Theory of short periodic orbits for partially open quantum maps

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We extend the semiclassical theory of short periodic orbits [M. Novaes *et al.*, Phys. Rev. E **80**, 035202(R) (2009)] to partially open quantum maps, which correspond to classical maps where the trajectories are partially bounced back due to a finite reflectivity R. These maps are representative of a class that has many experimental applications. The open scar functions are conveniently redefined, providing a suitable tool for the investigation of this kind of system. Our theory is applied to the paradigmatic partially open tribaker map. We find that the set of periodic orbits that belongs to the classical repeller of the open map ($R = 0$) is able to support the set of long-lived resonances of the partially open quantum map in a perturbative regime. By including the most relevant trajectories outside of this set, the validity of the approximation is extended to a broad range of R values. Finally, we identify the details of the transition from qualitatively open to qualitatively closed behavior, providing an explanation in terms of short periodic orbits.

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I. INTRODUCTION

In completely open quantum systems the fractal Weyl law [1] states that the number of long-lived states (the so-called resonances) scales with the Planck constant as $\hbar^{-d/2}$, where $d+1$ is the fractal dimension of the classical repeller. This has been thoroughly tested [2] and it is a well established result [3,4]. At the center stage we find the classical invariant distribution consisting of all the trajectories that do not escape either in the past or in the future: the repeller. It plays a fundamental role in the determination of the quantum spectral features. However, a finite reflectivity R is required to describe many experimental situations, as it is the case in optical cavities [5,6], for example. This means that the classical trajectories arriving at the opening are partially reflected. This leads to partially open systems, which can be very well represented with partially open maps. In these cases we have that the fractal dimension is the phase space dimension and then the usual fractal Weyl law no longer applies. Nevertheless, the classical fractal character that remains through the multifractal measures [7] leads to quantum mechanical signatures. In fact, in a very recent work [8] the case of the partially open tribaker map was analyzed. It was found that the number of long-lived resonances follows a nontrivial scaling.

On the other hand, the semiclassical theory of short periodic orbits (POs) has been successfully applied to a number of problems including closed quantum chaotic systems, scarring phenomena, and more recently to open quantum maps [9]. In this approach the main ingredients are the shortest POs contained in the repeller; they provide all the necessary information to construct a basis set of scar functions in which the quantum nonunitary operators can be written. The number of trajectories needed to reproduce the quantum repeller [10,11] is related to the fractal Weyl law [12].

In this work we extend the short-PO theory to partially open quantum maps where a fraction of the quantum probability is reflected. We apply it to the partially open tribaker map. For this purpose we modify the definition of the scar functions and define the way in which the trajectories inside and outside the repeller at $R = 0$ are taken into account. We find that there are several regimes as a function of the reflectivity. First is a perturbative one in which by just considering the shortest POs inside the repeller at $R = 0$ the long-lived resonances and the quantum distribution associated with the invariant classical measure can be obtained. By suitably incorporating the shortest POs outside of this set we are able to extend the validity of the semiclassical calculations well beyond this perturbative regime. In doing so we keep some of its efficiency in terms of reducing the dimension of the Hilbert space needed for the calculations. Finally, we find that a transition from an openlike to a closedlike behavior takes place and this is clearly characterized by its effect on the semiclassical calculations.

This paper adopts the following organization. In Sec. II we define the classical and quantum partially open tribaker maps and all the relevant quantities associated with them. In Sec. III we extend the short-PO theory to this kind of system. In Sec. IV we apply it to our map and discuss the results. We conclude in Sec. V

II. PARTIALLY OPEN MAPS AND THE TRIBAKER EXAMPLE

The study of maps, which constitute simple examples exhibiting a rich dynamics, has a long and fruitful history in the classical and quantum chaos literature [13–15]. In this respect, open maps on the 2-torus are transformations that represent the evolution of trajectories that disappear when they reach an open region in the bidimensional phase space. An invariant set is formed by the remaining trajectories, i.e., those that do not escape either in the past or in the future. These trajectories build the forward and backward trapped sets, respectively, and

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the intersection of both is what is called the repeller, which has a fractal dimension.

Partially open maps can be defined as those maps in which the opening does not absorb all the trajectories that arrive at it, but reflects back a certain amount. This amount is essentially given by the reflectivity $R \in (0:1)$. Here we exclude $R = 0$ since this corresponds to completely open maps and $R = 1$ since it represents a closed one. This is the simplest choice of the reflection mechanism. In general, one considers a function of the phase space points, $R(q,p)$ in bidimensional examples. Nevertheless, our simple model captures the main features of realistic systems of interest such as microcavities [8].

In contrast to what happens in an open map, in partially open ones the relevant measure is not uniformly distributed on a fractal, showing multifractality instead. We closely follow the definition found in [8]. In each phase space region X_i (where i individually labels them) this measure depends on the average intensity I_t with $t \to \infty$ of a number $N_{\rm IC}$ of random initial conditions taken inside X_i. For each trajectory associated with them the initial intensity is $I_0 = 1$ and is modified by $I_{t+1} = RI_t$ at each iteration of the map in the case in which it falls inside the opening. The finite time measure for X_i can be defined as $\mu^b_{t,i} = \langle I_{t,i}\rangle / \sum_i \langle I_{t,i}\rangle$, where the average is over the initial conditions in the given phase space region. In fact, this measure is the analog of the backward trapped set of open maps. If we evolve backward we obtain $\mu^f_{t,i}$, the analog of the forward trapped set, and the intersection gives what we call the partial repeller $\mu_{t,i}$.

The usual quantization scheme for maps on the torus proceeds in the following way: In the first place we impose boundary conditions for both the position and momentum representations by taking $\langle q+1|\psi\rangle = e^{i2\pi\chi_q}\langle q|\psi\rangle$ and $\langle p+1|\psi\rangle = e^{i2\pi\chi_p}\langle p|\psi\rangle$, with $\chi_q,\chi_p \in [0,1)$. Thus the Hilbert space is of finite dimension $N = (2\pi\hbar)^{-1}$ and the semiclassical limit corresponds to $N \to \infty$. The system's propagator is given by an $N \times N$ matrix. Position and momentum eigenstates are given by $|q_j\rangle = |(j+\chi_q)/N\rangle$ and $|p_j\rangle = |(j+\chi_p)/N\rangle$ with $j \in \{0,\ldots,N-1\}$. A discrete Fourier transform gives $\langle p_k|q_j\rangle = \frac{1}{\sqrt{N}} e^{-2i\pi(j+\chi_q)(k+\chi_p)/N} \equiv (G_N^{\chi_q,\chi_p})$.

When $R = 0$ the opening can be easily quantized as a projection operator P on its complement. We usually take a finite strip parallel to the p axis, so if U is the propagator for the closed system, then $\widetilde{U} = PUP$ stands for the open one. Here we take a partial opening, so we modify this projector by replacing the zero block of the opening by $\sqrt{R} \times \mathbb{1}$, where the identity has the dimension associated with the escape region. The resulting partially open quantum map has N right eigenvectors $|\Psi^R_j\rangle$ and N left ones $\langle\Psi^L_j|$, which are mutually orthogonal $\langle\Psi^L_j|\Psi^R_k\rangle = \delta_{jk}$ and are associated with resonances z_j. We choose $\langle\Psi^R_j|\Psi^R_j\rangle = \langle\Psi^L_j|\Psi^L_j\rangle$ for the norm.

We make all the calculations of this work on the tribaker map, whose classical expression is given by

$$\mathcal{B}(q,p) = \begin{cases} (3q, p/3) & \text{if } 0 \leqslant q < 1/3 \\ (3q-1, (p+1)/3) & \text{if } 1/3 \leqslant q < 2/3 \\ (3q-2, (p+2)/3) & \text{if } 2/3 \leqslant q < 1. \end{cases} \quad (1)$$

This is an area-preserving, uniformly hyperbolic, piecewise-linear, and invertible map with Lyapunov exponent $\lambda = \ln 3$.

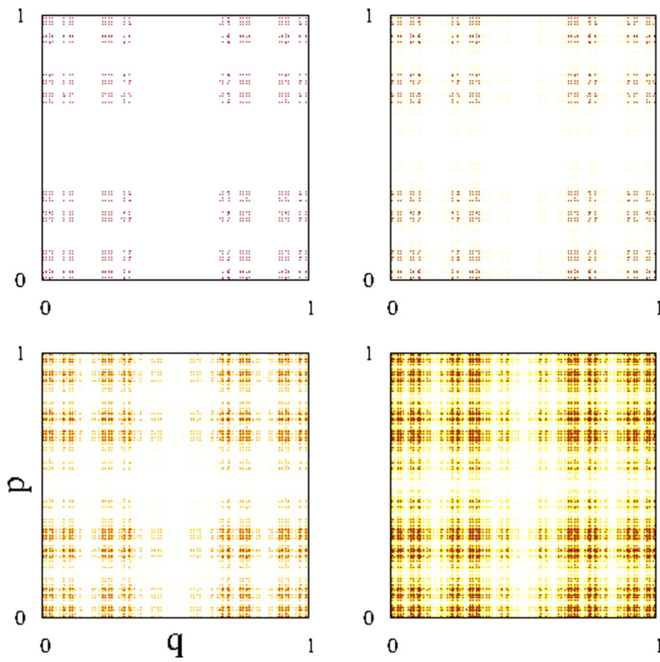

FIG. 1. Classical measure $\mu_{t,i}$, i.e., the partial repeller on the 2-torus for the partially open tribaker map, for four different values of the reflectivity R. In the top left panel we show the $R = 0$ case. In the top right panel we can observe the $R = 0.01$ case, which shows no appreciable differences with respect to the previous value of the reflectivity. The bottom left panel corresponds to $R = 0.07$, where we can find a finite measure outside of the repeller, and the lower right panel corresponds to $R = 0.2$, where the measure now extends to almost all regions of phase space.

An opening has been placed in the region $1/3 < q < 2/3$, where the reflectivity is given by R as explained above.

The quantum version of the tribaker map can be defined by means of the discrete Fourier transform $G_N^{1/2,1/2}$ (for brevity we omit superscripts in the following). Taking antiperiodic boundary conditions, i.e., $\chi_q = \chi_p = 1/2$, amounts to preserving time reversal and parity. In a position representation this map reads (this is a widely accepted quantization and details can be seen, for example, in [16,17])

$$U^{\mathcal{B}} = G_N^{-1} \begin{pmatrix} G_{N/3} & 0 & 0 \\ 0 & G_{N/3} & 0 \\ 0 & 0 & G_{N/3} \end{pmatrix}. \quad (2)$$

The partially open quantum tribaker map is then given by means of the operator

$$P = \begin{pmatrix} \mathbb{1}_{N/3} & 0 & 0 \\ 0 & \sqrt{R}\mathbb{1}_{N/3} & 0 \\ 0 & 0 & \mathbb{1}_{N/3} \end{pmatrix}, \quad (3)$$

applied to Eq. (2), thus obtaining

$$\widetilde{U}^{\mathcal{B}} = PU^{\mathcal{B}}P. \quad (4)$$

In Fig. 1 we show the finite time partial repeller $\mu_{t,i}$ at time $t = 10$. We have selected four representative examples for the reflectivity. In the top left panel we take $R = 0$ for comparison purposes. Next, in the top right panel we display the $R = 0.01$ case, which is almost indistinguishable from the completely open one; this suggests that there should be a perturbative regime at least up to these reflectivity values. In the bottom left panel we represent the case $R = 0.07$, where it is clear that the measure starts to be non-negligible outside the repeller ($R = 0$). Finally, in the bottom right panel the $R = 0.2$ case underlines the need to consider a much different scenario with widespread finite measure over the 2-torus.

III. SHORT-PERIODIC-ORBIT THEORY FOR PARTIALLY OPEN QUANTUM MAPS

Following the ideas of the short-PO theory for closed systems [18], we have recently developed a similar theory of short POs for open quantum maps [9,12]. In this theory the repeller has a central role and the short POs that belong to it provide all the essential information needed to recover the quantum long-lived eigenvalues and the quantum repeller (with some exceptions [11]). The fundamental tools in this approach are the open scar functions associated with each one of these trajectories. To make the paper self-contained, we give a brief description of the partially open scar function construction, which is a very natural adaptation to the case of partially open maps.

Let γ be a PO of fundamental period L that belongs to a partially open map. We can define coherent states $|q_j,p_j\rangle$ associated with each point of the orbit (it has a total of L points, all in the partial repeller). We then construct a linear combination with them:

$$|\phi_\gamma^m\rangle = \frac{1}{\sqrt{L}} \sum_{j=0}^{L-1} \exp\left\{-2\pi i\left(j A_\gamma^m - N\theta_j\right)\right\}|q_j,p_j\rangle, \quad (5)$$

where $m \in \{0,\ldots,L-1\}$ and $\theta_j = \sum_{l=0}^{j} S_l$. In this expression S_l is the action acquired by the lth coherent state in one step of the map. The total action is $\theta_L \equiv S_\gamma$ and $A_\gamma^m = (NS_\gamma + m)/L$. Finally, the right and left scar functions for the periodic orbit are defined through the propagation of these linear combinations under the partially open map $\widetilde{U}$ (up to approximately the system's Ehrenfest time τ):

$$|\psi_{\gamma,m}^R\rangle = \frac{1}{\mathcal{N}_\gamma^R} \sum_{t=0}^{\tau} \widetilde{U}^t e^{-2\pi i A_\gamma^m t} \cos\left(\frac{\pi t}{2\tau}\right)|\phi_\gamma^m\rangle \quad (6)$$

and

$$\langle\psi_{\gamma,m}^L| = \frac{1}{\mathcal{N}_\gamma^L} \sum_{t=0}^{\tau} \langle\phi_\gamma^m|\widetilde{U}^t e^{-2\pi i A_\gamma^m t} \cos\left(\frac{\pi t}{2\tau}\right). \quad (7)$$

Normalization $\mathcal{N}_\gamma^{R,L}$ is chosen in such a way that $\langle\psi_{\gamma,m}^R|\psi_{\gamma,m}^R\rangle = \langle\psi_{\gamma,m}^L|\psi_{\gamma,m}^L\rangle$ and $\langle\psi_{\gamma,m}^L|\psi_{\gamma,m}^R\rangle = 1$. These functions are suitable tools for the investigation of the morphology of the eigenstates.

In Fig. 2 we illustrate the partially open scar functions by means of a representation [10] that clearly shows the quantum probability that can be associated with the classical partial repeller. We define the symmetrical operator $\hat{h}_j$ related to the right $|\psi_j^R\rangle$ and left $\langle\psi_j^L|$ states (in this case scar functions, where we have collapsed both subscripts to just one for simplicity)

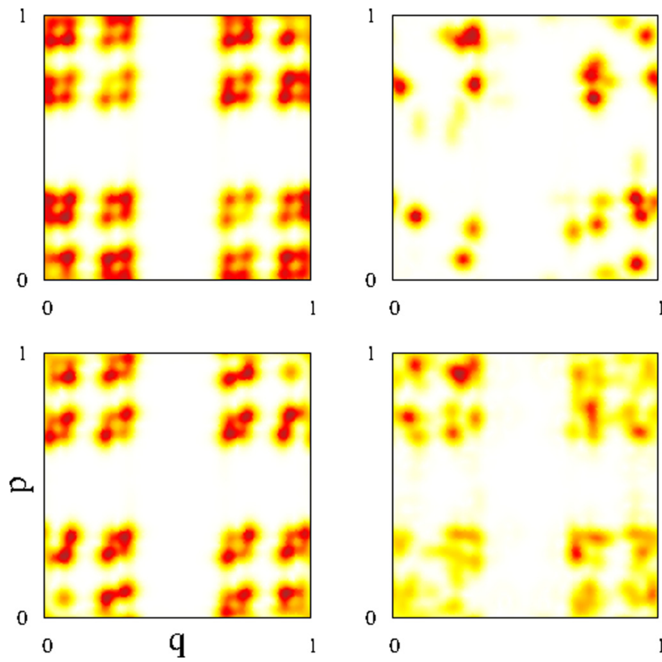

FIG. 2. Sum of $\hat{h}_j$ over the partially open scar functions. The top panels correspond to $R = 0.07$ and the bottom ones to $R = 0.2$. In the left column we show the scar function set associated with orbits γ inside the repeller at $R = 0$, in the right one the set associated with orbits outside of it.

$$\hat{h}_j = \frac{|\psi_j^R\rangle\langle\psi_j^L|}{\langle\psi_j^L|\psi_j^R\rangle}, \quad (8)$$

which is associated with the orbit γ. By calculating the sum over all these projectors [11] corresponding to the sets of scar functions used for a given semiclassical calculation we can see how different parts of the phase space are represented in the basis. In the top panels of Fig. 2 we show these sums for $R = 0.07$, while in the lower ones we show the sums for $R = 0.2$. In the left column we find open scar functions associated with short POs inside the repeller and in the right column those outside of it. These sets were actually used for some of the calculations to be described in Sec. IV. It can be noticed that the contribution from the orbits that are outside the repeller increases with R.

As a matter of fact, we now arrive at the other important point in adapting the theory to the partially open case: how to select the orbits that take part in the calculation. For this purpose we choose the following criterion: Select a given number of POs, N^{POs}, from the whole set up to a period L, which approximately covers the partial repeller. The selected degree in the approximation of this covering translates into the fraction of long-lived resonances that can be obtained. In practical terms this means allowing all POs up to period L that are inside the repeller into a first list, since they have a uniform weight. Those that are outside must have the greatest values of μ. We select them by fixing the maximum number allowed $N_{\max}^{\text{outPO}}$ (again, from all of them up to period L), which establishes a μ cutoff value from the list of these orbits ordered by decreasing weight. Here $N_{\max}^{\text{outPO}}$ is established having in mind that it should increase with R, starting from zero at $R = 0$. For that purpose $N_{\rm max}^{\rm outPO}$ grows with the reflectivity (note that this also depends on the particular map considered). Hence, the preliminary list of all the orbits incorporates them by increasing value of their period and contains the ones with the largest weights. We notice that the number of orbits in this list is typically much larger than the number needed for our approximation. As the next step we optimize the set by a reordering to provide the most uniform covering possible of the partial repeller. We finish the selection by cutting the list at $N^{\rm POs}$. This criterion blindly selects the most appropriate POs based solely on their weight and distribution in phase space.

Finally, we construct an appropriate basis in which we can write the partially open evolution operators associated with partially open maps as $\langle \psi_{\alpha,i}^L | \widetilde{U} | \psi_{\beta,j}^R \rangle$. This expression is the short-PO approximation to the partially open propagator $\widetilde{U}$ on the partial repeller. Equipped with $\langle \psi_n^L | \psi_m^R \rangle \neq \delta_{nm}$, we solve a generalized eigenvalue problem that provides the eigenstates of this matrix. The long-lived resonances [9] are constructed by a linear combination of the eigenvector's coefficients and the corresponding scar functions of the basis.

IV. RESULTS

We use the short-PO theory to construct a semiclassical approximation of the partially open quantum tribaker map in the $N = 243$ case, for several values of R, considering POs up to period $L = 7$. In order to quantify the behavior of the short-PO method we define its performance P [11] as the fraction of long-lived eigenvalues that it is able to reproduce within an error given by $\epsilon = \sqrt{[\mathrm{Re}(z_i^{\rm ex}) - \mathrm{Re}(z_i^{\rm sc})]^2 + [\mathrm{Im}(z_i^{\rm ex}) - \mathrm{Im}(z_i^{\rm sc})]^2}$, where $z_i^{\rm ex}$ and $z_i^{\rm sc}$ are the exact eigenvalues and those given by the semiclassical theory, respectively. We restrict our analysis to the number of exact eigenvalues with modulus greater than ν_c, which is a critical value that depends on R. In our calculations we have tried to keep the number constant at around $n_c = 60$ since for low values of R this represents the outer ring of eigenvalues that is a typical feature of the open quantum baker maps. This is very useful since it provides a natural separation between short- and long-lived resonances. We calculate the number of scar functions $N_{\rm SF}$ as a fraction of N that are needed in order to obtain as many semiclassical eigenvalues inside the $\epsilon = 0.001$ vicinity of the corresponding exact ones in order to reach $P \geqslant 0.8$. The fraction $N_{\rm SF}/N$ is a good indicator of the efficiency of the method. In fact, one of its main advantages is reducing the effective dimension of the matrices that one needs to diagonalize in order to obtain the resonances of the quantum system. However, it also reveals if there are quantum signatures of (multi)fractal dimensions when partially opening the map (like the fractal Weyl law). It is worth mentioning that all the threshold values considered guarantee a reasonably good performance of the approximation and the evaluation of a meaningful number of eigenvalues in the whole range of situations that we have studied. Stricter thresholds could be chosen, but this would not lead to significantly better results. In particular, the quantum repeller is guaranteed to be reproduced in an extremely acceptable way, as will be shown in the following. A note of caution is in order here. We have chosen to measure the performance in terms of the

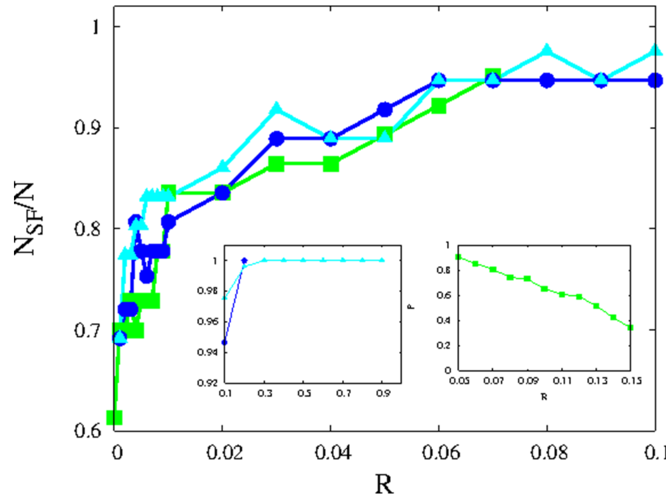

FIG. 3. Fraction of scar functions $N_{\rm SF}/N$ needed to reach $P = 0.8$ as a function of the reflectivity R. The green (gray) line with squares corresponds to the case where only POs inside the repeller have been considered. Blue (black) and cyan (light gray) lines with circles and triangles correspond to considering $N_{\rm max}^{\rm outPO} = 5$ and 50, respectively. The left inset shows the behavior for greater R values. In the right one we show the drop in the performance P as a function of R when the POs used in the calculation are all inside the repeller up to period $L = 7$.

size of the basis needed for the calculation. However, even in the completely closed case where a full size basis is always needed, this theory is highly advantageous with respect to trace formula methods [19], which require an exponentially growing number of trajectories. We should emphasize that the latter methods are extremely important in quantum chaos theory for many other reasons.

In Fig. 3 we show the fraction $N_{\rm SF}/N$ needed to reach $P \geqslant 0.8$ as a function of $R \in [0 : 0.1]$ for three different scenarios derived from our POs selection criterion. The line with squares corresponds to the case in which we only take POs that belong to the repeller. We have also considered $N_{\rm max}^{\rm outPO} = 5$ (POs outside of it), results that are represented by the line with circles. Finally, the case with $N_{\rm max}^{\rm outPO} = 50$ is shown through a line with triangles. We point out that there is no improvement in the calculations when considering more POs outside the repeller.

The results in Fig. 3 clearly show that the behavior of the short-PO theory for the partially open tribaker map can be divided into four regimes. First we notice that there is a perturbative regime in which the open scar functions associated with the POs inside the repeller at $R = 0$ are the only ones needed in order to accurately reproduce the long-lived resonances. This regime extends up to approximately $R = 0.01$. It is also clear that by incorporating up to five POs outside the repeller with the greatest μ values the performance is worse. This amount represents around 1% of the available POs up to period $L = 7$. Moreover, if we include up to 50 of these POs the result is much worse (a significant amount of the POs inside the repeller are now replaced by them). Beyond $R = 0.01$ the POs in the repeller are still enough to reproduce the long-lived resonances, but the overall amount of scar functions needed steadily increases up to $R = 0.07$, though at a slower pace than before. In fact, for $R \in [0.01 : 0.07]$ we identify a semiperturbative regime, in the sense that the classical information of the repeller is still enough but becomes less efficient. Above $R = 0.07$ the repeller is not enough to adequately treat the problem and other POs outside of it are needed. As a matter of fact, we have included all the POs inside the repeller up to period $L = 7$ and they were not sufficient to expand the number of eigenstates necessary to reach $P = 0.8$. This is the reason why the corresponding curve stops there. The right inset of Fig. 3 shows the behavior of P as a function of R, when all the POs up to period $L = 7$ inside the repeller are the only ones used in the calculation. It is clear that the performance drops to approximately one-third of the long-lived eigenstates for $R = 0.15$, but also that only up to a 60% of them can be reproduced at around $R = 1.0$. From $R = 0.07$ on, we also see that by incorporating a greater amount of POs not inside the repeller, we obtain a similar performance of the method. That is, the proper partially open regime extends up to $R = 0.3$ approximately. Beyond that we find the fourth and last regime, as can be seen in the inset of Fig. 3. Here a saturation occurs and we need $N_{\mathrm{SF}} = N$, leaving our method with no advantage with respect to a direct diagonalization. Also, the curve corresponding to $N_{\max}^{\mathrm{outPO}} = 5$ [blue (black) line with circles] stops at $R = 0.2$ since this number is not enough to satisfy our P criterion. Again, this does not invalidate many other advantages of our theory, which are related to the ability to express the eigenstates in terms of just the shortest POs of the system. In turn, it is a clear sign that the system has become effectively closed from our perspective, i.e., we need to expand all of the phase space no matter what the local value of the measure μ is. This relevant result has been found due to the implementation of our theory. We have verified that in the case $N = 729$, $N_{\mathrm{SF}}/N = 0.69$ for $R = 0.01$, obtained with all the POs inside the repeller, and $N_{\mathrm{SF}}/N = 0.96$ for $R = 0.1$, with POs mainly outside of it. This is in agreement with the previous results, though we leave the study of this scaling for future work.

To illustrate this behavior we use the projectors $\hat{h}_j$ of Sec. III now associated with the right $|\Psi_j^R\rangle$ and left $\langle\Psi_j^L|$ eigenstates, which are related to the eigenvalue z_j. We calculate the sum of the first j of these projectors [11], ordered by decreasing modulus of the corresponding eigenvalues ($|z_j| \geqslant |z_{j'}|$ with $j \leqslant j'$) up to completing the set of long-lived resonances:

$$\hat{Q}_j \equiv \sum_{j'=1}^{j} \hat{h}_{j'}. \tag{9}$$

Their phase space representation by means of coherent states $|q,p\rangle$ is given by

$$h_j(q,p) = |\langle q,p|\hat{h}_j|q,p\rangle|, \tag{10}$$

$$Q_j(q,p) = |\langle q,p|\hat{Q}_j|q,p\rangle|. \tag{11}$$

This is our formal definition of the partial quantum repeller, which we denote by Q_{n_c}.

In Fig. 4 we show Q_{n_c} for the exact resonances in the left column and the ones given by the short-PO approach, i.e., $Q_{n_c}^{\mathrm{sc}}$, in the right one. In the top panels the case $R = 0.07$ has been obtained by just using the POs that are in the repeller. This shows that the POs inside the repeller are enough to reproduce the partial quantum repeller; little probability is found outside of it. In the bottom panels we display the $R = 0.2$ case where the contribution from the orbits outside the repeller is crucial, as can be noticed from the greater nonzero probability found in other regions of phase space. The overlap of the normalized distributions, calculated as $O = \iint Q_{n_c}(q,p) Q_{n_c}^{sc}(q,p)dq\,dp$, is $O = 0.994$ for $R = 0.07$ and $O = 1.0$ for $R = 0.2$, confirming the excellent agreement between the exact and the semiclassical results.

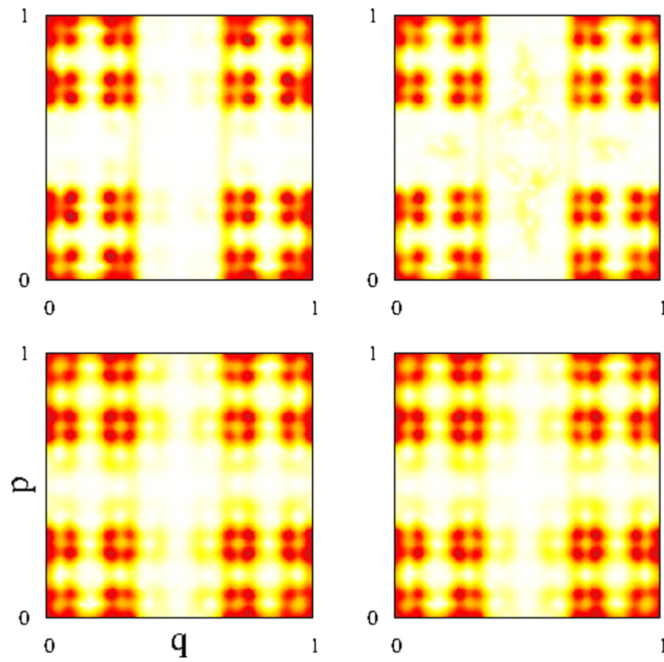

FIG. 4. Partial quantum repeller Q_{n_c} for the exact resonances (left column) and $Q_{n_c}^{\mathrm{sc}}$ for the short-PO-theory results (right column) for $R = 0.07$ (top) and $R = 0.2$ (bottom).

V. CONCLUSION

We have extended the short-PO theory to partially open quantum maps and have applied it to the particular case of the partially open tribaker map. It turns out that for low reflectivities, the long-lived resonances and the partial quantum repeller can be reproduced up to a very good accuracy with just the classical information of the repeller ($R = 0$). Moreover, we identify four different regimes in which the role played by the orbits that exist inside and outside the repeller changes. They are the perturbative, semiperturbative, proper partially open, and effectively closed regimes. This translates directly into the partial quantum repeller, which performs a nontrivial transition from an openlike to a closedlike shape as a function of R.

One of the main features of our theory when applied to open maps is that the size of the matrices needed in the final diagonalization is drastically reduced. This is due to the fact that the number of necessary scar functions scales according to the fractal Weyl law [12]. In the case of the partially open tribaker map we have found that for $R < 0.01$ (perturbative regime) the reduction in the calculation effort is very important. This result is clearly relevant since not only the striking efficiency of the method is almost fully kept, but also the spectral statistics already shows the effects of multifractality [8]. The behavior in this region of R offers a very interesting research line for the future, from the theoretical point of view and also from the perspective of applications [20]. However, this advantage is gradually lost as the reflectivity R increases and completely disappears for reflectivity values $R \gtrsim 0.2$. However, this is related to the very nature of the open map, which performs a transition that up to now could only be noticed by the change in the shape of the spectrum [8]. This underlines the importance that the probability distributed in each region of phase space has in this kind of map (despite the fact that we have selected the orbits in terms of their weights expressed by μ). Moreover, from the semiclassical point of view, all the POs are progressively more interconnected as a function of R and each detail of them becomes more and more relevant for a precise diagonalization.

In fact, our theory is a natural way to bridge the gap between the short-PO theories for completely open and completely closed maps. In doing so, we were able to show how the relevant phase space structures perform this transition. This opens many possible applications for studying the connections between orbits and properties of this kind of system. In this sense, the partially open scar function is a very interesting tool for the study of the morphology of the eigenstates. We think that our semiclassical theory can significantly contribute to the study of emission problems in microcavities. We must underline that, compared to trace formulas methods, our theory gives all quantum information in terms of just the shortest POs. In closed systems this approach is already a great advantage for many studies [21]. We think that the present work allows extension of these benefits to partially open maps for all R. To name a few specifically, we can say that the scarring of eigenstates could be better understood, for example. One way to do this would be to follow a given short PO along the spectrum for different values of the reflectivity. On the other hand, the interplay among POs could be influenced by the gradual recovering of the heteroclinic circuits connecting them. Understanding this should provide a different perspective to interpret the spectral behavior.

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