Ms. Ref. No.: IJMS-111337
Title: A general Ritz formulation for the free vibration analysis of thick trapezoidal and triangular laminated plates resting on elastic supports
International Journal of Mechanical Sciences

Dear Dr Quintana,

Comments on your work have now been received from referees. They feel the work has merit but have made a number of criticisms. If you feel able to deal with the points they have raised, I would be pleased to send a revision to them for further assessment. When responding please list the changes you have made in light of the referees' comments.

For your guidance, referees' comments are appended below.

I hope you feel able to do the further work involved and look forward to hearing from you in due course.

When submitting your revised manuscript, please ensure that you upload the source files (e.g. Word). Uploading only a PDF file at this stage will create delays should your manuscript be finally accepted for publication. If your revised submission does not include the source files, we will contact you to request them.

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Yours sincerely,

Professor M Wiercigroch
Journal Editor
International Journal of Mechanical Sciences

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Cover Letter

Salta, August 14, 2012

Editor International Journal of Mechanical Sciences

S. R. Reid

Dept. of Engineering, University of Aberdeen, Kings College, Aberdeen, AB24 3UE, UK

Dear Professor:

Please find attached the file of the manuscript A GENERAL RITZ FORMULATION FOR THE FREE VIBRATION ANALYSIS OF THICK TRAPEZOIDAL AND TRIANGULAR LAMINATED PLATES RESTING ON ELASTIC SUPPORTS, by V. Quintana and L. G. Nallim, to be considered for possible publication in International Journal of Mechanical Sciences.

Thank you very much.

Yours sincerely,

Dr. Virginia Quintana

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Reply to reviewers´ comments

Paper Ref.
Title: A general Ritz formulation for the free vibration analysis of thick trapezoidal and triangular laminated plates resting on elastic supports.*International Journal of Mechanical Sciences.*

Dear Editor

According to the comments and suggestions raised by the reviewers, the original manuscript has been carefully revised. The authors agree with several of the reviewers´comments, which are quite adequate and will surely help to enhance the paper.

In the following, you will find a detailed reply to each issue.

Comments by Reviewer #1

Of the Manuscript: A general Ritz formulation for the free vibration analysis of thick trapezoidal and triangular laminated plates resting on elastic supports.
By M. Virginia Quintana and Liz G. Nallim.

This paper presents an original approach to study the free vibrations of trapezoidal and triangular plates.According to me the manuscript is well written and all consideration seems to be correct.The numerical examples are also convincing.I have only minor suggestions which, according to me, could improve the presentation.

1. Authors could consider whether to change the title of Section 4.1 from "Comparison of results" to "Verification of the method".

Response:
The title of Section 4.1 has been changed *from "Comparison of results"* to *"Verification of the method".*

2. In many places in this and the following subsections the authors use the term "frequency coefficient". I think that a more appropriate name of the expression connected with the natural frequencies is "frequency parameter".

Response:
The term *"frequency coefficient"* has been replaced by *"frequency parameter"* in all the manuscript.

3. It seems to me that the expression "upper bounds of the exact values" at the second line (row 6)on page 14 is not very precise. The exact values of the frequencies *are just numbers and they don't have upper and lower bounds. The authors should write this sentence more.*

Response:

The authors agree with of the reviewers´ comment so the sentence bellow mentioned has been rewritten.

I think that the manuscript could be published in IJMS. In my opinion the suggested minor revisions could improve the paper.

Comments by Reviewer #2:

This paper deals with the application of the Rayleigh-Ritz method in conjunction with a mapping technique to the free vibration analysis of moderately thick laminated composite trapezoidal and triangular plates with edges elastically restrained against translation and rotation. The first order shear deformation plate theory is considered which means that shear deformation and rotary inertia are taken into account. The analysis is limited to a single ply laminate. A so-called algorithm is used to generate the polynomial bases for different combinations of boundary conditions. The number of results presented is very limited. Some comparisons are made with results available in the literature for thin and moderately thick plates. The effects of aspect ratio, number of layers, fiber orientation angle, and boundary conditions on the frequencies of free vibration are investigated. The mapping technique presented in this paper is not new (see reference [24,25]. There are some syntax and technical errors that need to be corrected and a number of points that should be addressed and made clearer before the manuscript can be accepted for publication in International Journal of Mechanical Sciences. Major revisions in the line of the following remarks are required.

1. The method used in this paper is actually Rayleigh-Ritz (not Ritz). Also, it is not clear what the authors mean by the word "general".

Response

The word “general” is mainly referred to the analysis of plates with general anisotropy, any combination of boundary conditions, aspect thickness ratio and any trapezoidal and triangular planform shape.

2. The title needs to be modified to read "A Rayleigh-Ritz formulation for the free vibration analysis of moderately thick laminated composite trapezoidal and triangular plates with edges elastically restrained against translation and rotation".

Response:

A discussion about this topic has been presented by A.W. Leissa in the paper “The historical bases of the Rayleigh and Ritz methods”, Journal of Sound and Vibration 287 (2005) 961–978. The author examines the both methods in detail and concludes that Rayleigh’s name should not be attached to the Ritz method; that is, the ‘‘Rayleigh–Ritz method’’ is an improper designation. For this reason we decided used only the Ritz method.

3. The authors state in the paper that they used an analytical method. In fact, they used a numerical method. The Rayleigh-Ritz method is based on a variational approach. Boundary conditions are exactly satisfied but the equations of motion are not.

Response:

We said that the Ritz’s method is an analytical method because an approximate solution is proposed in the form of a finite linear combination of polynomials functions. Then when minimizing the functional a final analytical expression, which includes the coefficient frequency, is obtained.

4. The literature review is incomplete. Papers dealing with the application of other approaches to vibrating laminated composite trapezoidal and triangular plates should be included in the list of references. The following references are examples:

- Rakesh K, Kapania, Andrew E. Lovejoy. "Free vibration of thick generally laminated cantilever quadrilateral plates", AIAA Journal, Vol. 34, No. 7 (1996), pp. 1474-1486.

- Liew KM. "Vibration of symmetrically laminated cantilever trapezoidal composite plates", International Journal of Mechanical Sciences, Vol. 34, No. 4 (1992), pp. 299-308.

- Qatu MS. "Vibrations of laminated composite completely free triangular and trapezoidal plates", International Journal of Mechanical Sciences, Vol. 36, No. 9 (1994), pp. 797-809.

- Kenji / Xie, Jimin / Sakata, Toshiyuki. "Free Vibration Analysis of Cantilevered Laminated Trapezoidal Plates", Science and Engineering of Composite Materials, Vol. 8, No. 1 (1999), pp. 1-10.

Response:

All the references have been included in the list of references and also they have been mentioned in the main text.

5. The generation of polynomial bases (see Appendix B) from the so-called algorithm should be explained and made clearer.

Response:

A more clear explanation has been included into the main text and in the Appendix

6. The application of boundary conditions for elastic restraints should be briefly explained.

Response:

A more clear explanation of the application of boundary conditions for elastic restraint has been included in page 8.

7. A convergence study as a function of the numbers of terms M and N in the series should be provided in order to evaluate correctly the efficiency of the proposed method.

Response:
A convergence study has been included in Table 1.

8. Two different thickness ratios (h/b and h/l) are adopted in the results section. The reviewer thinks that the ratio h/l in contrast to the ratio h/b does not describe correctly the type of laminate as far as the thickness is concerned (i.e. very thin, thin, moderately thick, or thick). It is therefore preferable to use h/b in all tables.

Response:
The corresponding tables have been modified. Only the *h/b* thickness ratio is adopted in the results section.

9. Additional results should be supplied as well as for symmetric and antisymmetric laminates with restrained edges in order to study the effect of stacking sequence on the frequencies.

Response:

Additional results for multiple-ply laminates whit different stacking sequence have been included in Table 5 and Figure 3.
Results for antisymmetric laminates have not been included because this paper only deals with symmetric laminated plates.

10. Additional results should also be supplied for very thin (h/b=0.001) and thin (h/b=0.01) laminates in order to examine the applicability of the proposed method to such structures.

Response:

Additional results for multiple-ply laminates whit different thickness ratio *(h/b=0.001, 0.01)* have been included in Table 5.

11. The effect of elastic restraints on the mode shapes should be explained in section 4.3.

Response:

The section 4.3 has been removed. Some representative mode shapes for triangular and trapezoidal plates have been included in section 4.2.

12. The abstract and conclusion should be rewritten in order to reflect the content of the paper.

Response:
The abstract and conclusion have been rewritten.

13. The abbreviation "FSFDT" in line 46 of page 2 should be replaced by "FSDT". The word "chapter" in line 56 of page 3 should be replaced by "paper". The sentence ".it is consisting by layers ." in line 26 of page 4 should read ". it consists of layers .". The sentence ". is symmetric respect to ." in line 28 of page 4 should read ". is symmetric with respect to .". In Equation 2, dA should be replaced by dxdy. The factor 1/3 in equation 6 should be removed. Time should be included in Equations (16-18).The vector in Equation 20 should be defined. Equations 22-30 should be moved to a new appendix. Indices representing frequency numbers should be added to ?in Figures 3 and 4.

Response:

The time is excluded in Eqs (19-21) because the components of the deflection field are split in a part that depend on the time and another that depend on the spatial coordinates. The following has been inserted into the manuscript, page 3:

“For free plate vibration, the displacement and rotations are given by harmonic functions of the time, i.e.

$$w\ \ x,y,t\ \ = w\ \ x,y\ \ \cos\omega t, \tag{1}$$

$$\phi_y\ \ x,y,t\ \ = \phi_y\ \ x,y,t\ \ \cos\omega t, \tag{2}$$

$$\phi_x\ \ x,y,t\ \ = \phi_x\ \ x,y,t\ \ \cos\omega t, \tag{3}$$

Where w is the radian frequency of the plate. “

Highlights

- In this work a variational formulation, based on the Ritz method, for the dynamic study of thick trapezoidal and triangular laminated plates is proposed
- The formulation includes the treatment of edges elastically restrained against rotation and translation in this kind of thick laminates.
- Simple polynomials, independent in each direction, automatically generated are used to build the approximate functions.
- Some new results are presented that can be useful for validation purposes.

A general Ritz formulation for the free vibration analysis of thick trapezoidal and triangular laminated plates resting on elastic supports

M. Virginia Quintana[a,b] and Liz G. Nallim[a]

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Abstract:

A general variational formulation for the determination of natural frequencies and mode shapes of free vibration of symmetric laminated plates of trapezoidal and triangular shapes is presented in this work. The kinematics corresponding to the first order shear deformation plate theory (FSDT) is used to take into account the effects of shear deformation and rotatonial inertia in the analysis.

The developed approach is based on the Ritz method and the plate geometry is approximated by non-orthogonal right triangular co-ordinates. The transverse deflection and two rotations of the laminate are independently approximated by sets of simple polynomials.

The algorithm allows obtaining approximate analytical solutions for laminated plates with different shapes, aspect ratio, number of layers, stacking sequence, angle of fiber orientation and boundary conditions including translational and rotational elastically restrained edges. The algorithm is simple to program and numerically stable.

Keywords: Vibrations, Trapezoidal plates, Laminated, FSDT, Ritz, Elastically restrained edges

1. Introduction

Anisotropic plates, especially those consisting of fibre reinforced composite materials, are widely used in various technological applications in many industrial and engineering fields such as mechanical, aerospace, automotive, etc. In many cases, the rapid and efficient determination of the natural vibration frequencies and associated mode shapes is fundamental in their design and performance evaluation. It is also important to include in the study elastic partial restrictions along the edges to include a significant group of practical problems. This is because the classical boundary conditions can not be applied to all real situations and elastic modelling constraints can be likened more rationally the actual restraint conditions. In particular, the flexibility of the edges has a significant influence on the plate vibrations.

Published papers about vibrations of trapezoidal plates are mostly based on the theory of thin plates [1-3]. Excellent sources of references are the works of the Leissa [4-7]. The classical laminated plate theory (CLPT) neglects the effects of shear deformation and rotational inertia, and this leads to results that overestimate the frequencies of vibration. This error is greater when the thickness of the plate increases. The simplest alternative to consider the above mentioned effects is the use of the first order shear deformation theory for moderately thick plates, proposed by Reissner [8] and Mindlin [9], which incorporates the effect of rotational inertia. This theory also requires the use of a correction factor to compensate the error resulting from the approximation made with respect to the non-uniform distribution of strains and shear stresses. A complete analysis of the theoretical basis and the main advantages, application areas and limitations of using FSDT or CLPT theories can be found, for example, in reference [11]. Liew et al. [10] presents a review of works on the vibration of thick plates which mainly use the first-order theory and are referred to rectangular plates.

Particularly, the study of moderately thick trapezoidal plates through approximate analytical methods presents the difficulty of the construction of simple and adequate approximation functions that can be applied to the entire domain of the plate. When these plates also have elastically restrained edges, the mathematical structure of the boundary conditions becomes complex and the generation of approximating functions becomes very difficult. To overcome this difficulty, several techniques have been developed and perfected. The global pb-2 Rayleigh-Ritz method was used to study the free vibration behavior of trapezoidal Mindlin plates [12] and cantilever triangular Mindlin plate [13]. Subsequently, the above methodology was extended to different combinations of classical boundary conditions [14]. Wu and Liu [15] used the differential cubature method for analysis of thick plates of arbitrary shape. Zhong [16] also analyzed the free vibrations of triangular plates by differential quadrature method. Dozio and Carrera [17] proposed a variable kinematic Ritz formulation for vibration study of arbitrary quadrilateral thick plates. All these works are referred to plates made of isotropic material.

A few studies can be found in the literature for the free vibration analysis of laminated thick trapezoidal plates. For instance, the Rayleigh– Ritz procedure has been applied by Kapania and Lovejoy [18] in the analysis of quadrilateral, thick, generally laminated plates having arbitrary edge supports together with Chebychev polynomials as trial functions. The boundary conditions have been enforced by the appropriate use of distributed linear and rotational spring along the edges, but the method only has been applied to cantilever plates. Chen et al. [19] studied the free vibration of cantilevered symmetrically laminated thick trapezoidal plates using *p*-Ritz method incorporating third-order shear deformation theory. Haldar and Manna [20] proposed a high precision triangular element with shear strain for free vibration analysis of composite trapezoidal plates. Gürses et al. [21] used the method of discrete singular convolution (DSC) for free vibration analysis of laminated trapezoidal plates. Zamani et al. [22] obtained the governing equations and boundary conditions for the free vibration of trapezoidal plate using the first order shear deformation theory (FSDT) together with proper transformation from Cartesian system into trapezoidal coordinates. Then the generalized differential quadrature (GDQ) method is employed to obtain solutions. All these papers consider only classical boundary conditions.

Elastic restraints along the edges of thick plates have been considered by some authors. But, in general, the cases considered correspond to rectangular plates [23-27]. According to the statement in the preceding paragraphs, the objective of this paper is to propose a general algorithm that allows obtaining approximate analytical solutions to study the free vibrations of moderately thick trapezoidal and triangular laminated plates, with edges elastically restrained against rotation and translation. To this end, a methodology based on an extension and generalization of previous works [28-29] is presented. The procedure is based on the Ritz method and covers two aspects. The first one is the approximation of the plate geometry through triangular coordinates and the second aspect is the approximation of the displacement field components with simple polynomials generated automatically from a basis polynomial.

2. Mathematical formulation

2.1 Geometrical and mechanical characteristics of the plate

The general scheme of the analyzed composite plate is shown in Fig.1. The laminate thickness is h and, in general, it consists by layers of unidirectional fibers composite material (Fig. 2b). The lamination scheme is symmetric with respect to the midplane. The angle of fibers orientation is denoted by β, measured from x- axis to the fibers direction as shown in Fig. 2a. The rotational and translational restraints are characterized by springs constants c_{R_i} and c_{T_i} $(i = 1,\ldots,4)$, respectively.

The present study is based on the first order plate theory (FSDT). The components of the displacements field in x, y, z directions, at any time t, are given by

$$\begin{aligned} u(x,y,z,t) &= u_0(x,y,t) - z\phi_y(x,y,t) \\ v(x,y,z,t) &= v_0(x,y,t) - z\phi_x(x,y,t) \\ w(x,y,z,t) &= w_0(x,y,t) \end{aligned} \qquad (1)$$

where $w(x,y,t)$ are the deflections of midplane points, $\phi_y(x,y,t)$ and $\phi_x(x,y,t)$ are the rotations of the cross sections with respect to the coordinates x and y respectively.

For free plate vibration, the displacement and rotations are given by harmonic functions of the time, i.e.

$$w\ x,y,t\ = w\ x,y\ \cos\omega t, \tag{2}$$

$$\phi_y\ x,y,t\ = \phi_y\ x,y,t\ \cos\omega t, \tag{3}$$

$$\phi_x\ x,y,t\ = \phi_x\ x,y,t\ \cos\omega t, \tag{4}$$

where w is the radian frequency of the plate.

According to Eqs (2) and (4) the maximum kinetic energy of the freely vibrating plate expressed in artesian co-ordinates, is given by

$$T_{\max} = \frac{\rho h\omega^2}{2}\iint_A \left[w^2 + \frac{h^2}{12}\ \phi_x^2 + \phi_y^2\ \right]dxdy \tag{5}$$

where ρ is the mass density of the plate material, ω is the circular frequency and the integration is carried out over the entire plate domain A.

The maximum strain energy of the mechanical system is given by

$$U_{\max} = U_{P,\max} + U_{R,\max} + U_{T,\max} \tag{6}$$

where $U_{P,\max}$ is the maximum strain energy due to plate bending, which in Cartesian co-ordinates is given by

$$\begin{aligned} U_{P,\max} = \frac{1}{2}\iint_A \Bigg\{ & D_{11}\left(\frac{\partial\phi_x}{\partial x}\right)^2 + D_{22}\left(\frac{\partial\phi_y}{\partial y}\right)^2 + 2D_{12}\frac{\partial\phi_x}{\partial x}\frac{\partial\phi_y}{\partial y} + 2D_{16}\left(\frac{\partial\phi_x}{\partial x}\frac{\partial\phi_x}{\partial y} + \frac{\partial\phi_x}{\partial x}\frac{\partial\phi_y}{\partial x}\right) + \\ & + 2D_{26}\left(\frac{\partial\phi_y}{\partial y}\frac{\partial\phi_x}{\partial y} + \frac{\partial\phi_y}{\partial y}\frac{\partial\phi_y}{\partial x}\right) + D_{66}\left(\frac{\partial\phi_x}{\partial y} + \frac{\partial\phi_y}{\partial x}\right)^2 + A_{44}\left(\frac{\partial w}{\partial y} + \phi_y\right)^2 + \\ & + A_{55}\left(\frac{\partial w}{\partial x} + \phi_x\right)^2 + 2A_{45}\left(\frac{\partial w}{\partial x}\frac{\partial w}{\partial y} + \phi_x\frac{\partial w}{\partial y} + \phi_y\frac{\partial w}{\partial x} + \phi_x\phi_y\right)\Bigg\}\, dx\, dy \end{aligned} \tag{7}$$

where the coefficients $D_{ij},\ i,j = 1,2,6$ are the bending, the twisting and the bending-twisting coupling rigidities and are given by

$$D_{ij} = \frac{1}{3}\sum_{k=1}^{N_c} \bar{Q}_{ij}^{(k)} \;\; z_{k+1}^3 - z_k^3 \tag{8}$$

where as $A_{ij},\; i,j = 4,5$ are the shear rigidities coefficients given by

$$A_{ij} = \kappa \sum_{k=1}^{N_c} \bar{Q}_{ij}^{(k)} \;\; z_{k+1} - z_k \tag{9}$$

where κ is the shear correction factor, the coordinates z_{k-1}, z_k are depicted in Fig. 2b, N_c is the total number of layers in the laminate and $\bar{Q}_{ij}$ are the reduced transformed rigidities (see for instance Ref. [11]) which depend of the mechanical properties of each lamina and the angle of fiber orientation.

The maximum strain energy stored in rotational and translational springs at the plate edges are, respectively

$$U_{T,\max} = \frac{1}{2}\oint_{\partial A} c_T(s) w^2 ds = \frac{1}{2}\sum_{i=1}^{4}\int_0^{l_i} c_{T_i} w^2 ds, \tag{10}$$

$$U_{R,\max} = \frac{1}{2}\oint_{\partial A} c_R(s) \phi_n{}^2 ds = \frac{1}{2}\sum_{i=1}^{4}\int_0^{l_i} c_{R_i} \phi_{ni}{}^2 ds, \tag{11}$$

where ϕ_{ni} denotes the rotation of the cross section about the corresponding co-ordinate and l_i denotes the length of $\partial A_i \;\; i = 1,...,4$.

2.2 Geometric mapping: Triangular non-orthogonal coordinates.

The actual plate of trapezoidal plan-form is mapped onto a rectangular one, using a coordinate transformation between the rectangular Cartesian and triangular non-orthogonal coordinates, according to the following expressions [28, 29]:

$$x = ul, \qquad y = uvl \tan\alpha_1 \tag{12}$$

where $\tan\alpha_1$ is the slope of the upper side of the plate.

The relationships between the partial derivatives in both coordinates systems are given by

$$\begin{bmatrix} \dfrac{\partial(\cdot)}{\partial x} \\ \dfrac{\partial(\cdot)}{\partial y} \end{bmatrix} = \mathbf{J}^{-1} \begin{bmatrix} \dfrac{\partial(\cdot)}{\partial u} \\ \dfrac{\partial(\cdot)}{\partial v} \end{bmatrix} = \begin{bmatrix} \dfrac{J_{22}}{|\mathbf{J}|} & -\dfrac{J_{12}}{|\mathbf{J}|} \\ -\dfrac{J_{21}}{|\mathbf{J}|} & \dfrac{J_{11}}{|\mathbf{J}|} \end{bmatrix} \begin{bmatrix} \dfrac{\partial(\cdot)}{\partial u} \\ \dfrac{\partial(\cdot)}{\partial v} \end{bmatrix} \tag{13}$$

where $\mathbf{J}$ is the Jacobian matrix of the geometrical mapping given by

$$\mathbf{J} = \begin{bmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{bmatrix} = \begin{bmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial y}{\partial u} \\ \dfrac{\partial x}{\partial v} & \dfrac{\partial y}{\partial v} \end{bmatrix} = \begin{bmatrix} l & vl\tan\alpha_1 \\ 0 & ul\tan\alpha_1 \end{bmatrix} \tag{14}$$

and $|\mathbf{J}|$ is the Jacobian determinant of the coordinate change.

The maximum kinetic and strain energies of the vibrating laminated plate can now be expressed in the non-orthogonal triangular coordinates by replacing Eqs. (9) and (10) into Eqs. (2) and (3) as follows:

$$T_{\max} = \frac{\rho h \omega^2}{2} \int_{c_l}^{1} \int_{v_0}^{1} \left[w^2 + \frac{h^2}{12} \left(\phi_x^2 + \phi_y^2 \right) \right] |\mathbf{J}| \, du\, dv \tag{15}$$

$$\begin{aligned} U_{P,\max} = \frac{1}{2} \int_{c_l}^{1} \int_{v_0}^{1} \Bigg\{ & S_1 \left(\frac{\partial w}{\partial u} \right)^2 + 2S_2 \frac{\partial w}{\partial u} \frac{\partial w}{\partial v} + S_3 \left(\frac{\partial w}{\partial v} \right)^2 + 2S_4 \frac{\partial w}{\partial u} \phi_x + 2S_5 \frac{\partial w}{\partial v} \phi_x + \\ & + 2S_6 \frac{\partial w}{\partial u} \phi_y + 2S_7 \frac{\partial w}{\partial v} \phi_y + S_8 \left(\frac{\partial \phi_x}{\partial u} \right)^2 + 2S_9 \frac{\partial \phi_x}{\partial u} \frac{\partial \phi_x}{\partial v} + S_{10} \left(\frac{\partial \phi_x}{\partial v} \right)^2 + \\ & + S_{11} \left(\phi_x \right)^2 + 2S_{12} \frac{\partial \phi_x}{\partial u} \frac{\partial \phi_y}{\partial v} + 2S_{13} \frac{\partial \phi_x}{\partial v} \frac{\partial \phi_y}{\partial u} + 2S_{14} \frac{\partial \phi_x}{\partial v} \frac{\partial \phi_y}{\partial v} + 2S_{15} \phi_x \phi_y + \\ & + 2S_{16} \frac{\partial \phi_y}{\partial u} \frac{\partial \phi_y}{\partial u} + 2S_{17} \frac{\partial \phi_y}{\partial u} \frac{\partial \phi_y}{\partial v} + 2S_{18} \frac{\partial \phi_y}{\partial u} \frac{\partial \phi_y}{\partial v} + 2S_{19} \frac{\partial \phi_y}{\partial v} \frac{\partial \phi_y}{\partial v} + S_{20} \left(\phi_y \right)^2 \Bigg\} |\mathbf{J}| \, du\, dv \end{aligned} \tag{16}$$

where S_i $i = 1,\ldots,20$, are functions that depend on the parameters of the problem, eg., geometry and material properties, and are defined in Appendix A.

Using the change of variables (9) and substituting Eqs. (10) and (11) into Eqs. (7) and (8) the maximum strain energies stored in the translational and rotational springs at the plate edges become

$$U_{T,\max} = \frac{1}{2} l \left(c_{T_1} \tan\alpha_1 c_l \int_{v_0}^{1} w^2 \Big|_{u=c_l} dv + c_{T_2} \tan\alpha_1 \int_{v_0}^{1} w^2 \Big|_{u=1} dv + \right. \\ \left. + \frac{c_{T_3}}{\cos\alpha_2} \int_{c_l}^{1} w^2 \Big|_{v=v_0} du + \frac{c_{T_4}}{\cos\alpha_1} \int_{c_l}^{1} w^2 \Big|_{v=1} du \right), \tag{17}$$

$$U_{R,\max} = \frac{1}{2} l \left(c_{R_1} \tan\alpha_1 c_l \int_{v_0}^{1} {\phi_x}^2 \Big|_{u=c_l} dv + c_{R_2} \tan\alpha_1 \int_{v_0}^{1} {\phi_x}^2 \Big|_{u=1} dv + \right. \\ \left. \frac{c_{R_3}}{\cos\alpha_2} \int_{c_l}^{1} {\phi_y}^2 \Big|_{v=v_0} du + \frac{c_{R_4}}{\cos\alpha_1} \int_{c_l}^{1} {\phi_y}^2 \Big|_{v=1} du \right), \tag{18}$$

where $c_l = c/l$ y $v_0 = \tan\alpha_2 \cot\alpha_1$.

2.3 Approximating functions

The transverse deflection and the rotations are expressed by products of simple one-dimensional polynomials in each of the triangular coordinates, as follows:

$$w(u,v) = \sum_{i=1}^{M} \sum_{j=1}^{N} c_{ij}^{(w)} p(u)_i^{(w)} q(v)_j^{(w)} \tag{19}$$

$$\phi_x(u,v) = \sum_{i=1}^{M} \sum_{j=1}^{N} c_{ij}^{(\phi_x)} p(u)_i^{(\phi_x)} q(v)_j^{(\phi_x)} \tag{20}$$

$$\phi_y(u,v) = \sum_{i=1}^{M} \sum_{j=1}^{N} c_{ij}^{(\phi_y)} p(u)_i^{(\phi_y)} q(v)_j^{(\phi_y)} \tag{21}$$

where $c_{ij}^{(w)}$, $c_{ij}^{(\phi_u)}$ and $c_{ij}^{(\phi_v)}$ are the unknown coefficients to be determined by the Ritz method.

The first polynomial of each set $p(u)_i^{(w)}$, $q(v)_j^{(w)}$, $p(u)_i^{(\phi_x)}$, $q(v)_j^{(\phi_x)}$, $p(u)_i^{(\phi_y)}$ and $q(v)_j^{(\phi_y)}$ is obtained as the simplest polynomial that satisfies the essential boundary conditions of the equivalent beam in each co-ordinate. In the case of beams involving free edges, simplest starting members of order zero, one or two, are used [30]. It is well known that it is not necessary to subject the co-ordinate functions to the natural boundary conditions. It is sufficient that they satisfy the geometrical conditions since, as the number of co-ordinate functions approaches infinity, the natural boundary conditions will be exactly satisfied [31]. Consequently, when the edges have rotational or translational restraints all boundary conditions are natural [32] so it is possible to ignore the boundary conditions in the construction of the first polynomial of each set.

The higher polynomials of each set are generated from the first polynomial, using the following procedure:

$$p(u)_i^{(w)} = p(u)_1^{(w)} u^{i-1}, \quad i = 2,\ldots,M,$$

$$p(u)_i^{(\phi_x)} = p(u)_1^{(\phi_x)} u^{i-1}, \quad i = 2,\ldots,M,$$

$$p(u)_i^{(\phi_y)} = p(u)_1^{(\phi_y)} u^{i-1}, \quad i = 2,\ldots,M,$$

The polynomials set along the v direction are generated using the same procedure. In this work, laminated plates with different boundary conditions are analyzed and the basis polynomials are given in Appendix B.

3. Application of Ritz method.

Application of the Ritz method requires the minimization of the following energy functional:

$$\Pi = U_{\max} - T_{\max} \tag{22}$$

where x and $U_{\max}$ are respectively given by Eqs. (15) and (16)-(18).

Minimization of functional (19) leads to the following equations system

$$\left[K\right] - \omega^2 \left[M\right] \quad \bar{c} \;=\; 0 \;, \tag{23}$$

where

$$\left[K\right]=\begin{bmatrix}\left[K^{ww}\right] & \left[K^{w\phi_x}\right] & \left[K^{w\phi_y}\right]\\ & \left[K^{\phi_x\phi_x}\right] & \left[K^{\phi_x\phi_y}\right]\\ \text{sym.} & & \left[K^{\phi_y\phi_y}\right]\end{bmatrix} \quad \text{and} \quad \left[M\right]=\begin{bmatrix}\left[M^{ww}\right] & \left[0\right] & \left[0\right]\\ & \left[M^{\phi_x\phi_x}\right] & \left[0\right]\\ \text{sym.} & & \left[M^{\phi_y\phi_y}\right]\end{bmatrix} \tag{24}$$

The elements of the rigidity matrix $\left[K\right]$ and mass matrix $\left[M\right]$ are given in Appendix C.

4. Numerical results.

Numerical results useful to appreciate the variation of dimensionless frequency parameters for plates with different geometries (triangular and trapezoidal plan form) and several edge support conditions are presented in this section. When treating with classical boundary conditions the nomenclature CSFS, for example, identifies a trapezoidal plate with edge 1 clamped, 2 simply supported, 3 free and 4 simply supported. For triangular plates edge 1 disappears and the nomenclature starts from edge 2. When the plate’s edges have rotational and/or translational restraints, the restraints parameters are specifically indicated in each case.

4.1 Verification of the method.

To validate the proposed approach, comparisons with numerical values provided by other researchers obtained by other methods, are carried out and also convergence studies have been implemented.

Results of a convergence study of the frequency parameter $\Omega = \omega b^2 \sqrt{\rho h / D_0} / 2\pi$ with $D_0 = E_1 h^3 / 12 \; 1-\nu_{12}\nu_{21}$ are shown in Table 1. The first five values of Ω are presented for triangular and trapezoidal thick plates $\alpha_1 = -\alpha_2 = 20^\circ$ simply supported at the sides. The plate consists in a single boron - epoxy layer $\beta = 45^\circ$, with $E_1 = 207\mathrm{GPa}$, $E_2 = 21\mathrm{GPa}$, $\nu_{12} = 0.3$,

$G_{12} = G_{13} = 7\text{GPa}$, $G_{23} = 4.2\text{Gpa}$ and $\kappa = 5/6$. The convergence of the mentioned frequency parameters is studied by gradually increasing the number of polynomial in the approximate functions w, ϕ_x and ϕ_y which are respectively given by M and N. It can be observed that the frequency parameters converge monotonically from above as the number of terms increases.

Table 2 shows the values of fundamental frequency parameter $\Omega = \omega b^2 \sqrt{\rho h/D}/2\pi$ for different isotropic trapezoidal plates simply supported at the four sides. On the other hand, frequency parameters $\Omega^* = \omega b^2 \sqrt{\rho/E_2}/h$ for laminated trapezoidal plates are depicted in Table 3. The frequency parameters are compared with those of Haldar and Manna [20] who employed high precision triangular elements including shear strains and they are also compared with results of Gürses et al. [21] who used DSC method. Gürses et al. [21] also present a convergence study increasing the number of grid points. The numerical values obtained by these authors depicted in Tables 2 and 3 have been computed using 15×15 grid points. It is important to point out that the values of the frequency parameters obtained by the methodology proposed in this work are obtained using seven terms in the co-ordinate functions in each direction $M = N = 7$. It should be noted that the Ritz method produces approximations from above for each eigenvalue with respect to the exact eigenvalues. It is important when the exact solution cannot be obtained. In Table 3 each layer of the laminated has the following material properties: $E_1 = 40E_2, G_{12} = G_{13} = 0.6E_2$, $G_{23} = 0.5E_2, \nu_{12} = 0.25$ and $\kappa = 0.833$. Results for two different boundary conditions (SSSS, CCCC), thickness ratios $h/b = 0.1, 0.2$ and several geometric configurations, have been included in this table. In all cases a very good agreement in the numerical values can be observed, indicating the accuracy that can be achieved through the application of this methodology, which uses simple polynomials to construct the shape functions.

4.2 New numerical results

In this section new numerical results that can serve as a supplement to the existing data base on vibration characteristics of moderately thick trapezoidal and triangular plates are presented. In particular, results for plates with different spring parameters of elastically restrained edges are presented. Table 4 shows the variation of the frequency parameters $\Omega^{**} = \omega b^2 \sqrt{\rho h / D_0}$ with $D_0 = E_1 h^3 / 12\ \ 1 - \nu_{12}\nu_{21}$, for isosceles triangular plates $\alpha_1 = -\alpha_2 = 15°$ with all edges elastically restrained against rotation and translation. Two thickness ratios $h/b = 0.1, 0.2$ and two values of the fibers orientation angle $\beta = 0°, \beta = 30°$ have been considered. The plate consists in a single boron - epoxy layer with the following mechanical properties $E_1 = 207\mathrm{GPa}$, $E_2 = 21\mathrm{GPa}$, $\nu_{12} = 0.3$, $G_{12} = G_{13} = 7\mathrm{GPa}$, $G_{23} = 4.2\mathrm{Gpa}$ and $\kappa = 5/6$. The dimensionless parameters characterizing the elastic constraints are given by: $R_i = c_{R_i} l / D_0$, $T_i = c_{T_i} l^3 / D_0\ \ i = 1,2,3$.

To asses the influence of the number of layers (for the same total plate thickness) and thickness ratio h/b in the response of cross-play laminated trapezoidal plates, values of the fundamental frequency parameter $\Omega = \omega b^2 \sqrt{\rho h / D_0} / 2\pi$, $D_0 = E_2 h^3 / 12\ \ 1 - \nu_{12}\nu_{21}$ are depicted in Table 5. The plate is elastically restrained against translation $T_1 = T_3 = T_4 = T, T_2 = \infty,$ $R_i = 0, i = 1, \ldots, 4$ and the study has been carried out for increasing values of the translational restraint parameter T.

To evaluate the effect of different fiber orientation angles β and as well as the influence of the thickness ratio h/b on the dynamic properties of trapezoidal laminated plates

$\tan\alpha_2 = -\tan\alpha_1 = 0.4, c_l = 0.2$, the variation of the values of the fundamental frequency parameter Ω for four layers laminate are plotted in Figure 3. The plate edges are elastically restrained $R_i = R, T_i = \infty, i = 1,...,4,$ and two different values of the rotational restraint parameter R are considered. The elastic properties of each layer of the laminated used in the Table 5 and Figure 3 are $E_1 = 40E_2, G_{12} = G_{13} = 0.6E_2, G_{23} = 0.5E_2, \nu_{12} = 0.25$ and $\kappa = 0.833$.

Finally, some representative mode shapes for triangular and trapezoidal plates with different boundary conditions obtained with the proposed methodology are presented in Fig. 4 and Fig. 5 respectively. All results considered here correspond to single layer boron- epoxy plates. In both cases the presence of elastically restrained edges is considered.

5. CONCLUSION

A simple, accurate and general algorithm for the free transverse vibration analysis of trapezoidal and triangular symmetrically laminated plates is proposed in this study. The developed methodology is based on the Ritz method and on the first order shear deformation theory, and used non-orthogonal right triangular co-ordinates to express the geometry of the plate in a simple form. The transvers deflection and the two rotations are approximated by means of simple polynomials. The algorithm allows a unified treatment of symmetrically laminated plates with several trapezoidal or triangular planform, different thickness ratios and boundary conditions, including edges elastically restrained against rotation and translation.

From the convergence studies and the comparisons with results available in the literature it is observed that the approach presented is reliable and accurate. Sets of numerical results are given in tabular and graphical form illustrating the influence of different number of layer, fibre stacking sequences and edge conditions.

Finally, it is important to note that the proposed method can be easily extended for application to static and stability analysis. It can also be generalized to study thick trapezoidal plates with non-symmetrical stacking sequence about the midplane.

6. ACKNOWLEDGEMENTS

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APPENDIX A

$$S_1 = \frac{1}{|\mathbf{J}|^2} \ A_{44} J_{21}^2 - 2A_{45} J_{21} J_{22} + A_{55} J_{22}^2 \ ,$$

$$S_2 = \frac{1}{|\mathbf{J}|^2} \Big[-A_{44} J_{21} J_{11} + A_{45} (J_{12} J_{21} + J_{22} J_{11} \ - A_{55} J_{12} J_{22} \Big],$$

$$S_3 = \frac{1}{|\mathbf{J}|^2} \ A_{44} J_{11}^2 - 2A_{45} J_{12} J_{11} + A_{55} J_{12}^2 \ ,$$

$$S_4 = \frac{1}{|\mathbf{J}|} \ -A_{45} J_{21} + A_{55} J_{22} \ ,$$

$$S_5 = \frac{1}{|\mathbf{J}|} \ A_{45} J_{22} - A_{55} J_{12} \ ,$$

$$S_6 = \frac{1}{|\mathbf{J}|} \ -A_{44} J_{21} + A_{45} J_{22} \ ,$$

$$S_7 = \frac{1}{|\mathbf{J}|} \ A_{44} J_{11} - A_{45} J_{12} \ ,$$

$$S_8 = \frac{1}{|\mathbf{J}|^2} \ D_{11} J_{22}^2 - 2D_{16} J_{21} J_{22} + D_{66} J_{21}^2 \ ,$$

$$S_9 = \frac{1}{|\mathbf{J}|^2} \Big[-D_{11} J_{12} J_{22} + D_{16} (J_{12} J_{21} + J_{22} J_{11} \ - D_{66} J_{21} J_{11} \Big],$$

$$S_{10} = \frac{1}{\left|\mathbf{J}\right|^{2}} \; D_{11}J_{12}^{2} - 2D_{16}J_{12}J_{11} + D_{66}J_{11}^{2} \; ,$$

$$S_{11} = A_{55},$$

$$S_{12} = \frac{1}{\left|\mathbf{J}\right|^{2}} \; -D_{12}J_{22}J_{21} + D_{16}J_{22}^{2} + D_{26}J_{21}^{2} - D_{66}J_{21}J_{22} \; ,$$

$$S_{13} = \frac{1}{\left|\mathbf{J}\right|^{2}} \; D_{12}J_{22}J_{11} - D_{16}J_{22}J_{12} - D_{26}J_{21}J_{11} + D_{66}J_{21}J_{12} \; ,$$

$$S_{14} = \frac{1}{\left|\mathbf{J}\right|^{2}} \; D_{12}J_{12}J_{21} - D_{16}J_{22}J_{12} - D_{26}J_{21}J_{11} + D_{66}J_{11}J_{22} \; ,$$

$$S_{15} = \frac{1}{\left|\mathbf{J}\right|^{2}} \; -D_{12}J_{12}J_{11} + D_{16}J_{12}^{2} + D_{26}J_{11}^{2} - D_{66}J_{12}J_{11} \; ,$$

$$S_{16} = A_{45},$$

$$S_{17} = \frac{1}{\left|\mathbf{J}\right|^{2}} \; D_{66}J_{22}^{2} - 2D_{26}J_{21}J_{22} + D_{22}J_{21}^{2} \; ,$$

$$S_{18} = \frac{1}{\left|\mathbf{J}\right|^{2}}\left[-D_{66}J_{12}J_{22} + D_{16}(J_{12}J_{21} + J_{22}J_{11} \; - D_{66}J_{21}J_{11}\right],$$

$$S_{19} = \frac{1}{\left|\mathbf{J}\right|^{2}} \; D_{66}J_{12}^{2} - 2D_{26}J_{12}J_{11} + D_{22}J_{11}^{2} \; ,$$

$$S_{20} = A_{44}.$$

APPENDIX B

First polynomials in the coordinates u and v for different combinations of boundary conditions.

		Boundary conditions (*)					
Sides		Free (F): $w \neq 0, \quad \phi_n \neq 0, \quad \phi_s \neq 0$ Simply Supported (S): $w = 0, \quad \phi_n \neq 0, \quad \phi_s = 0$ (**) Clamped (C): $w = 0, \quad \phi_n = 0, \quad \phi_s = 0$					
$u = c_l$ $v = v_0$	$u = 1$ $v = 1$	$p(u)_1^{(w)}$	$q(v)_1^{(w)}$	$p(u)_1^{(\phi_x)}$	$q(v)_1^{(\phi_x)}$	$p(u)_1^{(\phi_y)}$	$q(v)_1^{(\phi_y)}$
S	F	$u-c_l$	$v-v_0$	1	$v-v_0$	$u-c_l$	1
C	F	$u-c_l$	$v-v_0$	$u-c_l$	$v-v_0$	$u-c_l$	$v-v_0$
S	S	$u-c_l\ (u-1)$	$(v-v_0)(v-1)$	1	$(v-v_0)(v-1)$	$u-c_l\ (u-1)$	1
S	C	$u-c_l\ (u-1)$	$(v-v_0)(v-1)$	$u-1$	$(v-v_0)(v-1)$	$u-c_l\ (u-1)$	$v-1$
C	C	$u-c_l\ (u-1)$	$(v-v_0)(v-1)$	$u-c_l\ (u-1)$	$(v-v_0)(v-1)$	$u-c_l\ (u-1)$	$(v-v_0)(v-1)$
F	F	1	1	1	1	1	1
F	S	$u-1$	$v-1$	1	$v-1$	$u-1$	1
F	C	$u-1$	$v-1$	$u-1$	$v-1$	$u-1$	$v-1$
C	S	$u-c_l\ (u-1)$	$(v-v_0)(v-1)$	$u-c_l$	$(v-v_0)(v-1)$	$u-c_l\ (u-1)$	$v-v_0$

(*) For elastically restrained sides using the same polynomial basis that the free sides.

(**) ϕ_s denotes the rotation with respect to the normal co-ordinate *n*.

APPENDIX C

$$
\begin{aligned}
K_{ijkh}^{ww} = & \int_{c_l}^{1}\int_{v_0}^{1}\Big[S_1 P_{ik}^{(w,w)(1,1)} Q_{jh}^{(w,w)(0,0)} + S_2\ P_{ik}^{(w,w)(1,0)} Q_{jh}^{(w,w)(0,1)} + P_{ik}^{(w,w)(0,1)} Q_{jh}^{(w,w)(1,0)} + \\
& + S_3 P_{ik}^{(w,w)(0,0)} Q_{jh}^{(w,w)(1,1)}\Big]\left|\mathbf{J}\right| du dv + c_{T_1} c_l \tan\alpha_1 \int_{v_0}^{1} P_{ik}^{(w,w)(0,0)} Q_{jh}^{(w,w)(0,0)}\Big|_{u=c_l} dv + \\
& + c_{T_2} \tan\alpha_1 \int_{v_0}^{1} \left(P_{ik}^{(w,w)(0,0)} Q_{jh}^{(w,w)(0,0)}\right)_{u=1} dv + \frac{c_{T_3}}{\cos\alpha_2}\int_{c_l}^{1}\left(P_{ik}^{(w,w)(0,0)} Q_{jh}^{(w,w)(0,0)}\right)_{v=v_0} du + \\
& + \frac{c_{T_4}}{\cos\alpha_1}\int_{c_l}^{1}\left(P_{ik}^{(w,w)(0,0)} Q_{jh}^{(w,w)(0,0)}\right)_{v=1} du
\end{aligned}
$$

$$K_{ijkh}^{w\phi_x} = \int_{c_l}^{1}\int_{v_0}^{1} S_4 P_{ik}^{(w,\phi_x)(1,0)} Q_{jh}^{(w,\phi_x)(0,0)} + S_5 P_{ik}^{(w,\phi_x)(0,0)} Q_{jh}^{(w,\phi_x)(1,0)} \left|\mathbf{J}\right| du dv$$

$$K_{ijkh}^{w\phi_y} = \int_{c_l}^{1}\int_{v_0}^{1} S_6 P_{ik}^{(w,\phi_y)(1,0)} Q_{jh}^{(w,\phi_y)(0,0)} + S_7 P_{ik}^{(w,\phi_x)(0,0)} Q_{jh}^{(w,\phi_x)(1,0)} \left|\mathbf{J}\right| du dv$$

$$\begin{aligned} K_{ijkh}^{\phi_x\phi_x} = \int_{c_l}^{1}\int_{v_0}^{1} \Big[& S_8 P_{ik}^{(\phi_x,\phi_x)(1,1)} Q_{jh}^{(\phi_x,\phi_x)(0,0)} + S_9 \; P_{ik}^{(\phi_x,\phi_x)(1,0)} Q_{jh}^{(\phi_x,\phi_x)(0,1)} + P_{ik}^{(\phi_x,\phi_x)(0,1)} Q_{jh}^{(\phi_x,\phi_x)(1,0)} + \\ & + S_{10} P_{ik}^{(\phi_x,\phi_x)(0,0)} Q_{jh}^{(\phi_x,\phi_x)(1,1)} + S_{11} P_{ik}^{(\phi_x,\phi_x)(0,0)} Q_{jh}^{(\phi_x,\phi_x)(0,0)} \Big] \left|\mathbf{J}\right| du dv + \\ & + c_{R_1} c_l \tan\alpha_1 \int_{v_0}^{1} \left(P_{ik}^{(\phi_x,\phi_x)(0,0)} Q_{jh}^{(\phi_x,\phi_x)(0,0)} \right)_{u=c_l} dv + c_{R_2} \tan\alpha_1 \int_{v_0}^{1} \left(P_{ik}^{(\phi_x,\phi_x)(0,0)} Q_{jh}^{(\phi_x,\phi_x)(0,0)} \right)_{u=1} dv \end{aligned}$$

$$\begin{aligned} K_{ijkh}^{\phi_x\phi_y} = \int_{c_l}^{1}\int_{v_0}^{1} \Big[& S_{12} P_{ik}^{(\phi_x,\phi_y)(1,1)} Q_{jh}^{(\phi_x,\phi_y)(0,0)} + S_{13} P_{ik}^{(\phi_x,\phi_y)(1,0)} Q_{jh}^{(\phi_x,\phi_y)(0,1)} + \\ & + S_{14} P_{ik}^{(\phi_x,\phi_y)(0,1)} Q_{jh}^{(\phi_x,\phi_y)(1,0)} + S_{15} P_{ik}^{(\phi_x,\phi_y)(0,0)} Q_{jh}^{(\phi_x,\phi_y)(1,1)} + S_{16} P_{ik}^{(\phi_x,\phi_y)(0,0)} Q_{jh}^{(\phi_x,\phi_y)(0,0)} \Big] \left|\mathbf{J}\right| du dv \end{aligned}$$

$$\begin{aligned} K_{ijkh}^{\phi_y\phi_y} = \int_{c_l}^{1}\int_{v_0}^{1} \Big[& S_{17} P_{ik}^{(\phi_y,\phi_y)(1,1)} Q_{jh}^{(\phi_y,\phi_y)(0,0)} + S_{18} \; P_{ik}^{(\phi_y,\phi_y)(1,0)} Q_{jh}^{(\phi_y,\phi_y)(0,1)} + P_{ik}^{(\phi_y,\phi_y)(0,1)} Q_{jh}^{(\phi_y,\phi_y)(1,0)} + \\ & + S_{19} P_{ik}^{(\phi_y,\phi_y)(0,0)} Q_{jh}^{(\phi_y,\phi_y)(1,1)} + S_{20} P_{ik}^{(\phi_y,\phi_y)(0,0)} Q_{jh}^{(\phi_y,\phi_y)(0,0)} \Big] \left|\mathbf{J}\right| du dv + \\ & + \frac{c_{R_3}}{\cos\alpha_2} \int_{c_l}^{1} \left(P_{ik}^{(\phi_y,\phi_y)(0,0)} Q_{jh}^{(\phi_y,\phi_y)(0,0)} \right)_{v=v_0} du + \frac{c_{R_4}}{\cos\alpha_1} \int_{c_l}^{1} \left(P_{ik}^{(\phi_y,\phi_y)(0,0)} Q_{jh}^{(\phi_y,\phi_y)(0,0)} \right)_{v=1} du \end{aligned}$$

$$M_{ijkh}^{ww} = \rho h \int_{c_l}^{1}\int_{v_0}^{1} P_{ik}^{(w,w)(0,0)} Q_{jh}^{(w,w)(0,0)} \left|\mathbf{J}\right| du dv$$

$$M_{ijkh}^{\phi_x\phi_x} = \frac{\rho h^3}{12} \int_{c_l}^{1}\int_{v_0}^{1} P_{ik}^{(\phi_x,\phi_x)(0,0)} Q_{jh}^{(\phi_x,\phi_x)(0,0)} \left|\mathbf{J}\right| du dv$$

$$M_{ijkh}^{\phi_y\phi_y} = \frac{\rho h^3}{12} \int_{c_l}^{1}\int_{v_0}^{1} h^2 P_{ik}^{(\phi_y,\phi_y)(0,0)} Q_{jh}^{(\phi_y,\phi_y)(0,0)} \left|\mathbf{J}\right| du dv$$

with

$$P_{ik}^{(\alpha,\beta)(r,s)} = \frac{\partial^r p_i^{(\alpha)}(u)}{\partial u^r} \frac{\partial^s p_k^{(\beta)}(u)}{\partial u^s},\; Q_{jh}^{(\alpha,\beta)(r,s)} = \frac{\partial^r q_j^{(\alpha)}(v)}{\partial v^r} \frac{\partial^s q_h^{(\beta)}(v)}{\partial v^s},$$

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Figure(s)

List of Fig.

Fig. 1. General description of the plate model $c_l = c/l$.

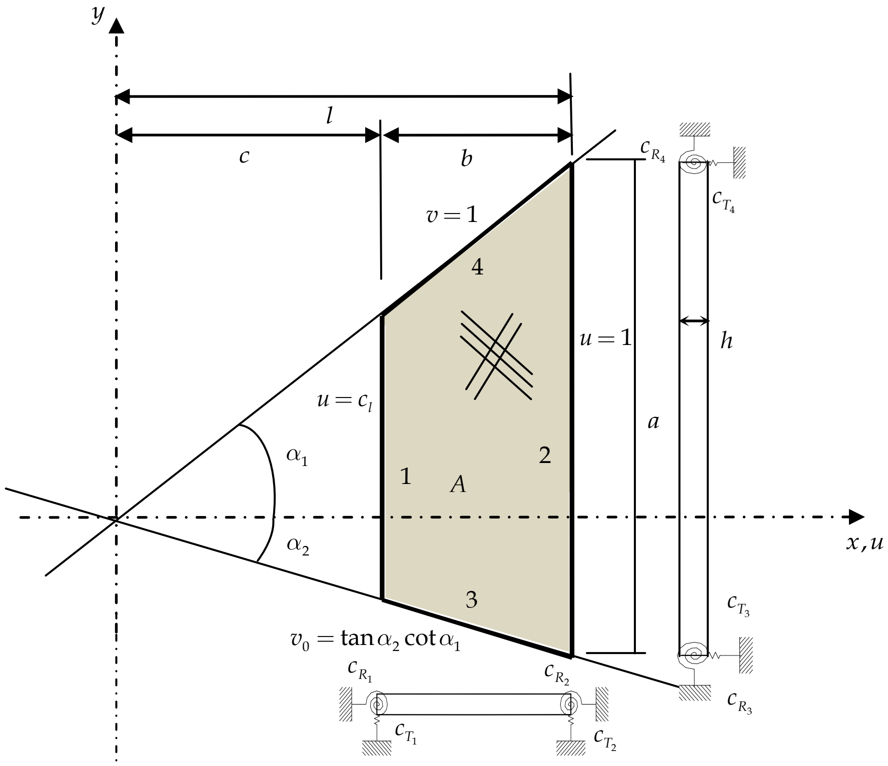
Fig. 2. Mechanical system. a) Elastic restraints and angle of fibers orientation. b) Profile and laminate stacking sequence.

Fig. 3. Effect of the fiber orientation on the fundamental frequency parameter $\Omega = \omega b^2 \sqrt{\rho h / D_0} / 2\pi$, $D_0 = E_2 h^3 / 12 \ 1 - \nu_{12}\nu_{21}$ of trapezoidal laminated $[\beta / -\beta / -\beta / \beta]$ plates $\tan\alpha_2 = -\tan\alpha_1 = 0.4, c_l = 0.2$ for two different thickness ratio h/b and two different values of the rotational restraint parameter R $R_i = R, T_i = \infty, i = 1,\ldots,4,$.

Fig. 4. First six values of the frequency parameter $\Omega^{**} = \omega l^2 \sqrt{\rho h / D_0}, D_0 = E_1 h^3 / 12 \ 1 - \nu_{12}\nu_{21}$ and mode shapes for an isosceles triangular plate $\alpha_1 = -\alpha_2 = 20°$, of a single layer boron-epoxy composite material $\beta = 20°$ with $h/b = 0.1$. Edge 1 elastically restrained against rotation $R_1 = 10, T_1 = \infty$, edge 2 free and edge 3 clamped $R_3 = T_3 = \infty$.

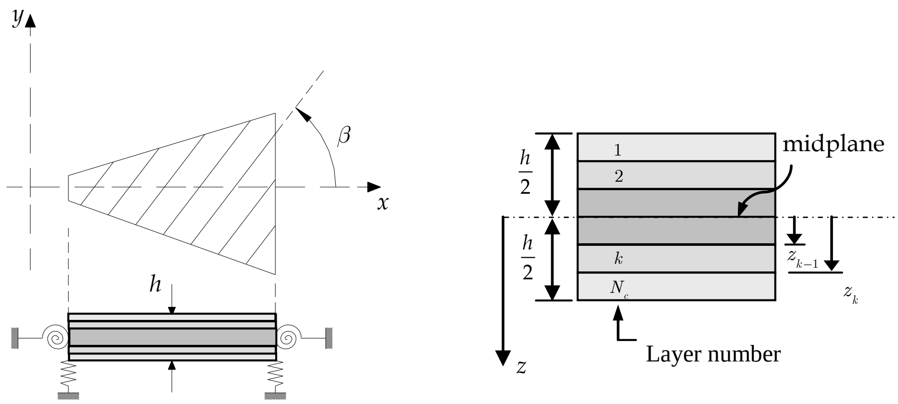
Fig. 5. First six values of the frequency parameter $\Omega^{**} = \omega l^2 \sqrt{\rho h / D_0}, D_0 = E_1 h^3 / 12 \ 1 - \nu_{12}\nu_{21}$ and mode shapes for a trapezoidal plate $\alpha_1 = -\alpha_2 = 20°, c_l = 0.2$, of a single layer boron-epoxy composite material $\beta = 20°$ with $h/b = 0.125$. Edges 1 and 3 free, edge 2 elastically restrained against rotation $R_2 = 10, T_2 = \infty$, edge 4 clamped $R_4 = T_4 = \infty$.

Fig. 1



y
l
c
b
c_{R_4}
c_{T_4}
$v=1$
4
$u=1$
h
$u=c_l$
a
α_1
1
A
2
α_2
x, u
3
c_{T_3}
$v_0=\tan\alpha_2\cot\alpha_1$
c_{R_1}
c_{R_2}
c_{R_3}
c_{T_1}
c_{T_2}

Fig. 2



y
β
x
h
$\frac{h}{2}$
1
2
midplane
$\frac{h}{2}$
k
z_{k-1}
N_c
z_k
Layer number
z

a) b)

Fig. 3

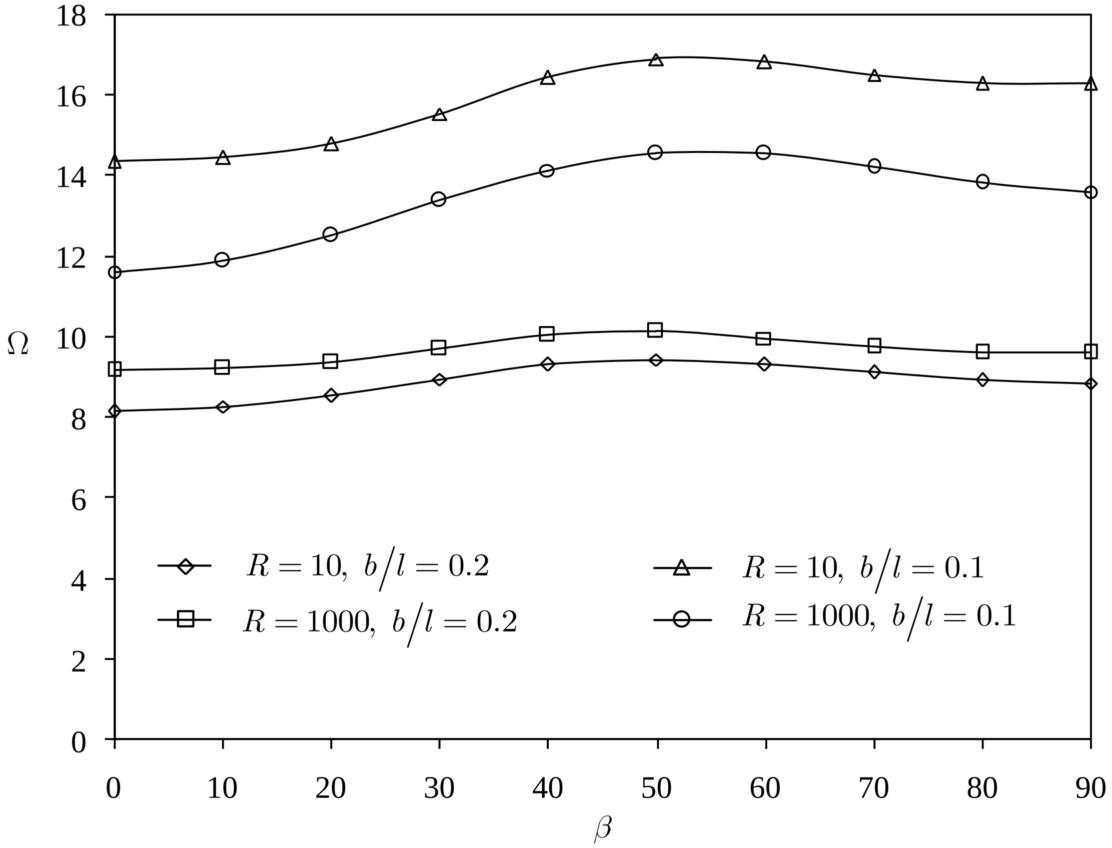

18
16
14
12
10
8
6
4
2
0
Ω
0
10
20
30
40
50
60
70
80
90
β
$R = 10,\ b/l = 0.2$
$R = 1000,\ b/l = 0.2$
$R = 10,\ b/l = 0.1$
$R = 1000,\ b/l = 0.1$

Fig. 4

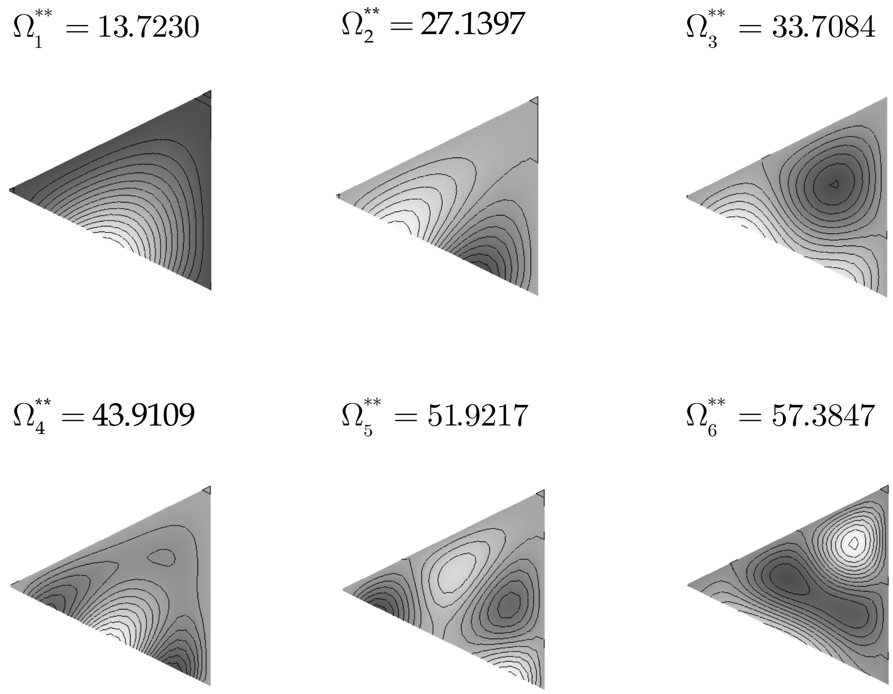
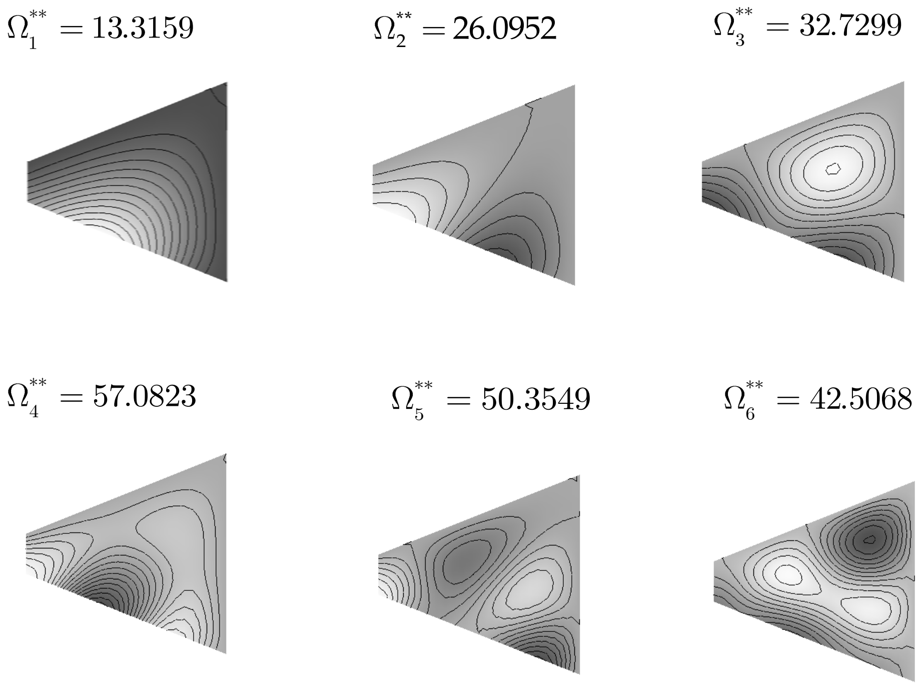

$\Omega_1^{**} = 13.7230$
$\Omega_2^{**} = 27.1397$
$\Omega_3^{**} = 33.7084$
$\Omega_4^{**} = 43.9109$
$\Omega_5^{**} = 51.9217$
$\Omega_6^{**} = 57.3847$

Fig. 5


$\Omega_1^{**} = 13.3159$
$\Omega_2^{**} = 26.0952$
$\Omega_3^{**} = 32.7299$
$\Omega_4^{**} = 57.0823$
$\Omega_5^{**} = 50.3549$
$\Omega_6^{**} = 42.5068$

Table(s)

List of Table

Table 1. Convergence study of the first five values of the frequency parameter $\Omega = \omega b^2 \sqrt{\rho h / D_0} / 2\pi$ for boron-epoxy $\beta = 45^{\circ}$ SSSS triangular $c_l = 0$ and trapezoidal $c_l = 0.3$ thick plate $\alpha_1 = -\alpha_2 = 20^{\circ}$.

Table 2. Dimensionless parameters of fundamental frequency $\Omega = \omega b^2 \sqrt{\rho h / D} / 2\pi$ for simply supported isotropic trapezoidal plates $\nu = 0.3, \kappa = 0.833, \tan \alpha_2 = -\tan \alpha_1$.

Table 3. Dimensionless parameters of fundamental frequency $\Omega^* = \omega b^2 \sqrt{\rho / E_2} / h$ for cross ply $[90 / 0 / 0 / 90]$ trapezoidal plates $\tan \alpha_2 = -\tan \alpha_1$.

Table 4. Values of natural frequency parameters $\Omega^{**} = \omega l^2 \sqrt{\rho h / D_0}$ for boron-epoxy triangular plates $\alpha_1 = -\alpha_2 = 15^{\circ}$, with their three edges elastically restrained against rotation and translation.

Table 5. Values of natural frequency parameter $\Omega = \omega b^2 \sqrt{\rho h / D_0} / 2\pi$, $D_0 = E_2 h^3 / 12\ 1 - \nu_{12}\nu_{21}$ for trapezoidal cross-ply laminated plates $\tan \alpha_2 = -\tan \alpha_1 = 0.4, c_l = 0.2$ with elastically restrained edges $T = T_1 = T_3 = T_4, T_2 = \infty, R_i = 0, i = 1, \ldots, 4$.

Table 1:

c_l	h/b	$M=N$	Ω_1	Ω_2	Ω_3	Ω_4	Ω_5
0	0.1	4	4,57615	7,86156	9,42001	11,84275	13,10450
		5	4,41491	7,34312	8,81653	11,29107	12,55692
		6	4,37969	7,00625	8,67149	10,35167	11,74176
		7	4,37223	6,93214	8,59743	9,86022	11,45510
		8	4,37051	6,90159	8,56727	9,74434	11,30503
		9	4,36996	6,89547	8,55656	9,65633	11,23624
0.3	0.143	4	2,18274	3,72624	4,54314	5,66471	6,30604
		5	2,16069	3,48989	4,27373	5,27008	5,92628
		6	2,15694	3,46432	4,24574	4,95582	5,67386
		7	2,15570	3,45332	4,23488	4,93235	5,62836
		8	2,15539	3,45142	4,23159	4,90765	5,60461
		9	2,15514	3,45095	4,23087	4,90531	5,59954

Table 2:

$\tan\alpha_1$	c_l	h/b	Present	Ref. [16]	Ref. [17]
0.4	0.2	0.01	6.0214	5.99	6.02
0.3	0.4		4.9160	4.90	4.91
0.2	0.6		4.0834	4.06	4.09
0.1	0.8		3.5199	3.51	3.52

Table 3:

$\tan\alpha_1$	c_l	h/b	SSSS			CCCC		
			Present	Ref. [21]	Ref. [20]	Present	Ref. [21]	Ref. [20]
0.4	0.2	0.1	27.4992	27.54	27.50	34.7488	34.76	34.74
0.3	0.4		23.8160	24.06	23.91	30.9378	31.08	30.95
0.2	0.6		20.3205	20.48	20.35	27.5372	27.62	27.53
0.1	0.8		17.3748	18.41	17.39	24.7568	25.12	24.73
0.4	0.2	0.2	17.5348	17.63	17.54	19.4716	19.51	19.45
0.3	0.4		15.3903	15.46	15.44	17.4697	17.56	17.45
0.2	0.6		13.4812	13.51	13.49	15.7885	15.88	15.76
0.1	0.8		11.9789	11.99	11.97	14.4980	14.59	14.46

Table 4.

h/b	β	$T_1 = T_2 = T_3$	Ω_1^{**}	Ω_2^{**}	Ω_3^{**}	Ω_4^{**}	Ω_5^{**}
0.1	0	$R_1 = R_2 = R_3 = 0$					
		10	8.8596	12.9869	13.1944	21.1688	25.6800
		50	16.6428	24.9218	27.2887	35.2862	37.8434
		100	20.4885	31.5216	35.4297	45.3232	45.3450
		1000	28.2317	48.1133	55.2800	67.2016	76.9101
		∞	29.6561	51.5719	59.4189	72.8520	84.4889
0.1	0	$R_1 = R_2 = R_3 = 1$					
		10	9.0223	14.0184	17.4090	26.7242	31.5405
		50	17.5331	25.7223	28.6378	38.4587	40.6999
		100	21.9828	32.4106	35.9700	45.9584	48.4773
		1000	31.6724	50.5033	56.7538	69.0430	77.6589
		∞	33.5935	54.6418	61.7398	75.5628	86.2242
0.1	0	$R_1 = R_2 = R_3 = 10$					
		10	9.0941	15.4940	19.4886	28.5265	38.0426
		50	18.1062	26.8141	29.3164	39.5838	44.8331
		100	23.1215	33.3869	36.2963	46.9344	51.1007
		1000	35.2387	52.8613	58.3365	70.9982	78.3059
		∞	37.9004	57.9476	64.3810	78.7914	87.8862
0.1	0	$R_1 = R_2 = R_3 = 100$					
		10	9.1062	15.9436	19.9217	29.1932	39.7094
		50	18.2196	27.1863	29.4455	40.0202	46.0003
		100	23.3725	33.6914	36.3565	47.2954	51.8967
		1000	36.2406	53.3853	58.8000	71.4490	78.4411
		∞	39.1749	58.7696	65.1827	79.6195	88.2719
0.2	0	$R_1 = R_2 = R_3 = 0$					
		10	8.43846	11.68331	11.95309	16.18746	20.16124
		50	14.2904	20.8137	22.3420	24.7234	28.7000
		100	16.3718	24.4269	24.4269	28.8137	32.7070

		1000	19.1780	29.5791	34.6356	37.6990	38.6119
		∞	19.5634	30.2703	35.5133	38.8310	39.6686
0.2	0	$R_1 = R_2 = R_3 = 1$					
		10	8.5791	12.8568	15.3042	20.6018	23.7328
		50	14.9078	21.4401	23.9642	29.1790	31.8166
		100	17.2746	25.0784	28.0774	33.2062	36.6520
		1000	20.5828	30.5738	34.7952	40.3052	45.8582
		∞	21.0475	31.3518	35.7947	41.4686	47.1516
0.2	0	$R_1 = R_2 = R_3 = 10$					
		10	8.6409	13.7729	16.2979	21.0267	26.1726
		50	15.2366	21.9583	24.3045	29.4527	33.0979
		100	17.8002	25.5285	28.2332	33.4969	37.3869
		1000	21.5070	31.2175	34.8829	40.8731	45.9550
		∞	22.0394	32.0661	35.9134	42.1279	47.4044
0.2	0	$R_1 = R_2 = R_3 = 100$					
		10	8.6511	14.0000	16.4498	21.1303	26.6210
		50	15.2921	22.0808	24.3501	29.5092	33.3274
		100	17.8937	25.6218	28.2540	33.5515	37.5098
		1000	21.6860	31.3318	34.9047	40.9802	45.9623
		∞	22.2340	32.1942	35.9423	42.2535	47.4143
0.1	30	$R_1 = R_2 = R_3 = 0$					
		10	8.73751	12.14035	13.23303	18.29803	26.45221
		50	16.7875	23.4818	27.7043	31.4541	39.4186
		100	20.9779	30.1204	36.3415	39.7185	48.1492
		1000	30.0225	47.5531	58.7151	65.8267	78.6892
		∞	31.8314	51.4623	63.8552	72.7126	87.2826
0.1	30	$R_1 = R_2 = R_3 = 1$					
		10	8.91611	13.22669	18.05694	21.99265	30.96322
		50	17.4849	24.2220	29.4137	33.7908	41.8628
		100	22.0766	22.0766	37.0513	41.3052	49.8181
		1000	32.6446	49.1759	59.8659	66.9877	79.4023
		∞	34.9160	53.6775	65.7739	74.7189	88.8657

0.1	30	$R_1 = R_2 = R_3 = 10$					
		10	8.98240	13.98581	20.62641	23.86004	35.16434
		50	17.9936	24.7733	30.4128	35.3900	44.3311
		100	23.0342	31.3161	37.5211	42.5245	51.3740
		1000	35.5067	50.9568	61.1492	68.2745	80.1734
		∞	38.3970	56.2707	68.0200	77.2354	90.6653
0.1	30	$R_1 = R_2 = R_3 = 100$					
		10	8.99299	14.18899	21.14055	24.27771	36.03792
		50	18.1018	24.9130	30.6242	35.7161	44.9263
		100	23.2532	31.4374	37.6237	42.7808	51.7734
		1000	36.2168	51.3848	61.4730	68.5849	80.3697
		∞	39.2721	56.9193	68.5943	77.8872	91.1171
0.2	30	$R_1 = R_2 = R_3 = 0$					
		10	8.34632	11.30107	11.75267	15.92984	18.42826
		50	14.4336	20.0515	22.6937	24.5901	25.8684
		100	16.7065	23.8385	27.5723	28.3677	30.7296
		1000	19.8761	29.4859	35.9567	37.6609	38.6158
		∞	20.3219	30.2607	36.9434	38.8309	40.0943
0.2	30	$R_1 = R_2 = R_3 = 1$					
		10	8.4942	12.2121	15.6017	18.2678	23.7010
		50	14.9239	20.6128	24.5005	27.2872	32.6665
		100	17.4013	24.3112	28.8761	31.7198	37.6163
		1000	20.9736	30.1320	36.2098	39.8301	46.0916
		∞	21.4886	30.9860	37.3155	41.1944	46.2905
0.2	30	$R_1 = R_2 = R_3 = 10$					
		10	8.54591	12.71674	16.77830	18.95096	25.09848
		50	15.2083	20.9008	24.9223	27.7358	33.3775
		100	17.8452	24.5718	29.0678	32.0244	38.0373
		1000	21.7381	30.5547	36.3341	40.1811	47.2048
		∞	22.3080	31.4654	37.4803	41.6183	48.4176
0.2	30	$R_1 = R_2 = R_3 = 100$					

		10	8.55396	12.83284	16.96173	19.06037	25.32183
		50	15.2590	20.9620	24.9853	27.8006	33.4988
		100	17.9264	24.6238	29.0965	32.0701	38.1057
		1000	21.8811	30.6324	36.3564	40.2459	47.2425
		∞	22.4616	31.5529	37.5093	41.6969	48.5827

Table 5

h/b	T	$[0/90/0]$	$[0/90/0/90/0]$	$[0/90/0/90/0/90/0]$
0.2	10	3.1046	3.0400	2.9843
	100	4.0786	4.0825	4.0539
	1000	6.9214	7.3599	7.4081
0.1	10	4.1300	3.9743	3.8506
	100	5.0290	4.9525	4.8645
	1000	9.0714	9.4264	9.4360
0.05	10	4.6432	4.4384	4.2744
	100	5.5095	5.3818	5.2555
	1000	10.1222	10.3558	10.3229
0.01	10	5.5095	4.7161	4.5334
	100	5.7555	5.6247	5.4802
	1000	10.5368	10.7234	10.6740
0.001	10	4.9448	4.7472	4.5621
	100	5.7779	5.6498	5.5032
	1000	10.5560	10.7427	10.693