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EXISTENCE OF OPTIMAL SUBSPACES IN REFLEXIVE BANACH SPACES

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Abstract. Given a finite set Y in a reflexive Banach space F and a family $\mathcal{C}$ of closed subspaces of F, we study the problem of finding a subspace V_0 in $\mathcal{C}$ that best approximates the data Y in the sense that $\sum_{f\in Y} d(f,V_0) = \min_{V\in\mathcal{C}} \sum_{f\in Y} d(f,V)$, where d is the distance function on F. In this paper, we give necessary conditions and sufficient conditions over $\mathcal{C}$ for which such a best approximation exists. In particular, when F has finite dimension a characterization on $\mathcal{C}$ is given.

1. Introduction

Let $(F, \|\cdot\|)$ be a reflexive Banach space and let $\mathcal{C}$ be a family of closed subspaces of F. Given a finite set $Y \subset F$, we consider the problem to find $V_0 \in \mathcal{C}$ minimizing

$$E(Y,V) := \sum_{f\in Y} d(f,V), \tag{1.1}$$

over $V \in \mathcal{C}$, where $d(g,V) = \inf_{h\in V} \|g-h\|$.

Kolmogorov was the first to address this type of questions in [10]. This problem recently regained attention due to its connection to signal processing (see for instance [1]-[4]). In [1], the authors introduced the following definition when F is a Hilbert space.

Definition 1.1. *A family of subspaces $\mathcal{C}$ of a Banach space F has the Minimum Subspace Approximation Property (MSAP) if for every finite subset $Y \subset F$ there*

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exists an element $V \in \mathcal{C}$ that minimizes (1.1). Such an element will be called an optimal subspace.

There are some cases for which it is known that the MSAP is satisfied. In [1, 9] the authors established that $\mathcal{C} = \{V \subset F : \dim(V) \leq m\}$ satisfies MSAP for $F = \mathbb{C}^d$ and $F = L^2(\mathbb{R}^d)$. For this family and a broad class of normed spaces, results in [6, 7] imply that $\mathcal{C}$ satisfies the MSAP.

Necessary and sufficient conditions on the family $\mathcal{C}$ of closed subspaces in a separable Hilbert space such that $\mathcal{C}$ satisfies MSAP were given in [4]. The technic used there consisted to identify $\mathcal{C}$ with the set of maps $I - P$, where I is the identity map and P is the orthogonal projection on $V \in \mathcal{C}$. In this way, using the weak operator topology, the problem was to characterize MSAP in terms of a compactness property.

In this paper we give a generalization of this approach to any reflexive Banach spaces, replacing orthogonal projections by any metric selection. In Section 4, we obtain several sufficient conditions for MSAP in terms of the weak operator topology. Here, we introduce the concept of contact set, which generalize the set of contact half-spaces introduced in [4]. In Section 5, we get a necessary condition for MSAP in terms of the strong operator topology. Further, in finite dimensional spaces we characterize families of subspaces $\mathcal{C}$ with MSAP.

2. Definitions and notations

In this section we give some definitions and notations necessary to develop this paper.

A map $T : F \to F$, is called *bounded* if there exists a constant K such that $\|T(h)\| \leq K\|h\|$ for all $h \in F$. We denote $\mathcal{B}(F)$ the set of all bounded maps. As usual, if $T \in \mathcal{B}(F)$ we define the norm of T by

$$\|T\| = \sup\left\{\frac{\|T(h)\|}{\|h\|} : h \in F, h \neq 0\right\}.$$

Clearly, $(\mathcal{B}(F), \|\cdot\|)$ is a normed space.

An element $v \in V \subset F$ is called a *best approximant of $f \in F$ from V* if $d(f, v) = d(f, V)$. As F is reflexive, then any closed subspace V is proximinal, i.e., for all $f \in F$ there exists a best approximant of f from V (see [12], p. 99). We recall that a *metric selection* P on V is any map from F onto V for which $P(f)$ is a best approximant to f from V (see [11], p. 25).

Now, the above problem can be reformulated in terms of certain maps in

$$\mathcal{B}_1(F) := \{T \in \mathcal{B}(F) : \|T\| \leq 1\}.$$

In fact, let I be the identity map on F and $V \in \mathcal{C}$. We consider

$$\Gamma_V := \{Q \in \mathcal{B}(F) : Q = I - P \text{ with } P \text{ a metric selection on } V\} \subset \mathcal{B}_1(F) \quad (2.1)$$

and

$$\pi(\mathcal{C}) := \{Q \in \Gamma_V : V \in \mathcal{C}\}.$$

If $Q \in \Gamma_V$, then $\|Q(f)\| = d(f, V) \leq \|f\|$, $f \in F$, so we can write $E(Y, V)$ in (1.1) as

$$E(Y, V) = \phi_Y(Q), \quad (2.2)$$

where $\phi_Y : \mathcal{B}(F) \to \mathbb{R}$ is defined by

$$\phi_Y(T) = \sum_{f \in Y} \|T(f)\|, \quad T \in \mathcal{B}(F). \tag{2.3}$$

The original problem is reduced to find $T_0 \in \pi(\mathcal{C})$ minimizing (2.3) over $\pi(\mathcal{C})$.

Since we are looking for problems of existence of minimizers, compactness in some topology will be of great help. For this purpose we introduce the weak operator topology (WOT) and the strong operator topology (SOT) in $\mathcal{B}(F)$. We denote F' the dual of F.

Definition 2.1. *Given a net $T_\alpha \in \mathcal{B}(F)$ and $T \in \mathcal{B}(F)$, we say that*

(a) T_α WOT-converges to T if and only if $T_\alpha(h)$ weakly converges to $T(h)$, for all $h \in F$ (i.e. $\varphi(T_\alpha(h)) \to \varphi(T(h))$, for all $h \in F$, $\varphi \in F'$).

(b) T_α SOT-converges to T if and only if $T_\alpha(h)$ strongly converges to $T(h)$, for all $h \in F$ (i.e. $\|T_\alpha(h) - T(h)\| \to 0$, for all $h \in F$).

Clearly, the SOT is stronger than the WOT. We observe that $\mathcal{B}(F)$ is a Hausdorff topological space with respect to each of these topologies.

3. Compactness of the unit ball

It is well known that the unit ball of the space of all linear continuous operators from F to F is WOT-compact when F is reflexive (see [8], p. 512). In this section, we show that if F is reflexive, $B_1(F)$ also is WOT-compact.

Let (X, τ) be a topological space and let $G : X \to \mathcal{B}(F)$ be a function. For each $\varphi \in F'$ and $h \in F$, we consider the function $H_{\varphi,h} : X \to \mathbb{R}$ defined by $H_{\varphi,h}(x) = \varphi(G(x)(h))$.

Lemma 3.1. *If $H_{\varphi,h}$ is a continuous function for all $\varphi \in F'$, $h \in F$, then $G : (X, \tau) \to (\mathcal{B}(F), \text{WOT})$ is a continuous function.*

Proof. Let $\{x_\alpha\}$ be a net in X such that $x_\alpha \to x \in X$. Given $h \in F$, by hypothesis we have $H_{\varphi,h}(x_\alpha) \to H_{\varphi,h}(x)$ for all $\varphi \in F'$, so $G(x_\alpha)(h)$ weakly converges to $G(x)(h)$. As h is arbitrary, $G(x_\alpha)$ WOT-converges to $G(x)$. □

Let $Z = \mathbb{R}^{F' \times F}$. For $w \in Z$, we write $w_{\varphi,f} = w(\varphi, f)$, $\varphi \in F'$, $f \in F$. Let τ_π be the product topology on Z, induced by the Euclidean topology in $\mathbb{R}$. Set $\Gamma : \mathcal{B}(F) \to Z$ defined by $\Gamma(T)_{\varphi,f} = \varphi(T(f))$.

Lemma 3.2. *The function $\Gamma : (B_1(F), \text{WOT}) \to (\Gamma(B_1(F)), \tau_\pi)$ is a homeomorphism.*

Proof. Clearly, Γ is a continuous function. On the other hand, if $\Gamma(T_1) = \Gamma(T_2)$, $T_1, T_2 \in \mathcal{B}(F)$, then $T_1 = T_2$. In fact, if $T_1(f) \neq T_2(f)$ for some $f \in F$, the Hahn–Banach Theorem implies there exists $\varphi \in F'$ such that $\varphi(T_1(f)) \neq \varphi(T_2(f))$, a contradiction. Finally, by Lemma 3.1 the inverse function of Γ is continuous. This finishes the proof. □

Theorem 3.3. *$B_1(F)$ is WOT-compact.*

Proof. By Lemma 3.2, it will be sufficient to prove that $\Gamma(B_1(F))$ is a τ_π-compact set. If $T \in B_1(F)$, $|\Gamma(T)_{\varphi,f}| \leq \|\varphi\|\|f\|$ for all $\varphi \in F'$, $f \in F$. In addition, $\Gamma(T)_{\varphi,f}$ is a linear function in the variable φ, for any f fix. So,

$$\begin{aligned}\Gamma(B_1(F)) \subset \{w \in Z : |w_{\varphi,f}| \leq \|\varphi\|\|f\|, \quad w_{\varphi+\psi,f} = w_{\varphi,f} + w_{\psi,f},\\ w_{\lambda\varphi,f} = \lambda w_{\varphi,f}, \text{ for all } \varphi,\psi \in F', f \in F, \lambda \in \mathbb{R}\} =: D\end{aligned} \tag{3.1}$$

Next, we will prove that the inclusion in (3.1) is an equality.
Let $w \in D$ and $f \in F$. We consider the function $\chi : F' \to \mathbb{R}$ defined by $\chi(\varphi) = w_{\varphi,f}$. As $|\chi(\varphi)| \leq \|\varphi\|\|f\|$, χ is a continuous function. Clearly, χ is linear, so $\chi \in F''$. As F is reflexive, there exists an element $T(f) \in F$ such that $\chi(\varphi) = \varphi(T(f))$, $\varphi \in F'$. We observe that $T \in B_1(F)$. In fact, $|\varphi(T(f))| \leq \|\varphi\|\|f\|$, $\varphi \in F'$, i.e., $\left|\frac{\varphi}{\|\varphi\|}(T(f))\right| \leq \|f\|$, $\varphi \in F'$, $\varphi \neq 0$. It is well known (see [5], p. 4) that

$$\|T(f)\| = \max\{|\psi(T(f))| : \|\psi\| \leq 1, \psi \in F'\},$$

thus $\|T(f)\| \leq \|f\|$. Since by definition of Γ, $\Gamma(T) = w$, we get $\Gamma(B_1(F)) = D$.
By Tihonov Theorem, the set

$$\begin{aligned}K_1 : &= \{w \in Z : |w_{\varphi,f}| \leq \|\varphi\|\|f\| \text{ for all } \varphi \in F', f \in F\}\\ &= \prod_{\varphi\in F', f\in F} [-\|\varphi\|\|f\|, \|\varphi\|\|f\|]\end{aligned}$$

is compact. On the other hand, for $f \in F$, the set

$$\begin{aligned}K_{2,f} :&= \{w \in Z : w_{\varphi,f} \text{ is a linear function respect to } \varphi\}\\ &= \bigcap_{\varphi,\psi\in F'} \{w \in Z : w_{\varphi+\psi,f} - w_{\varphi,f} - w_{\psi,f} = 0\}\\ &\cap \bigcap_{\varphi\in F',\lambda\in\mathbb{R}} \{w \in Z : w_{\lambda\varphi,f} - \lambda w_{\varphi,f} = 0\}\end{aligned}$$

is τ_π-closed, because it is an intersection of τ_π-closed sets. Finally, $\Gamma(B_1(F)) = K_1 \cap \bigcap_{f\in F} K_{2,f}$ is τ_π-closed and consequently it is τ_π-compact. □

4. Sufficient conditions for MSAP

In this section we give sufficient conditions for MSAP on families $\mathcal{C}$ of closed subspaces in a reflexive Banach space.

Lemma 4.1. *For each finite set $Y \subset F$, the function $\phi_Y : (\mathcal{B}(F), \text{WOT}) \to \mathbb{R}$, defined in (2.3), is convex and lower semicontinuous.*

Proof. Since the norm function is convex, ϕ_Y is a convex function. Let $T_\alpha, T \in \mathcal{B}(F)$ be such that T_α WOT-converges to T. Since $T_\alpha(f)$ weakly converges to $T(f)$ for all $f \in Y$ and the norm is a lower weak semicontinuous function, we have $\|T(f)\| \leq \liminf_\alpha \|T_\alpha(f)\|$. So, ϕ_Y is lower semicontinuous. □

Let (X, τ) be a Hausdorff topological space, let $\phi : X \to \mathbb{R}$ be lower semicontinuous function, and let K be a compact subset of X. By Weierstrass Theorem it is well known that ϕ achieves its infimum over K. Since $(\mathcal{B}(F), \text{WOT})$ is a Hausdorff topological space, the next theorem follows from Theorem 3.3 and Lemma 4.1.

Theorem 4.2. *Let $Y \subset F$ be a finite set and $\mathcal{M} \subset B_1(F)$. If $\mathcal{M}$ is* WOT*-closed, then the function ϕ_Y achieves its infimum over $\mathcal{M}$. Consequently, if $\pi(\mathcal{C})$ is* WOT*-closed, then $\mathcal{C}$ satisfies MSAP.*

It is not necessary that $\pi(\mathcal{C})$ is WOT-closed to get MSAP, as it shows the following example given in ([4], p. 369).

Example 4.3. Let F be the Hilbert space $\mathbb{R}^3$ and $\mathcal{C} = \{\text{span}\{e_1, e_2\}\} \cup \{\text{span}\{e_3 + c\, e_2\} : c \in \mathbb{R}\}$. This family $\mathcal{C}$ has MSAP, but $\pi(\mathcal{C})$ is not WOT-closed.

With the purpose to obtain another result over families which satisfy MSAP, we need to introduce the following sets. Given $k \in \mathbb{N}$ and a family $\mathcal{C}$ of closed subspaces, let

$$\pi(\mathcal{C})^* := \{Q \in \Gamma_V : \exists\, W \in \mathcal{C},\, V \subset W \text{and } V \text{ is a closed subspace}\},$$

where Γ_V was defined in (2.1), and let

$$\pi(\mathcal{C})^*_k := \{Q \in \Gamma_V : \exists\, W \in \mathcal{C},\, V \subset W \text{and } \dim(V) \le k\} \subset \pi(\mathcal{C})^*.$$

Lemma 4.4. *Let $Y \subset F$ be such that $\#(Y) = k \in \mathbb{N}$. Then $\inf\limits_{Q \in \pi(\mathcal{C})} \phi_Y(Q) = \inf\limits_{T \in \pi(\mathcal{C})^*_k} \phi_Y(T)$.*

Proof. We denote $a = \inf\limits_{Q \in \pi(\mathcal{C})} \phi_Y(Q)$ and $b = \inf\limits_{T \in \pi(\mathcal{C})^*_k} \phi_Y(T)$. By the definition of $\pi(\mathcal{C})^*_k$, it follows immediately that $a \le b$. Now, given $W \in \mathcal{C}$ and P a selection metric on W, it is easy to see that if $V := \text{span}\{P(f) : f \in Y\}$ and $Q \in \Gamma_V$, then $Q \in \pi(\mathcal{C})^*_k$ and $b \le \phi_Y(Q) \le \phi_Y(I - P)$. Therefore $b \le a$. □

We will need the following lemma which was proved in ([12], p. 273).

Lemma 4.5. *Let F be a Banach space of dimension n. Then there exist n linearly independent elements $e_1, \cdots, e_n \in F$ and n functionals $g_1, \cdots, g_n \in F'$ such that $\|e_k\| = \|g_k\| = 1$, $g_i(e_k) = 1$ if $i = k$, and $g_i(e_k) = 0$ if $i \ne k$, $1 \le i, k \le n$. Consequently, for every $e = \sum\limits_{i=1}^{n} \alpha_i e_i \in F$ we have then $|\alpha_i| \le \|e\|$, $1 \le i \le n$.*

Proposition 4.6. *Let $k \in \mathbb{N}$ and $\mathcal{D}$ be a family of subspaces of F with dimension at most k. If $T \in \overline{\pi(\mathcal{D})}^{\text{WOT}}$ then there exists a subspace $V \subset F$ such that $\dim(V) \le k$ and $(I - T)(F) \subset V$.*

Proof. By hypothesis, there is a net $\{V_\alpha\} \subset \mathcal{D}$ such that $Q_\alpha = I - P_\alpha \in \Gamma_{V_\alpha}$ WOT-converges to T. As $\dim(V_\alpha) \le k$ for all α, there exists a subsequence of $\{V_\alpha\}$, say $\{V_{\alpha_s}\}$, such that $\dim(V_{\alpha_s}) = r \le k$. By Lemma 4.5, for each $s \in \mathbb{N}$ there exists a basis $\{e_{si}\}_{i=1}^r$ of V_{α_s} such that $\|e_{si}\| = 1$, and $|c_i| \le \|g\|$ for $g = \sum\limits_{i=1}^{r} c_i e_{si}$. Since $\|e_{si}\| = 1$, there are a subsequence of $\{V_{\alpha_s}\}$, which we denote in the same way, and $e_j \in F$, $1 \le j \le r$, such that e_{sj} weakly converges to e_j, as $s \to \infty$ (see [5], p. 50). Set $W := \text{span}\{e_1, \dots, e_r\}$. Given $h \in F$, we write $P_{\alpha_s}(h) = \sum\limits_{i=1}^{r} c_{si}(h) e_{si}$. It is easy to see that $|c_{si}(h)| \le 2\|h\|$. Then there exists a subsequence of $\{V_{\alpha_s}\}$, which we denote in the same way, and $c_i(h) \in \mathbb{R}$, $1 \le i \le r$, such that $\lim_{s\to\infty} c_{si}(h) = c_i(h)\|h\|$. So, $P_{\alpha_s}(h)$ weakly converges to $\sum_{i=1}^{r} c_i(h)\|h\|e_i$, as $s \to \infty$, and $h - T(h) = \sum_{i=1}^{r} c_i(h)\|h\|e_i$. As h is arbitrary, $(I-T)(F) \subset W$, with $\dim(W) \le r \le k$. □

The following definition extends the concept of contact half-space introduced in ([4], Definition 2.7).

Definition 4.7. *Let $\mathcal{E} \subset \mathcal{B}(F)$. A set $\mathcal{M} \subset \mathcal{B}(F)$ will be called a contact set of $\mathcal{E}$ if $\mathcal{E} \subset \mathcal{M}$ and $\mathcal{E} \cap \partial(\mathcal{M}) \neq \emptyset$, where ∂ is the* WOT*-boundary. We denote $\mathcal{T}(\mathcal{E})$ the collection of the contact sets of $\mathcal{E}$.*

Given $a \in \mathbb{R}$ and a function $\phi : \mathcal{B}(F) \to \mathbb{R}$, we write

$$\mathcal{H}_{\phi,a} := \{T \in \mathcal{B}(F) : \phi(T) \ge a\}.$$

Lemma 4.8. *Let $\mathcal{E} \subset \mathcal{B}(F)$ and let $\phi : (\mathcal{B}(F), \text{WOT}) \to \mathbb{R}$ be a convex and lower semicontinuous function. Then ϕ has a minimum a on $\mathcal{E}$ if and only if $\mathcal{H}_{\phi,a} \in \mathcal{T}(\mathcal{E})$.*

Proof. Assume $a = \min_{T\in\mathcal{E}} \phi(T)$, hence $\mathcal{E} \subset \mathcal{H}_{\phi,a} =: \mathcal{H}$. Let $T_0 \in \mathcal{E}$ be such that $\phi(T_0) = a$ and $T_1 \notin \mathcal{H}$. Writing $T_\alpha = \alpha T_1 + (1-\alpha)T_0$, $0 < \alpha \le 1$, we have T_α WOT-converges to T_0, as $\alpha \to 0$. But $T_\alpha \notin \mathcal{H}$ since $\phi(T_\alpha) \le \alpha\phi(T_1) + (1-\alpha)\phi(T_0) < a$. Then $T_0 \in \partial(\mathcal{H})$ and therefore $\mathcal{H}$ is a contact set of $\mathcal{E}$.
Suppose $\mathcal{H} \in \mathcal{T}(\mathcal{E})$, then $\phi(T) \ge a$, for all $T \in \mathcal{E}$. By assumption there exist a net $\{T_\alpha\} \subset \mathcal{B}(F) \setminus \mathcal{H}$ and $T_0 \in \mathcal{E}$ such that T_α WOT-converges to T_0. From the lower semicontinuity of ϕ we get $a \le \phi(T_0) \le \liminf_{\alpha\to 0} \phi(T_\alpha) \le a$. So, $\phi(T_0) = \min_{T\in\mathcal{E}} \phi(T) = a$. □

In the following theorem we give sufficient conditions for MSAP.

Theorem 4.9. *We consider the following statements:*
(a) $\pi(\mathcal{C})^*$ *is* WOT*-closed;*
(b) $\pi(\mathcal{C})^*_k$ *is* WOT*-closed for all* $k \in \mathbb{N}$*;*
(c) $\mathcal{T}\left(\pi(\mathcal{C})^*_k\right) = \mathcal{T}\left(\overline{\pi(\mathcal{C})^*_k}^{\text{WOT}}\right)$ *for all* $k \in \mathbb{N}$*;*
(d) $\mathcal{C}$ *satisfies MSAP.*
It verifies $(a) \Rightarrow (b) \Rightarrow (c) \Rightarrow (d)$*.*

Proof. $(a) \Rightarrow (b)$. Let $k \in \mathbb{N}$ and $T \in \overline{\pi(\mathcal{C})^*_k}^{\text{WOT}}$. As $\overline{\pi(\mathcal{C})^*_k}^{\text{WOT}} \subset \overline{\pi(\mathcal{C})^*}^{\text{WOT}} = \pi(\mathcal{C})^*$, there exist a closed subspace S of F and $W \in \mathcal{C}$ such that $T \in \Gamma_S$ and $S \subset W$. Since $(I-T)(F) = S$, Proposition 4.6 implies that $\dim(S) \le k$ and consequently $T \in \pi(\mathcal{C})^*_k$.
$(b) \Rightarrow (c)$ is obvious.
$(c) \Rightarrow (d)$. Let $Y \subset F$ be a finite set and let $\#(Y) = k$. From Theorem 4.2 there exists $T_0 \in \overline{\pi(\mathcal{C})^*_k}^{\text{WOT}} \subset B_1(F)$ such that $\phi_Y(T_0) = \min_{T\in\overline{\pi(\mathcal{C})^*_k}^{\text{WOT}}} \phi_Y(T) =: a$. From Lemma 4.8, $\mathcal{H}_{\phi_Y,a} \in \mathcal{T}\left(\overline{\pi(\mathcal{C})^*_k}^{\text{WOT}}\right)$. Since $\mathcal{T}\left(\pi(\mathcal{C})^*_k\right) = \mathcal{T}\left(\overline{\pi(\mathcal{C})^*_k}^{\text{WOT}}\right)$, again

Lemma 4.8 implies that ϕ_Y achieves the minimum on $\pi(\mathcal{C})^*_k$, i.e., there exist a subspace S of F, $W \in \mathcal{C}$ and an map $\overline{Q}$ in Γ_S such that $\phi_Y(\overline{Q}) = a$, $S \subset W$ and $\dim(S) \leq k$. Thus, $\|Q(f)\| \leq \|\overline{Q}(f)\|$, $f \in Y$, where $Q \in \Gamma_W$, and so $\phi_Y(Q) \leq a$. By Lemma 4.4 we get, $\phi_Y(Q) = \min_{T\in\pi(\mathcal{C})} \phi_Y(T)$. □

The implication $(d) \Rightarrow (b)$ it is not true in general, as it shows the next example. However, if F has finite dimension, we will prove in Theorem 5.4 that (b) and (d) are equivalent.

Example 4.10. Let F be the Hilbert space $l^2(\mathbb{R})$ and $m \in \mathbb{N}$. We consider the family $\mathcal{C} = \{V \subset F : V \neq 0 \text{ and } \dim\,(V) \leq m\}$. For each $k \in \mathbb{N}$, $\pi(\mathcal{C})^*_k$ is not WOT-closed. On the contrary from ([4], Proposition 3.1), $I \in \pi(\mathcal{C})^*_k \subset \pi(\mathcal{C})$, a contradiction. However, as it was showed in ([7], p. 92), $\mathcal{C}$ satisfies *MSAP*.

5. Necessary conditions for MSAP

In this section we give a necessary condition for MSAP on families $\mathcal{C}$ of closed subspaces in a reflexive Banach space. Further, a characterization for MSAP in finite dimensional spaces is proved.

Lemma 5.1. *Let $Y \subset F$ be such that $\#(Y) = k \in \mathbb{N}$. Then*

$$\inf_{Q\in\pi(\mathcal{C})^*_k} \phi_Y(Q) = \inf_{T\in\overline{\pi(\mathcal{C})^*_k}^{\text{SOT}}} \phi_Y(T).$$

Proof. Since $\phi_Y : (\mathcal{B}(F), \text{SOT}) \to \mathbb{R}$ is a continuous function, the lemma immediately follows. □

Proposition 5.2. *Let $k \in \mathbb{N}$ and $\mathcal{D}$ a family of subspaces of F with dimension at most k. If $T \in \overline{\pi(\mathcal{D})}^{\text{SOT}}$ then $T \in \Gamma_V$, where $V = span\{h - T(h) : h \in F\}$ and $dim(V) \leq k$.*

Proof. By hypothesis, there is a net $\{V_\alpha\} \subset \mathcal{D}$ such that $Q_\alpha = I - P_\alpha \in \Gamma_{V_\alpha}$ SOT-converges to T, i.e.,

$$\lim_\alpha \|Q_\alpha(h) - T(h)\| = \lim_\alpha \|h - T(h) - P_\alpha(h)\| = 0, \quad h \in F. \tag{5.1}$$

For $h \in F$, let $a = d(h, V)$. There is a sequence $\{h_n\} \subset V$ such that $\|h - h_n\| < a + \frac{1}{n}$. We write $h_n = \sum_{i=1}^{s_n} c_{ni}(h_{ni} - T(h_{ni}))$. Since

$$\|Q_\alpha(h)\| = \|h - P_\alpha(h)\| \leq \left\| h - \sum_{i=1}^{s_n} c_{ni} P_\alpha(h_{ni}) \right\|,$$

from (5.1) we have

$$\begin{aligned}\|h - (h - T(h))\| = \lim_\alpha \|Q_\alpha(h)\| &\leq \left\| h - \sum_{i=1}^{s_n} c_{ni}(h_{ni} - T(h_{ni})) \right\| \\ &= \|h - h_n\| < a + \frac{1}{n}.\end{aligned}$$

Letting $n \to \infty$, we have $\|h - (h - T(h))\| \leq a$. Since $h - T(h) \in V$, $h - T(h)$ is a best approximant to h from V. As $h \in F$ is arbitrary, $P = I - T$ is a metric selection on V, and $T \in \Gamma_V$. Clearly $\overline{\pi(\mathcal{D})}^{\text{SOT}} \subset \overline{\pi(\mathcal{D})}^{\text{WOT}}$, then $T \in \overline{\pi(\mathcal{D})}^{\text{WOT}}$. Proposition 4.6 implies that there exists a subspace $V_1 \subset F$ such that $\dim(V_1) \leq k$ and $P(F) \subset V_1$. Now, $P(F) = V$, so $\dim(V) \leq k$. □

In the following theorem we give a necessary condition for MSAP.

Theorem 5.3. *If $\mathcal{C}$ satisfies MSAP, then $\pi(\mathcal{C})^*_k$ is* SOT*-closed for all $k \in \mathbb{N}$.*

Proof. Let $k \in \mathbb{N}$ and $T_0 \in \overline{\pi(\mathcal{C})^*_k}^{\text{SOT}}$. By Proposition 5.2 we have $T_0 \in \Gamma_V$ where $V = \operatorname{span}\{h - T_0(h) : h \in F\}$ and $\dim(V) \leq k$. Let $Y = \{f_1, \ldots, f_r\}$ be a basis of V. Since $\phi_Y(T_0) = \sum_{i=1}^{r} \|T_0(f_i)\| = 0$, from Lemmas 4.4 and 5.1, $\inf_{Q \in \pi(\mathcal{C})} \phi_Y(Q) = \inf_{T \in \overline{\pi(\mathcal{C})^*_k}^{\text{SOT}}} \phi_Y(T) = 0$. As $\mathcal{C}$ satisfies MSAP, there is $W \in \mathcal{C}$ such that $\phi_Y(\overline{Q}) = 0$, with $\overline{Q} \in \Gamma_W$. Thus $Y \subset W$, and consequently $V \subset W$. Therefore $T_0 \in \pi(\mathcal{C})^*_k$. □

Now, we give a characterization of families with the MSAP in finite dimensional spaces. We remark that it generalizes ([4], Theorem 2.4).

Theorem 5.4. *Let F be a normed space such that $\dim(F) = n$. Then $\mathcal{C}$ satisfies MSAP if and only if $\pi(\mathcal{C})^*_n$ is* WOT*-closed.*

Proof. Assume $\mathcal{C}$ satisfies MSAP. By Theorem 5.3, $\pi(\mathcal{C})^*_n$ is SOT-closed. Since $\dim(F) = n$ we have SOT = WOT. Therefore $\pi(\mathcal{C})^*_n$ is WOT-closed.
Conversely, let $Y \subset F$ be a finite set and suppose $\pi(\mathcal{C})^*_n$ WOT-closed. From Theorem 4.2, there exists $Q \in \pi(\mathcal{C})^*_n$ such that $\phi_Y(Q) = \min_{T \in \pi(\mathcal{C})^*_n} \phi_Y(T)$. By definition of $\pi(\mathcal{C})^*_n$ there exist two subspaces of F, say V and W, with $V \subset W$, $W \in \mathcal{C}$, and $Q \in \Gamma_V$. Therefore, for $\overline{Q} \in \Gamma_W$, we have $\phi_Y(\overline{Q}) \leq \phi_Y(Q) \leq \phi_Y(T)$, for all $T \in \pi(\mathcal{C})^*_n$. Since $\pi(\mathcal{C}) \subset \pi(\mathcal{C})^*_n$, we get $\phi_Y(\overline{Q}) = \min_{T \in \pi(\mathcal{C})} \phi_Y(T)$. □

As a consequence of Theorems 4.9 and 5.4 we obtain the following theorem.

Theorem 5.5. *Let F be a normed space of finite dimension. Then statements $(a), (b), (c)$ and (d) of Theorem 4.9 are equivalents.*

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