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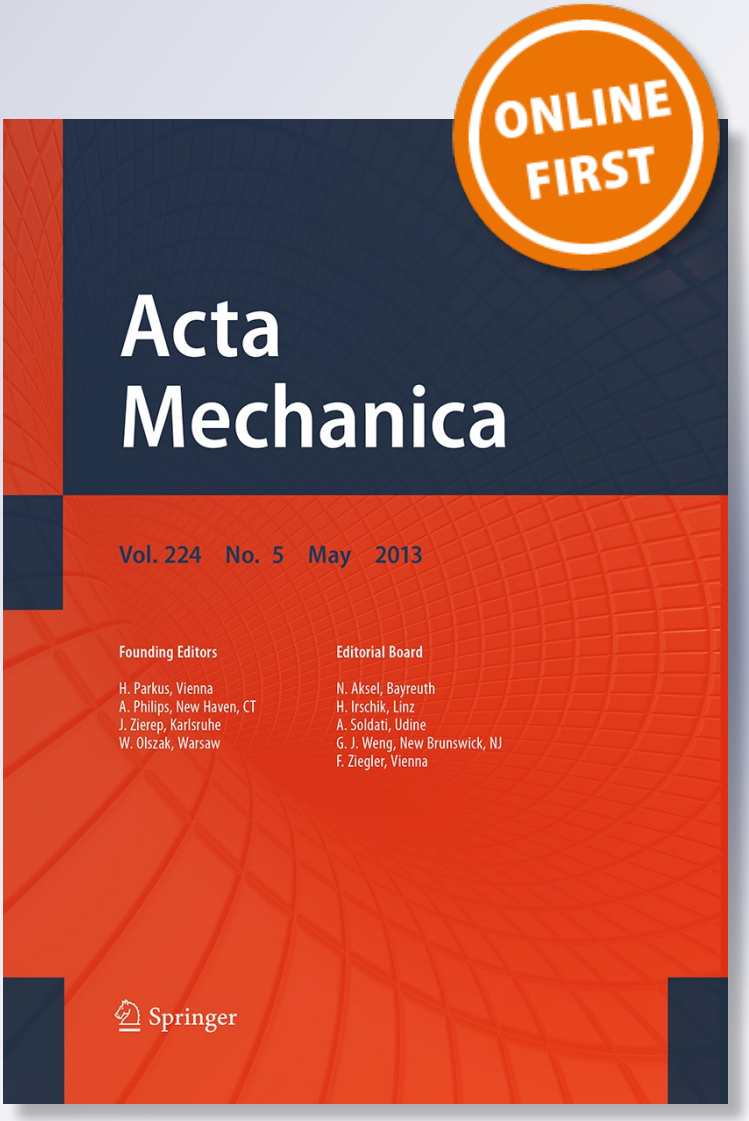

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Abstract This paper deals with the free transverse vibration of anisotropic plates with several arbitrarily located internal line hinges and piecewise smooth boundaries, elastically restrained against rotation and translation. The equations of motion and the associated boundary and transition conditions are derived using Hamilton's principle in a rigorous framework. A new analytical manipulation based on a condensed notation is used to compact the corresponding analytical expressions. A combination of the Ritz method and the Lagrange multipliers method with polynomials as coordinate functions is used to obtain tables of the nondimensional frequencies and the corresponding mode shapes, for rectangular plates with different boundary conditions and restraint conditions in the internal line hinges. The cases not previously treated of two- and three line hinges are particularly analyzed.

1 Introduction

Substantial literature has been devoted to the formulation, by means of the calculus of variations of boundary value problems of mathematical physics [1–14]. Several books treated the study of isotropic and anisotropic plates including the determination of static, buckling, and vibrations characteristics [15–21]. It is not the intention to review the literature consequently; only some of the published papers related to the present work will be cited. A great number of articles treated the dynamical behavior of plates with complicating effects such as elastically restrained boundaries, presence of elastically or rigidly connected masses, variable thickness, anisotropic material, points and lines supports, presence of holes, etc. A review of the literature has shown that there is only a limited amount of information for the vibration of plates with line hinges. A line hinge in a plate can be used to facilitate folding of gates and to simulate a through crack along the interior of the plate, among other applications. Wang et al. [22] studied the buckling and vibration of plates with an internal line hinge by using the Ritz method. Gupta and Reddy [23] presented the exact buckling loads and vibration frequencies of orthotropic rectangular plates with an internal line hinge by employing an analytical method which applies the Levy solution and the domain decomposition technique. Xiang and Reddy [24] provided the first-known solutions based on the first-order shear deformation theory for vibration of rectangular plates with an internal

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line hinge. The Lévy method and the state-space technique were employed to solve the vibration problem. Huang et al. [25] developed a discrete method for analyzing the free vibration problem of thin and moderately thick rectangular plates with a line hinge and various classical boundary conditions. Quintana and Grossi [26] dealt with the study of free transverse vibrations of isotropic rectangular plates with an internal line hinge and elastically restrained boundaries. The problem was solved employing a combination of the Ritz method and the Lagrange multiplier method.

All of these studies have considered rectangular plates with only one free internal line hinge. However, there is no previous study for the vibration of anisotropic plates with generally restrained piecewise smooth boundaries and with several internal line hinges, elastically restrained against rotation and translation.

Engineers and applied mathematicians increasingly used the techniques of calculus of variations to solve a large number of problems, and in this discipline, the "operator" δ has been assigned special properties and handled using heuristic procedures. Commonly, the domain of definition of a functional and the space of admissible directions of the variation of this functional are not clearly stated; thus, most of the analytical manipulations are confusing and not mathematically precise. Grossi [27] formulated an analytical model for the dynamic behavior of anisotropic plates, with an arbitrarily located internal line hinge and piecewise smooth boundaries among other complicating effects. By introducing an adequate change of variables, the energies which correspond to the different elastic restraints were handled in a rigorous framework. In the same manner, in the present paper, a complete rigorous application of the Hamilton's principle is developed for the derivation of equations of motion and its associated boundary and transition conditions, for anisotropic plates with several arbitrarily located internal line hinges and piecewise smooth boundaries, elastically restrained against rotation and translation. Also, a methodology based on a combination of the Ritz method and the Lagrange multipliers method with polynomials as coordinate functions is used to investigate the natural frequencies and mode shapes. To demonstrate the validity and efficiency of the proposed algorithm, results of a convergence study are included, several numerical examples not previously treated are presented, and some particular cases are compared with results obtained by other authors. Tables are given for frequencies, and two-dimensional plots for mode shapes are included.

The consideration of special characteristics such as elastically restrained piecewise smooth boundary, variable thickness, anisotropic material and particularly several internal line hinges, leads to complicated analytical expressions and tedious algebraic manipulations. For this reason, in this paper, a new analytical manipulation based on a condensed notation is used. The compact analytical expressions substantially lower the analytical effort and the amount of information.

This paper is organized in the following way. In Sect. 2, a rigorous treatment of techniques of the calculus of variations to obtain the governing differential equations and the boundary conditions is presented. In Sect. 3, the transition conditions at the line hinges are determined. In Sect. 4, a combination of the Ritz method and the Lagrange multipliers method is used for the determination of frequencies and mode shapes of rectangular anisotropic plates. Verifications and numerical applications are also included. Finally, Sect. 5 contains the conclusions of this paper.

2 The variation of the energy functional

Let us consider an anisotropic plate that in the equilibrium position covers the two-dimensional domain G, with piecewise smooth boundary ∂G elastically restrained against rotation and translation. The plate has $N-1$ intermediate line hinges elastically restrained against rotation and translation, as it is shown in Fig. 1. In order to analyze the transverse displacements of the system under study, we suppose that the vertical position of the plate at any time t is described by the function $w = w(x, t)$, where $x = (x_1, x_2) \in \bar{G}, \quad \bar{G} = G \cup \partial G$ and that the domain G is divided by the lines $\Gamma^{(c_k)},\ k = 1, 2, \ldots, N-1$, into N parts $G^{(k)}$ with boundaries $\partial G^{(k)}$, (see Fig. 1). Different rigidities and mass density of the anisotropic material correspond to each subdomain $G^{(k)}$. The extreme points a_k and b_k of the lines $\Gamma^{(c_k)},\ k = 1, 2, \ldots, N-1$, divide the boundary curve ∂G such that (see Fig. 2):

$$\partial G = \Gamma^{(1)} \cup \Gamma^{(2)} \cdots \cup \Gamma^{(N)} \tag{1}$$

and

$$\Gamma^{(k)} = \partial G^{(k)} - \Gamma^{(c_{k-1})} - \Gamma^{(c_k)}, \quad k = 1, 2, \ldots, N, \tag{2}$$

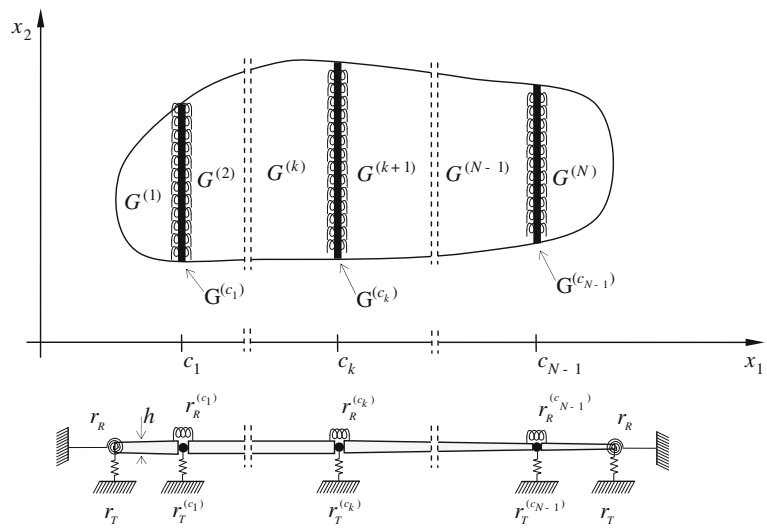


Fig. 1 Mechanical system under study

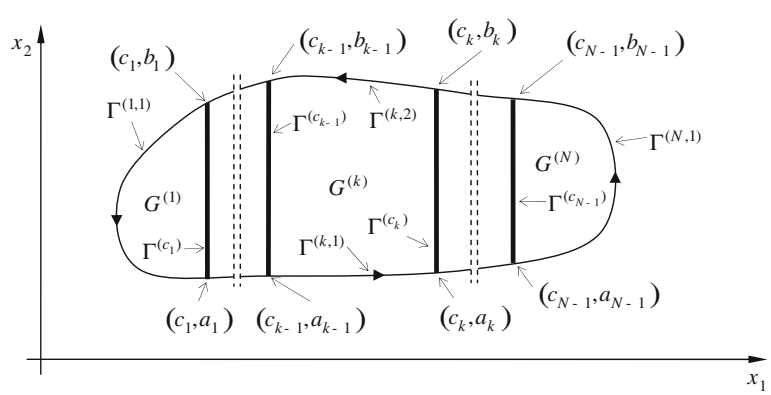


Fig. 2 Domains and boundaries

where

$$\Gamma^{(k)} = \Gamma^{(k,1)} \cup \Gamma^{(k,2)}, \quad k = 1, 2, \ldots, N \tag{3}$$

with $\Gamma^{(c_0)} = \Gamma^{(c_N)} = \Gamma^{(1,2)} = \Gamma^{(N,2)} = \emptyset$, where equality is understood in the sense of set theory and $\emptyset$ denotes the empty set.

Let us assume that the boundary curve ∂G is described by a smooth path γ in $\mathbb{R}^2$ defined in the compact interval $[0, l]$, where $l = l(\partial G)$ is the length of the path γ. The image of $[0, l]$ under γ (the graph of γ) is the boundary curve ∂G and will be denoted by $im(\gamma)$. We also assume that the curves $\Gamma^{(k)}$ given by Eq. (3) are described, respectively, by the smooth paths

$$\gamma^{(k,m)} : \left[0, l^{(k,m)}\right] \to \mathbb{R}^2$$

with

$$\gamma^{(k,m)}(s) = \left(\gamma_1^{(k,m)}(s), \gamma_2^{(k,m)}(s)\right), s \in \left[0, l^{(k,m)}\right], \quad m = 1, 2, \quad k = 1, 2, \ldots, N, \tag{4}$$

where $l^{(k,m)} = l\left(\Gamma^{(k,m)}\right)$ is the length of the path $\gamma^{(k,m)}$ and $l^{(1,2)} = l^{(N,2)} = 0$. From (3) and (4), it follows that $\gamma^{(k,m)}$ describes the arc $\Gamma^{(k,m)}$. For $k = 2, \ldots, N$, in Eq. (4), s denotes the arc length measured from the point (c_{k-1}, a_{k-1}) of the curve $\Gamma^{(k,1)}$ and from the point (c_k, b_k) of the curve $\Gamma^{(k,2)}$,

where $a_{k-1} = \gamma_2^{(k-1,1)}(l^{(k-1,1)})$, $\quad b_k = \gamma_2^{(k,2)}(0)$ (see Fig. 2). In the case of the curve $\Gamma^{(1,1)}$, the arc length is measured from the point (c_1, b_1). The path which describes the boundary ∂G can be expressed as:

$$\gamma(s) = \begin{cases} \gamma^{(1,1)}(s) & \text{if } s \in \left[0, l^{(1,1)}\right], \\ \gamma^{(2,1)}\left(s - l^{(1,1)}\right) & \text{if } s \in \left[l^{(1,1)}, l^{(1,1)} + l^{(2,1)}\right], \\ \dots & \\ \gamma^{(N-1,1)}\left(s - A_{N-2}\right) & \text{if } s \in \left[A_{N-2}, A_{N-1}\right], \\ \gamma^{(N,1)}\left(s - A_{N-1}\right) & \text{if } s \in \left[A_{N-1}, A_N\right], \\ \gamma^{(N-1,2)}\left(s - A_N\right) & \text{if } s \in \left[A_N, A_N + B_1\right], \\ \dots & \\ \gamma^{(2,2)}\left(s - A_N - B_{N-3}\right) & \text{if } s \in \left[A_N + B_{N-3}, A_N + B_{N-2}\right], \end{cases} \tag{5}$$

where

$$A_j = \sum_{i=1}^{j} l^{(i,1)}, \quad B_j = \sum_{i=1}^{j} l^{(N-i,2)}, \quad l = \sum_{i=1}^{N} \left(l^{(i,1)} + l^{(i,2)}\right).$$

It is well known that if $f : im(\beta) \to \mathbb{R}$ is a continuous function defined on the image Γ of a piecewise smooth path $\beta : [c, d] \to \mathbb{R}^2$, the curvilinear integral of f along Γ is given by the following:

$$\int_{\Gamma} f(x_1, x_2)\, \mathrm{d}s = \int_c^d (f \circ \beta)(r) \left\|\beta'(r)\right\| \mathrm{d}r, \tag{6}$$

where $(f \circ \beta)(r) = f(\beta(r))$ and the norm $\left\|\beta'(r)\right\|$ is given by $\left(\beta_1'^2(r) + \beta_2'^2(r)\right)^{1/2}$. In the case of a real continuous function, f defined on the image of the path γ [given by (5)], i.e., the boundary curve $\partial\Omega$, the definition (6) when s is taken as the parameter r, leads to

$$\int_{\partial G} f(x_1, x_2)\, \mathrm{d}s = \int_0^l (f \circ \gamma)(s) \left\|\gamma'(s)\right\| \mathrm{d}s = \int_0^l (f \circ \gamma)(s)\, \mathrm{d}s, \tag{7}$$

and

$$\int_{\partial G} f(x_1, x_2)\, \mathrm{d}s = \sum_{m=1}^{2} \sum_{k=1}^{N} \int_{\Gamma^{(k,m)}} f(x_1, x_2)\, \mathrm{d}s, \tag{8}$$

where, in view of Eq. (4) and the conditions $l^{(1,2)} = l^{(N,2)} = 0$, we have

$$\int_{\Gamma^{(1,2)}} f(x_1, x_2)\, \mathrm{d}s = 0 \quad \text{and} \quad \int_{\Gamma^{(N,2)}} f(x_1, x_2)\, \mathrm{d}s = 0.$$

The additive property (8) will prove be valuable in the definitions of functions and functionals over $\partial\Omega$, since they can be set-up independently in each $\Gamma^{(k)}$ and by using Eq. (4). Thus, we assume that the rotational rigidities of the elastic restraints along the boundary and the line hinges are, respectively, given by the functions

$$r_R^{(k,m)} : im(\gamma^{(k,m)}) \to \mathbb{R}, \quad m = 1, 2, k = 1, 2, \dots, N, \tag{9a}$$

$$r_R^{(c_k)} : im(\gamma^{(c_k)}) \to \mathbb{R}, \quad k = 1, 2, \dots, N-1, \tag{9b}$$

where $\gamma^{(c_k)}$ is the path which describes the line $\Gamma^{(c_k)}$. In the same manner, the translational rigidities are, respectively, given by the functions

$$r_T^{(k,m)} : im(\gamma^{(k,m)}) \to \mathbb{R}, \quad m = 1, 2, \quad k = 1, 2, \dots, N, \tag{10a}$$

$$r_T^{(c_k)} : im(\gamma^{(c_k)}) \to \mathbb{R}, \quad k = 1, 2, \dots, N-1. \tag{10b}$$

It is obvious that

$$r_R^{(1,2)} = r_R^{(N,2)} = r_T^{(1,2)} = r_T^{(N,2)} \equiv 0. \tag{11}$$

The presence of several complicating effects [particularly the existence of $N-1$ line hinges which requires a careful analysis similar to the developed procedure to obtain the path which describes the boundary ∂G by Eq. (5)] leads to complicated analytical expressions and tedious algebraic manipulations. In order to obtain a compact analytical scheme for the derivation of the boundary value problem which describes the dynamical behavior of the mechanical system, we consider the following new procedure for the manipulation of derivatives introduced in [28].

Let us consider the following notation: $C^n(S)$ denotes the set of all real functions $u : S \to \mathbb{R}$ that have continuous partial derivatives of order n and $C^n\left(\bar{S}\right)$ denotes the set of all $u \in C^n(S)$ for which all partial derivatives of order n can be extended continuously to the closure $\bar{S}$ of S. Consider the well-known notation

$$D^\alpha u(\bar{x}) = \frac{\partial^{|\alpha|} u(\bar{x})}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \partial x_3^{\alpha_3}}, \tag{12}$$

where $u : A \to \mathbb{R}, \quad u \in C^{|\alpha|}(A), \quad A \subset \mathbb{R}^3$, and $\bar{x} = (x_1, x_2, x_3)$. The vector $\alpha = (\alpha_1, \alpha_2, \alpha_3)$ is a multi-index whose coordinates are non-negative integers and $|\alpha|$ is the sum $|\alpha| = \sum_{i=1}^{3} \alpha_i$. In the case of the described plate, we consider a sufficiently smooth function $v : A \to \mathbb{R}$, defined on $A = G \times [0, T]$ for some fixed time $T > 0$, with $x = (x_1, x_2) \in G,\ x_3 = t,\ G \subset \mathbb{R}^2$.

A new compact notation is generated if we introduce the following multi-indices:

$$\begin{aligned} &\alpha^{(1)} = (2,0,0), \quad \alpha^{(2)} = (0,2,0), \quad \alpha^{(3)} = (1,1,0), \quad \alpha^{(4)} = (0,0,2), \\ &\mathbf{1}^{(i)} = (\delta_{1i}, \delta_{2i}, \delta_{3i}), \quad i = 1,2,3, \end{aligned} \tag{13}$$

where δ_{ji} is the Kronecker delta, $\delta_{ji} = 1$ if $j = i$ and $\delta_{ji} = 0$ if $j \neq i$. The use of (12)–(13) leads to

$$\begin{aligned} &D^{\mathbf{1}^{(i)}} v(x,t) = \frac{\partial v}{\partial x_i}(x,t), \quad i = 1,2, \quad D^{\mathbf{1}^{(3)}} v(x,t) = \frac{\partial v}{\partial t}(x,t), \\ &D^{\alpha^{(i)}} v(x,t) = \frac{\partial^2 v}{\partial x_i^2}(x,t), \quad i = 1,2, \\ &D^{\alpha^{(3)}} v(x,t) = \frac{\partial^2 v}{\partial x_1 \partial x_2}(x,t), \quad D^{\alpha^{(4)}} v(x,t) = \frac{\partial^2 v}{\partial t^2}(x,t). \end{aligned} \tag{14}$$

The multi-indices (13) verify the following algebraic rules [28]:

$$\mathbf{1}^{(i)} + \mathbf{1}^{(i)} = \alpha^{(i)}, \quad \mathbf{1}^{(3-i)} + \mathbf{1}^{(i)} = \alpha^{(3)}, \quad \forall i \in \{1,2\}, \quad \mathbf{1}^{(3)} + \mathbf{1}^{(3)} = \alpha^{(4)}. \tag{15}$$

These rules will probe be valuable in the algebraic manipulation of the variation of the energy functional. An essential step to compact analytical expressions is the derivation of formulae needed to transform the terms which involves derivatives of variations. Let us suppose that

$$\begin{aligned} &S_i^{(k)} : A \to \mathbb{R}, \quad v : A \to \mathbb{R}, \quad A = G \times [0,T], \quad S_i^{(k)}(\bullet, t), \quad v(\bullet, t) \in C^2(\bar{G}), \\ &n_i^{(k)} : \partial\Omega^{(k)} \to \mathbb{R}, \quad i \in \{1,2\}, k \in \{1,2,\ldots,N\}. \end{aligned}$$

Then, the following formulas are valid:

$$\int_{G^{(k)}} S_i^{(k)}(x,t) D^{\alpha^{(i)}} v(x,t)\mathrm{d}x = \int_{\partial G^{(k)}} \Big[S_i^{(k)}(x,t) \left(D^{\mathbf{1}^{(i)}} v(x,t) \right) n_i^{(k)}(x) - \left(D^{\mathbf{1}^{(i)}} S_i^{(k)}(x,t) \right) v(x,t)\, n_i^{(k)}(x) \Big] \mathrm{d}s + \int_{G^{(k)}} \left(D^{\alpha^{(i)}} S_i^{(k)}(x,t) \right) v(x,t)\,\mathrm{d}x, \quad i = 1,2, \tag{16}$$

$$\int_{G^{(k)}} S_3^{(k)}(x,t)\, D^{\alpha^{(3)}} v(x,t)\,\mathrm{d}x = \frac{1}{2}\sum_{i=1}^{2} \left\{ \int_{\partial G^{(k)}} \Big[S_3^{(k)}(x,t) \left(D^{\mathbf{1}^{(3-i)}} v(x,t) \right) n_i^{(k)}(x) - \left(D^{\mathbf{1}^{(i)}} S_3^{(k)}(x,t) \right) v(x,t)\, n_{3-i}^{(k)}(x) \Big] \mathrm{d}s \right\} + \int_{G^{(k)}} \left(D^{\alpha^{(3)}} S_3^{(k)}(x,t) \right) v(x,t)\,\mathrm{d}x, \tag{17}$$

where $n_i^{(k)}$ denotes the i-th component of the outward unit normal $\vec{n}^{(k)}$ to the boundary $\partial\Omega^{(k)}$. The demonstrations of (16) and (17) are a direct consequence of the proposition 2 of Grossi [28].

Hamilton's principle requires that between times t_0 and t_1, at which the positions of the mechanical system are known, it should execute a motion which makes stationary the functional

$$F(w) = \int_{t_0}^{t_1} (E_K - E_D)\,\mathrm{d}t,$$

on the space of admissible functions, where E_K denotes the kinetic energy and E_D the total potential energy. From the well-known expressions of the kinetic and potential energies of the mechanical system under study, it follows that the energy functional is given by

$$\begin{aligned} F(w) = \frac{1}{2}\int_{t_0}^{t_1} \Bigg\{ \sum_{k=1}^{N} \Bigg[\int_{G^{(k)}} \Bigg(& (\rho h)^{(k)} \left(\frac{\partial w}{\partial t} \right)^2 - C_{11}^{(k)} \left(\frac{\partial^2 w}{\partial x_1^2} \right)^2 \\ & - 2C_{12}^{(k)} \frac{\partial^2 w}{\partial x_1^2} \frac{\partial^2 w}{\partial x_2^2} - C_{22}^{(k)} \left(\frac{\partial^2 w}{\partial x_2^2} \right)^2 - 4\frac{\partial^2 w}{\partial x_1 \partial x_2} \left(C_{16}^{(k)} \frac{\partial^2 w}{\partial x_1^2} + C_{26}^{(k)} \frac{\partial^2 w}{\partial x_2^2} \right) \\ & - 4C_{66}^{(k)} \left(\frac{\partial^2 w}{\partial x_1 \partial x_2} \right)^2 \Bigg) \mathrm{d}x - \sum_{m=1}^{2} \int_{\Gamma^{(k,m)}} \left(r_T^{(k,m)} w^2 + r_R^{(k,m)} \left(\frac{\partial w}{\partial \vec{n}^{(k,m)}} \right)^2 \right) \mathrm{d}s \\ & - \int_{\Gamma^{(c_k)}} \left(r_T^{(c_k)} w^2 + r_R^{(c_k)} \left([\partial w/\partial x_1]_{c_k} \right)^2 \right) \mathrm{d}s \Bigg] \Bigg\} \mathrm{d}t, \end{aligned} \tag{18}$$

where $w(x,t)$ is given by $w^{(k)}(x,t)$ when $x \in G^{(k)}$; $(\rho h)^{(k)}$ is the mass density, and $C_{ij}^{(k)} = C_{ij}^{(k)}(x)$ are the rigidities of the anisotropic material [18], which correspond to the subdomain $G^{(k)}$; $\partial w/\partial \vec{n}^{(k,m)}$ is the directional derivative of w with respect to the outward normal unit vector $\vec{n}^{(k,m)}$ to the curve $\Gamma^{(k,m)}$, and finally, the symbol $[\partial w/\partial x_1]_{c_k}$ denotes the difference of lateral derivatives

$$\left[\frac{\partial w}{\partial x_1} \right]_{c_k} = \frac{\partial w}{\partial x_1}\left(c_k^+, x_2, t \right) - \frac{\partial w}{\partial x_1}\left(c_k^-, x_2, t \right). \tag{19}$$

Since the number of line hinges is $N-1$, it is necessary to adopt $r_R^{(c_N)} = r_T^{(c_N)} \equiv 0$ in Eq. (18), and in view of conditions (11), we have the following:

$$\int_{\Gamma^{(1,2)}} (\bullet)\mathrm{d}s = \int_{\Gamma^{(N,2)}} (\bullet)\mathrm{d}s = \int_{\Gamma^{(c_N)}} (\bullet)\mathrm{d}s = 0.$$

The definition of the variation of F at w in the direction v is given by

$$\delta F\left(w;v\right)=\left.\frac{\mathrm{d}F}{\mathrm{d}\varepsilon}\left(w+\varepsilon v\right)\right|_{\varepsilon=0}, \tag{20}$$

and the condition of stationary functional requires that

$$\delta F\left(w;v\right)=0,\quad \forall v\in D_a, \tag{21}$$

where D_a is the space of admissible directions at w for the domain D of this functional. In order to make the mathematical developments required by the application of the techniques of the calculus of variations, we assume the following:

$$\begin{aligned}&(\rho h)^{(k)}\in C\left(\bar{G}^{(k)}\right),\quad C_{ij}^{(k)}\in C^2\left(\bar{G}^{(k)}\right),\quad w\left(x,\bullet\right)\in C^2\left[t_0,t_1\right],\quad w\left(\bullet,t\right)\in C\left(\bar{G}\right),\\&w\left(\bullet,t\right)|_{\bar{G}^{(k)}}\in C^4\left(\bar{G}^{(k)}\right),\quad \bar{G}^{(k)}=G^{(k)}\cup\partial G^{(k)},\quad k=1,2,\ldots,N.\end{aligned}$$

It must be noted that as a consequence of the presence of the line hinges, the derivative $\partial w/\partial x_1$ and the corresponding derivatives of greater order do not necessarily exist in the domain G, so it is necessary to impose the conditions $w\left(\bullet,t\right)|_{\bar{G}^{(k)}}\in C^4\left(\bar{G}^{(k)}\right), k=1,2,\ldots,N$.

In view of all these observations and since Hamilton's principle requires that at times t_0 and t_1 the positions are known, the space D is given by

$$\begin{aligned}D=\Big\{&w;\,w\left(x,\bullet\right)\in C^2\left[t_0,t_1\right],\,w\left(\bullet,t\right)\in C\left(\bar{G}\right),\,w\left(\bullet,t\right)|_{\bar{G}^{(k)}}\in C^4\left(\bar{G}^{(k)}\right),\\&k=1,\ldots,N,\,w(x,t_0),\,w\left(x,t_1\right)\text{ prescribed}\Big\}.\end{aligned} \tag{22}$$

The only admissible directions v at $w\in D$ are those for which $w+\varepsilon v\in D$ for all sufficiently small ε and $\delta F(w;v)$ exists. In consequence, and in view of (22), v is an admissible direction at w for D if, and only if, $v\in D_a$ where

$$\begin{aligned}D_a=\Big\{&v;\,v\left(x,\bullet\right)\in C^2\left[t_0,t_1\right],\,v\left(\bullet,t\right)\in C\left(\bar{G}\right),\,v\left(\bullet,t\right)|_{\bar{G}^{(k)}}\in C^4\left(\bar{G}^{(k)}\right),\\&k=1,\ldots,N,\,v\left(x,t_0\right)=v\left(x,t_1\right)=0,\forall x\in\bar{G}\Big\}.\end{aligned} \tag{23}$$

Performing the corresponding analytical developments by using the compact notation in Eq. (20), we have

$$\begin{aligned}&\delta F\left(w;v\right)\\&=\int_{t_0}^{t_1}\Bigg\{\sum_{k=1}^{N}\Bigg[\int_{G^{(k)}}\left((\rho h)^{(k)}\left(D^{1^{(3)}}w\right)\left(D^{1^{(3)}}v\right)-\sum_{i=1}^{3}S_i^{(k)}D^{\alpha^{(i)}}v\right)\mathrm{d}x\\&\quad-\sum_{m=1}^{2}\int_{\Gamma^{(k,m)}}\left(r_T^{(k,m)}wv+r_R^{(k,m)}\frac{\partial w}{\partial\vec{n}^{(k,m)}}\frac{\partial v}{\partial\vec{n}^{(k,m)}}\right)\mathrm{d}s\\&\quad-\int_{\Gamma^{(c_k)}}\left(r_T^{(c_k)}wv+r_R^{(c_k)}\left[\partial w/\partial x_1\right]_{c_k}\left[\partial v/\partial x_1\right]_{c_k}\right)\mathrm{d}s\Bigg]\Bigg\}\,\mathrm{d}t,\end{aligned} \tag{24}$$

where

$$S_i^{(k)}=\sum_{j=1}^{3}A_{ij}^{(k)}\left(x\right)D^{\alpha^{(j)}}w\left(x,t\right), \tag{25}$$

with the coefficients $A_{ij}^{(k)}$ as elements of the symmetric matrix

$$\mathbf{A} = \begin{pmatrix} C_{11}^{(k)} & C_{12}^{(k)} & 2C_{16}^{(k)} \\ C_{12}^{(k)} & C_{22}^{(k)} & 2C_{26}^{(k)} \\ 2C_{16}^{(k)} & 2C_{26}^{(k)} & 4C_{66}^{(k)} \end{pmatrix}. \tag{26}$$

It is convenient from now on to introduce a change of variables in order to deal with the points which correspond to the curves $\Gamma^{(k)}$. Let us consider the new variables (y_1, y_2) where y_1 is a distance measured from the boundary and along the normal to ∂G and y_2 is the arc length measured from the point (c_1, b_1) of the boundary ∂G, (see Fig. 2). This problem has been addressed in [27].

The mentioned change of variables transforms $w = w(x, t)$ into $\tilde{w} = \tilde{w}(y, t)$ with $y = (y_1, y_2)$. It also transforms v into $\tilde{v}$ and leads to the following relation between the original derivatives $D^{\mathbf{1}^{(i)}} v$ (i.e. $\partial v/\partial x_i$) and the new ones $D^{\mathbf{1}^{(i)}} \tilde{v}$ (i.e. $\partial \tilde{v}/\partial y_i$):

$$D^{\mathbf{1}^{(i)}} v = \left(D^{\mathbf{1}^{(1)}} \tilde{v}\right) \tilde{n}_i^{(k)} + (-1)^i \left(D^{\mathbf{1}^{(2)}} \tilde{v}\right) \tilde{n}_{3-i}^{(k)}, \quad i = 1, 2, \quad \text{in } \partial G^{(k)}. \tag{27}$$

Integrating by parts with respect to t the first term of (24) and applying the conditions

$$v(x, t_0) = v(x, t_1) = 0, \quad \forall x \in \bar{G},$$

imposed in (23), we obtain

$$\begin{aligned} \int_{t_0}^{t_1} \int_{G^{(k)}} (\rho h)^{(k)} \left(D^{1^{(3)}} w\right) \left(D^{1^{(3)}} v\right) \mathrm{d}x\mathrm{d}t &= \int_{G^{(k)}} (\rho h)^{(k)} \left(D^{1^{(3)}} w\right) v \Big|_{t_0}^{t_1} \mathrm{d}x \\ - \int_{t_0}^{t_1} \int_{G^{(k)}} (\rho h)^{(k)} \left(D^{1^{(3)}} \left(D^{1^{(3)}} w\right)\right) v \mathrm{d}x\mathrm{d}t &= - \int_{t_0}^{t_1} \int_{G^{(k)}} (\rho h)^{(k)} \left(D^{\alpha^{(4)}} w\right) v \mathrm{d}x\mathrm{d}t, \end{aligned} \tag{28}$$

where the last algebraic rule from (15) has been applied.

To transform the terms $S_i^{(k)} D^{\alpha^{(i)}} v$ of (24), the formulae (16)–(17) and the change of variables described above must be employed. This procedure and (28) lead to

$$\begin{aligned} &\delta F(w; v) \\ &= - \int_{t_0}^{t_1} \Bigg\{ \sum_{k=1}^{N} \Bigg[\int_{G^{(k)}} \left((\rho h)^{(k)} \left(D^{\alpha^{(4)}} w\right) + \sum_{i=1}^{3} \left(D^{\alpha^{(i)}} S_i^{(k)}\right) \right) v \mathrm{d}x \\ &\quad + \int_{\partial G^{(k)}} \Bigg(\sum_{i=1}^{2} \left(S_i^{(k)} n_i^{(k)} \left(D^{\mathbf{1}^{(i)}} v\right) - \left(D^{\mathbf{1}^{(i)}} S_i^{(k)}\right) n_i^{(k)} v \right. \\ &\quad \left. + 0.5 \left(S_3^{(k)} n_i^{(k)} \left(D^{\mathbf{1}^{(3-i)}} v\right) - \left(D^{\mathbf{1}^{(i)}} S_3^{(k)}\right) n_{3-i}^{(k)} v\right)\right) \mathrm{d}s \\ &\quad + \sum_{m=1}^{2} \Bigg(\int_0^{l^{(k,m)}} \tilde{r}_T^{(k,m)}(0, y_2)\, \tilde{w}(0, y_2, t)\, \tilde{v}(0, y_2, t)\, \mathrm{d}y_2 \\ &\quad + \int_0^{l^{(k,m)}} \tilde{r}_R^{(k,m)}(0, y_2)\, D^{\mathbf{1}^{(1)}} \tilde{w}(0, y_2, t)\, D^{\mathbf{1}^{(1)}} \tilde{v}(0, y_2, t)\, \mathrm{d}y_2 \Bigg) \\ &\quad + \int_{\Gamma^{(c_k)}} \left(r_T^{(c_k)} w v + r_R^{(c_k)} \left[\partial w/\partial x_1\right]_{c_k} \left[\partial v/\partial x_1\right]_{c_k} \right) \mathrm{d}s \Bigg] \Bigg\} \mathrm{d}t, \end{aligned} \tag{29}$$

where

$$\tilde{r}_R^{(1,2)} = \tilde{r}_R^{(N,2)} = r_R^{(c_N)} = \tilde{r}_T^{(1,2)} = \tilde{r}_T^{(N,2)} = r_T^{(c_N)} \equiv 0.$$

From Eqs. (2)–(3), we have the following:

$$\begin{aligned} &\partial G^{(k)} = \Gamma^{(k,1)} \cup \Gamma^{(k,2)} \cup \Gamma^{(c_{k-1})} \cup \Gamma^{(c_k)}, \text{ when } k = 2, \ldots, N-1, \\ &\partial G^{(1)} = \Gamma^{(1,1)} \cup \Gamma^{(c_1)} \quad \text{and} \quad \partial G^{(N)} = \Gamma^{(N,1)} \cup \Gamma^{(c_{N-1})}, \end{aligned}$$

and in consequence, in Eq. (29), it is possible to write (see Fig. 2)

$$\begin{aligned} &\sum_{k=1}^{N} \int_{\partial G^{(k)}} S^{(k)} \mathrm{d}s \\ &= \sum_{k=1}^{N} \left(\sum_{m=1}^{2} \int_{\Gamma^{(k,m)}} S^{(k,m)} \mathrm{d}s + \int_{\Gamma^{(c_{k-1})}} S^{(k)} \mathrm{d}s + \int_{\Gamma^{(c_k)}} S^{(k)} \mathrm{d}s \right) \end{aligned} \tag{30}$$

where

$$\int_{\Gamma^{(1,2)}} S^{(1,2)} \mathrm{d}s = \int_{\Gamma^{(N,2)}} S^{(N,2)} \mathrm{d}s = \int_{\Gamma^{(c_0)}} S^{(1)} \mathrm{d}s = \int_{\Gamma^{(c_N)}} S^{(N)} \mathrm{d}s = 0,$$

and

$$S^{(k)} = \sum_{i=1}^{2} \left[S_i^{(k)} n_i^{(k)} \left(D^{\mathbf{1}^{(i)}} v \right) - \left(D^{\mathbf{1}^{(i)}} S_i^{(k)} \right) n_i^{(k)} v + 0.5 \left(S_3^{(k)} n_i^{(k)} \left(D^{\mathbf{1}^{(3-i)}} v \right) - \left(D^{\mathbf{1}^{(i)}} S_3^{(k)} \right) n_{3-i}^{(k)} v \right) \right]. \tag{31}$$

It must be noted that $S^{(k,m)}$ denotes the expression $S^{(k)}$ when $n_i^{(k)}$ is replaced by $n_i^{(k,m)}$. The compact notation has the following useful property which allows an adequate collection of terms, [28]:

$$\sum_{i=1}^{2} \left(D^{\mathbf{1}^{(3-i)}} v \right) n_i^{(k,m)} = \sum_{i=1}^{2} \left(D^{\mathbf{1}^{(i)}} v \right) n_{3-i}^{(k,m)},$$

then from Eq. (31), the expression of $S^{(k,m)}$ is given by

$$S^{(k,m)} = \sum_{i=1}^{2} \left[\left(S_i^{(k)} n_i^{(k,m)} + 0.5 S_3^{(k)} n_{3-i}^{(k,m)} \right) D^{\mathbf{1}^{(i)}} v - \left(\left(D^{\mathbf{1}^{(i)}} S_i^{(k)} \right) n_i^{(k,m)} + 0.5 \left(D^{\mathbf{1}^{(i)}} S_3^{(k)} \right) n_{3-i}^{(k,m)} \right) v \right]. \tag{32}$$

Now, if the line integrals along $\Gamma^{(k,m)}$ are replaced by the corresponding ordinary integrals according to (7) and the new variables (y_1, y_2), we have the following:

$$\int_{\Gamma^{(k,m)}} S^{(k,m)} \mathrm{d}s = \int_{0}^{l^{(k,m)}} \tilde{S}^{(k,m)} (0, y_2, t) \, \mathrm{d}y_2.$$

The expression of the derivatives $D^{\mathbf{1}^{(i)}} v$ in Eq. (32) can be expressed in the new variables (y_1, y_2) by using Eqs. (27), and then, when the boundary ∂G is smooth, the Eq. (29) reduces to

$$\begin{aligned} &\delta F(w; v) \\ &= - \int_{t_0}^{t_1} \left\{ \sum_{k=1}^{N} \left[\int_{G^{(k)}} \left((\rho h)^{(k)} \left(D^{\alpha^{(4)}} w \right) + \sum_{i=1}^{3} \left(D^{\alpha^{(i)}} S_i^{(k)} \right) \right) v \mathrm{d}x \right. \right. \end{aligned}$$

$$+\sum_{m=1}^{2}\int_{0}^{l^{(k,m)}}\left(\tilde{H}^{(k,m)}+\tilde{r}_T^{(k,m)}\tilde{w}\tilde{v}+\tilde{r}_R^{(k,m)}D^{\mathbf{1}^{(1)}}\tilde{w}D^{\mathbf{1}^{(1)}}\tilde{v}\right)\mathrm{d}y_2$$

$$+\int_{\Gamma^{(c_k-1)}}S^{(k)}\mathrm{d}x_2+\int_{\Gamma^{(c_k)}}S^{(k)}\mathrm{d}x_2+\int_{\Gamma^{(c_k)}}\left(r_T^{(c_k)}wv+r_R^{(c_k)}\left[\partial w/\partial x_1\right]_{c_k}\left[\partial v/\partial x_1\right]_{c_k}\right)\mathrm{d}s\Bigg]\Bigg\}\,\mathrm{d}t, \quad (33)$$

where $\tilde{H}^{(k,m)}$ is obtained by the following procedure. Let us consider

$$H^{(k)}=Q^{(k)}D^{\mathbf{1}^{(1)}}v+R^{(k)}D^{\mathbf{1}^{(2)}}v-P^{(k)}v \quad (34)$$

and

$$\tilde{H}^{(k)}=\tilde{Q}^{(k)}D^{\mathbf{1}^{(1)}}\tilde{v}+\tilde{R}^{(k)}D^{\mathbf{1}^{(2)}}\tilde{v}-\tilde{P}^{(k)}\tilde{v}, \quad (35)$$

where the expressions of $\tilde{P}^{(k)}$, $\tilde{Q}^{(k)}$, and $\tilde{R}^{(k)}$ is obtained by introducing the new variables (y_1, y_2), respectively, in

$$P^{(k)}=\sum_{i=1}^{2}\left[\left(D^{\mathbf{1}^{(i)}}S_i^{(k)}\right)n_i^{(k)}+0.5\left(D^{\mathbf{1}^{(i)}}S_3^{(k)}\right)n_{3-i}^{(k)}\right], \quad (36)$$

$$Q^{(k)}=\sum_{i=1}^{2}\left[S_i^{(k)}\left(n_i^{(k)}\right)^2+0.5S_3^{(k)}n_i^{(k)}n_{3-i}^{(k)}\right] \quad (37)$$

and

$$R^{(k)}=\sum_{i=1}^{2}\left[(-1)^i\left(S_i^{(k)}n_i^{(k)}n_{3-i}^{(k)}+0.5S_3^{(k)}\left(n_{3-i}^{(k)}\right)^2\right)\right]. \quad (38)$$

Finally, the expressions of $\tilde{P}^{(k,m)}$, $\tilde{Q}^{(k,m)}$, and $\tilde{R}^{(k,m)}$, needed in $\tilde{H}^{(k,m)}$, are obtained by replacing $\tilde{n}_i^{(k)}$ by $\tilde{n}_i^{(k,m)}$ and $\tilde{n}_{3-i}^{(k)}$ by $\tilde{n}_{3-i}^{(k,m)}$, respectively, in $\tilde{P}^{(k)}$, $\tilde{Q}^{(k)}$ and $\tilde{R}^{(k)}$. The integration by parts

$$\int_{0}^{l^{(k,m)}}\left(\tilde{R}^{(k,m)}D^{\mathbf{1}^{(2)}}\tilde{v}\right)\mathrm{d}y_2=\tilde{R}^{(k,m)}\tilde{v}\Big|_0^{l^{(k,m)}}-\int_{0}^{l^{(k,m)}}\left(D^{\mathbf{1}^{(2)}}\tilde{R}^{(k,m)}\right)\tilde{v}\mathrm{d}y_2 \quad (39)$$

transforms Eq. (33) into

$$\delta F(w;v)=-\int_{t_0}^{t_1}\Bigg\{\sum_{k=1}^{N}\Bigg[\int_{G^{(k)}}\left((\rho h)^{(k)}\left(D^{\alpha^{(4)}}w\right)+\sum_{i=1}^{3}\left(D^{\alpha^{(i)}}S_i^{(k)}\right)\right)v\mathrm{d}x$$

$$+\sum_{m=1}^{2}\Bigg(\int_{0}^{l^{(k,m)}}\left(D^{\mathbf{1}^{(1)}}\tilde{v}\left(\tilde{Q}^{(k,m)}+\tilde{r}_R^{(k,m)}D^{\mathbf{1}^{(1)}}\tilde{w}\right)-\left(\tilde{P}^{(k,m)}+D^{\mathbf{1}^{(2)}}\tilde{R}^{(k,m)}-\tilde{r}_T^{(k,m)}\tilde{w}\right)\tilde{v}\right)\mathrm{d}y_2$$

$$+\tilde{R}^{(k,m)}\tilde{v}\Big|_0^{l^{(k,m)}}\Bigg)+\int_{\Gamma^{(c_k-1)}}S^{(k)}\mathrm{d}x_2+\int_{\Gamma^{(c_k)}}S^{(k)}\mathrm{d}x_2$$

$$+\int_{\Gamma^{(c_k)}}\left(r_T^{(c_k)}wv+r_R^{(c_k)}\left[\partial w/\partial x_1\right]_{c_k}\left[\partial v/\partial x_1\right]_{c_k}\right)\mathrm{d}s\Bigg]\Bigg\}\,\mathrm{d}t. \quad (40)$$

According to the condition of stationary functional (21), the expression (40) must vanish for the function w corresponding to the actual motion of the plate for all admissible directions v, and in particular for those admissible v, (or $\tilde{v}$) and $D^{\mathbf{1}^{(1)}}\tilde{v}$, for which the one-dimensional integrals in (40) vanish. Then, the variation reduces to

$$\delta F(w;v) = -\int_{t_0}^{t_1}\int_{G^{(k)}}\left[\sum_{k=1}^{N}\left((\rho h)^{(k)}\left(D^{\alpha^{(4)}}w\right)+\sum_{i=1}^{3}\left(D^{\alpha^{(i)}}S_i^{(k)}\right)\right)\right]v\mathrm{d}x\mathrm{d}t. \tag{41}$$

If the fundamental lemma of the calculus of variations is applied, it is concluded that the function w must satisfy the following differential equations:

$$(\rho h)^{(k)}\left(D^{\alpha^{(4)}}w\right)+\sum_{i=1}^{3}\left(D^{\alpha^{(i)}}S_i^{(k)}\right)=0,\quad \forall x\in G^{(k)},\quad t\geq 0,\quad k=1,2,\ldots,N. \tag{42}$$

Next, we remove the conditions for which the one-dimensional integrals in (40) vanish, and since the function w must satisfy Eqs. (42), the functional (40) reduces to

$$\begin{aligned}\delta F(w;v) = -\int_{t_0}^{t_1}\Bigg\{\sum_{k=1}^{N}\Bigg[\sum_{m=1}^{2}\Bigg(\int_0^{l^{(k,m)}}\Big(D^{\mathbf{1}^{(1)}}\tilde{v}\left(\tilde{Q}^{(k,m)}+\tilde{r}_R^{(k,i)}D^{\mathbf{1}^{(1)}}\tilde{w}\right)\\ -\left(\tilde{P}^{(k,m)}+D^{\mathbf{1}^{(2)}}\tilde{R}^{(k,m)}-\tilde{r}_T^{(k,i)}\tilde{w}\right)\tilde{v}\Big)\mathrm{d}y_2+\tilde{R}^{(k,m)}\tilde{v}\Big|_0^{l^{(k,m)}}\Bigg)+\int_{\Gamma^{(c_{k-1})}}S^{(k)}\mathrm{d}x_2+\int_{\Gamma^{(c_k)}}S^{(k)}\mathrm{d}x_2\\ +\int_{\Gamma^{(c_k)}}\left(r_T^{(c_k)}wv+r_R^{(c_k)}\left[\partial w/\partial x_1\right]_{c_k}\left[\partial v/\partial x_1\right]_{c_k}\right)\mathrm{d}s\Bigg]\Bigg\}\mathrm{d}t.\end{aligned} \tag{43}$$

When the boundary ∂G is smooth, using admissible directions v, for which the curvilinear integrals along $\Gamma^{(c_{k-1})}$ and $\Gamma^{(c_k)}$ vanish, the condition of stationarity of (43) leads to the following natural boundary conditions:

$$\begin{aligned}\tilde{r}_R^{(k,m)}(0,y_2)D^{\mathbf{1}^{(1)}}\tilde{w}(0,y_2,t) = -\sum_{i=1}^{2}\Big(&\tilde{S}_i^{(k)}(0,y_2,t)\left(\tilde{n}_i^{(k,m)}(0,y_2)\right)^2\\ &+0.5\tilde{S}_3^{(k)}(0,y_2,t)\,\tilde{n}_i^{(k,m)}(0,y_2,)\,\tilde{n}_{3-i}^{(k,m)}(0,y_2)\Big),\\ &y_2\in\left[0,l^{(k,m)}\right],\quad k=1,\ldots,N,\quad m=1,2,\end{aligned} \tag{44}$$

$$\begin{aligned}\tilde{r}_T^{(k,m)}(0,y_2)\tilde{w}(0,y_2,t) = \sum_{i=1}^{2}\Big[&\left(D^{1^{(i)}}\tilde{S}_i^{(k)}(0,y_2,t)+0.5D^{1^{(3-i)}}\tilde{S}_3^{(k)}(0,y_2,t)\right)\tilde{n}_i^{(k,m)}(0,y_2)\\ &+D^{1^{(2)}}\Big((-1)^i\left(\tilde{S}_i^{(k)}(0,y_2,t)\,\tilde{n}_i^{(k,m)}(0,y_2)\,\tilde{n}_{3-i}^{(k,m)}(0,y_2)\right)\\ &+0.5\tilde{S}_3^{(k)}(0,y_2,t)\left(n_{3-i}^{(k,m)}(0,y_2)\right)^2\Big)\Big],\\ &y_2\in\left[0,l^{(k,m)}\right],\quad k=1,\ldots,N,\quad m=1,2.\end{aligned} \tag{45}$$

When the boundary is piecewise smooth, additional corner conditions are generated as a consequence of the terms, [27]:

$$\tilde{R}^{(k,m)}\tilde{v}\Big|_0^{l^{(k,m)}}.$$

3 The transition conditions at the line hinges

If we remove the conditions from which the curvilinear integrals along $\Gamma^{(c_{k-1})}$ and $\Gamma^{(c_k)}$ vanish and since the function w must satisfy the natural boundary conditions (44)–(45), the functional (43) reduces to

$$\delta F(w;v) = -\int_{t_0}^{t_1}\left\{\sum_{k=1}^{N}\left[\int_{\Gamma^{(c_{k-1})}} S^{(k)}\mathrm{d}x_2 + \int_{\Gamma^{(c_k)}} S^{(k)}\mathrm{d}x_2 + \int_{\Gamma^{(c_k)}}\left(r_T^{(c_k)} wv + r_R^{(c_k)}\left[\partial w/\partial x_1\right]_{c_k}\left[\partial v/\partial x_1\right]_{c_k}\right)\mathrm{d}s\right]\right\}\mathrm{d}t. \tag{46}$$

Now, let us consider the curvilinear integrals over $\Gamma^{(c_{k-1})}$ and $\Gamma^{(c_k)}$. A path that describes the line $\Gamma^{(c_k)}$, when it is considered as a part of $\partial G^{(k)}$, $k = 1, 2, \ldots, N-1$, is given by the parametric representation

$$\gamma^{(c_k^-)}(x_2) = \left(\gamma_1^{(c_k^-)}(x_2), \gamma_2^{(c_k^-)}(x_2)\right) = (c_k, x_2), x_2 \in [a_k, b_k], \quad k = 1, 2, \ldots, \quad N-1, \tag{47}$$

where $a_k = \gamma_2^{(k+1,1)}(0)$, $b_k = \gamma_2^{(k,2)}(0)$, (see Fig. 2). In this case, we have $\vec{n}^{(k)} = (1, 0)$. A path that describes the line $\Gamma^{(c_{k-1})}$, when it is considered as a part of $\partial G^{(k)}$, with $k = 2, \ldots, N$, is given by the parametric representation

$$\gamma^{(c_{k-1}^+)}(x_2) = \left(\gamma_1^{(c_{k-1}^+)}(x_2), \gamma_2^{(c_{k-1}^+)}(x_2)\right) = (c_{k-1}, b_{k-1} - x_2), x_2 \in \left[0, b_{k-1} - a_{k-1}\right], \tag{48}$$

and $\vec{n}^{(k-1)} = (-1, 0)$, when $k = 2, \ldots, N$. It must be noted that the notations $\gamma^{(c_{k-1}^+)}$ and $\gamma^{(c_k^-)}$ clearly indicate that for a given k, the lines $\Gamma^{(c_{k-1})}$ and $\Gamma^{(c_k)}$ belong to $\partial G^{(k)}$. If we use (47) and (48) in (46) and integrate by parts, we obtain

$$\begin{aligned}
\delta F(w;v) &= \int_{t_0}^{t_1}\left\{\sum_{k=1}^{N-1}\left(\int_{a_k}^{b_k}\left[\left(\left(D^{1^{(1)}}S_1^{(k)} + D^{1^{(2)}}S_3^{(k)}\right)v\right)(c_k^-, x_2, t)\right.\right.\right.\\
&\quad \left.-\left(S_1^{(k)}D^{1^{(1)}}v\right)(c_k^-, x_2, t)\right]\mathrm{d}x_2 + \sum_{k=2}^{N}\int_0^{b_{k-1}-a_{k-1}}\left[-\left(\left(D^{1^{(1)}}S_1^{(k)} + D^{1^{(2)}}S_3^{(k)}\right)v\right)(c_{k-1}^+, b_{k-1} - x_2, t)\right.\\
&\quad \left.+\left(S_1^{(k)}D^{1^{(1)}}v\right)(c_{k-1}^+, b_{k-1} - x_2, t)\right]\mathrm{d}x_2 - 0.5\sum_{k=1}^{N-1}\left(S_3^{(k)}v\right)(c_k^-, x_2, t)\Big|_{a_k}^{b_k}\\
&\quad - 0.5\sum_{k=2}^{N}\left(S_3^{(k)}v\right)(c_{k-1}^+, b_{k-1} - x_2, t)\Big|_0^{b_{k-1}-a_{k-1}} - \sum_{k=1}^{N-1}\int_{a_k}^{b_k}\left(r_T^{(c_k)}(c_k, x_2)\, w(c_k, x_2, t)\, v(c_k, x_2, t)\right.\\
&\quad \left.\left.+ r_R^{(c_k)}(c_k, x_2)\left[D^{1^{(1)}}w\right]_{c_k}\left[D^{1^{(1)}}v\right]_{c_k}\right)\mathrm{d}x_2\right\}\mathrm{d}t.
\end{aligned} \tag{49}$$

It is remarkable that both the integral over $\Gamma^{(c_{k-1})}$ and that over $\Gamma^{(c_k)}$ must be computed twice when all the subdomains $G^{(k)}$ have been considered. The properties

$$\int_0^{b_{k-1}-a_{k-1}} S^{(k)}\left(c_{k-1}^+, b_{k-1} - x_2, t\right)\mathrm{d}x_2 = \int_{a_{k-1}}^{b_{k-1}} S^{(k)}\left(c_{k-1}^+, x_2, t\right)\mathrm{d}x_2$$

and

$$\sum_{k=2}^{N}\left(\int_{a_{k-1}}^{b_{k-1}} S^{(k)}\left(c_{k-1}^{+}, b_{k-1}-x_2, t\right)\mathrm{d}x_2\right)=\sum_{k=1}^{N-1}\left(\int_{a_k}^{b_k} S^{(k)}\left(c_k^{+}, x_2, t\right)\mathrm{d}x_2\right)$$

allow to collect terms, and consequently, from (49), in the manner of achieving Eqs. (44)–(45), we obtain:

$$\begin{aligned} & r_R^{(c_k)}\left(c_k, x_2\right)\left(D^{1^{(1)}} w\left(c_k^{+}, x_2, t\right)-D^{1^{(1)}} w\left(c_k^{-}, x_2, t\right)\right)=S_1^{(k)}\left(c_k^{-}, x_2, t\right) \\ & \quad=S_1^{(k)}\left(c_k^{+}, x_2, t\right), \ x_2 \in\left[a_k, b_k\right], 1 \leq k \leq N-1, \end{aligned} \tag{50}$$

$$\begin{aligned} & r_T^{(c_k)}\left(c_k, x_2\right) w\left(c_k, x_2, t\right)=\left(D^{1^{(1)}} S_1^{(k)}+D^{1^{(2)}} S_3^{(k)}\right)\left(c_k^{-}, x_2, t\right) \\ & \quad-\left(D^{1^{(1)}} S_1^{(k)}+D^{1^{(2)}} S_3^{(k)}\right)\left(c_k^{+}, x_2, t\right), x_2 \in\left[a_k, b_k\right], 1 \leq k \leq N-1. \end{aligned} \tag{51}$$

Since the domain of definition of the problem is G, and this is an open set in $\mathbb{R}^2$, given by $G=\bigcup_{k=1}^{N} G^{(k)} \cup \bigcup_{k=1}^{N-1} \Gamma^{(c_k)}$ with boundary ∂G given by Eq. (1), only the Eqs. (44)–(45) correspond to the boundary conditions. All the points of the lines $\Gamma^{(c_k)}$ with $1 \leq k \leq N-1$ are interior points of G, and the equations formulated on each $\Gamma^{(c_k)}$ can be called *transition conditions*. Then, Eqs. (50)–(51) correspond to the transition conditions of the problem. Since $w(\bullet, t) \in C\left(\bar{G}\right)$, there exists continuity of deflection at the points (c_k, x_2), and this generates the additional transition conditions

$$w\left(c_k^{-}, x_2, t\right)=w\left(c_k^{+}, x_2, t\right)=w\left(c_k, x_2, t\right), x_2 \in\left[a_k, b_k\right], \quad k=1,2, \ldots, N-1. \tag{52}$$

Different situations can be generated by substituting values and/or limiting values of the restraint parameters $r_T^{(c_k)}$ and $r_R^{(c_k)}$ in (50) and (51). It is remarkable that since all the points of the lines $\Gamma^{(c_k)}$ are interior points of G, the change of variables to (y_1, y_2) has not been implemented in the integrals $\int_{\Gamma^{(c_{k-1})}} S^{(k)}\mathrm{d}x_2$ and $\int_{\Gamma^{(c_k)}} S^{(k)}\mathrm{d}x_2$ of Eq. (33).

4 The Ritz and Lagrange multipliers methods in rectangular anisotropic plates

Let us consider a rectangular plate with

$$G=\left\{\left(x_1, x_2\right), 0<x_1<a, 0<x_2<b\right\}, \tag{53}$$

and three internal line hinges parallel to the x_2 axis. Consequently, the corresponding sub-domains are given by the following:

$$G^{(k)}=\left\{\left(x_1, x_2\right), c_{k-1}<x_1<c_k, 0<x_2<b\right\}, \quad k=1, \ldots, 4, \tag{54}$$

where $c_0=0$ and $c_4=a$. The curves

$$\Gamma^{(k)}=\Gamma^{(k,1)} \cup \Gamma^{(k,2)}, \quad k=2,3,$$

are described by the smooth paths:

$$\begin{aligned} \gamma^{(k,1)} & =\left\{\left(c_{k-1}+x_1, 0\right), \ x_1 \in\left[0, c_k-c_{k-1}\right]\right\}, \\ \gamma^{(k,2)} & =\left\{\left(c_k-x_1, b\right), \ x_1 \in\left[0, c_k-c_{k-1}\right]\right\}, \quad k=2,3. \end{aligned} \tag{55}$$

In this case, the curves $\Gamma^{(1)}$ and $\Gamma^{(4)}$ are given by (see Fig. 3) the following:

$$\Gamma^{(k)}=\Gamma^{(k,1)} \cup \Gamma^{(k,2)} \cup \Gamma^{(k,3)}, \quad k=1,4,$$

and are described by the smooth paths:

$$\begin{aligned}
&\gamma^{(1,1)} = \{(c_1 - x_1, b)\,, x_1 \in [0, c_1]\}\,, \quad \gamma^{(1,2)} = \{(0, b - x_2)\,,\ x_2 \in [0, b]\}\,,\\
&\gamma^{(1,3)} = \{(x_1, 0)\,, x_1 \in [0, c_1]\}\,,\\
&\gamma^{(4,1)} = \{(c_3 + x_1, 0)\,, x_1 \in [0, a - c_3]\}\,, \quad \gamma^{(4,2)} = \{(a, x_2)\,, x_2 \in [0, b]\}\,,\\
&\gamma^{(4,3)} = \{(a - x_1, b)\,, x_1 \in [0, a - c_3]\}\,.
\end{aligned}$$

It must be noted that the upper sides of the plate are given by (see Fig. 3) the following:

$$\Gamma^{(k,m)},\ (k, m) \in \{(1, 1)\,, (2, 2)\ (3, 2)\,, (4, 3)\}\,,$$

and to these sides correspond $n_1^{(k)} = 0, n_2^{(k)} = 1$. Meanwhile, the lower sides of the plate are given by

$$\Gamma^{(k,m)},\ (k, m) \in \{(1, 3)\,, (2, 1)\,, (3, 1)\,, (4, 1)\}\,,$$

and $n_1^{(k)} = 0, n_2^{(k)} = -1$. Finally, to the side $\Gamma^{(1,2)}$ corresponds $n_1^{(1)} = -1, n_2^{(1)} = 0$, and to the side $\Gamma^{(4,2)}$ corresponds $n_1^{(4)} = 1, n_2^{(4)} = 0$.

It is immediate that Eq. (27) with v replaced by w lead to the following relations:

$$\begin{aligned}
D^{\mathbf{1}^{(1)}} w\,(x_1, x_2, t)\Big|_{\Gamma^{(k,m)}} &= -D^{\mathbf{1}^{(2)}} \tilde{w}\,(0, y_2, t)\,,\\
D^{\mathbf{1}^{(2)}} w\,(x_1, x_2, t)\Big|_{\Gamma^{(k,m)}} &= D^{\mathbf{1}^{(1)}} \tilde{w}\,(0, y_2, t)
\end{aligned} \tag{56}$$

when $(k, m) \in \{(1, 1)\,, (2, 2)\ (3, 2)\,, (4, 3)\}$ and

$$\begin{aligned}
D^{\mathbf{1}^{(1)}} w\,(x_1, x_2, t)\Big|_{\Gamma^{(k,m)}} &= D^{\mathbf{1}^{(2)}} \tilde{w}\,(0, y_2, t)\,,\\
D^{\mathbf{1}^{(2)}} w\,(x_1, x_2, t)\Big|_{\Gamma^{(k,m)}} &= -D^{\mathbf{1}^{(1)}} \tilde{w}\,(0, y_2, t)
\end{aligned} \tag{57}$$

when $(k, m) \in \{(1, 3)\,, (2, 1)\,, (3, 1)\,, (4, 1)\}$. Finally, we have

$$\begin{aligned}
D^{\mathbf{1}^{(1)}} w\,(x_1, x_2, t)\Big|_{\Gamma^{(1,2)}} &= -D^{\mathbf{1}^{(1)}} \tilde{w}\,(0, y_2, t)\,,\\
D^{\mathbf{1}^{(2)}} w\,(x_1, x_2, t)\Big|_{\Gamma^{(1,2)}} &= -D^{\mathbf{1}^{(2)}} \tilde{w}\,(0, y_2, t)\,,\\
D^{\mathbf{1}^{(1)}} w\,(x_1, x_2, t)\Big|_{\Gamma^{(4,2)}} &= D^{\mathbf{1}^{(1)}} \tilde{w}\,(0, y_2, t)\,,\\
D^{\mathbf{1}^{(2)}} w\,(x_1, x_2, t)\Big|_{\Gamma^{(4,2)}} &= D^{\mathbf{1}^{(2)}} \tilde{w}\,(0, y_2, t)\,.
\end{aligned} \tag{58}$$

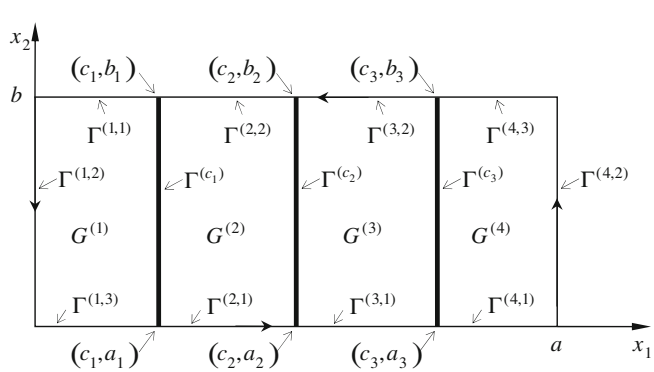


Fig. 3 Rectangular plate with three internal lines hinges

From Eq. (44), we have

$$\tilde{r}_R^{(k,m)}(0, y_2)\, D^{\mathbf{1}^{(1)}} \tilde{w}(0, y_2, t) = -\tilde{S}_2^{(k)}(0, y_2, t), \;\; y_2 \in \left[0, l^{(k,m)}\right] \tag{59}$$

when $(k, m) \in \{(1, 1), (2, 2)\,(3, 2), (4, 3)\}$ and

$$\tilde{r}_R^{(k,m)}(0, y_2) D^{\mathbf{1}^{(1)}} \tilde{w}(0, y_2, t) = -\tilde{S}_2^{(k)}(0, y_2, t), \;\; y_2 \in \left[0, l^{(k,m)}\right] \tag{60}$$

when $(k, m) \in \{(1, 3), (2, 1), (3, 1), (4, 1)\}$. Finally, for the remaining two sides, we have the following:

$$\tilde{r}_R^{(1,2)}(0, y_2)\, D^{\mathbf{1}^{(1)}} \tilde{w}(0, y_2, t) = -\tilde{S}_1^{(1)}(0, y_2, t), \; y_2 \in [0, b] \tag{61}$$

and

$$\tilde{r}_R^{(4,2)}(0, y_2)\, D^{\mathbf{1}^{(1)}} \tilde{w}(0, y_2, t) = -\tilde{S}_1^{(4)}(0, y_2, t), \;\; y_2 \in [0, b]. \tag{62}$$

Let us consider the first of Eqs. (59). From Eq. (56) and the relations between the derivatives of order two, we obtain the following boundary condition:

$$r_R^{(1,1)}(x_1, b)\; D^{\mathbf{1}^{(2)}} w(x_1, b, t)\Big|_{\Gamma^{(1,1)}} = -\sum_{j=1}^{3} A_{2j}^{(1)}(x_1, b)\, D^{\alpha^{(j)}} w(x_1, b, t), \; x_1 \in [0, c_1]. \tag{63}$$

From Eqs. (25)–(26), it is immediate that Eq. (63) in the classical notation is given by the following:

$$\begin{aligned} r_R^{(1,1)}(x_1, b)\, \frac{\partial w}{\partial x_2}(x_1, b, t) = &-C_{12}^{(1)}(x_1, b)\, \frac{\partial^2 w}{\partial x_1^2}(x_1, b, t) - C_{22}^{(1)}(x_1, b)\, \frac{\partial^2 w}{\partial x_2^2}(x_1, b, t) \\ &- 2C_{26}^{(1)}(x_1, b)\, \frac{\partial^2 w}{\partial x_1 \partial x_2}(x_1, b, t), \;\; x_1 \in [0, c_1]. \end{aligned} \tag{64}$$

In an analog form can be obtained the remaining boundary conditions and it must be noted that in this case, the boundary is composed of four smooth arcs and has four corner points, and then, there exist four corner conditions.

The transition conditions (52) ensure the continuity of the transverse deflection along the internal line hinges. Since it is difficult to construct a simple and adequate deflection function which can be applied to the entire plate and to show the continuity of displacement and the discontinuities of the slope crossing the line hinges, the Ritz method is used in conjunction with the Lagrange multipliers method to force the continuity along the line hinges by means of suitable multipliers. When the plate makes free vibrations, its displacement is given by an harmonic function of the time, i.e.,

$$w(x_1, x_2, t) = W(x_1, x_2) \cos \omega t, \tag{65}$$

where ω is the radian frequency of the plate. Substituting Eq. (65) into Eq. (18) leads to its maximum expression $F_{\max}$, and the Lagrange multipliers method requires the stationarity of the functional

$$L = F_{\max} + F_\lambda, \tag{66}$$

where

$$F_\lambda = \sum_{k=1}^{N-1} \int_{\Gamma^{(c_k)}} \lambda^{(k)}(x_2) \left(W\left(c_k^-, x_2\right) - W\left(c_k^+, x_2\right)\right) \mathrm{d}x_2, \tag{67}$$

is the subsidiary condition which imposes the transition conditions (52). In this case, the Lagrange multipliers are functions which can be represented by a set of polynomials as:

$$\lambda^{(k)}(x_2) = \sum_{i=1}^{r_k} d_i^{(k)} x_2^{i-1}, \tag{68}$$

where $d_i^{(k)}$ are unknown coefficients.

In the present paper, the transverse deflection of the rectangular plate is represented by means of

$$W(x_1, x_2) = W^{(k)}(x_1, x_2) \quad \text{if } (x_1, x_2) \in \bar{G}^{(k)}, k = 1, 2, \ldots, N,$$

where

$$W^{(k)}(x_1, x_2) = \sum_{i=1}^{m_k} \sum_{j=1}^{n_k} a_{ij}^{(k)} p_i^{(k)}(x_1/a) q_j^{(k)}(x_2/b), \quad k = 1, 2, \ldots, N, \tag{69}$$

and $p_i^{(k)}, q_i^{(k)}$ are polynomial functions. The application of the Ritz method in conjunction with the Lagrange multipliers method leads to the governing eigenvalue equation:

$$\left([\mathbf{K}] - \Omega^2 [\mathbf{M}]\right) \{\mathbf{d}\} = \{\mathbf{0}\}, \tag{70}$$

where $\Omega = \omega b^2 \sqrt{\rho h / C_{11}}$ is the nondimensional frequency parameter. For the sake of simplicity, the following has been adopted: $C_{11}^{(k)} = C_{11}, h^{(k)} = h, m_k = n_k = M$ for $k = 1, 2, \ldots, N$ and $r_k = M$ for $k = 1, 2, \ldots, N-1$.

4.1 Convergence and comparison of eigenvalues and modal shapes

The terminology to be used throughout the remainder of the paper for describing the boundary conditions of the plate considered will now be introduced. In all Tables and Figures, the symbol F, S, and C denote free, simply supported and clamped edges, and, for example, in the designation CSFS, the first symbol indicates the boundary condition at $x_1 = 0$, the second at $x_2 = 0$, the third at $x_1 = a$, and the fourth at $x_2 = b$.

In order to establish the accuracy and applicability of the approach developed and discussed in the previous sections, numerical results were computed for a number of plate problems for which comparison values were available in the literature and also convergence studies have been implemented. Additionally, new numerical results were generated for rectangular plates with one, two, and three internal line hinges and different boundary conditions. All calculations have been performed taking Poisson's ratio $\mu = 0.3$.

Results of a convergence study of the values of the frequency parameter $\Omega = \omega b^2 \sqrt{\rho h / C}$ of a rectangular isotropic plate are presented in Table 1. The isotropy is characterized by

$$C_{11} = C_{22} = C, \quad C_{16} = C_{26} = 0, \quad C_{12} = \mu C, \quad C_{66} = 0.5(1-\mu)C,$$

where C denotes the flexural rigidity of the isotropic plate. The first ten values of Ω are presented for a CSCS plate with aspect ratio $b/a = 1/2$ and with a free internal line hinge located at two different positions, namely,

Table 1 Convergence study of the first ten values of the frequency parameter Ω for an isotropic rectangular CSCS plate with aspect ratio $b/a = 1/2$, and with a free internal line hinge located a $\bar{c}_1 = 1/3$ and $\bar{c}_1 = 1/2$

P		Mode sequence									
$\bar{c}_1$	M	1	2	3	4	5	6	7	8	9	10
1/3	4	13.08704	21.34970	41.24297	42.19241	50.52497	57.45938	68.12335	87.31363	88.83035	115.50618
	5	13.02938	21.29404	38.87133	42.10803	50.51505	53.58910	66.54726	83.70899	88.57164	92.42562
	6	13.02747	21.24199	38.48675	41.93049	50.29860	53.31515	66.25574	75.30024	82.89380	92.36946
	7	13.02725	21.23979	38.41950	41.92656	50.29003	52.94558	66.21742	75.29608	82.45892	90.88579
	8	13.02725	21.23977	38.41588	41.92540	50.28853	52.94515	66.21506	74.60979	82.45845	90.88127
	9	13.02725	21.23977	38.41574	41.92535	50.28843	52.94376	66.21504	74.60933	82.45819	90.85921
	10	13.02725	21.23977	38.41574	41.92534	50.28842	52.94374	66.21504	74.60406	82.45801	90.85909
	11	13.02725	21.23977	38.41574	41.92534	50.28842	52.94373	66.21504	74.60406	82.45799	90.85897
1/2	4	12.70627	23.72724	33.17153	41.93788	51.85000	61.06898	63.33109	78.11975	88.07950	108.15864
	5	12.68772	23.64758	33.11678	41.86940	51.81540	58.93268	63.23609	73.34897	86.37351	92.13940
	6	12.68738	23.64638	33.06603	41.70366	51.67531	58.67674	63.01938	72.54043	86.15765	92.10829
	7	12.68736	23.64632	33.06510	41.70266	51.67489	58.64840	63.01580	72.40644	86.13690	90.63356
	8	12.68736	23.64632	33.06509	41.70193	51.67428	58.64640	63.01485	72.39779	86.13450	90.63286
	9	12.68736	23.64632	33.06509	41.70193	51.67428	58.64637	63.01483	72.39760	86.13446	90.61148
	10	12.68736	23.64632	33.06509	41.70193	51.67427	58.64636	63.01483	72.39756	86.13446	90.61148
	11	12.68736	23.64632	33.06509	41.70193	51.67427	58.64636	63.01483	72.39756	86.13446	90.61137

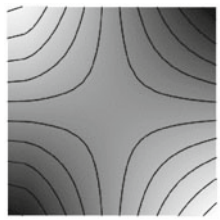
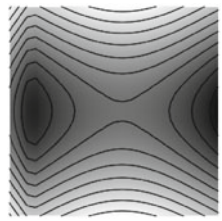
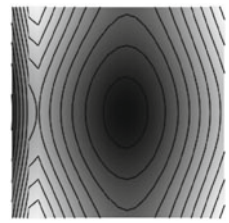
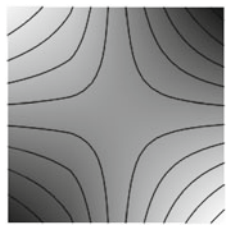
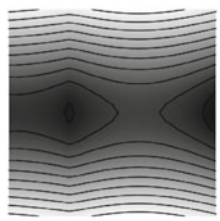
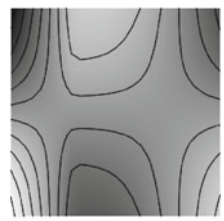
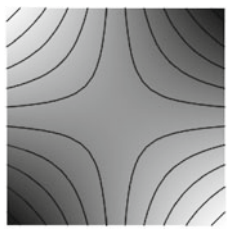
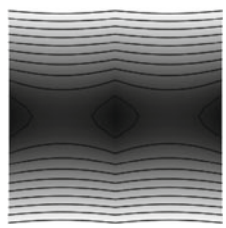
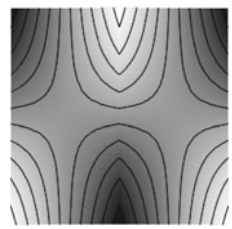
$\bar{c}_1$	Reference	Mode sequence		
		1	2	3
0.1	Present	13.18790	20.71524	26.94803
	Ref.[26]	13.18792	20.71540	26.94847
				
0.3	Present	13.24791	21.36125	30.42226
	Ref.[26]	13.24793	21.36147	30.42241
				
0.5	Present	13.46820	21.46315	26.57492
	Ref.[26]	13.46822	21.46337	26.57505
				

Fig. 4 Comparison of the first three values of the frequency parameter Ω and their modal shapes contour lines of an isotropic FFFF square plate with a free internal line hinge located at different positions

Table 2 First four values of the frequency parameter Ω for an anisotropic CCCC square plate with two elastically restrained internal line hinges with different values of the rotational restrictions $R_R^{(c_k)} = r_R^{(c_k)} a/C_{11}, k = 1, 2$, located at $\bar{c}_1 = 0.25$ and $\bar{c}_2 = 0.75$

$R_R^{(c_1)}$	$R_R^{(c_2)}$	Mode sequence			
		1	2	3	4
0	0	23.45956	30.56252	37.85855	44.15721
10	0	23.61956	30.75000	44.68048	45.98426
1000	0	23.64241	30.77948	45.97679	46.06113
∞	0	23.64268	30.77983	45.98424	46.07002
∞	10	23.92261	31.09512	46.34815	59.35300
∞	1,000	23.96589	31.14802	46.40945	62.73310
∞	∞	23.96640	31.14865	46.41019	62.77503
Ref. [29]		23.96642	31.14868	46.4672	62.77512

$\bar{c}_1 = 1/3$, and $\bar{c}_1 = 1/2$, where $\bar{c}_1 = c_1/a$. The convergence of the mentioned frequency parameters is studied by gradually increasing the number of polynomial in the approximate functions $W^{(1)}$, $W^{(2)}$ and the Lagrange multiplier $\lambda^{(1)} = \lambda^{(1)}(x_2)$, given by $m_1 = n_1 = m_2 = n_2 = r_1 = M$, as stated above. It can be seen that $M = 10$ is adequate to reach a stable convergence in almost all cases.

Figure 4 shows a comparison of the first three values of the frequency parameter $\Omega = \omega b^2 \sqrt{\rho h/C}$, and their modal shapes contour lines of an isotropic FFFF square plate with a free internal line hinge located at $\bar{c}_1 = 0.1$, $\bar{c}_1 = 0.3$ and $\bar{c}_1 = 0.5$. The comparison of results with those of Quintana and Grossi [26] shows

$\bar{c}_1$	$\bar{c}_2$	Mode sequence				
		1	2	3	4	5
1/3	2/3	22.31765	23.52980	25.75968	31.52952	37.18707
						
0.2	0.4	22.40032	23.45263	26.25576	29.48197	36.75353
						
0.4	0.8	22.28405	23.47255	26.65831	30.28586	34.97443
						

Fig. 5 First five values of the frequency parameter Ω and modal shapes of an isotropic CCFC rectangular plate with $b/a = 1/3$ and two free internal line hinges with different locations

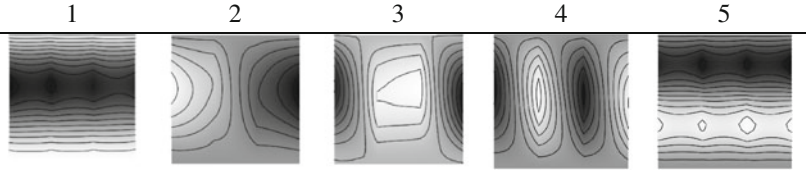
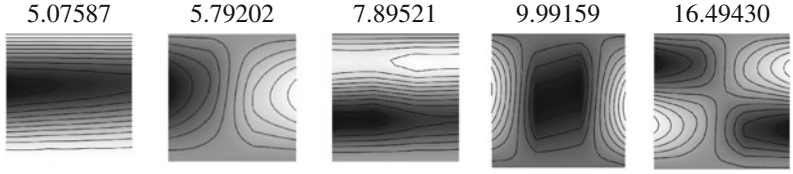
b/a	Mode sequence				
	1	2	3	4	5
1/2					
	5.07587	5.79202	7.89521	9.99159	16.49430
2					
	4.99314	10.00064	16.26355	19.68378	23.75562

Fig. 6 First five values of the frequency parameter Ω and modal shapes contour lines of an anisotropic FCFS rectangular plate with two free internal line hinges located at $\bar{c}_1 = 1/3$ and $\bar{c}_2 = 2/3$ for $b/a = 1/2$ and $b/a = 2$

that the present values are slightly lower, in consequence more accurate, since the Ritz method gives upper bounds for eigenvalues.

Table 2 gives the first four values of the frequency parameter $\Omega = \omega b^2\sqrt{\rho h/C_{11}}$ for an anisotropic CCCC square plate with two internal line hinges elastically restrained against rotation. Different values of the rotational restrictions $R_R^{(c_k)} = r_R^{(c_k)} a/C_{11}, k = 1, 2$, located at $\bar{c}_1 = 0.25$ and $\bar{c}_2 = 0.75$ are considered. The values which correspond to $R_R^{(c_k)} = \infty, k = 1, 2$, are compared with those obtained in [29]. The plate anisotropy is characterized by

$$\bar{C}_{22} = 0.1, \bar{C}_{66} = 0.0247750, \bar{C}_{12} = 0.03, \bar{C}_{16} = \bar{C}_{26} = 0,$$

where $\bar{C}_{ij}$ denotes the quotient C_{ij}/C_{11}.

$\bar{c}_1$	$\bar{c}_2$	Mode sequence				
		1	2	3	4	5
1/3	2/3	14.40984	14.69129	15.66289	16.54287	22.41634
						
0.2	0.4	14.41714	14.68934	15.42291	17.88748	19.44216
						
0.4	0.8	14.40627	14.67245	15.52800	17.86573	21.52009
						

Fig. 7 First five values of the frequency parameter Ω and modal shapes of an isotropic rectangular plate elastically restrained against rotation and translation with two free internal line hinges for different locations and aspect ratio $b/a = 1/3$. Edge $x_1 = 0 : R_R = r_R a/C_{11} = 10$, $R_T = r_T a^3/C_{11} = 100$; edge $x_2 = 0 : R_R = r_R b/C_{11} = 1000$, $R_T = r_T b^3/C_{11} = 100$; edge $x_1 = a : R_R = r_R a/C_{11} = 100$, $R_T = r_T a^3/C_{11} = 10$; edge $x_2 = b : R_R = r_R b/C_{11} = 100$, $R_T = r_T b^3/C_{11} = 1000$

Table 3 First ten values of the frequency parameter Ω for an isotropic SSSS square plate with three internal line hinges elastically restrained against rotation, for different values of the rotational restrictions $R_R^{(c_k)} = r_R^{(c_k)} a/C, k = 1, 2, 3$ located at $\bar{c}_1 = 0.1, \bar{c}_2 = 0.3$ and $\bar{c}_3 = 0.5$

			Mode Sequence									
$R_R^{(c_1)}$	$R_R^{(c_2)}$	$R_R^{(c_3)}$	1	2	3	4	5	6	7	8	9	10
0	0	0	15.621165	31.424942	45.810014	55.555108	68.911856	70.699358	94.685532	100.372229	120.8326357	127.2018414
												
100	0	0	15.659686	33.084877	45.899001	67.436921	70.634302	94.834955	106.078408	122.404505	144.903509	161.598162
1000	0	0	15.659953	33.094441	45.900913	67.452516	70.666660	94.840425	106.152974	122.456922	147.762890	161.744345
∞	0	0	15.659983	33.095511	45.901130	67.454243	70.670329	94.841058	106.161355	122.462959	148.090485	161.761063
∞	100	0	16.129347	46.714194	49.022783	74.937776	78.665061	95.988678	110.796045	128.011627	164.529706	164.701207
∞	1000	0	16.134228	46.735689	49.315058	75.247587	78.926764	96.035115	111.001479	128.274188	164.678417	164.779776
∞	∞	0	16.134777	46.738147	49.348022	75.283383	78.956835	96.040505	111.025379	128.304857	164.695911	164.789017
∞	∞	100	19.657690	49.255831	49.348022	78.956835	97.871077	98.567425	127.548259	128.304857	167.602232	167.783275
∞	∞	1000	19.730888	49.338503	49.348022	78.956835	98.610977	98.682601	128.226215	128.304857	167.766261	167.783275
∞	∞	∞	19.739209	49.348022	49.348022	78.956835	98.696044	98.696044	128.304857	128.304857	167.783275	167.785666
												

The mode shapes correspond to $R_R^{(c_k)} = 0$ and $R_R^{(c_k)} = \infty, k = 1, 2, 3$

4.2 New numerical results

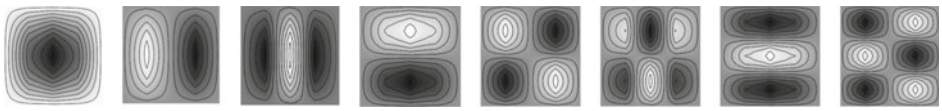
Figure 5 shows the first five values of the frequency parameter $\Omega = \omega b^2 \sqrt{\rho h/C}$ and modal shapes of an isotropic CCFC rectangular plate with $b/a = 1/3$ and two free internal line hinges with different locations.

In Fig. 6, it can be observed the first five values of the frequency parameter $\Omega = \omega b^2 \sqrt{\rho h/C_{11}}$, and their modal shapes contour lines of an anisotropic FCFS rectangular plate with two free internal line hinges located at $\bar{c}_1 = 1/3$ and $\bar{c}_2 = 2/3$ for two different aspect ratios. The plate anisotropy considered is characterized by

$$\bar{C}_{22} = 0.115202317, \bar{C}_{66} = 0.0948810, \ \bar{C}_{12} = 0.100812496, \bar{C}_{16} = -0.24333539, \bar{C}_{26} = -0.0120837.$$

Figure 7 shows the first five values of the frequency parameter $\Omega = \omega b^2 \sqrt{\rho h/C}$ and the modal shapes of an isotropic rectangular plate elastically restrained against rotation and translation, with two free internal line

Table 4 First eight values of the frequency parameter Ω for an isotropic SSSS square plate with three internal line hinges elastically restrained against rotation, for different values of the rotational restrictions $R_R^{(c_k)} = r_R^{(c_k)} a/C$, $k = 1, 2, 3$, located at $\bar{c}_1 = 0.25$, $\bar{c}_2 = 0.5$ and $\bar{c}_3 = 0.75$

$R_R^{(c_1)}$	$R_R^{(c_2)}$	$R_R^{(c_3)}$	Mode Sequence							
			1	2	3	4	5	6	7	8
0	0	0	15.36329	27.52213	42.77809	45.42986	64.53911	91.18849	94.24992	116.83206
										
100	0	0	15.69608	31.96383	45.99468	64.65612	69.54496	95.01830	102.58982	121.44341
1000	0	0	15.69950	32.00540	46.00921	65.42236	69.68312	95.04829	103.13919	121.64389
∞	0	0	15.69988	32.01005	46.01087	65.51062	69.69890	95.05177	103.20331	121.66719
∞	100	0	17.15566	37.11687	47.73096	70.87514	88.20107	97.07621	119.98941	122.10969
∞	1000	0	17.17572	37.20665	47.78292	70.90971	88.99991	97.16574	120.70381	122.12872
∞	∞	0	17.17799	37.21680	47.78889	70.91367	89.09194	97.17618	120.78681	122.13093
∞	∞	100	19.69815	48.99297	49.30165	78.63597	98.28650	98.63132	127.92899	127.98137
∞	∞	1000	19.73505	49.31163	49.34326	78.92361	98.65354	98.68932	128.26556	128.27096
∞	∞	∞	19.73921	49.34802	49.34802	78.95684	98.69604	98.69604	128.30486	128.30486

The mode shapes correspond to $R_R^{(c_k)} = 0$, $k = 1, 2, 3$

hinges for different locations and aspect ratio $b/a = 1/3$. The edges are elastically restrained according to the following:

$$\begin{aligned}
&\text{edge } x_1 = 0: && R_R = r_R a/C = 10, && R_T = r_T a^3/C = 100,\\
&\text{edge } x_2 = 0: && R_R = r_R b/C = 1{,}000, && R_T = r_T b^3/C = 100,\\
&\text{edge } x_1 = a: && R_R = r_R a/C = 100, && R_T = r_T a^3/C = 10,\\
&\text{edge } x_2 = b: && R_R = r_R b/C = 100, && R_T = r_T b^3/C = 1{,}000.
\end{aligned}$$

Table 3 depicts the first ten values of the frequency parameter $\Omega = \omega b^2 \sqrt{\rho h/C}$ for an isotropic SSSS square plate with three internal line hinges elastically restrained against rotation, for different values of the rotational restrictions $R_R^{(c_k)} = r_R^{(c_k)} a/C$, $k = 1, 2, 3$ located at $\bar{c}_1 = 0.1$, $\bar{c}_2 = 0.3$ and $\bar{c}_3 = 0.5$. The mode shapes correspond to $R_R^{(c_k)} = 0$ and $R_R^{(c_k)} = \infty$, $k = 1, 2, 3$.

Table 4 gives the first eight values of the frequency parameter $\Omega = \omega b^2 \sqrt{\rho h/C}$ for an isotropic SSSS square plate with three internal line hinges elastically restrained against rotation, for different values of the rotational restrictions $R_R^{(c_k)} = r_R^{(c_k)} a/C$, $k = 1, 2, 3$, located at $\bar{c}_1 = 0.25$, $\bar{c}_2 = 0.5$ and $\bar{c}_3 = 0.75$. The mode shapes correspond to $R_R^{(c_k)} = 0$, $k = 1, 2, 3$.

5 Concluding remarks

This paper presents the formulation of an analytical model for the dynamic behavior of anisotropic plates, with several arbitrarily located internal line hinges with elastics supports and piecewise smooth boundaries elastically restrained against rotation and translation. The equations of motion and the associated boundary and transition conditions were derived handling Hamilton's principle in a rigorous framework. The presence of a generic number of line hinges constitutes a complicating effect in the analysis and development of the variational treatment, so a new analytical manipulation based on a condensed notation is used to compact the corresponding analytical expressions.

An approach to the solution of the natural vibration problems of the mentioned plates by a direct variational method has been presented. A simple computationally efficient and accurate algorithm has been developed for the determination of frequencies and modal shapes of natural vibrations. The approach is based on a combination of the Ritz method and the Lagrange multipliers method. Sets of parametric studies have been performed to show the influence of the line hinges and it locations on the vibration behavior. Although numerical results are presented for rectangular plates with one, two, and three line hinges, the algorithm developed is applicable for any number of line hinges. It is worth noting that when $r_r^{(c_k)} \to \infty$, $r_T^{(c_k)} \to \infty$, the $k-$th line hinge is transformed into a line support. In the consequence, the results obtained in this paper can also be utilized to study the internal support optimization.

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