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A polyhedral study of the maximum stable set problem with weights on vertex-subsets[☆]

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ABSTRACT

Given a graph $G = (V, E)$, a family of nonempty vertex-subsets $\mathcal{S} \subseteq 2^V$, and a weight $w : \mathcal{S} \to \mathbb{R}_+$, the *maximum stable set problem with weights on vertex-subsets* consists in finding a stable set I of G maximizing the sum of the weights of the sets in $\mathcal{S}$ that intersect I. This problem arises within a natural column generation approach for the vertex coloring problem. In this work we perform an initial polyhedral study of this problem, by introducing a natural integer programming formulation and studying the associated polytope. We address general facts on this polytope including some lifting results, we provide connections with the stable set polytope, and we present three families of facet-inducing inequalities.

1. Introduction

In this work we address a generalization of the maximum weighted stable set problem which arises in the solution of the vertex coloring problem via column generation techniques. Given a graph $G = (V, E)$, a family of nonempty vertex-subsets $\mathcal{S} \subseteq 2^V$, and a weight $w : \mathcal{S} \to \mathbb{R}_+$, we define the *maximum stable set problem with weights on vertex-subsets* (STABws) as the problem of finding a stable set I of G (i.e., a set $I \subseteq V$ such that no two vertices from I are adjacent) that maximizes the sum of the weights of the sets in $\mathcal{S}$ that intersect I. Formally, STABws consists in finding a stable set $I \subseteq V$ maximizing $\sum\{w(S) : S \in \mathcal{S} \text{ and } S \cap I \neq \emptyset\}$. In this context, the vertex subsets in $\mathcal{S}$ are called *structures*. For $v \in V$, define $S(v) = \{S \in \mathcal{S} : v \in S\}$.

STABws naturally arises within the column generation procedure of a straightforward algorithm for the classical vertex coloring problem [2]. In this setting, let $\mathcal{S} \subseteq 2^V$ be the set of maximal stable sets of G. We have a binary variable z_I for every $I \in \mathcal{S}$, and the constraints

$$\sum_{I \in \mathcal{S} : v \in I} z_I \geq 1, \quad v \in V. \tag{1}$$

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In a standard column generation approach, the pricing problem reduces to finding a maximum weighted stable set [3–5]. However, if additional cutting planes are added then the column generation problem no longer corresponds to a classical weighted stable set problem, since the objective function now includes the dual variables corresponding to the added inequalities.

Indeed, let $S_1, \ldots, S_k \subseteq V$ be subsets of vertices and assume that the following cuts have been added to the original formulation:

$$\sum_{I \in \mathcal{S}: I \cap S_i \neq \emptyset} z_I \geq b_i, \quad i = 1, \ldots, k, \tag{2}$$

where $b_1, \ldots, b_k \in \mathbb{R}$ are suitable values making these inequalities valid, i.e., b_i is a lower bound on the minimum number of colors needed to color the subgraph induced by S_i. If λ_v is the dual variable corresponding to Eq. (1), for $v \in V$, and μ_i is the dual variable corresponding to the inequality (2), for $i = 1, \ldots, k$, then the column generation problem consists in finding a stable set $I \subseteq V$ maximizing $\sum\{\lambda_v : v \in I\} + \sum\{\mu_i : S_i \cap I \neq \emptyset, i = 1, \ldots, k\}$. This problem reduces to STABws, where the structures are $\mathcal{S} := \{S_i\}_{i=1}^{k} \cup \{\{v\}\}_{v \in V}$. Note that it is not necessary to consider vertex weights in the definition of STABws, since we can consider a singleton structure for each vertex, thus modeling such weights.

This pricing problem is addressed in [2], where the authors point out that the pricing problem is not exactly a maximum weight stable set problem anymore, give some insights on how the two problems relate to each other, and use these insights in order to improve the bounds of the coloring formulation. Section 6.4 from [2] points out the difficulty to deal with cut generation approaches for the vertex coloring problem (based on the original column generation algorithm in [5]). In [2], the authors attempt to circumvent this difficulty by stating an optimization problem that is "a good approximation of the pricing problem". In practice, such an approximation leads to small improvements in the lower bound, at the cost of a moderate increase in the running times. However, the authors state that the enumeration tree is in most cases significantly smaller. Thus, one can expect that more efficient strategies to solve the pricing problem when cuts are present could help to improve the performance of the whole procedure for solving the vertex coloring problem.

There are two possible approaches for tackling STABws, namely searching for combinatorial algorithms and studying its polyhedral structure with the objective of implementing an algorithm based on integer programming techniques. Combinatorial algorithms for the maximum stable set problem are more effective than integer-programming-based procedures [6,7], but it is not clear how such algorithms can be applied to STABws. This motivates the present work, which aims at providing more details on the structure of the generalized problem and its similarities and main differences with the weighted maximum stable set problem.

In this work we address STABws from an integer programming point of view. We are interested in partial descriptions of the polytope associated with a natural integer programming formulation of this problem. We provide general results on this polytope, including some properties of general facet-inducing valid inequalities, relations between facets of this polytope and facets of the stable set polytope, a straightforward lifting lemma, and a lifting procedure for generating more complex facet-inducing inequalities. We also show how STABws can be reduced to a stable set problem on a larger graph. Although this reduction may not be useful from a practical point of view, it shows additional connections between the polytope studied in this work and the standard stable set polytope. Finally, we present three families of facet-inducing inequalities, two of them being of a quite general nature.

This paper is organized as follows. In Section 2 we present an integer programming formulation of STABws and provide some initial results on the associated polytope $P(G, \mathcal{S})$ (to be defined in Section 2). Section 3 presents a procedure for deriving strong inequalities for $P(G, \mathcal{S})$ from the stable set polytope. Section 4 presents relations between $P(G, \mathcal{S})$ and the stable set polytope, and in Section 5 we explore general families of facets for $P(G, \mathcal{S})$. Finally, Section 6 closes the paper with concluding remarks and open problems.

2. Integer programming formulation

In this section we present an integer programming formulation for STABws. For each vertex $v \in V$, we introduce the binary *vertex variable* x_v such that $x_v = 1$ if and only if the vertex v belongs to the solution. For each structure $S \in \mathcal{S}$, we introduce the binary *structure variable* y_S such that $y_S = 1$ only if the solution intersects S. With these definitions, STABws can be formulated as follows.

$$\max \sum_{S \in \mathcal{S}} w_S y_S$$

$$x_u + x_v \leq 1 \qquad \forall\, uv \in E \tag{3}$$

$$y_S \leq \sum_{v \in S} x_v \qquad \forall\, S \in \mathcal{S} \tag{4}$$

$$x_v \in \{0, 1\} \qquad \forall\, v \in V \tag{5}$$

$$y_S \in \{0, 1\} \qquad \forall\, S \in \mathcal{S}. \tag{6}$$

The objective function asks for the total weight to be maximized. Constraints (3) assert that no two adjacent vertices can be selected (hence x is the characteristic vector of a stable set), whereas constraints (4) assert that $y_S = 0$ if no vertex from S is selected. Note that we do not constrain y_S to take value 1 when some vertex from S is selected in the current solution, since we assume the weights to be non-negative.

Definition 1. Given a graph $G = (V, E)$ and a family of vertex-subsets $\mathcal{S} \subseteq 2^V$, we define $P(G, \mathcal{S})$ to be the convex hull of the vectors $(x, y) \in \mathbb{R}^{|V|+|\mathcal{S}|}$ satisfying constraints (3)–(6).

We now collect some straightforward facts on $P(G, \mathcal{S})$. For $v \in V$, we denote by $N_G(v)$ the neighborhood of v in G, and we simply write $N(v)$ when the graph is clear from the context.

Theorem 1. (i) *The polytope $P(G, \mathcal{S})$ is full-dimensional.*
(ii) *For each $uv \in E$, the model constraint (3) induces a facet of $P(G, \mathcal{S})$ if and only if uv is not contained in a larger clique, i.e., $N(u) \cap N(v) = \emptyset$.*
(iii) *For each $S \in \mathcal{S}$, the model constraint (4) induces a facet of $P(G, \mathcal{S})$.*
(iv) *For each $v \in V$, the inequality $x_v \geq 0$ induces a facet of $P(G, \mathcal{S})$. For each $S \in \mathcal{S}$, the inequality $y_S \geq 0$ induces a facet of $P(G, \mathcal{S})$.*

The following lemma states a property that will be useful throughout the paper. A similar property holds for the x-variables in the stable set polytope $STAB(G)$, defined as the convex hull of the vectors $x \in \mathbb{R}^{|V|}$ satisfying constraints (3) and (5).

Lemma 1. *Let $\pi x + \mu y \leq \pi_0$ be a facet-inducing inequality of $P(G, \mathcal{S})$, $S \in \mathcal{S}$ and $v \in V$. If the inequality differs from $y_S \geq 0$, then $\mu_S \geq 0$. If the inequality differs from $x_v \geq 0$ and $\mu_S = 0$ for all $S \in S(v)$, then $\pi_v \geq 0$.*

Proof. Since $\pi x + \mu y \leq \pi_0$ is different from $y_S \geq 0$ and is facet-inducing, there exists a solution (x, y) satisfying the inequality with equality and such that $y_S = 1$ (otherwise, $y_S = 0$ for every point in the face induced by the inequality, contradicting facetness). Construct a new feasible solution (x, y') only differing from (x, y) in $y'_S = 0$. If $\mu_S < 0$ then this new solution violates $\pi x + \mu y \leq \pi_0$, a contradiction. Hence, $\mu_S \geq 0$ for every $S \in \mathcal{S}$.

Now assume that the inequality is different from $x_v \geq 0$ and $\mu_S = 0$ for all $S \in S(v)$. Therefore, there exists $(x, y) \in P(G, \mathcal{S})$ satisfying $\pi x + \mu y = \pi_0$, $x_v = 1$ and $y_S = 0$, for all $S \in S(v)$. Changing the x_v to 0 leads to another feasible solution that does not violate the inequality only if $\pi_v \geq 0$. □

The polytope $P(G, \mathcal{S})$ admits facet-inducing inequalities with negative coefficients for x-variables, in contrast to $STAB(G)$, whose facets have nonnegative coefficients—with the exception of $x_v \geq 0$ for $v \in V$. This can only happen for vertices that belong to at least one structure, as the following corollary to Lemma 1 shows.

Corollary 1. *If $v \in V$ does not belong to any structure in $\mathcal{S}$ and $\pi x + \mu y \leq \pi_0$ is a facet-inducing inequality of $P(G, \mathcal{S})$ different from $x_v \geq 0$, then $\pi_v \geq 0$.*

The particular structure of the formulation (3)–(6) provides a simple lifting result for the structure variables, as the following proposition shows.

Lemma 2 (*Lifting Lemma*). *Every facet-inducing inequality for $P(G, \mathcal{S})$ is also valid and facet-inducing for $P(G, \mathcal{S} \cup \{S\})$, for any $S \subseteq V$, $S \notin \mathcal{S}$, $S \neq \emptyset$.*

Proof. If the facet-inducing inequality for $P(G, \mathcal{S})$ is $x_v \geq 0$, then the lemma holds by Theorem 1. Otherwise, let $\pi x + \mu y \leq \pi_0$ be an inequality inducing the facet F of $P(G, \mathcal{S})$ and $S \subseteq V$, $S \notin \mathcal{S}$, $S \neq \emptyset$. Select affinely independent points $(x^1, y^1), \ldots, (x^{|V|+|\mathcal{S}|}, y^{|V|+|\mathcal{S}|})$ in F. Since these selected points cannot be all in the intersection with the facet $\{(x, y) \in P(G, \mathcal{S}) : x_v = 0\}$ for any $v \in V$, let us consider a point (x^i, y^i), for some $i \in \{1, \ldots, |V|+|\mathcal{S}|\}$, such that there exists a $v \in S$ with $x^i_v = 1$. Hence, $(x^i, y^i, 1), (x^1, y^1, 0), \ldots, (x^{|V|+|\mathcal{S}|}, y^{|V|+|\mathcal{S}|}, 0)$ are $|V| + |\mathcal{S}| + 1$ affinely independent points in $\{(x, y, y_S) \in P(G, \mathcal{S} \cup \{S\}) : \pi x + \mu y = \pi_0\}$. □

The projection of $P(G, \mathcal{S})$ onto the space of the vertex variables coincides with the stable set polytope $STAB(G)$, so every valid inequality for $STAB(G)$ is also valid for $P(G, \mathcal{S})$. Furthermore, Lemma 2 shows that facetness is also preserved.

Corollary 2. *Every facet-inducing inequality of $STAB(G)$ is also valid and facet-inducing for $P(G, \mathcal{S})$.*

On the other hand, as $x \in STAB(G)$ implies $(x, 0) \in P(G, \mathcal{S})$ and $(x, y) \in P(G, \mathcal{S})$ implies $x \in STAB(G)$, we get a converse for the above statement.

Proposition 1. *If $\pi x \leq \pi_0$ is facet-inducing for $P(G, \mathcal{S})$, it is also valid and facet-inducing for $STAB(G)$.*

Lifting Lemma implies that all the facets for $P(G, \mathcal{S})$ translate to $P(G, \mathcal{S} \cup \{S\})$, for any $S \notin \mathcal{S}$. There exist additional facets, in particular facets involving y_S with a nonzero coefficient (which furthermore must be positive, by Lemma 1). In those cases, we have the following result.

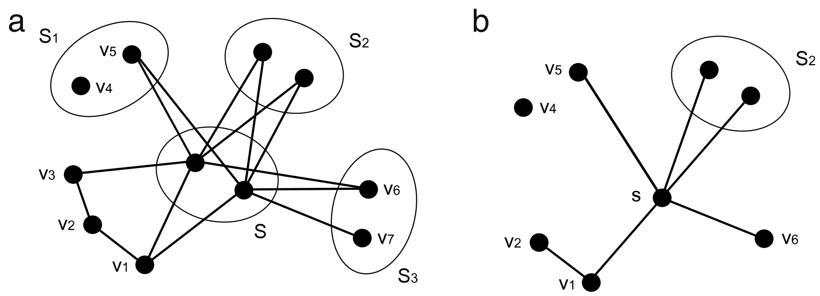

Fig. 1. (a) Example instance $(G, \mathcal{S})$ of STABws with $\mathcal{S} = \{S_1, S_2, S_3\}$ and (b) the instance $(G', \mathcal{S}')$ constructed by the procedure in Section 3. The vertices v_3 and v_7 are adjacent to just one vertex from S, so they are deleted. Structure S_1 has a vertex not adjacent to any vertex from S and structure S_3 gets at least one vertex deleted, so they are removed from $\mathcal{S}$.

Proposition 2. *If $S \in \mathcal{S}$ induces a clique, then a valid inequality $\pi x + \mu y \leq \pi_0$ with $\mu_S > 0$ is facet-inducing if and only if it coincides with the model constraint (4).*

Proof. The "if" part is a direct rephrasing of Theorem 1(iii). For the converse direction, we claim that every feasible solution (x, y) satisfying the inequality with equality also is in the facet $\{(x, y) \in P(G, \mathcal{S}) : y_S = \sum_{v \in S} x_v\}$. Since S induces a clique, by (3)–(4) we have $y_S \leq \sum_{v \in S} x_v \leq 1$. If $y_S = 0$ and $\sum_{v \in S} x_v = 1$, then by setting $y_S = 1$ we still have a feasible solution which violates $\pi x + \mu y \leq \pi_0$ (since $\mu_S > 0$), a contradiction. We conclude that $y_S = \sum_{v \in S} x_v$ and, since (x, y) is an arbitrary solution satisfying $\pi x + \mu y = \pi_0$, the claim holds. □

In the facetness results throughout this paper we will ask for the structures participating in the valid inequalities to not induce cliques of G (or, alternatively, stronger hypotheses implying this condition). Proposition 2 ensures that these hypotheses are indeed necessary for achieving facetness.

3. A procedure for generating strong valid inequalities

Lifting Lemma (Lemma 2) shows that every facet-inducing inequality of $STAB(G)$ is also facet-inducing for $P(G, \mathcal{S})$. In this section we show that facets of $P(G, \mathcal{S})$ involving the structure variables (i.e., the y-variables) can also be generated from facets of the stable set polytope in certain particular situations, and we provide an iterative procedure for accomplishing this task. If $A, B \subseteq V$ are two vertex sets, we denote $E(A, B) = \{uv \in E : u \in A, v \in B\}$ to be the set of edges with one endpoint in A and the other endpoint in B.

Procedure. Let $S \subseteq V$, $S \notin \mathcal{S}$, $S \neq \emptyset$.

1. Remove from G all vertices $v \in V \backslash S$ such that $E(S, \{v\}) \neq S \times \{v\}$ and $E(S, \{v\}) \neq \emptyset$. Call $R \subseteq V$ the removed vertices and $\tilde{G} = (\tilde{V}, \tilde{E})$ the resulting graph.
2. Remove from $\mathcal{S}$ any structure S' such that $S' \cap R \neq \emptyset$ or $S' \cap \{v : E(S, \{v\}) = \emptyset\}$ or $S' \cap S \neq \emptyset$. Call $\mathcal{S}'$ the resulting family of structures.
3. Identify the vertices in S onto a single vertex s. This identification operation consists in deleting all vertices in S from $\tilde{V}$ (and its incident edges), adding the new vertex s, and adding an edge sv for every $v \in \tilde{V} \setminus S$ such that $E(S, \{v\}) = S \times \{v\}$. Call $G' = (V', E')$ the resulting graph (see Fig. 1 for an example).
4. Find a facet $\pi x' + \mu y' \leq \pi_0$ of $P(G', \mathcal{S}')$, and output the inequality

$$\sum_{v \in V' \backslash \{s\}} \pi_v x_v + \pi_s y_S + \sum_{S' \in \mathcal{S}'} \mu_{S'} y_{S'} \leq \pi_0. \tag{7}$$

Recall that Corollary 1 implies that $\pi_s \geq 0$, which is required by Lemma 1 for (7) to define a facet.

Theorem 2. *The inequality (7) generated by the procedure is valid for $P(G, \mathcal{S} \cup \{S\})$. In addition, if S is not a clique, then (7) is facet-inducing for $P(\tilde{G}, \mathcal{S}' \cup \{S\})$.*

Proof. Let $(x, y) \in P(G, \mathcal{S} \cup \{S\})$ be an arbitrary integer solution, and let $(x', y') \in \{0, 1\}^{|V'|+|\mathcal{S}'|}$ be an associated solution (to be shown to belong to $P(G', \mathcal{S}')$), defined by $x'_v = x_v$ for $v \in V' \backslash \{s\}$, $x'_s = y_S$, and $y'_{S'} = y_{S'}$ for $S' \in \mathcal{S}'$. Since every $v \in N_{G'}(s)$ has $E(S, \{v\}) = S \times \{v\}$ by construction, then $x'_s + x'_v = y_S + x_v \leq 1$ holds. Moreover, $N_{G'}(v) \subseteq N_G(v) \cup \{s\}$, for all $v \neq s$. Then, x' induces a stable set in G'. Also, if $y'_{S'} = 1$ for some $S' \in \mathcal{S}'$ then $x'_v = 1$ for some $v \in S'$, since all the vertices in S' remain in G'. Hence, $(x', y') \in P(G', S')$. Since $\pi x' + \mu y' \leq \pi_0$ is valid for $P(G', \mathcal{S}')$ and $x'_s = y_S$, we conclude that the inequality generated by the procedure is valid for $P(G, \mathcal{S} \cup \{S\})$.

In order to prove that (7) induces a facet of $P(\tilde{G}, \mathcal{S}' \cup \{S\})$, we construct $|\tilde{V}| + |\mathcal{S}'| + 1$ affinely independent points in the face induced by the inequality. To this end, fix some $u \in S$ and define $\Phi : P(G', \mathcal{S}') \to P(\tilde{G}, \mathcal{S}' \cup \{S\})$ mapping a point (x', y')

into a point $\Phi(x', y') = (x, y)$ as follows. We have $x_v = x'_v$ for all $v \in V' \setminus \{s\}$, $x_u = y_S = x'_s$, $x_v = 0$ for all $v \in S \setminus \{u\}$, and $y_{S'} = y'_{S'}$ for all $S' \in \mathcal{S}'$. Equipped with this mapping, we construct the following affinely independent points satisfying (7) with equality:

- Let $\mathcal{A} = \{(x^i, y^i)\}_{i=1}^{|V'|+|\mathcal{S}'|}$ be a set of affinely independent points satisfying $\pi x' + \mu y' = \pi_0$, and construct $\{\Phi(x^i, y^i)\}_{i=1}^{|V'|+|\mathcal{S}'|}$. These new points are affinely independent and satisfy the new inequality with equality.
- Assume w.l.o.g. $x_s^1 = 1$ and construct $|S| - 1$ new points from $\Phi(x^1, y^1)$ by alternatively replacing u by the remaining vertices in S. The new points are feasible by construction, satisfy (7) with equality, and are affinely independent with the preceding points since the new vertices are not present in the previously constructed feasible solutions. This way we get $|S| - 1$ additional points.
- Let $v, w \in S$ such that $vw \notin E$ (such vertices exist since S is not a clique), and construct a new point from $\Phi(x^1, y^1)$ by replacing u by v and w. Since every neighbor of u in G' is also a neighbor of v and w in G, then this new solution is feasible, and satisfies (7) with equality. Differently from the previous points, this point is not included in the hyperplane $y_S = \sum_{v \in S} x_v$.

Since there exist $|\tilde{V}| + |\mathcal{S}'| + 1$ affinely independent points satisfying (7) at equality, we conclude that this inequality induces a facet of $P(\tilde{G}, \mathcal{S}' \cup \{S\})$. □

4. Relations to the stable set polytope

In this section we explore the close relationship that exists between $P(G, \mathcal{S})$ and the stable set polytope, although on a different graph. Fix a graph G and a set of structures $\mathcal{S}$, and consider the following two operations. The first operation constructs an equivalent instance with no intersecting structures.

Splitting. If there exists some vertex $v \in V$ such that $S(v) = \{S_1, \ldots, S_k\}$ with $k \geq 2$, let D_1, D_2 be a nontrivial partition of $\{1, \ldots, k\}$ and define a new graph $G' := (V \cup \{v'\}, E \cup \{v'w : w \in N_G(v)\})$ (i.e., vertex v' is added to G, in such a way that v and v' are *false twins*), and consider the structure set $\mathcal{S}' := \left[\mathcal{S} \backslash S(v)\right] \cup \{S_i : i \in D_1\} \cup \{(S_i \backslash \{v\}) \cup \{v'\} : i \in D_2\}$. In other words, a false twin of v is created and the structures in $S(v)$ are "divided" between these twins, namely the structures indexed by D_1 remain with v and the structures indexed by D_2 are removed of vertex v and receive vertex v' instead. Define the weights $w' : \mathcal{S}' \to \mathbb{R}_+$ as $w'(S) = w(S)$ for $S \in \mathcal{S} \backslash S(v)$, $w'(S_i) = w(S_i)$ for $i \in D_1$, and $w'((S_i \backslash \{v\}) \cup \{v'\}) = w(S_i)$ for $i \in D_2$. We call $I_1 := (G_1, \mathcal{S}_1, w) := (G, \mathcal{S}, w)$ the original instance of STABws, and $I'_1 = (G', \mathcal{S}', w')$ the instance obtained after this splitting operation.

Mappings that preserve the objective function value of the solutions of an instance of STABws and the one derived from the splitting operation are as follows. Let $\mathcal{I}_1$ and $\mathcal{I}'_1$, respectively, be the families of all solutions of the two instances I_1 and I'_1 mentioned above.

Definition 2 ($\Phi_{1,1'} : \mathcal{I}_1 \to \mathcal{I}'_1$). A solution for I_1 encoded as (x, y) maps to a solution $\Phi_{1,1'}(x, y) = (x', y')$ of I'_1 such that $x'_v = x'_{v'} = x_v$, $x'_u = x_u$ for all $u \in V \setminus \{v\}$, and $y' = y$.

Definition 3 ($\Phi_{1',1} : \mathcal{I}'_1 \to \mathcal{I}_1$). A solution $(x', y') \in \mathcal{I}'_1$ maps to a solution $\Phi_{1',1}(x', y') = (x, y)$ of I_1 such that $x_v = \max\{x'_v, x'_{v'}\}$, $x_u = x'_u$ for all $u \in V \setminus \{v\}$, and $y' = y$.

The mappings $\Phi_{1,1'}$ and $\Phi_{1',1}$ yield an equivalence between the two problems, and imply the following result.

Proposition 3. *The optimal values of I_1 and I'_1 coincide.*

After a finite number of applications of this operation, the resulting instance of STABws does not have intersecting structures. We call $I_2 = (G_2, \mathcal{S}_2, w^2)$ the instance obtained as a fixed point of this splitting operation. Let $\Phi_{1,2}$ represent the composition of the several applications of $\Phi_{1,1'}$ that transform a solution of I_1 into a solution of I_2. Similarly, let us define $\Phi_{2,1}$ as the converse transformation.

The following operation converts I_2 into an instance I_3 of the maximum weighted stable set problem.

Conversion into cliques. Let $G_2 = (V_2, E_2)$ and construct $I_3 = (G_3, w^3)$ by $G_3 = (V_2, E_2 \cup \left[\cup_{S \in \mathcal{S}_2} \{uv : u, v \in S\}\right])$. Furthermore, construct vertex weights $w^3 : V_2 \to \mathbb{R}_+$ defined by $w_v^3 = w_S^2$ if $v \in S$ for some $S \in \mathcal{S}_2$, and $w_v^3 = 0$ otherwise, for every $v \in V_2$. Since each vertex belongs to at most one structure in $\mathcal{S}_2$, then w^3 is well defined. In other words, we add all the edges among the vertices of each structure S (so it becomes a clique in G_3) and assign the structure weight w_S to each vertex in S. Note that the weight of any stable set I in G_3 coincides with the weight of the corresponding feasible solution of I_2, where all the structures intersecting I are selected. This is the basis of the following result.

Again, an equivalence between the two problems is established by means of a definition of mappings that preserve the value of the objective function. To this end, let $\mathcal{I}_3$ be the family of all solutions of instance I_3 defined above.

Definition 4 ($\Phi_{2,3} : \mathcal{I}_2 \to \mathcal{I}_3$). A solution for I_2 encoded as (x, y) maps to a solution $\Phi_{2,3}(x, y) = x'$ of I_3 such that $0 \leq x'_v \leq x_v$, for all $v \in V$, and $\sum_{v \in S} x'_v = y_S$, for all $S \in \mathcal{S}_2$.

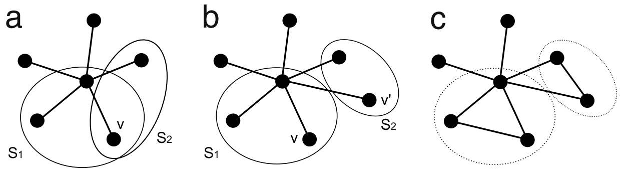

Fig. 2. (a) Example instance I_1 of STABws, (b) splitted instance I_2, and (c) equivalent instance I_3 of the maximum weighted stable set problem.

Definition 5 ($\Phi_{3,2} : \mathcal{I}_3 \to \mathcal{I}_2$). A solution $x' \in \mathcal{I}_3$ maps to a solution $\Phi_{3,2}(x') = (x, y)$ of I_2 such that $x_v = x'_v$, for all $v \in V$, and $y_S = \sum_{v \in S} x'_v$, for all $S \in \mathcal{S}_2$.

Again, the existence of these mappings establishes the following result.

Proposition 4. *The optimal values of I_2 and I_3 coincide.*

This way, starting from the original instance I_1 of STABws, we construct an equivalent instance I_2 of STABws and an equivalent instance I_3 of the maximum weighted stable set problem (see Fig. 2 for an example). Let $P_1 := P(G_1, \mathcal{S}_1)$, $P_2 := P(G_2, \mathcal{S}_2)$, and $P_3 := STAB(G_3)$ be the corresponding polytopes. We are interested in the relations among these three polytopes, and in this section we provide partial results on this issue.

It is worth remarking that in practice it is not generally a good idea to search for a maximum-weight stable set in I_3 instead of directly tackling I_1, since in the setting that motivated this work the number of structures (corresponding to the number of previously generated cutting planes) may be quite large. Due to this fact, I_3 will generally be much larger than I_1 if the cutting planes have intersecting supports. Nevertheless, the relations between the corresponding polytopes are interesting and – in our opinion – are worth investigating.

4.1. Relations between P_1 and P_2

We first consider relations between P_1 and P_2. For the sake of simplicity, we assume that G_1 has exactly one vertex v contained in more than one structure, and furthermore that v is contained in exactly two structures, called S_1 and S_2 respectively. This is not too strong an assumption, since if there are more than two structures containing v or there are more vertices in this situation, we can iteratively apply the results we obtain for this basic case. In the new instance I_2, we assume S_1 to include v, and S_2 to include the false twin v'.

Given a vector z with entries indexed by a set U, and a subset $U' \subseteq U$, let $z_{U'}$ denote the subvector comprising the entries indexed by U'.

Proposition 5 (*Facets from P_1 to P_2*). *If $\pi x + \mu y \le \pi_0$ is valid for P_1 and $\pi_v \ge 0$, then this inequality is also valid for P_2. If, furthermore,* (a) *the inequality is facet-inducing for P_1 and* (b) *$\mu_{S_2} = 0$ or there exists a feasible solution (x, y) satisfying $\pi x + \mu y = \pi_0$, $x_v = 1$, and $\sum_{w \in S_2} x_w \ge 2$, then $\pi x + \mu y \le \pi_0$ is facet-inducing for P_2.*

Proof. Let (x', y') be an integer point in P_2 and $(x, y) = \Phi_{2,1}(x', y')$. Since $(x, y) \in P_1$, we get $\pi x + \mu y \le \pi_0$. In addition, Definition 3 states that $y = y'$ and $x'_w \le x_w$ holds for all $w \in V$, being satisfied at equality if $w \ne v$. Thus, $\pi_v \ge 0$ leads to $\pi x'_V + 0x'_{v'} + \mu y' \le \pi x + \mu y \le \pi_0$, which implies validity.

To show facetness, let $(x^1, y^1), \dots, (x^{|V|+|\mathcal{S}|}, y^{|V|+|\mathcal{S}|})$ be affinely independent points in $F = \{(x, y) \in P_1 : \pi x + \mu y = \pi_0\}$. The points $\mathcal{A} = \{\Phi_{1,2}(x^i, y^i)\}_{i=1}^{|V|+|\mathcal{S}|}$ are also affinely independent, and satisfy $\pi x + \mu y = \pi_0$. Let $(x, y) \in F$ with $x_v = 1$, which exists because $\pi_v \ge 0$ and so the facet-inducing inequality is not $-x_v \le 0$. By hypothesis (b), $\mu_{S_2} = 0$ or we can assume that $x_u = 1$ for some $u \in S_2 \backslash \{v\}$. In the first case, construct the feasible solution $(\bar{x}, \bar{y}) \in P_2$, where $\bar{x}_V = x$, $\bar{x}_{v'} = x_v$, $\bar{y}_{S_2} = 1 - y_{S_2}$ and $\bar{y}_S = y_S$, for all $S \ne S_2$. In the second case, construct the feasible solution $(\bar{x}, \bar{y}) \in P_2$, where $\bar{x}_V = x$, $\bar{x}_{v'} = 0$ and $\bar{y} = y$. In both cases, the new solution is feasible and is affinely independent w.r.t. the points in $\mathcal{A}$, since $x_v = x_{v'}$ and $y = y'$ hold for every $(x, y) \in \mathcal{A}$. Hence, $\mathcal{A} \cup \{(\bar{x}, \bar{y})\}$ is a set of $|V| + |\mathcal{S}| + 1$ affinely independent points in P_2 satisfying $\pi x + \mu y \le \pi_0$ with equality, so this inequality induces a facet of P_2. □

Some words on the hypothesis (b) are in order. Assume $\mu_{S_2} \ne 0$. Since $y_{S_2} \le \sum_{w \in S_2} x_w$ and $\mu_{S_2} > 0$ by Lemma 1, then there is no feasible solution in the face of P_1 with $\sum_{w \in S_2} x_w \ge 1$ and $y_{S_2} = 0$ (notice that setting $y_{S_2} = 1$ would provide a larger LHS, thus violating the inequality). This implies that there must exist some feasible solution in this face with $\sum_{w \in S_2} x_w \ge 2$, since otherwise we would get $y_{S_2} = \sum_{w \in S_2} x_w$, contradicting the fact that $\pi x + \mu y \le \pi_0$ induces a facet and $\pi_v \ge 0$. The hypothesis (b) asks for the existence of such a solution, where moreover one of the selected vertices is v.

Proposition 5 implies that the symmetrical inequality $\pi_v x_{v'} + \sum_{u \ne v} \pi_u x_u + \mu y \le \pi_0$ (i.e., the inequality obtained from $\pi x + \mu y \le \pi_0$ by replacing x_v by $x_{v'}$) is also valid for P_2, and induces a facet under the same assumptions.

For the converse direction, note that P_1 consists of the intersection of P_2 with the hyperplane $\{(x, y) : x_v = x_{v'}\}$, then projected down in order to eliminate the variable x'_v. If we denote $\text{proj}_{v'}(P)$ to be the polytope obtained from P by projecting

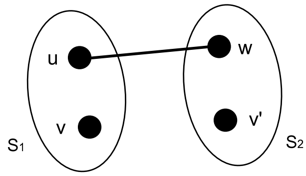

Fig. 3. Instance showing that the inclusion (8) is strict.

out the variable $x_{v'}$, then

$$P_1 \subseteq \text{proj}_{v'}\left[P_2 \cap \{(x, y) : x_v = x_{v'}\}\right], \tag{8}$$

and this inclusion can be strict. Indeed, consider the instance depicted in Fig. 3. The point (x, y) with $x = (1/2, 1/2, 1/2)$ and $y = (1, 1)$ does not belong to P_1 (since, e.g., it violates $y_{S_1} + y_{S_2} \le x_v + 1$, which is valid for P_1) but the corresponding point in P_2, i.e., $x = (1/2, 1/2, 1/2, 1/2)$ and $y = (1, 1)$ is a convex combination of $(x^1_v, x^1_{v'}, x^1_u, x^1_w, y^1_{S_1}, y^1_{S_2}) = (1, 0, 0, 1, 1, 1)$ and $(x^2_v, x^2_{v'}, x^2_u, x^2_w, y^2_{S_1}, y^2_{S_2}) = (0, 1, 1, 0, 1, 1)$.

Lemma 3. *Let $U = \{1, 2, \ldots, p\}$ and $U' = U \cup \{p + 1\}$. If $\{z^i\}_{i\in U}$ is a linearly independent set in $\mathbb{R}^{p+1}$, then $\{z^i_U\}_{i\in U}$ has a linearly independent subset with $p - 1$ vectors in $\mathbb{R}^p$. If $\{z^i\}_{i\in U'}$ is an affinely independent set in $\mathbb{R}^{p+1}$, then $\{z^i_U\}_{i\in U'}$ has an affinely independent subset with p vectors in $\mathbb{R}^p$.*

Proof. Let $A \in \mathbb{R}^{(p+1)\times p}$ and $A_U \in \mathbb{R}^{p\times p}$ be the matrices whose columns are the vectors $\{z^i\}_{i\in U}$ and $\{z^i_U\}_{i\in U}$, respectively. If $\text{rank}(A) = p$, then $\text{rank}(A_U) \ge p - 1$. This proves the first part. To show the second one, it is enough to recall that $\{z^i\}_{i\in U'}$ is an affinely independent set if, and only if, $\{z^i - z^{p+1}\}_{i\in U}$ is a linearly independent set. □

Proposition 6. *If $\pi x + \mu y \le \pi_0$ is valid for P_2, then $\sum_{u\ne v,v'} \pi_u x_u + (\pi_v + \pi_{v'})x_v + \mu y \le \pi_0$ is valid for P_1. Furthermore, it is facet-inducing for P_1 if $\pi x + \mu y \le \pi_0$ is facet-inducing for P_2, $\pi_{v'} = 0$ and $\pi_v > 0$.*

Proof. Let (x, y) be an arbitrary point in P_1, and consider $(x', y') = \Phi_{1,2}(x, y)$. Since this new point belongs to P_2, $\pi x' + \mu y' \le \pi_0$ holds. By the definition of $\Phi_{1,2}$, we get $\pi x' + \mu y' = \sum_{u\ne v,v'} \pi_u x_u + (\pi_v + \pi_{v'})x_v + \mu y$, and the first part of the proposition follows.

To show facetness, let $(x^1, y^1), \ldots, (x^{|V|+|\mathcal{S}|+1}, y^{|V|+|\mathcal{S}|+1})$ be affinely independent integer points in $F = \{(x, y) \in P_2 : \pi x + \mu y = \pi_0\}$. Let $i = 1, 2, \ldots, |V| + |\mathcal{S}| + 1$. If $x^i_{v'} = 1$, then $x^i_v = 1$—otherwise, setting $x^i_v = 1$ would lead to another point in P_2 violating the inequality, since $\pi_v > 0$. Therefore, $\max\{x^i_{v'}, x^i_v\} = x^i_v$. This implies $(\bar{x}^i, \bar{y}^i) := \Phi_{2,1}(x^i, y^i) = (x^i_V, y^i)$ and $\sum_{u\ne v,v'} \pi_u \bar{x}^i_u + (\pi_v + \pi_{v'})\bar{x}^i_v + \mu\bar{y}^i = \pi x^i + \mu y^i = \pi_0$, provided that $\pi_{v'} = 0$. Finally, Lemma 3 ensures that the set $\mathcal{A} = \{(x^i_V, y^i)\}_{i=1}^{|V|+|\mathcal{S}|+1}$ has an affinely independent subset with $|V| + |\mathcal{S}|$ points. □

4.2. Relations between P_2 and P_3

We now consider relations between P_2 and P_3. We will assume that $\mathcal{S}_2 = \{S\}$, so the following results apply for single-structure instances, but they can be iteratively applied for general instances. In general, P_3 is a projected face of P_2, since $P_2^{=} = \{(x, y) \in P_2 : y_S = \sum_{v\in S} x_v\}$ is a facet of P_2, and

$$P_3 = \text{proj}_S(P_2^{=}),$$

where $\text{proj}_S(P)$ denotes the polytope obtained from P by projecting out the variable y_S (see Fig. 4).

Proposition 7. *If $\pi x + \beta y_S \le \pi_0$ is a valid inequality for P_2, then $\pi x + \beta\sum_{v\in S} x_v \le \pi_0$ is valid for P_3. Furthermore, if $\pi x + \beta y_S \le \pi_0$ induces a facet of $P_2^{=}$, then $\pi x + \beta\sum_{v\in S} x_v \le \pi_0$ is facet-inducing for P_3.*

Proof. Let $x \in P_3$ be an arbitrary point in P_3, and construct $(x, y) = \Phi_{3,2}(x)$, which is feasible for P_2. Since $\pi x + \beta y_S \le \pi_0$ and $y_S = \sum_{v\in S} x_v$, then $\pi x + \beta\sum_{v\in S} x_v \le \pi_0$ also holds. Since x is arbitrary, then this last inequality is valid for P_3.

Now, suppose that the inequality is facet-inducing for $P_2^{=}$. Since $dim(P_2^{=}) = dim(P_2) - 1 = |V|$, there is an affinely independent set $\mathcal{A} := \{(x^i, y^i_S) : i = 1, 2, \ldots, |V|\}$ of integer points in $P_2^{=}$ satisfying the inequality at equality. Let $i \in \{1, 2, \ldots, |V|\}$. Since $y^i_S = \sum_{v\in S} x^i_v \in \{0, 1\}$, we have that $\Phi_{2,3}(x^i, y^i) = x^i$ and $\pi x^i + \beta\sum_{v\in S} x^i_v = \pi x^i + \beta y^i_S = \pi_0$. To show that $\{x^i : i = 1, 2, \ldots, |V|\}$ is affinely independent, consider a combination $\sum_{i=1}^{|V|} \alpha_i x^i = 0$. Notice that $\sum_{i=1}^{|V|} \alpha_i y^i_S = \sum_{v\in S}\sum_{i=1}^{|V|} \alpha_i x^i_v = 0$. The desired result follows by the affine independence of $\mathcal{A}$. □

Proposition 8. *If $\pi x \le \pi_0$ is a valid inequality for P_3 with $\pi_i = \beta$ for every $i \in S$, then $\pi_{V_2\setminus S} x_{V_2\setminus S} + \beta y_S \le \pi_0$ is valid for P_2. If, furthermore, the inequality induces a facet of P_3, then $\pi_{V_2\setminus S} x_{V_2\setminus S} + \beta y_S \le \pi_0$ is facet-inducing for $P_2^{=}$.*

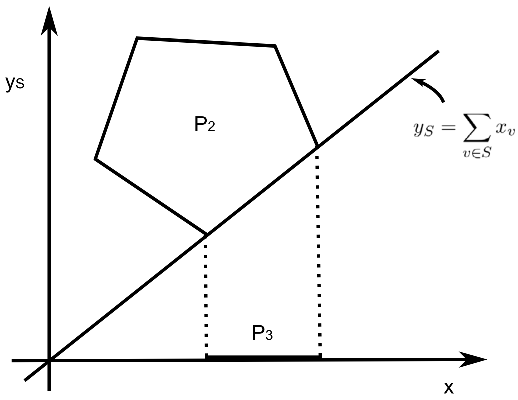

Fig. 4. The polytope P_3 is a projected face of P_2.

Propositions 8 and 6 allow us to perform a separation procedure in *STAB*(G_3) and convert the obtained inequalities into valid inequalities for $P(G, \mathcal{S})$, as long as the separated inequalities for *STAB*(G_3) have the same coefficient for all the vertices in each structure (and this can be achieved by, e.g., separating rank inequalities as in [8]).

Overall, these results do not provide a complete characterization of either polytope in terms of the other ones, but still point to the close relationship that exists between STABws and the maximum weighted stable set problem.

5. General facets of $P(G, \mathcal{S})$

We now present general families of facet-inducing inequalities for $P(G, \mathcal{S})$. The emphasis on this section lies on generality, so we shall provide quite general definitions, and when suitable we will provide particular cases corresponding to standard graph structures.

5.1. The α-inequalities

Let $\mathcal{S}' = \{S_1, \ldots, S_k\} \subseteq \mathcal{S}$ be a family of structures such that all edges between every pair of structures S_i, S_j, $i, j = 1, \ldots, k$, $i \neq j$, exist, i.e., $E(S_i, S_j) = S_i \times S_j$. If $\mathcal{S}' \neq \emptyset$, define $\mathbb{S} = \cup_{i=1}^{k} S_i$ and $N(\mathcal{S}') = \bigcap_{v \in \mathbb{S}} N(v) \setminus \mathbb{S}$, i.e., $N(\mathcal{S}')$ is the common neighborhood outside $\mathcal{S}'$ of the vertices in $\mathcal{S}'$. For consistency, if $\mathcal{S}' = \emptyset$, define $\mathbb{S} = \emptyset$ and $N(\mathcal{S}') = V$. Let $U \subseteq N(\mathcal{S}')$. As depicted in the example of Fig. 5, this configuration is such that $E(S_i, S_j) = S_i \times S_j$ and $E(S_i, U) = S_i \times U$, for all $i, j = 1, \ldots, k$, $i \neq j$. Finally, for $u \in U$, denote by α_u the size of the largest stable set of U containing u. We define

$$\sum_{u \in U} \frac{1}{\alpha_u} x_u + \sum_{S \in \mathcal{S}'} y_S \leq 1 \tag{9}$$

to be the α-*inequality* associated with $\mathcal{S}'$ and U. These inequalities are inspired by the *external inequalities* in [1] for the *asymmetric representatives formulation* for the classical vertex coloring problem.

Proposition 9. *The* α*-inequality* (9) *is valid for* $P(G, \mathcal{S})$.

Proof. In every feasible solution, if a structure $S_i \in \mathcal{S}'$ is chosen, then some $v \in S_i$ must be chosen as well, which implies that no vertices in $S_j \in \mathcal{S}'$, $j \neq i$, or in U can be chosen. Consequently, $\sum_{u \in U} \frac{1}{\alpha_u} x_u + \sum_{S \in \mathcal{S}', S \neq S_i} y_S = 0$ holds for such a solution. On the other hand, if none of the structures in $\mathcal{S}'$ is chosen, then a stable set $I \subseteq U$ can be selected. Since $|I| \leq \alpha_u$, for all $u \in I$, we get $\sum_{u \in I} \frac{1}{\alpha_u} x_u \leq 1$ and (9) holds. $\square$

We now address the facetness properties of these inequalities. Following usual notation, we call $\alpha(T)$ the size of the largest stable set in $G[T]$ (i.e., the subgraph of G induced by T). We say that U is α-*maximal in* $N(\mathcal{S}')$ if $U \neq \emptyset$ and $\alpha(U \cup \{v\}) > \alpha(U)$, for every vertex $v \in N(\mathcal{S}') \backslash U$. Call $E(U) := \{uv \in E : u, v \in U\}$. We say that the edge $uv \in E(U)$ is *safe* if there exist two stable sets $I_u, I_v \subseteq U$ such that

- $I_u \backslash I_v = \{u\}$,
- $I_v \backslash I_u = \{v\}$, and
- $|I_u| = |I_v| = \alpha_z$, for all $z \in I_u \cup I_v$.

Define $G^{\text{safe}} = (U, E')$ to be the graph with vertex set U and edge set $E' = \{uv \in E(U) : uv \text{ is safe}\}$.

If $A \subseteq V$, then we define $x^A \in \{0, 1\}^{|V|}$ to be the *characteristic vector of* A within V, i.e., $x_v^A = 1$ if $v \in A$ and $x_v^A = 0$ otherwise. If $v \in V$, we write $x^v := x^{\{v\}}$ as a notational shorthand. Similarly, if $B \subseteq \mathcal{S}$, then we denote by y^B the characteristic vector of B within $\mathcal{S}$, and we define $y^S := y^{\{S\}}$ for $S \in \mathcal{S}$.

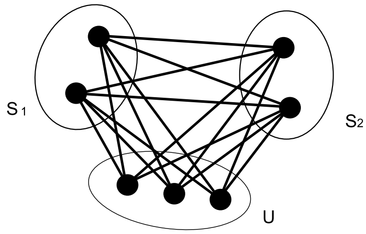

Fig. 5. Example of the support for the α-inequalities: $E(S_1, S_2) = S_1 \times S_2$, $E(S_1, U) = S_1 \times U$, and $E(S_2, U) = S_2 \times U$.

Theorem 3. *If* (a) $N(\mathcal{S}') = \emptyset$ *or* U *is* α*-maximal in* $N(\mathcal{S}')$, (b) $G[C]$ *contains a stable set of size* α_u*, for every connected component* C *of* G^{safe} *and* $u \in C$, *and* (c) *no structure in* $\mathcal{S}'$ *induces a clique, then the* α*-inequality* (9) *induces a facet of* $P(G, \mathcal{S})$.

Proof. Let F be the face of $P(G, \mathcal{S})$ defined by the inequality (9), and let $(\pi, \mu, \pi_0) \in \mathbb{R}^{|V|+|\mathcal{S}|+1}$ be such that $\pi x + \mu y = \pi_0$ for every $(x, y) \in F$. We will show that (π, μ) is a multiple of the coefficient vector of (9), thus proving that this inequality induces a facet of $P(G, \mathcal{S})$.

Claim 1: *$\pi_v = 0$ for $v \in V \backslash (U \cup \mathbb{S})$. If $v \notin N(\mathcal{S}')$, then there are $S \in \mathcal{S}'$ and $u \in S$ such that $uv \notin E$. The points (x^u, y^S) and $(x^{\{u,v\}}, y^S)$, both in F, show that $\pi_v = 0$. If $v \in N(\mathcal{S}')$, by (a) U is α-maximal in $N(\mathcal{S}')$, and so there exists a maximum stable set $I \subseteq U$ of $G[U]$ such that $I \cup \{v\}$ is also stable. The feasible solutions $(x^I, 0)$ and $(x^{I\cup\{v\}}, 0)$ both satisfy (9) with equality and show that $\pi_v = 0$.*

Claim 2: *$\pi_v = 0$ for every $v \in \mathbb{S}$. Let $i \in \{1, \ldots, k\}$. Since S_i is not a clique, there exist $u, v \in S_i$ such that $uv \notin E$. The points (x^u, y^{S_i}), (x^v, y^{S_i}), and $(x^{\{u,v\}}, y^{S_i})$ show that $\pi_u = \pi_v = 0$. For every $w \in S_i \backslash \{u, v\}$, the solutions (x^u, y^{S_i}) and (x^w, y^{S_i}) imply $\pi_w = \pi_u$, hence $\pi_w = 0$. We conclude that $\pi_t = 0$ for every $t \in S_i$.*

Claim 3: *$\pi_u = \pi_v$ for $uv \in E'$. This equality is derived from points $(x^{I_u}, 0)$ and $(x^{I_v}, 0)$, where I_u and I_v are stable sets that establish that uv is safe.*

Claim 4: *$\alpha_u \pi_u = \mu_{S_i}$ for $u \in U$ and $i = 1, \ldots, k$. Let C be the connected component of G^{safe} including u. Let $v \in S_i$ and $I \subseteq U$ be a maximum stable set of $G[C]$ containing u (given by hypothesis* (b)*). Claim 3 implies $\pi_r = \pi_s$ for every $r, s \in C$, hence $\pi_v = \pi_u$ for all $v \in I$. Moreover, we have $|I| = \alpha_u$. The points $(x^I, 0)$ and (x^v, y^{S_i}) establish the claim.*

Claim 5: *$\mu_{S_i} = \mu_{S_j}$ for $i, j = 1, \ldots, k$. Regarding Claim 4, it suffices to consider the case where $U = \emptyset$. The points (x^u, y^{S_i}) and (x^v, y^{S_j}), for $u \in S_i$ and $v \in S_j$, show the desired equality.*

By combining these claims, we get that (π, μ) is a multiple of the coefficient vector of (9), hence this inequality induces a facet of $P(G, \mathcal{S})$. $\square$

Condition (b) in Theorem 3 is satisfied when U is a clique, odd cycle, or odd wheel, among others. Finding a violated α-inequality such that U is a clique and $k = 0$ is equivalent to finding a clique of maximum weight. So if we restrict ourselves to the case where the only violated α-inequalities have this property, the separation problem of these inequalities is NP-complete. This suggests that the general separation problem for this family of inequalities is computationally hard.

5.2. Outer-set inequalities

Let $S \in \mathcal{S}$ be a structure, and let $T \subseteq V \backslash S$ be a vertex subset. Let $k \in \mathbb{Z}_+$ such that $\alpha(T) \leq k + 1$ and every $u \in S$ satisfies $\alpha(T \backslash N(u)) \leq k$. In this setting, we define

$$y_S + \sum_{u \in T} x_u \;\leq\; 1 + k \tag{10}$$

to be the *outer-set inequality* associated with S, T, and k. Fig. 6 shows an example of such a structure, with $k = 1$. Note that the anti-neighborhood in T of every vertex $u \in S$ (i.e., the set $T \backslash N(u)$) has stability number $k = 1$, whereas $\alpha(T) = k + 1 = 2$. This is the key observation for the following result.

Proposition 10. *The outer-set inequality* (10) *is valid for* $P(G, \mathcal{S})$.

Proof. Let (x, y) be an arbitrary feasible solution. If $y_S = 1$ then some vertex $v \in S$ has $x_v = 1$, hence $\sum_{u \in T} x_u = \sum_{u \in T \backslash N(v)} x_u \leq \alpha(T \setminus N(v)) \leq k$, and (10) is satisfied. On the other hand, if $y_S = 0$ then inequality (10) holds trivially because $\sum_{u \in T} x_u \leq \alpha(T) \leq k + 1$. Since (x, y) is arbitrary, the inequality (10) is valid for $P(G, \mathcal{S})$. $\square$

In order to analyze facetness, define a pair $u, v \in S \cup T$ to be *safe* if there exist two stable sets $I_u, I_v \subseteq S \cup T$ such that

- $|I_u| = |I_v| = k + 1$,
- $|I_r \cap T| \geq k$, for $r = u, v$,
- $I_u \backslash I_v = \{u\}$, and
- $I_v \backslash I_u = \{v\}$.

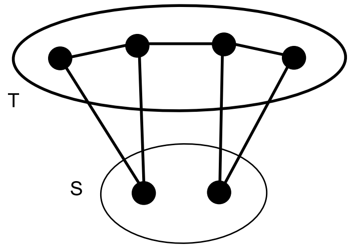

Fig. 6. Example of the support for the outer-set inequalities.

Define $G^{\text{safe}} = (S \cup T, E')$ to be the graph with vertex set $S \cup T$ and edge set $E' = \{uv : uv \text{ is safe}\}$. Notice that, if $u, v \in T$ is a safe pair, then $uv \in E$. However, a safe pair with a vertex from S may be a non-edge.

Theorem 4. *If*

(a) $\alpha(T) = k + 1$,
(b) $\alpha(T \backslash N(u)) = k$ *for every* $u \in S$,
(c) *there is a safe pair belonging to S that defines a non-edge,*
(d) *T is maximal with respect to properties* (a) *and* (b), *i.e., for every vertex* $v \notin T \cup S$, $\alpha(T \cup \{v\}) = k+2$ *or* $\alpha((T \cup \{v\}) \backslash N(u)) = k + 1$, *for some* $u \in S$, *and*
(e) *T belongs to a connected component of* G^{safe},

then the outer-set inequality (10) *induces a facet of* $P(G, \mathcal{S})$.

Proof. Assume $\mathcal{S} = \{S\}$ by the Lifting Lemma. Let $\mathcal{F}$ be the face of $P(G, \mathcal{S})$ defined by the inequality (10), and let $(\pi, \mu, \pi_0) \in \mathbb{R}^{|V|+|\mathcal{S}|+1}$ be such that $\pi x + \mu y = \pi_0$ for every $(x, y) \in \mathcal{F}$. Again, we will show that (π, μ) is a multiple of the coefficient vector of (10).

Claim 1: $\pi_u = \pi_v$ *for* $u, v \in T$. *Let* $wz \in E'$ *such that* $w, z \in S \cup T$ *is a safe pair and take the corresponding stable sets* I_w *and* I_z *with* $k + 1$ *vertices. Consider three cases:*

- $w, z \in T$: *we must have* $|I_w \cap T| = |I_z \cap T| = k' \in \{k, k+1\}$. *If* $k' = k+1$, *the points* $(x^{I_w}, 0)$ *and* $(x^{I_z}, 0)$ *are in* $\mathcal{F}$. *If* $k' = k$, *then there is exactly one vertex from S in both* I_w *and* I_z, *and so* (x^{I_w}, y^S) *and* (x^{I_z}, y^S) *are in* $\mathcal{F}$. *In both cases, we conclude that* $\pi_w = \pi_z$;
- $w, z \in S$: *in this case,* $|I_w \cap T| = |I_z \cap T| = k$. *Again* (x^{I_w}, y^S) *and* (x^{I_z}, y^S) *are in* $\mathcal{F}$, *thus showing that* $\pi_w = \pi_z$;
- $w \in T, z \in S$: *now,* $|I_w \cap T| = k + 1$ *and* $|I_z \cap T| = k$. *The solutions* $(x^{I_w}, 0)$ *and* (x^{I_z}, y^S) *are in* $\mathcal{F}$. *Then,* $\pi_w = \pi_z + \mu_S$.

These three items imply that $\pi_r = \pi_s$ *for every pair of vertices* $r, s \in S \cup T$ *belonging to the same connected component of* G^{safe}. *Since T itself is included in a single connected component of* G^{safe}, *the claim follows.*

Claim 2: $\pi_u = \pi_v$ *for* $u, v \in S$. *By the hypothesis (b), there exist stable sets* $I \subseteq T \backslash N(u)$ *and* $I' \subseteq T \backslash N(v)$, *both of size k. The solutions* $(x^{I \cup \{u\}}, y^S)$ *and* $(x^{I' \cup \{v\}}, y^S)$ *satisfy the inequality at equality and, together with Claim 1, imply* $\pi_u = \pi_v$.

Claim 3: $\pi_u = 0$ *for* $u \in S$. *Let* $v, w \in S$ *be a safe pair such that* $vw \notin E$, *which exists by the hypothesis (c). The solutions* (x^{I_w}, y^S) *and* $(x^{I_w \cup \{v\}}, y^S)$ *show that* $\pi_v = 0$. *Together with Claim 2, this shows* $\pi_u = 0$ *for every* $u \in S$.

Claim 4: $\mu_S = \pi_v$ *for some* $v \in T$. *Pick a vertex* $w \in S$. *By the hypothesis (b), there exists a stable set* $I \subseteq T \backslash N(w)$ *of size k. Also, the hypothesis (a) asserts that there exists a stable set* $I' \subseteq T$ *with size* $k + 1$. *The points* $(x^{I \cup \{w\}}, y^S)$ *and* $(x^{I'}, 0)$ *satisfy the inequality with equality, implying* $\mu_s + \pi_w + \sum_{u \in I} \pi_u = \sum_{u \in I'} \pi_u$. *Claim 1 and Claim 3 imply* $\mu_S = \pi_v$, *for some* $v \in I' \subseteq T$.

Claim 5: $\pi_v = 0$ *for* $v \notin T \cup S$. *By the hypothesis (d), either (i) there exist some vertex* $u \in S$ *and some stable set* $I \subseteq (T \cup \{v\}) \backslash N(u)$ *with* $|I| = k + 1$ *or (ii) there exists a stable set* $I' \subseteq T \cup \{v\}$ *with* $|I'| = k + 2$. *Notice that both I and I' contain v. If (i) holds, then the solutions* $(x^{I \cup \{u\}}, y^S)$ *and* $(x^{I \cup \{u\} \backslash \{v\}}, y^S)$ *establish the claim. If (ii) holds, then the solutions* $(x^{I \cup \{u\}}, 0)$ *and* $(x^{I \cup \{u\} \backslash \{v\}}, 0)$ *establish the claim.*

By combining these claims, we get that (π, μ) is a multiple of the coefficient vector of (10), hence this inequality induces a facet of $P(G, \mathcal{S})$. □

Again, if $k = 0$ then the outer-set inequality (10) amounts to finding a clique of maximum weight in the common neighborhood of the vertices in S, hence its separation is NP-complete. We leave open the computational complexity of the separation problem for general values of k, but we conjecture such problem to be computationally difficult, due to this observation for $k = 0$.

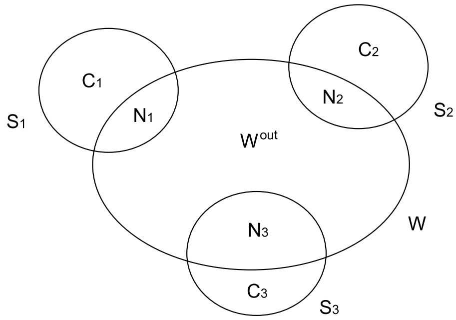

Fig. 7. Support for the cell inequalities.

5.3. Cell inequalities

Let $\mathcal{S}' = \{S_1, \ldots, S_k\} \subseteq \mathcal{S}$, $\mathcal{S}' \neq \emptyset$. Let $W \subseteq V$, $W \neq \emptyset$, and, for $i = 1, \ldots, k$, define $N_i := S_i \cap W$ and $C_i = S_i \backslash W$. Also define $W^{\text{out}} := W \backslash \cup_{i=1}^{k} S_i$ (see Fig. 7). An *$\mathcal{S}'$-partial transversal stable set* ($\mathcal{S}'$-PTSS) is a stable set $I \subseteq V$ such that $|I \cap S_i| \leq 1$ for every $i = 1, \ldots, k$. Call $\alpha^S(W)$ the size of the largest $\mathcal{S}'$-PTSS in $G[W]$. In this setting, we define

$$\sum_{S \in \mathcal{S}'} y_S + \sum_{v \in W^{\text{out}}} x_v \leq \alpha^S(W) + \sum_{i=1}^{k} \sum_{v \in C_i} x_v \tag{11}$$

to be the *cell inequality* associated with $\mathcal{S}'$ and W. We assume $N_i \neq \emptyset$ for $i = 1, \ldots, k$, as otherwise (11) is dominated by the model constraint (4) associated to S_i combined with the cell inequality for $\mathcal{S}' \backslash S_i$ and W.

Proposition 11. *The cell inequality (11) is valid for $P(G, \mathcal{S})$.*

Proof. Let (x, y) be an arbitrary feasible solution, and let $\mathcal{B} = \{S_i \in \mathcal{S}' : \sum_{v \in N_i} x_v \geq 1\}$. We claim

$$\sum_{S \in \mathcal{B}} y_S + \sum_{v \in W^{\text{out}}} x_v \leq \alpha^S(W),$$

since the LHS is at most the size of the largest $\mathcal{B}$-PTSS included in the solution. Furthermore, for $S_i \notin \mathcal{B}$ we get $y_{S_i} \leq \sum_{v \in C_i} x_v$, since $x_v = 0$ for every $v \in N_i$. By summing these inequalities and $0 \leq \sum_{S_i \in \mathcal{B}} \sum_{v \in C_i} x_v$, we get (11) which is, therefore, valid. □

For facetness, define a pair of vertices $(u, v) \in W \times W$ to be *safe* if there exist $\mathcal{S}'$-PTSSs I_u and I_v in $G[W]$ such that

- $|I_u| = |I_v| = \alpha^S(W)$,
- $I_u \backslash I_v = \{u\}$, and
- $I_v \backslash I_u = \{v\}$.

Again, define $G^{\text{safe}} = (W, E')$ to be the graph with vertex set W and edge set $E' = \{uv \in E(W) : uv \text{ is safe}\}$. Finally, we say that W is *α^S-maximal* if the addition of any vertex to W strictly increases $\alpha^S(W)$, i.e., $\alpha^S(W \cup \{v\}) > \alpha^S(W)$ for any $v \notin W$.

Theorem 5. *If* (a) *W is α^S-maximal,* (b) *N_i has a non-edge whose endpoints are a safe pair, for $i = 1, \ldots, k$, and* (c) *G^{safe} is connected, then the cell inequality (11) induces a facet of $P(G, \mathcal{S})$.*

Proof. Assume $\mathcal{S} = \mathcal{S}'$ by the Lifting Lemma. Let F be the face of $P(G, \mathcal{S})$ defined by the inequality (11), and let $(\pi, \mu, \pi_0) \in \mathbb{R}^{|V|+|\mathcal{S}|+1}$ such that $\pi x + \mu y = \pi_0$ for every $(x, y) \in F$. Again, we will show that (π, μ) is a multiple of the coefficient vector of (11). Let $\mathbb{S} = \cup_{i=1}^{k} S_i$ and $\mathcal{S}'_I = \{S \in \mathcal{S}' : S \cap I \neq \emptyset\}$, for $I \subseteq V$.

Claim 1: *$\pi_u = 0$ for $u \notin \mathbb{S} \cup W$. Since $W \neq \emptyset$ is α^S-maximal, there exists an $\mathcal{S}'$-PTSS $I \subseteq W$ such that $I \cup \{u\}$ is also an $\mathcal{S}'$-PTSS. We also get $|I| = \alpha^S(W)$, since otherwise the addition of u to W would keep $\alpha^S(W)$ unchanged. Then, the solutions $(x^I, y^{\mathcal{S}'_I})$ and $(x^{I \cup \{u\}}, y^{\mathcal{S}'_I})$ settle the claim.*

Claim 2: *$\pi_v = -\mu_{S_i}$ for $S_i \in \mathcal{S}$ and $v \in C_i$. The α^S-maximality of $W \neq \emptyset$ implies that there exists an $\mathcal{S}'$-PTSS $I \subseteq W$ such that $I \cup \{v\}$ is also an $\mathcal{S}'$-PTSS. Therefore, the stable set I cannot have a vertex in N_i. Then, the solutions $(x^{I \cup \{v\}}, y^{\mathcal{S}'_I \cup \{S_i\}})$ and $(x^I, y^{\mathcal{S}'_I})$ imply $\pi_v + \mu_{S_i} = 0$.*

Claim 3: *$\pi_u = 0$ for $u \in N_i$, $i = 1, \ldots, k$. By the hypothesis (b), N_i has a non-edge rs whose endpoints are a safe pair. The set $I := I_r \cup I_s$ is a stable set (although it is not an $\mathcal{S}'$-PTSS) and furthermore $(x^I, y^{\mathcal{S}'_I})$ satisfies (11) with equality. The solutions $(x^I, y^{\mathcal{S}'_I})$ and $(x^{I \backslash \{r\}}, y^{\mathcal{S}'_I})$ show $\pi_r = 0$. Since G^{safe} is connected then $\pi_r = \pi_t$ for every $t \in N_i$, and the claim follows.*

Claim 4: *$\mu_{S_i} = \mu_{S_j} = \pi_u$, for all $i, j = 1, \ldots, k$ and $u \in W^{out}$. Let $wz \in E'$ be a safe pair of vertices in W. Then, there exist $\mathcal{S}'$-PTSSs I_w and I_z in $G[W]$ such that the solutions $(x^{I_w}, y^{\mathcal{S}'_{I_w}})$ and $(x^{I_z}, y^{\mathcal{S}'_{I_z}})$ are in F. Since $I_w \backslash I_z = \{w\}$ and $I_z \backslash I_w = \{z\}$, we get $\pi_w + \mu y^{\mathcal{S}'_{I_w}} = \pi_z + \mu y^{\mathcal{S}'_{I_z}}$. Consider the following cases:*

- *If $w \in N_i$ and $z \in N_j$ for some $i, j \in \{1, \ldots, k\}$, $i \neq j$, we have that $\mathcal{S}'_{I_w} \backslash \mathcal{S}'_{I_z} = \{S_i\}$ and $\mathcal{S}'_{I_z} \backslash \mathcal{S}'_{I_w} = \{S_j\}$, so Claim 3 implies $\mu_{S_i} = \mu_{S_j}$.*
- *If $w \in N_i$ for some $i \in \{1, \ldots, k\}$ and $z \in W^{out}$, then $\mathcal{S}'_{I_w} = \mathcal{S}'_{I_z} \cup \{S_i\}$, which implies $\mu_{S_i} = \pi_z$.*
- *If $w, z \in W^{out}$, then $\mathcal{S}'_{I_w} = \mathcal{S}'_{I_z}$, hence $\pi_w = \pi_z$.*

Since $\mathcal{S}' \neq \emptyset$ and G^{safe} is connected, we get the desired equalities.

By combining these claims, we get that (π, μ) is a multiple of the coefficient vector of (11), hence this inequality induces a facet of $P(G, \mathcal{S})$. □

Concerning the separation of the cell inequalities, let us consider the case where every S_i is a clique and is contained in W, which implies that $C_i = \emptyset$ and $\alpha^S(W) = \alpha(W)$. Then the separation problem for the corresponding inequality is equivalent to finding a maximum weighted stable set in W. In this case, the inequality does not satisfy condition (b) and is not facet-defining. So, this does not imply that the separation of the cell inequalities is computationally hard, but it leads us to conjecture that this is indeed the case.

6. Concluding remarks

In this work we started an initial polyhedral study of the maximum stable set problem with weights on vertex subsets. The polytope associated to a natural integer programming formulation shares many properties with the stable set polytope, but has many additional features. Many open questions remain. On the one hand, it would be interesting to deepen the knowledge on the relations between the polytope associated to STABws and the stable set polytope, studied in Section 4. On the other hand, $P(G, \mathcal{S})$ shows intrinsic properties that do not seem to be inherited from the stable set polytope (as, e.g., facet-inducing inequalities with negative coefficients in the x-variables), and further studying such properties seems to be a promising line of research, specially in order to tackle real-life instances arising within column-generation procedures for the vertex coloring problem.

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