LM method and exact solution respectively of the first five values of the frequency parameter *W* of a Timoshenko beam for $R_c = 0$ and different values of T_c and R_{12} located at $c_1 = 0.5$ with $\sqrt{12}\ r/l$ 0.5 for S-S, C-C and F-F boundary conditions. Modal shapes shown in **Table 7** and **8** correspond to $T_c = 1000$ and $R_{12} = 0$.

Table 9 shows the first three values of the frequency parameter *W* of a uniform Timoshenko beam with F-F, S-S, C-C, S-F and C-F boundary conditions (obtained with the R&LM method with $N = M = 12$) and the exact solution for $T_c = R_c = R_{12} = 0$ at different locations for $\sqrt{12}\ r/l$ 0.001.

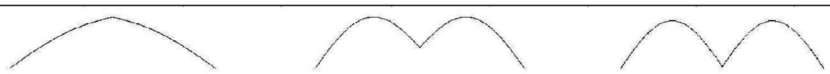
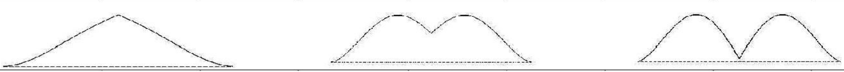
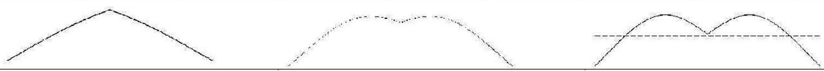
Table 10 shows the first six exact values of the frequency parameter $\Omega = \omega l^2 \sqrt{\rho_1 A_1 / E_1 I_1}$ for a cantilever (C-F) and simply supported (S-S) stepped beam with an internal free hinge ($T_c = R_c = R_{12} = 0$) located at different positions.

Table 5. Values of fundamental frequency parameter Ω of a uniform Timoshenko S-S, C-C and F-F beams with $\sqrt{12}\ r/l$ = 0.1, 0.3 and 0.6, obtained with R&LM method, for R_c = 0 and different values of T_c and R_{12} located at c_1 0.5

Boundary condition	R_{12}	$\sqrt{12}\ r/l$								
		0.1	0.3	0.6	0.1	0.3	0.6	0.1	0.3	0.6
		$T_c = 100$			$T_c = 1000$			$T_c = 10000$		
S-S	0	**16.1447**	14.9548	11.9502	**32.2988**	25.3839	17.0112	**36.5931**	27.1249	17.5932
	100	16.8473	15.5080	12.7744	37.0962	27.3922	17.0147	37.0962	27.3147	17.6462
	1000	16.8602	15.5162	12.7802	37.0962	27.4332	17.0148	37.0962	27.3147	17.6462
										
C-C	0	**23.3060**	19.9094	14.8909	**44.9941**	30.1018	17.5648	**52.8442**	31.6270	17.7768
	100	25.9774	20.5242	14.8967	47.4799	31.7852	17.7990	53.7468	31.7852	17.7990
	1000	26.0304	20.5327	14.8967	47.5536	31.7852	17.7990	53.7468	31.7852	17.7990
										
F-F	0	**19.0466**	16.7598	12.0666	**44.8242**	33.0958	17.4364	**54.9988**	36.9666	18.2488
	100	8.4242	7.8422	6.5132	12.6584	10.7290	7.8240	13.3122	11.1183	7.9777
	1000	8.4757	7.8866	6.5425	12.8530	10.8555	7.8803	13.5358	11.2587	8.0378
										

Note: Modal shapes shown correspond to $\sqrt{12}\ r/l = 0.1$ and $R_{12} = 0$

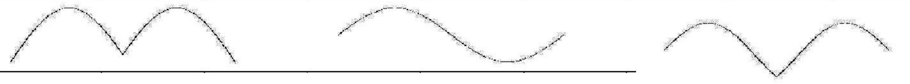
Table 6. Exact results of fundamental frequency parameter Ω of a uniform Timoshenko S-S, C-C and F-F beams with $\sqrt{12}\ r/l$ = 0.1, 0.3 and 0.6, for R_c = 0 and different values of T_c and R_{12} located at c_1 = 0.5

Boundary condition		$\sqrt{12}\ r/l$								
		0.1	0.3	0.6	0.1	0.3	0.6	0.1	0.3	0.6
	R_{12}	$T_c = 100$			$T_c = 1000$			$T_c = 10000$		
S-S	0	**16.1447**	14.9548	11.9502	**32.2988**	25.3839	17.0112	**36.5931**	27.1249	17.5932
	100	16.8473	15.5080	12.7744	37.0962	27.3147	17.0147	37.0962	27.3147	17.6462
	1000	16.8602	15.5162	12.7802	37.0962	27.3147	17.0148	37.0962	27.3147	17.6462
										
C-C	0	**23.3060**	19.9094	14.8909	**44.9941**	30.1018	17.5648	**52.8442**	31.6270	17.7768
	100	25.9774	20.5242	14.8967	47.4799	31.7852	17.7990	53.7468	31.7852	17.7990
	1000	26.0304	20.5327	14.8967	47.5536	31.7852	17.7990	53.7468	31.7852	17.7990
										
F-F	0	**19.0466**	16.7598	12.0666	**44.8242**	33.0957	17.4364	**54.9987**	36.9666	18.2488
	100	8.4242	7.8422	6.5132	12.6584	10.7290	7.8240	13.3122	11.1183	7.9777
	1000	8.4757	7.8866	6.5425	12.8530	10.8555	7.8803	13.5358	11.2587	8.0378
										

Note: Modal shapes shown correspond to $\sqrt{12}\ r/l = 0.1$ and $R_{12} = 0$

Table 7. First three values frequencies parameter Ω of a uniform Timoshenko beam obtained with R&LM method with $R_c = 0$ and different values of T_c and R_{12} located at $c_1 = 0.5$ with $\sqrt{12}\, r/l = 0.5$ for S-S, C-C and F-F boundary conditions

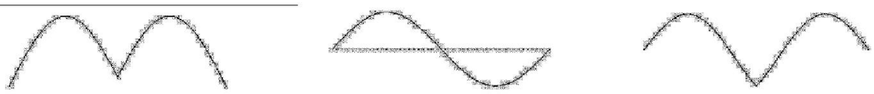
Boundary condition	R_{12}	$c/l = 0.5;\ \sqrt{12}\, r/l = 0.5$								
		$T_c = 100$			$T_c = 1000$			$T_c = 10000$		
		Ω_1	Ω_2	Ω_3	Ω_1	Ω_2	Ω_3	Ω_1	Ω_2	Ω_3
S-S	0	13.0863	20.1495	24.2519	**19.3340**	**20.1495**	**26.4141**	20.0728	20.1495	27.0917
	100	13.6916	20.1495	27.1746	19.6723	20.1495	27.1746	20.1495	20.8384	27.1746
	1000	13.6975	20.1495	27.1746	19.6753	20.1495	27.1746	20.1495	20.8451	27.1746
C-C	0	16.4047	20.9842	25.8990	**20.5666**	**20.9842**	**33.5814**	20.9448	20.9842	35.7162
	100	16.4239	20.9842	35.4292	20.9842	22.4179	35.9556	20.9842	23.5489	35.9556
	1000	16.4241	20.9842	35.5288	20.9842	22.4393	35.9556	20.9842	23.5816	35.9556
F-F	0	13.6452	23.5509	34.4165	**21.9081**	**23.5509**	**34.5278**	23.3820	23.5509	34.5427
	100	6.9676	17.3298	23.5509	8.6778	22.7043	23.5509	8.8852	23.5509	23.8724
	1000	7.0018	17.3904	23.5509	8.7513	22.7166	23.5509	8.9645	23.5509	23.8797



Note: Modal shapes shown correspond to $T_c = 1000$ and $R_{12} = 0$

Table 8. First three exact values frequencies parameter Ω of a uniform Timoshenko beam, with $R_c = 0$ and different values of T_c and R_{12} located at $c_1 = 0.5$ with $\sqrt{12}\, r/l = 0.5$ for S-S, C-C and F-F boundary conditions

Boundary condition	R_{12}	$c/l = 0.5;\ \sqrt{12}\, r/l = 0.5$								
		$T_c = 100$			$T_c = 1000$			$T_c = 10000$		
		Ω_1	Ω_2	Ω_3	Ω_1	Ω_2	Ω_3	Ω_1	Ω_2	Ω_3
SS-SS	0	13.0863	20.1495	24.2519	**19.3340**	**20.1495**	**26.4141**	20.0728	20.1495	27.0917
	100	13.6916	20.1495	35.1868	19.6723	20.1495	35.7661	20.1495	20.8384	35.7909
	1000	13.6975	20.1495	35.3370	19.6753	20.1495	35.9150	20.1495	20.8451	35.9370
C-C	0	16.4047	20.9842	25.8990	**20.5666**	**20.9842**	**33.5814**	20.9448	20.9842	35.7162
	100	16.4239	20.9842	35.4292	20.9842	22.4179	35.9556	20.9842	23.5489	35.9556
	1000	16.4241	20.9842	35.5288	20.9842	22.4393	35.9556	20.9842	23.5816	35.9556



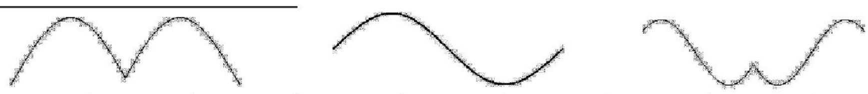
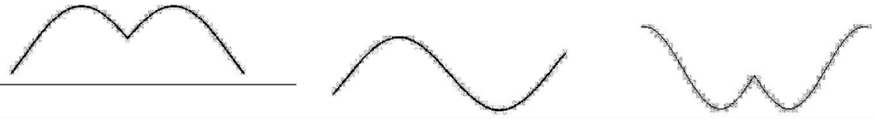


Table 8. Continued

Boundary condition	R_{12}	$c/l = 0.5$; $\sqrt{12}\,r/l = 0.5$								
		$T_c = 100$			$T_c = 1000$			$T_c = 10000$		
		Ω_1	Ω_2	Ω_3	Ω_1	Ω_2	Ω_3	Ω_1	Ω_2	Ω_3
F-F	0	13.6452	23.5509	34.4163	**21.9081**	**23.5509**	**34.5275**	23.3820	23.5509	34.5424
	100	6.9676	17.3298	23.5509	8.6778	22.7043	23.5509	8.8852	23.5509	23.8724
	1000	7.0018	17.3904	23.5509	8.7513	22.7166	23.5509	8.9645	23.5509	23.8797



Note: Modal shapes shown correspond to $T_c = 1000$ and $R_{12} = 0$

Table 9. First three values of frequencies parameter Ω of a uniform Timoshenko beam with different boundary conditions, obtained with the R&LM method with $N = M = 12$ and the exact solution, for $T_C = R_C = R_{12} = 0$ at different locations, for $\sqrt{12}\,r/l = 0.001$

Boundary condition	c_l	Ω_1		Ω_2		Ω_3	
		R&LM	Exact	R&LM	Exact	R&LM	Exact
F-F	0.5	61.6728	61.6728	89.4931	89.4931	199.8594	199.8594
	0.3	39.7090	39.7090	110.8393	110.8392	194.1497	194.1495
	0.1	26.3124	26.3123	72.7993	72.7993	143.1623	143.1604
S-S	0.5	39.4784	39.4784	61.6728	61.6728	157.9137	157.9136
	0.3	26.3352	26.3352	86.0614	86.0614	138.7773	138.7773
	0.1	17.8622	17.8621	58.2367	58.2367	122.1138	122.1137
C-C	0.5	14.0641	14.0640	61.6728	61.6728	88.1380	88.1379
	0.3	20.0983	20.0983	43.3521	43.3520	111.8010	111.8009
	0.1	18.9074	18.9073	60.1749	60.1749	120.7269	120.7268
S-F	0.9	18.1166	18.1165	58.9283	58.9283	123.3481	123.3480
	0.7	27.1872	27.1872	89.8417	89.8417	178.0350	178.0349
	0.5	46.0561	46.0560	79.6851	79.6850	171.3698	171.3697
	0.3	38.5079	38.5078	104.0083	104.0083	147.0098	147.0098
	0.1	25.9581	25.9581	71.9805	71.9804	141.7791	141.7774
C-F	0.9	4.1174	4.049998	25.9183	25.572091	72.8272	72.007581
	0.7	5.9945	5.710696	39.1706	38.022597	110.8688	104.015296
	0.5	9.8696	9.071125	61.6728	46.597025	88.8264	78.757613
	0.3	19.1292	5.095956	43.8673	34.049756	111.7699	91.098222
	0.1	18.8923	12.129691	60.1824	47.821706	120.7251	104.235587

Table 10. First six exact values of frequency parameter $\Omega = \omega l^2 \sqrt{\rho_1 A_1 / E_1 I_1}$ for C-F and S-S stepped beams with h2 / h1 = 0.8, r1 = 0.0267 with an internal free hinge located at c1 = 0.2, 0.5 and 0.7

Boundary condition	c_l	Ω_1	Ω_2	Ω_3	Ω_4	Ω_5	Ω_6
C-F	0.2	17.8904	45.9096	72.4613	123.1828	188.1842	252.5720
	0.5	10.1457	49.6080	72.0649	142.1003	175.8299	260.3349
	0.7	6.0781	36.7487	93.6803	137.4823	180.7118	261.1928
S-S	0.2	16.3327	52.3097	104.6555	167.4798	225.0180	262.1155
	0.5	32.6495	53.2655	117.1564	153.5805	231.8271	278.7503
	0.7	25.8961	75.1946	102.4467	162.4955	243.8653	289.2510

5. CONCLUSIONS

The free transverse vibration of a Timoshenko beam with ends elastically restrained against rotation and translation, and an arbitrarily located internal hinge including intermediate elastic constraints is studied. For this purpose, an exact model and a simple and accurate approach has been developed based on two models: a combination of the Ritz and the Lagrange multiplier methods for the determination of natural frequencies. The algorithm is general and it is characterized by a low computational cost and high accuracy. Close agreement with the results presented by previous investigators is demonstrated for some examples and for a crack model.

The obtained results may provide useful information for structural designers and engineers. Finally, the algorithms developed with the combined method can be easily extended to a beam with an arbitrary number of hinges and variable cross-section.

ACKNOWLEDGMENTS

The presented work has been sponsored by CIUNSa Project Nº 1899 and PID UTN Nº 1901.

NOTATION

A	= cross-sectional area
$c_1 = c/l$	= geometrical parameter
E	= Young's modulus
G	= transverse shear modulus
I	= moment of inertia
l	= length of the beam
h	= beam depth
a	= crack depth
n	= crack depth ratio
q	= non-dimensional crack sectional flexibility
$r = \sqrt{I / A}$	= radius of gyration of cross section
r_1, r_2	= rotational stiffness at the left and right ends respectively
r_{12}	= rotational stiffness at the internal hinge
r_c	= rotational stiffness at the point
R_c, R_{12}, R_i, $i = 1, 2$	= dimensionless rotational parameters
t	= time
t_1, t_2	= translational stiffness at the left and right ends respectively
t_c	= translational stiffness at the point
T	= kinetic energy
T_c, T_i, $i = 1, 2$	= dimensionless translational parameters
U	= strain energy
x	= dimensionless abscissa
$\bar{X}$	= abscissa
$\Omega = \omega l^2 \sqrt{\rho A / EI}$	= dimensionless natural frequency parameter
w	= radian frequency
r	= mass density

REFERENCES

[1] Timoshenko S. On the Correction of Shear of Differential Equations of Transverse Vibrations of Prismatic Bars. Phil Mag 1921;41(245):744-746.

[2] Timoshenko S. On the Transverse Vibrations of Bars of Uniform Cross Section. Phil Mag 1922;43(6):125-131.

[3] Quintana MV, Grossi RO. Eigenfrequencies of Generally Restrained Timoshenko Beams. Proc Inst Mech Eng Part K J Multi-body Dyn 2009;224(1):117-125.

[4] Abbas B. Vibrations of Timoshenko beams with elastically restrained ends. J Sound Vib 1984;97(4):541-548.

[5] Farghaly SH. Vibration and Stability Analysis of Timoshenko Beams with Discontinuities in Cross-section. J Sound Vib 1994;174(5):591-605.

[6] Kocatürk T, Simsek M. Free Vibration analysis of Timoshenko Beams under Various Boundary Conditions. Sigma J Eng Nat Sci 2005;1:30-44.

[7] Kocatürk T, Simsek M. Free Vibration Analysis of Elastically Supported Timoshenko Beams. Sigma J Eng Na. Sci 2005;3:79-93.

[8] Zhou D. Free Vibration of Multi-span Timoshenko Beams Using Static Timoshenko Beam Functions. J Sound Vib 2001;241(4):725-734.

[9] Grossi RO, Aranda A. Formulación Variacional de Problemas de Contorno Para Vigas Timoshenko. Rev Int Mét Num Cálc Dis Ing 1993;9(3):313-324.

[10] Han SM, Benaroya H, Wei T. Dynamic of Transversely Vibrating Beams Using Four Engineering Theories. J Sound Vib 1999;225(5):935-988.

[11] Ewing MS, Mirsafian S. Forced Vibrations of Two Beams Joined with a Non-linear Rotational Joint: Clamped and Simply Supported End Conditions. J Sound Vib 1996;193(2):483-496.

[12] Wang CY, Wang CM. Vibrations of a Beam with an Internal Hinge. Int J Struct Stab Dyn 2001;1(1):163-167.

[13] Chang TP, Lin GL, Chang E. Vibration Analysis of a Beam with an Internal Hinge Subjected to a Random Moving Oscillator. Int J Solids Struct 2006;43(21):6398-6412.

[14] Grossi RO, Quintana MV. The Transition Conditions in the Dynamics of Elastically Restrained Beams. J Sound Vib 2008;316(1-5):274-297.

[15] Lee YY, Wang CM, Kitipornchai S. Vibration of Timoshenko Beams with Internal Hinge. J Eng Mech 2003;129(3):293-301.

[16] Lele, S.P. and Maiti, S.K. Modelling of Transverse Vibration of Short Beams for Crack Detection and Measurement of Crack Extension. J Sound Vib 2002;257(3): 559-583.

[17] Khaji N, Shafiei M, Jalalpour M. Closed-form Solutions for Crack Detection Problem of Timoshenko Beams with Various Boundary Conditions. Int J Mech Sci 2009 51(9-10):667-681.

[18] Ostachowicz WM, and Krawczuk M. Analysis of the Effect of Cracks on the Natural Frequencies of a Cantilever Beam. J Sound Vib 1991;150(2):191-201.

[19] Haisty BS, Springer WT. A General Beam Element for Use in Damage Assessment of Complex Structures. ASME J Vib Acoust Stress: Reliab, Stress Ana Fail Preven 1988;110(3):389-394.

[20] Narkis Y. Identification of Crack Location in Vibrating Simply Supported Beams. J Sound Vib 1994;172(4):549-558.

[21] Quintana MV. Dynamic Analysis of Beams and Plates with Hinge: Analytical and Numerical Study. PhD Thesis. Universidad Nacional de Salta, Salta, 2012.

[22] Mikhiln SG. Variational methods in mathematical physics. Pergamon Press, Oxford, 1964. p. 109-116.

[23] Reddy JN. Applied functional analysis and variational methods in engineering. McGraw-Hill Inc., New York, 1986. p. 125-287.

[24] Maplesoft. Maple User Manual. Ontario, Canada, 2009.

[25] Rossi RE, Laura PAA, Gutierrez RH. A Note on Transverse Vibrations of a Timoshenko Beam of Non-uniform Thickness Clamped at One End and Carrying a Concentrated Mass at the Other. J Sound Vib 1990;143(3):491-502.

[26] Tong X, Tabarrok B. Vibration Analysis of Timoshenko Beams with Non-homogeneity and Varying Cross-section. J Sound Vib 1995;186(5):821-835.

APPENDIX

First members of the set of polynomials $\{p_i^{(k)}(X)\}$ and $\{q_j^{(k)}(X)\}$ for all possible combinations of classical boundary conditions and with intermediate elastic restraints.

Classical boundary conditions and intermediate elastic restraints hinge at. x-c_1	$p_1^{(1)}$	$q_1^{(1)}$	$p_1^{(2)}$	$q_1^{(2)}$
S-S	1	X	1	X-1
S-F	1	X	1	1
F-F	1	*1*	1	1
C-C	X	X	X-1	X-1
C-S	X	X	1	X-1
C-F	X	X	1	1
Classical boundary conditions with intermediate point support at. x-c_1	$p_1^{(1)}$	$q_1^{(1)}$	$p_1^{(2)}$	$q_1^{(2)}$
S-S	1	$x(x$-$c_1)$	1	$(x$-1$)(x$-$c_1)$
S-F	1	$x(x$-$c_1)$	1	x-c_1
F-F	1	x x-c_1	1	x-c_1
C-C	X	$x(x$-$c_1)$	X-1	$(x$-1$)(x$-$c_1)$
C-S	X	$x(x$-$c_1)$	1	$(x$-1$)(x$-$c_1)$
C-F	X	$x(x$-$c_1)$	1	x-c_1