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ON CONJUGACY OF CARTAN SUBALGEBRAS IN NON-FGC LIE TORI

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To H. Abels on his 75th birthday

Abstract. We establish the conjugacy of Cartan subalgebras for generic Lie tori "of type A". This is the only conjugacy problem of Lie tori related to Extended Affine Lie Algebras that remained open.

Introduction

Extended Affine Lie Algebras (EALAs for short) are a rich class of Lie algebras that were first conceived by the physicists R. Høegh-Krohn and B. Torresani and

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then brought to the attention of mathematicians by P. Slodowy (the reader should look at [Ne2] and [Ne3] for a comprehensive review of basic EALA theory and references. The original mathematical formulation is to be found in [AABGP]). An EALA has an invariant non-negative integer attached called its *nullity*. In nullity 0 EALAs are nothing but the finite-dimensional simple Lie algebras, while in nullity 1 EALAs are the celebrated affine Kac–Moody Lie algebras (we assume for simplicity in this Introduction that our base field is the complex numbers). Roughly speaking an EALA E is constructed from a class of Lie algebras called Lie tori by taking a central extension and adding a suitable space of derivations. In the case of the affine algebras, for example, the Lie torus is a loop algebra L based on a finite-dimensional simple Lie algebra $\mathfrak{g}$. Note that L is naturally a Lie algebra over the Laurent polynomial ring $\mathbb{C}[t^{\pm 1}]$). This ring is the centroid of L.The central extension of L is the universal one (which is one-dimensional). The space of derivations is also one-dimensional and corresponds to the degree derivation $t(d/dt)$.

An EALA, by definition, comes equipped with a so-called Cartan subalgebra (just like the affine algebras do, but unlike the finite-dimensional simple Lie algebras; see § 2 for details). In the setting of EALAs, a Cartan subalgebra is the same as a self-centralizing ad-diagonalizable subalgebra, as defined in § 2.7. With respect to the given Cartan subalgebra the EALA admits a root space decomposition. The structure of the resulting "root system" plays a fundamental role in understanding the structure as well as the representation theory of the given EALA. It is obvious that all of this would be of little use (or mathematically unnatural) if the nature of the root system was to depend on the choice of Cartan subalgebra. The most elegant way of dealing with this problem is by establishing "Conjugacy", i.e., by showing that all Cartan subalgebras are conjugate under the action of the group of automorphisms of the EALA (in all cases it is sufficient to use a precise subgroup of the full group of automorphisms. Conjugacy in the finite-dimensional case, in the spirit of the present work, is due to Chevalley. For the affine algebras the result is due to Peterson and Kac [PK]). For almost all EALAs (see below) conjugacy, hence the invariance of the root system, has been established in [CGP] (for Lie tori) and [CNPY] (for the full EALAs). One case, the so called non-fgc case (see §2 for definitions), remained open. The purpose of this paper is to establish conjugacy for the Lie tori (the analogue of [CGP] in the non-fgc case) underlying this remaining family of EALAs.

The centroid of a Lie torus L is always a Laurent polynomial ring R in finitely many variables. In all cases but one, L is an R-module of finite type. This is the fgc case (where fgc stands for finitely generated over the centroid). When this does not happen, the non-fgc case, the nature of L is perfectly understood:

$$L = \mathfrak{sl}_\ell(Q)$$

where Q is a quantum torus with at least one generic entry. We remind the reader that by definition Q is the complex unital associative algebra presented by generators $x_1, \ldots, x_n, x_1^{-1}, \ldots, x_n^{-1}$ and relations

$$x_i x_i^{-1} = 1_Q = x_i^{-1} x_i, \quad x_i x_j = q_{ij} x_j x_i,$$

where the q_{ij} are non-zero complex numbers. That "one entry be generic" means that one of the q_{ij} cannot be a root of unity. The centre $\mathcal{Z}(Q)$ of Q is always a Laurent polynomial ring. The fgc condition on the Lie algebra $\mathfrak{sl}_\ell(Q)$ is equivalent to Q being a $\mathcal{Z}(Q)$-module of finite type.

We can now state our

Main Theorem. *Let $f\colon \mathfrak{sl}_{\ell'}(Q') \to \mathfrak{sl}_\ell(Q)$ be an isomorphism of non-fgc Lie tori. Then $\ell = \ell'$ and if $\mathfrak{h}'$ and $\mathfrak{h}$ denote the given Cartan subalgebras of $\mathfrak{sl}_{\ell'}(Q')$ and $\mathfrak{sl}_\ell(Q)$ respectively,*[2] *then $f(\mathfrak{h}')$ and $\mathfrak{h}$ are conjugate under an automorphism of $\mathfrak{sl}_\ell(Q)$.*

In the fgc case the proof of conjugacy is naturally divided into two steps. One first establishes conjugacy at the level of Lie tori, and then extends this conjugacy "downstairs" to the full EALA. The fgc condition allows the Lie tori to be viewed as simple Lie algebras (in the sense of [SGA3]) over Laurent polynomial rings. Conjugacy in the fgc case makes heavy use of the powerful methods of [SGA3] and Bruhat–Tits theory. None of this is possible in the non-fgc case. New methods/ideas are needed. The crucial ingredient that we develop to deal with this new situation is a method that we call "specialization". The idea, roughly speaking, is to create a subring R of $\mathbb{C}$ with the property that

(i) our non-fgc Lie torus "exists" over R,
(ii) there exists a maximal ideal $\mathfrak{m}$ of R which after base change (reduction modulo $\mathfrak{m}$) yields an fgc Lie torus.

The catch is that the field $R/\mathfrak{m}$ is of positive characteristic! One does not even have a suitable definition of Lie tori in positive characteristic. Yet the resulting object and its group of automorphisms is explicit enough that we can establish conjugacy for them. The specialization method is invoked once again to show that conjugacy holds before the reduction modulo m.

Notation. Throughout, R is a commutative unital ring which often occurs as the base ring of some algebraic structure being considered; F is an arbitrary field, and k often denotes a field of characteristic 0. An R-algebra is an arbitrary algebra over R (in particular not necessarily associative or a Lie algebra). Group schemes are usually denoted with bold letters. For example, $\mathbf{PGL}_{m,R}$ denotes the R-group scheme of automorphisms of the (associative and unital) R-algebra $M_m(R)$.

Structure of the paper. In Section 1 we collect some basic results about centroids of (arbitrary) algebras. Particular attention is devoted to the case of quantum tori. Section 2 looks at the structure of the Lie algebra $\mathfrak{sl}_\ell(Q)$ where Q is a quantum torus. The definition and basic properties of maximal abelian diagonalizable (MAD) subalgebra are also given in this section (these are the subalgebras that play the role of the Cartan subalgebras in EALA theory). Section 3 is devoted to a detailed study of the group scheme $\mathbf{PGL}_\mathcal{A}$ where $\mathcal{A}$ is an Azumaya algebra over a ring R, and of the connection between this R-group scheme and the R-group scheme of automorphisms of $\mathfrak{sl}_\ell(\mathcal{A})$. Section 4 presents a detailed analysis of the

[2] As we have mentioned already, a distinguished "Cartan subalgebra" is part of the definition of a Lie torus.

automorphism group of the Lie algebra $\mathfrak{sl}_\ell(Q)$ when Q is an fgc quantum torus. Section 5 develops the method that we called "specialization" mentioned above. This is the key that allows us to deal with the non-fgc case by translating the problem into an fgc question, but now over fields of positive characteristic. Section 6 presents a collection of preliminary results to be used in the proof of the main Theorem, which is given in Section 7.

1. Some results on centroids and quantum tori

1.1. Centroids and base change

Let R be a commutative ring and let $\mathcal{A}$ be an arbitrary R-algebra.[3] Recall that the derived subalgebra of $\mathcal{A}$ is the additive subgroup of $\mathcal{A}$ generated by all products ab with $a, b \in \mathcal{A}$. It is trivial to see that this group is indeed an R-subalgebra of $\mathcal{A}$. The algebra $\mathcal{A}$ is called *perfect* if it equals to its derived algebra. Note that any unital algebra is perfect.

A crucial object for our work is the *centroid* $\mathrm{Ctd}(\mathcal{A})$ of the algebra $\mathcal{A}$. Recall that

$$\mathrm{Ctd}(\mathcal{A}) = \{\chi \in \mathrm{End}_R(\mathcal{A}) : \chi(a_1a_2) = \chi(a_1)a_2 = a_1\chi(a_2) \text{ for all } a_i \in \mathcal{A}\}.$$

Clearly $\mathrm{Ctd}(\mathcal{A})$ a unital commutative (if $\mathcal{A}$ is perfect) subalgebra of the associative R-algebra $\mathrm{End}_R(\mathcal{A})$. It is obvious that we can consider $\mathcal{A}$ as an algebra over $\mathcal{C} = \mathrm{Ctd}(\mathcal{A})$—it will be denoted $\mathcal{A}_{(\mathcal{C})}$.[4] We will say that $\mathcal{A}$ is *fgc* if $\mathcal{A}_{(\mathcal{C})}$ is a finitely generated $\mathcal{C}$-module.

More generally, if $S \in R$–**alg**, i.e., S is a unital associative commutative R-algebra, and if $\rho\colon S \to \mathrm{Ctd}(\mathcal{A})$ is a unital algebra homomorphism, $\mathcal{A}$ becomes an S-algebra by defining $s \cdot a = \rho(s)(a)$ for $s \in S$ and $a \in \mathcal{A}$. We will denote the algebra obtained in this way by $\mathcal{A}_{(\rho)}$ or $\mathcal{A}_{(S)}$ if ρ is clear from the context.

Example 1. Assume that $f\colon \mathcal{A}' \xrightarrow{\sim} \mathcal{A}$ is an isomorphism of perfect R-algebras. It is then easily seen (and well known) that

$$\mathrm{Ctd}(f)\colon \mathrm{Ctd}(\mathcal{A}) \xrightarrow{\sim} \mathrm{Ctd}(\mathcal{A}'), \quad \chi \mapsto f^{-1} \circ \chi \circ f \tag{1.1.1}$$

is an isomorphism of R-algebras. Since $\mathcal{C} = \mathrm{Ctd}(\mathcal{A}) \in R$–**alg**, we can use $\mathrm{Ctd}(f)$ to make $\mathcal{A}'$ a $\mathcal{C}$-algebra: $\chi \cdot a' = \big(\mathrm{Ctd}(f)(\chi)\big)(a') = (f^{-1} \circ \chi \circ f)(a')$. Then

$$f\colon \mathcal{A}'_{(\mathcal{C})} \to \mathcal{A}_{(\mathcal{C})} \textit{ is } \mathcal{C}\textit{-linear.} \tag{1.1.2}$$

Indeed,

$$f(\chi \cdot a') = f\big((f^{-1} \circ \chi \circ f)(a')\big) = \chi\big(f(a')\big) = \chi \cdot f(a')$$

for $\chi \in \mathcal{C}$ and $a' \in \mathcal{A}'$.

Example 2 (centre)**.** Let $\mathcal{A}$ be a unital associative R-algebra. As usual, $[a, b] = ab - ba$ for $a, b, \in \mathcal{A}$ denotes the (Lie) commutator. The *centre* $\mathcal{Z}(\mathcal{A}) = \mathcal{Z}$ consists

[3] It will not be sufficient in the following to consider only algebras over fields.

[4] We use $\mathcal{A}_{(\mathcal{C})}$ instead of $\mathcal{A}_{\mathcal{C}}$ since the latter usually denotes base change.

of all $z \in \mathcal{A}$ satisfying $[z,a]=0$. One easily checks that the left multiplication by a central element is an isomorphism of R-algebras:

$$\mathcal{Z}(\mathcal{A}) \xrightarrow{\sim} \mathrm{Ctd}(\mathcal{A}), \quad z \mapsto L_z. \tag{1.1.3}$$

Hence we can (and will) consider $\mathcal{A}$ as a $\mathcal{Z}$-algebra, denoted $\mathcal{A}_{(\mathcal{Z})}$.

Example 3. Let $\mathcal{A}$ be as above and let $\mathcal{L} = \mathfrak{sl}_\ell(\mathcal{A})$ be the special linear Lie R-algebra introduced in 2.2. If $\mathcal{Z} = \mathcal{Z}(\mathcal{A})$, we have an obvious R-algebra homomorphism

$$\zeta\colon \mathcal{Z} \to \mathrm{Ctd}(\mathcal{L}), \quad \zeta(z) = \big((x_{ij}) \mapsto (zx_{ij})\big)$$

for $z \in \mathcal{Z}$ and $x = (x_{ij}) \in \mathcal{L}$. We will show in Lemma 2.4 that ζ is an isomorphism if $\ell \geq 2$ and $\frac{1}{2} \in R$. It is easily seen that

$$\mathfrak{sl}_\ell(\mathcal{A})_{(\mathcal{Z})} = \mathfrak{sl}_\ell(\mathcal{A}_{(\mathcal{Z})}). \tag{1.1.4}$$

1.2. Some properties of quantum tori

We list some properties of quantum tori that we will use. Throughout, F is a field of arbitrary characteristic, and Λ is a free abelian group of rank n.

(a) (Definitions) By definition, a *quantum torus* (*with grading group* Λ) is an associative unital Λ-graded F-algebra $Q = \bigoplus_{\lambda\in\Lambda} Q^\lambda$ such that $\dim Q^\lambda = 1$ for all $\lambda \in \Lambda$ and that every $0 \neq a \in Q^\lambda$ is invertible.

After fixing a basis $\boldsymbol{\varepsilon} = (\varepsilon_i)$ of Λ, we can choose $0 \neq x_i \in Q^{\varepsilon_i}$ and then get a quantum matrix $q = (q_{ij}) \in \mathrm{M}_n(F)$ defined by $x_i x_j = q_{ij} x_j x_i$. Then, using $x_i^{-1} =$ the inverse of x_i, we define $x^\lambda = x_1^{\ell_1} \cdots x_n^{\ell_n}$ for $\lambda = \ell_1\varepsilon_1 + \cdots + \ell_n\varepsilon_n \in \Lambda$:

$$Q = \textstyle\bigoplus_{\lambda\in\Lambda} F x^\lambda. \tag{1.2.1}$$

One can then also realize a quantum torus as the unital associative F-algebra presented by generators $x_1, \dots, x_n, x_1^{-1}, \dots, x_n^{-1}$ and relations

$$x_i x_i^{-1} = 1_Q = x_i^{-1} x_i, \quad x_i x_j = q_{ij} x_j x_i.$$

We will refer to this view of Q as a *coordinatization.*

We point out that such a presentation is not unique: it depends on the chosen $\mathbb{Z}$-basis $\boldsymbol{\varepsilon}$ of Λ. In other words, for any integral matrix $A = (a_{ij}) \in \mathrm{GL}_n(\mathbb{Z})$ the set $\tilde{x} = \{\tilde{x}_1, \dots, \tilde{x}_n\}$ of invertible elements in Q, defined by

$$\tilde{x}_1 = x_1^{a_{11}} \cdots x_n^{a_{1n}}, \dots, \tilde{x}_n = x_1^{a_{n1}} \cdots x_n^{a_{nn}},$$

also generates Q and the associated quantum matrix $\tilde{q} = (\tilde{q}_{ij})$ is given by $\tilde{q}_{ij} = \prod_{s,t} q_{st}^{a_{is}a_{jt}}$.

(b) The centre of Q is a Λ-graded subalgebra,

$$\mathcal{Z}(Q) = \textstyle\bigoplus_{\xi\in\Xi} Q^\xi$$

where Ξ is the so-called *central grading group*:

$$\Xi = \{\lambda \in \Lambda : Q^\lambda \subset \mathcal{Z}(Q)\}.$$

This is a free abelian group of rank $z \le n$. Hence $\mathcal{Z}(Q)$ is a Laurent polynomial ring in z variables, which we may take as $t_1, \dots, t_z$ (these can be taken to be of the form x^λ for suitable λs).

(c) We define

$$[Q, Q] = \operatorname{span}_F\{[a, b] : a, b \in Q\},$$

a graded subspace of Q. One knows (see, e.g., [BGK, Prop. 2.44(iii)] for $F = \mathbb{C}$ or [NY, (3.3.2)] in general)

$$Q = \mathcal{Z}(Q) \oplus [Q, Q]. \tag{1.2.2}$$

(d) Q is a domain: $ab = 0$ for $a, b \in Q$ implies that $a = 0$ or $b = 0$, whence a nondegenerate and thus prime associative F-algebra. This implies that Q is connected: the only idempotents in Q are 0 and 1_Q.

(e) An element u of Q is invertible if and only if $0 \neq u \in Q^\lambda$ for some $\lambda \in \Lambda$.

(f) The grading properties of a quantum torus Q show that Q is fgc in the sense of 1.1 if and only if Ξ has finite index in Λ. Equivalently, for some (hence all) coordinatization all entries q_{ij} of the quantum matrix q have finite order. If this holds, then for every coordinatization the q_{ij} have finite order.

(g) We let $\mathcal{Z} = \mathcal{Z}(Q)$, and denote by $\widetilde{\mathcal{Z}}$ the quotient field of $\mathcal{Z}$, a rational function field. The $\widetilde{\mathcal{Z}}$-algebra

$$\widetilde{Q} = Q \otimes_{\mathcal{Z}} \widetilde{\mathcal{Z}}$$

is called the *central closure of* Q. It has the following properties: $\widetilde{Q}$ is a central $\widetilde{\mathcal{Z}}$-algebra, $\widetilde{Q}$ is a domain (since Q is a domain), and Q embeds into $\widetilde{Q}$. In particular, if Q is fgc, then $\widetilde{Q}$ is a finite-dimensional central domain over $\widetilde{\mathcal{Z}}$, whence a central division $\widetilde{\mathcal{Z}}$-algebra.

(h) (*Trace, $\mathbb{Z}$-grading, degree*) As in (a) we fix a basis $\boldsymbol{\varepsilon} = (\varepsilon_1, \dots, \varepsilon_n)$ of Λ and define the $\boldsymbol{\varepsilon}$-*trace* as $\operatorname{tr}_{\boldsymbol{\varepsilon}}(\lambda) = \sum_i \lambda_i$ for $\lambda = \sum_i \lambda_i \varepsilon_i$ with $\lambda_i \in \mathbb{Z}$. Since $\operatorname{tr}_{\boldsymbol{\varepsilon}} : \Lambda \to \mathbb{Z}$ is a group homomorphism, the Λ-grading of Q can be made into a $\mathbb{Z}$-grading

$$Q = \bigoplus\nolimits_{n \in \mathbb{Z}} Q_{(n, \boldsymbol{\varepsilon})} \quad \text{with} \quad Q_{(n, \boldsymbol{\varepsilon})} = \bigoplus\nolimits_{\operatorname{tr}_{\boldsymbol{\varepsilon}}(\lambda) = n} Q^\lambda. \tag{1.2.3}$$

Every $0 \neq q \in Q$ can be uniquely written as $q = \sum_{n \le m} q_{(n)}$ with $q_{(n)} \in Q_{(n, \boldsymbol{\varepsilon})}$ and $q_{(m)} \neq 0$. We call m the $\boldsymbol{\varepsilon}$-*degree of* q and denote it by $\deg_{\boldsymbol{\varepsilon}} q$. We put $\deg_{\boldsymbol{\varepsilon}} 0 = -\infty$.

In the following we may suppress the dependance on $\boldsymbol{\varepsilon}$ and just speak of the trace. Analogously for the $\mathbb{Z}$-grading (1.2.3).

1.3. Lemma. *Let Q be a quantum torus over the field F, and let $q, q_1, q_2 \in Q$.*

(a) *For any $\mathbb{Z}$-basis $\boldsymbol{\varepsilon}$ of Λ we have*

$$\deg_{\boldsymbol{\varepsilon}}(q_1 + q_2) \le \max\{\deg_{\boldsymbol{\varepsilon}}(q_1), \deg_{\boldsymbol{\varepsilon}}(q_2)\}, \tag{1.3.1}$$

$$\deg_{\boldsymbol{\varepsilon}}(sq) = \deg_{\boldsymbol{\varepsilon}}(q) \quad \text{for } 0 \neq s \in F, \tag{1.3.2}$$

$$\deg_{\boldsymbol{\varepsilon}}(q_1 q_2) = \deg_{\boldsymbol{\varepsilon}}(q_1) + \deg_{\boldsymbol{\varepsilon}}(q_2) \tag{1.3.3}$$

with the obvious rules in case one of q, q_1 or $q_2 = 0$.

(b) *For $0 \neq q \in Q$ we have $q \in Q^0 = F1_Q \iff \deg_{\boldsymbol{\varepsilon}}(q) = 0$ for all $\mathbb{Z}$-bases $\boldsymbol{\varepsilon}$ of Λ.*

Proof. (a) All formulas are easily verified in case one of q, q_1 or $q_2 = 0$ (using the convenience that 0 is of degree $-\infty$). Also, (1.3.1) and (1.3.2) follow immediately from the definition. For (1.3.3) and $q_1, q_2 \neq 0$ we have $q_1 q_2 \neq 0$ since Q is a domain. We write $q_1 = \sum_{n \le m} q_{(n)}$ with $\deg_{\boldsymbol{\varepsilon}} q_1 = m$ and $q_2 = \sum_{n \le p} q'_{(n)}$ with $\deg_{\boldsymbol{\varepsilon}}(q_2) = p$. Since $Q = \bigoplus_{n \in \mathbb{Z}} Q_{(n,\boldsymbol{\varepsilon})}$ is a $\mathbb{Z}$-grading, it follows that $q_1 q_2 = q_{(m)} q'_{(p)} +$ terms of lower degree in the $\mathbb{Z}$-grading with respect to $\boldsymbol{\varepsilon}$.

(b) Let $0 \neq q \in Q$. If $q \in Q^0$ then $\deg_{\boldsymbol{\varepsilon}}(q) = 0$ because $\operatorname{tr}_{\boldsymbol{\varepsilon}}(q) = 0$ for any $\mathbb{Z}$-basis $\boldsymbol{\varepsilon}$ of Λ. Conversely, we can assume $q \neq 0$ and write $q = \sum_{\lambda \in \operatorname{supp}(q)} q^\lambda$ with $q^\lambda \in Q^\lambda$ and $\operatorname{supp}(q) = \{\lambda \in \Lambda : q^\lambda \neq 0\}$. Then $\operatorname{tr}_{\boldsymbol{\varepsilon}}(\lambda) \le 0$ for every $\lambda \in \operatorname{supp}(q)$ by assumption. But since both $\boldsymbol{\varepsilon}$ and $-\boldsymbol{\varepsilon}$ are $\mathbb{Z}$-bases of Λ and $\operatorname{tr}_{-\boldsymbol{\varepsilon}} = -\operatorname{tr}_{\boldsymbol{\varepsilon}}$ we get $\operatorname{tr}_{\boldsymbol{\varepsilon}}(\lambda) = 0$ for every $\lambda \in \operatorname{supp}(q)$. For a fixed $\boldsymbol{\varepsilon}$ and $\lambda = \sum_{i=1}^n \ell_i \varepsilon_i \in \operatorname{supp}(q)$ we therefore have $\sum_{i=1}^n \ell_i = 0$. Our claim obviously holds if $\Lambda \cong \mathbb{Z}$. Thus we can assume that Λ has rank at least 2. With respect to the $\mathbb{Z}$-basis $\boldsymbol{\varepsilon}' = (\varepsilon_1 + 2\varepsilon_2, \varepsilon_2, \varepsilon_3, \ldots)$ we have $\lambda = \ell_1(\varepsilon_1 + 2\varepsilon_2) + (\ell_2 - 2\ell_1)\varepsilon_2 + \cdots$. Thus $0 = \sum_i \ell_i = \ell_1 + (\ell_2 - 2\ell_1) + \sum_{i \ge 3} \ell_i = 0$, and $\ell_1 = 0$ follows. Similarly, all $\ell_i = 0$, i.e., $\lambda = 0$. □

In the following lemma we will describe certain F-diagonalizable endomorphisms ϕ of a quantum torus Q over F. The term *F-diagonalizable* means of course that there exists an F-basis of the F-vector space Q consisting of eigenvectors of ϕ with eigenvalues in F.

1.4. Lemma. *Let $Q = \bigoplus_{\lambda \in \Lambda} Q^\lambda$ be a quantum torus over the field F, and let $d \in Q$.*

(a) *If $dq = \omega q$ for some $0 \neq q \in Q$ and $\omega \in F$, then $d \in Q^0$. In particular, the left multiplication L_d for $d \in Q$ is F-diagonalizable if and only if $d \in F1_Q = Q^0$.*

(b) *If $[d, q] = \omega q$ for some $0 \neq \omega \in F$, then $q = 0$. In particular, the endomorphism $\operatorname{ad} d \in \operatorname{End}_F(Q)$, defined by $(\operatorname{ad} d)(q) = [d, q]$, is F-diagonalizable if and only if $d \in \mathcal{Z}(Q)$.*

Proof. (a) follows from the fact that Q is a domain and $q \neq 0$.

(b) Suppose $0 \neq q$. Then for any $\mathbb{Z}$-basis $\boldsymbol{\varepsilon}$ of Λ we get, using the formulas of Lemma 1.3(a),

$$\begin{aligned} \deg_{\boldsymbol{\varepsilon}}(dq) &= \deg_{\boldsymbol{\varepsilon}}(d) + \deg_{\boldsymbol{\varepsilon}}(q) = \deg_{\boldsymbol{\varepsilon}}(d) + \deg_{\boldsymbol{\varepsilon}}(-q) = \deg_{\boldsymbol{\varepsilon}}(-qd), \\ \deg_{\boldsymbol{\varepsilon}}(q) &= \deg_{\boldsymbol{\varepsilon}}(\omega q) = \deg_{\boldsymbol{\varepsilon}}(dq - qd) \le \max\{\deg_{\boldsymbol{\varepsilon}}(dq), \deg_{\boldsymbol{\varepsilon}}(-qd)\} \\ &= \deg_{\boldsymbol{\varepsilon}}(dq) = \deg_{\boldsymbol{\varepsilon}}(d) + \deg_{\boldsymbol{\varepsilon}}(q) \end{aligned}$$

whence $\deg_{\boldsymbol{\varepsilon}}(d) \ge 0$. But since $-\boldsymbol{\varepsilon}$ is also a basis of Λ, we in fact have $\deg_{\boldsymbol{\varepsilon}}(d) = 0$. Thus, by Lemma 1.3(b), we have $d \in Q^0$. But then $[d, q] = 0$ yields a contradiction. This shows $q = 0$, and also that $\operatorname{ad} d$ does not have a non-zero eigenvalue. In particular, if $\operatorname{ad} d$ is F-diagonalizable, necessarily $\operatorname{ad} d = 0$, i.e., $d \in \mathcal{Z}(Q)$. □

In the remainder of this section we present some results which are special for fgc quantum tori.

1.5. Canonical presentation

Let Q be a quantum torus over the field F, coordinatized as $Q = \bigoplus_{\lambda \in \Lambda} F x^\lambda$ as in (1.2.1), and let $q = (q_{ij})$ be the associated quantum matrix. We will say that the coordinatization (or presentation) is *canonical* if all entries of the quantum matrix q outside of its diagonal blocks of size 2×2 are equal to 1. Equivalently, for every $i \geq 1$ the generators x_{2i-1}, x_{2i} of Q commute with all other generators x_j where $j \neq 2i-1, 2i$.

The following lemma is stated without proof in [CP, Rem. 7.2]. A proof is given in [Nee, Thm. 4.5].

1.6. Lemma. *Any fgc quantum torus Q has a canonical presentation.*

1.7. Example (Quantum 2-tori)**.** Let Q be a quantum torus whose grading group has rank 2. Hence Q is generated by two elements, say x_1, x_2, and the corresponding quantum matrix has the form

$$q = \begin{pmatrix} 1 & q_{12} \\ q_{12}^{-1} & 1 \end{pmatrix}$$

where $q_{12} \in F^\times$. By definition, this presentation of Q is canonical. The algebra Q is fgc if and only if q_{12} is a root of unity, say primitive of degree ℓ. Let us assume this in the following.The $\mathcal{Z}$-algebra $Q_{(\mathcal{Z})}$ will be called a *symbol algebra* of degree ℓ. Its centre $\mathcal{Z}$ is a Laurent polynomial ring (1.2(b))

$$\mathcal{Z} = k[t_1^{\pm 1}, t_2^{\pm 1}]$$

where $t_1 = x_1^\ell$, $t_2 = x_2^\ell$ (observe that ℓ is independent of the coordinatization, since $\Lambda/\Xi = \ell^2$). We will usually denote this $\mathcal{Z}$-algebra by $(t_1, t_2)_{\mathcal{Z}, q_{12}}$ or simply (t_1, t_2) if there is no risk of confusion. Note that $Q_{(\mathcal{Z})}$ has order ℓ in the Brauer group $\mathrm{Br}(\mathcal{Z})$.

If $\ell \in F^\times$ the subalgebra $E = \mathcal{Z}[x_1^{\pm 1}]$ is a maximal (abelian) étale subalgebra of $Q_{(\mathcal{Z})}$.

1.8. Example. Let $Q = \bigoplus_{\lambda \in \Lambda} F x^\lambda$ be an fgc quantum torus over a field F. By Lemma 1.6 we can assume that Q is canonically presented, say with quantum matrix $q = (q_{ij})$. Up to re-numbering (=re-coordinatization), we may assume that the elements $q_{12}, q_{34}, \ldots, q_{2s-1,2s} \neq 1$, but $q_{2i+1,2i+2} = 1$ for all $i \geq s$. Then Q admits a decomposition

$$Q_{(\mathcal{Z})} = Q_{1,\mathcal{Z}} \otimes_{\mathcal{Z}} \cdots \otimes_{\mathcal{Z}} Q_{s,\mathcal{Z}}$$

where the $Q_{i,\mathcal{Z}} = (t_{2i-1}, t_{2i})$ are the symbol algebras in degree ℓ_i corresponding to the nontrivial diagonal blocks of q of size 2×2, i.e., to the block diagonal sub-matrices

$$\begin{pmatrix} 1 & q_{2i-1,2i} \\ q_{2i,2i-1} & 1 \end{pmatrix}$$

of q where $i \leq s$. Here $t_{2i-1} = x_{2i-1}^{\ell_i}, t_{2i} = x_{2i}^{\ell_i}$ and $\ell_i = |q_{2i-1,2i}|$ is the order of $q_{2i-1,2i}$. Obviously,

$$\mathcal{Z} = F[t_1^{\pm 1}, \ldots, t_{2s}^{\pm 1}, t_{2s+1}^{\pm 1}, \ldots, t_n^{\pm 1}]$$

where $t_{2s+1} = x_{2s+1}, \ldots, t_n = x_n$.

1.9. *Remark.* In the Setting of 1.8 assume that all $l_i \in F^\times$. Then

$$E = \mathcal{Z}[x_1^{\pm 1}, x_3^{\pm 1}, \ldots, x_{2s-1}^{\pm 1}] \subset Q_{(\mathcal{Z})}$$

is a maximal étale $\mathcal{Z}$-subalgebra of $Q_{(\mathcal{Z})}$, and the pair $(Q_{(\mathcal{Z})}, E)$ gives rise to a reductive $\mathcal{Z}$-group scheme $\mathbf{GL}_{Q_{(\mathcal{Z})}}$ and its maximal torus $\mathbf{S} = R_{E/\mathcal{Z}}(\mathbf{G}_{m,E})$.

2. Results on $\mathfrak{sl}_\ell(Q)$ (mostly in good characteristic)

2.1. Associative and Lie algebras

For an arbitrary associative algebra $\mathcal{A}$ over a base ring R we denote by $\mathcal{A}^{\mathrm{op}}$ the opposite algebra: $\mathcal{A}^{\mathrm{op}} = \mathcal{A}$ as R-modules but the multiplication is given by $a \cdot_{\mathcal{A}^{\mathrm{op}}} b = b \cdot_{\mathcal{A}} a$. Note that for $\mathrm{M}_\ell(\mathcal{A})$, $\ell \in \mathbb{N}_+$, namely the associative R-algebra of $\ell \times \ell$-matrices over $\mathcal{A}$, we have

$$\mathrm{M}_\ell(\mathcal{A})^{\mathrm{op}} \simeq \mathrm{M}_\ell(\mathcal{A}^{\mathrm{op}}). \tag{2.1.1}$$

The algebra $\mathcal{A}$ becomes a Lie R-algebra, denoted $\mathrm{Lie}(\mathcal{A})$, with respect to the commutator $[a, b] = ab - ba$ as multiplication. We leave it to the reader to check that the map

$$\iota^{\mathrm{op}} \colon \mathrm{Lie}(\mathcal{A}^{\mathrm{op}}) \xrightarrow{\sim} \mathrm{Lie}(\mathcal{A}), \quad a \mapsto -a, \tag{2.1.2}$$

is an isomorphism of Lie R-algebras.

Abiding by the traditional notation we put

$$\mathrm{Lie}\big(\mathrm{M}_\ell(\mathcal{A})\big) = \mathfrak{gl}_\ell(\mathcal{A}).$$

Thus, combining (2.1.1) and (2.1.2) we obtain a Lie algebra isomorphism, also denoted ι^{op},

$$\iota^{\mathrm{op}} \colon \mathfrak{gl}_\ell(\mathcal{A}^{\mathrm{op}}) \xrightarrow{\sim} \mathfrak{gl}_\ell(\mathcal{A}), \quad x \mapsto -x.$$

2.2. The Lie algebra $\mathfrak{sl}_\ell(\mathcal{A})$

Let $\mathcal{A}$ be a unital associative R-algebra, and let $\ell \in \mathbb{N}$, $\ell \geq 2$. The derived algebra of the Lie algebra $\mathfrak{gl}_\ell(\mathcal{A})$ of 2.1, is called the *special linear Lie algebra* $\mathfrak{sl}_\ell(\mathcal{A})$:

$$\mathfrak{sl}_\ell(\mathcal{A}) = [\mathfrak{gl}_\ell(\mathcal{A}), \mathfrak{gl}_\ell(\mathcal{A})].$$

Whenever we consider $\mathfrak{sl}_\ell(\mathcal{A})$ in the future, it will implicitly be assumed that $\ell \geq 2$. Obviously, the restriction of the isomorphism ι^{op},

$$\iota^{\mathrm{op}} \colon \mathfrak{sl}_\ell(\mathcal{A}^{\mathrm{op}}) \xrightarrow{\sim} \mathfrak{sl}_\ell(\mathcal{A}), \quad x \mapsto -x \tag{2.2.1}$$

is an isomorphism of Lie R-algebras. We will later need the fine structure of $\mathfrak{sl}_\ell(\mathcal{A})$. First, we have

$$\mathfrak{sl}_\ell(\mathcal{A}) = \{x \in \mathfrak{gl}_\ell(\mathcal{A}) : \mathrm{tr}(x) \in [\mathcal{A}, \mathcal{A}]\},$$

where the trace $\mathrm{tr}(x)$ of $x \in \mathfrak{gl}_\ell(\mathcal{A})$ is defined as usual. We denote by E_{ij}, $1 \leq i, j \leq \ell$ the usual matrix units. It is easy to see that

$$\begin{aligned} \mathfrak{sl}_\ell(A) &= \mathcal{L}_0 \oplus \big(\textstyle\bigoplus_{1 \leq i \neq j \leq \ell} \mathcal{A}E_{ij}\big), \\ \mathcal{L}_0 &= [\mathcal{A}, \mathcal{A}]E_{11} \oplus \{\textstyle\sum_{i=1}^{\ell} a_i E_{ii} : a_i \in \mathcal{A}, \sum_i a_i = 0\}. \end{aligned} \tag{2.2.2}$$

In particular, for any unital subalgebra S of R the Lie S-algebra $\mathfrak{sl}_\ell(\mathcal{A})_{(S)}$ contains

$$\mathfrak{sl}_\ell(S) = \{x \in \mathfrak{gl}_\ell(S) : \mathrm{tr}(x) = 0\}$$

as subalgebra. We denote

$$\mathfrak{h}_S = \mathfrak{sl}_\ell(S) \cap \mathcal{L}_0 = \{\sum_{i=1}^{\ell} s_i E_{ii} : s_i \in S, \sum_i s_i = 0\} \tag{2.2.3}$$

the diagonal subalgebra of $\mathfrak{sl}_\ell(S)$.

We will say that a domain R has *good characteristic for* $\mathfrak{sl}_\ell(\mathcal{A})$ if the characteristic of the fraction field of R is either 0 or $p > 3$ and such that p does not divide ℓ. In that case, if F is a subfield of R, the subspace $\mathfrak{h}_F$ is an ad-diagonalizable subalgebra of $\mathfrak{sl}_\ell(\mathcal{A})_{(F)}$ in the sense of Subsection 2.7, and (2.2.2) is the joint eigenspace decomposition of $\mathfrak{h}_F$.

It will be useful later to have a coordinate-free approach to $\mathfrak{sl}_\ell(\mathcal{A})$. Namely, we let

$$V = V_{\mathcal{A}} = \mathcal{A} \oplus \cdots \oplus \mathcal{A} =: \mathcal{A}^{\oplus \ell} \tag{2.2.4}$$

be the free right $\mathcal{A}$-module of rank ℓ. We denote by $\mathsf{B} = \{e_1, \dots, e_\ell\}$ the *standard basis* of the $\mathcal{A}$-module V:

$$e_1 = (1, 0, \dots, 0), \quad \cdots \quad , e_\ell = (0, 0, \dots, 1), \tag{2.2.5}$$

so that

$$V = \bigoplus_{i=1}^{\ell} e_i \mathcal{A}. \tag{2.2.6}$$

We let the associative algebra $\mathrm{End}_{\mathcal{A}}(V)$ of $\mathcal{A}$-linear endomorphisms of V act on V from the left. Representing $f \in \mathrm{End}_{\mathcal{A}}(V)$ by the matrix $\mathrm{Mat}_{\mathsf{B}}(f)$ with respect to B provides us with an R-algebra isomorphism

$$\mathrm{Mat}_{\mathsf{B}} \colon \mathrm{End}_{\mathcal{A}}(V) \xrightarrow{\sim} \mathrm{M}_\ell(\mathcal{A}) \tag{2.2.7}$$

and thus also an isomorphism of the associated Lie algebras, $\mathfrak{gl}_{\mathcal{A}}(V) \simeq \mathfrak{gl}_\ell(\mathcal{A})$, and of their derived algebras,

$$\mathfrak{sl}_{\mathcal{A}}(V) := [\mathfrak{gl}_{\mathcal{A}}(V), \mathfrak{gl}_{\mathcal{A}}(V)] \xrightarrow{\sim} \mathfrak{sl}_\ell(\mathcal{A}).$$

Moreover, observe

$$e_i\mathcal{A} = E_{ii}(V) \quad \text{and} \quad E_{ij} \colon e_j\mathcal{A} \xrightarrow{\sim} e_i\mathcal{A} \text{ is an isomorphism of } \mathcal{A}\text{-modules.}$$

2.3. Lemma. *In the Setting of* 2.2 *assume* $\ell \cdot 1_R \in R^\times$ *and* $\mathcal{A} = \mathcal{Z}(\mathcal{A}) \oplus [\mathcal{A}, \mathcal{A}]$. *Then*

$$\mathfrak{gl}_\ell(\mathcal{A}) = \mathcal{Z}\big(\mathfrak{gl}_\ell(\mathcal{A})\big) \oplus \mathfrak{sl}_\ell(\mathcal{A}) \quad \textit{with} \quad \mathcal{Z}\big(\mathfrak{gl}_\ell(\mathcal{A})\big) = \mathcal{Z}(\mathcal{A})E_\ell$$

where $E_\ell \in \mathfrak{gl}_\ell(\mathcal{A})$ *is the* $\ell \times \ell$ *identity matrix.*

Proof. Straightforward. $\square$

The following result, determining the centroid of $\mathfrak{sl}_\ell(\mathcal{A})$, is folklore.

2.4. Lemma. *Let R be a commutative ring with $\frac{1}{2} \in R$, let $\mathcal{A}$ be a unital associative R-algebra, and let $\mathcal{L} = \mathfrak{sl}_\ell(\mathcal{A})$ with $\ell \geq 2$. Then for every $z \in \mathcal{Z}(\mathcal{A})$ the map $\zeta_z \colon \mathcal{L} \to \mathcal{L}$, $\zeta_z\big((x_{ij})\big) = (zx_{ij})$, is a centroidal transformation of $\mathcal{L}$. The map*

$$\zeta \colon \mathcal{Z}(\mathcal{A}) \xrightarrow{\sim} \mathrm{Ctd}(\mathcal{L}), \quad z \mapsto \zeta_z \tag{2.4.1}$$

is an isomorphism of associative algebras. In particular,

$$\mathfrak{sl}_\ell(\mathcal{A}) \text{ is fgc } \iff \mathcal{A} \text{ is fgc.} \tag{2.4.2}$$

Proof. Since $z[a,b] = [za,b]$ it is clear that ζ_z is an endomorphism of $\mathcal{L}$. It is also immediate that $\zeta_z \in \mathrm{Ctd}(\mathcal{L})$ and that ζ is an injective homomorphism of associative algebras. Thus it remains to prove surjectivity. Let $\chi \in \mathrm{Ctd}(\mathcal{L})$.

We first consider the case $\ell = 2$. For $a, b \in \mathcal{A}$ we define

$$e(a) = \begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix}, \qquad f(b) = \begin{pmatrix} 0 & 0 \\ b & 0 \end{pmatrix},$$
$$H(a,b) = [e(a), f(b)] = \begin{pmatrix} ab & 0 \\ 0 & -ba \end{pmatrix}, \qquad h = H(1,1) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Then

$$\{l \in \mathcal{L} : [h,l] = 2l\} = e(\mathcal{A}), \quad \{l \in \mathcal{L} : [h,l] = -2l\} = f(\mathcal{A}),$$
$$\{l \in \mathcal{L} : [h,l] = 0\} = \mathrm{span}\{H(a,b) : a,b \in \mathcal{A}\} =: \mathcal{L}_0.$$

It follows that χ leaves $e(\mathcal{A})$, $f(\mathcal{A})$ and $\mathcal{L}_0$ invariant. In particular, we can define $\chi_\pm \in \mathrm{End}(\mathcal{A})$ by

$$\chi\big(e(a)\big) = e\big(\chi_+(a)\big), \quad \chi\big(f(b)\big) = f\big(\chi_-(b)\big).$$

Denoting by $\{a\,b\,c\} = abc + cba$ the Jordan triple product of $\mathcal{A}$, we have the following multiplication rules of $\mathcal{L}$:

$$[H(a,b), e(c)] = e\big(\{a\ b\ c\}\big), \quad [H(a,b), f(c)] = f\big(-\{b\ a\ c\}\big).$$

They imply $\chi_\pm(\{a\ b\ c\}) = \{a\ b\ \chi_\pm(c)\}$. Define $z_\pm \in \mathcal{A}$ by $\chi_\pm(1_\mathcal{A}) = z_\pm$. Then, specializing $a = c = 1_\mathcal{A}$ we obtain

$$2\chi_\pm(b) = \chi_\pm\big(\{1_\mathcal{A}\ 1_\mathcal{A}\ b\}\big) = b\chi_\pm(1_\mathcal{A}) + \chi_\pm(1_\mathcal{A})b = 2z_\pm \circ b,$$

where $x \circ y = \frac{1}{2}(xy + yx)$ is the Jordan algebra product of $\mathcal{A}$. Thus $\chi_\pm(b) = z_\pm \circ b$. From $H(a,b) = [e(a), f(b)]$ we now get

$$H(z_\pm \circ a, b) = [\chi(e(a)),\ f(b)] = [e(a), \chi(f(b))] = H(a, z_- \circ b).$$

Comparing the (11)-entry of H, this proves

$$z_+ab + az_+b = az_-b + abz_-.$$

In particular, for $a = 1_{\mathcal{A}}$ we obtain $2z_+b = z_-b + bz_-$. Specializing $b = 1_{\mathcal{A}}$ in the formula above, this shows $z_+ = z_- =: z$, whence $2zb = zb + bz$, or $zb = bz$ for all $b \in \mathcal{A}$, i.e., $z \in \mathcal{Z}(\mathcal{A})$. Finally, $\chi\big(H(a,b)\big) = H(za,b) = zH(a,b)$ proves $\chi = \zeta_z$.

Let now $\ell \geq 3$, and $i \neq j$. From

$$\mathcal{A}E_{ij} = \{l \in \mathfrak{sl}_\ell(\mathcal{A}); [E_{ii} - E_{jj}, l] = 2l\}$$

we get $\chi(\mathcal{A}E_{ij}) = \mathcal{A}E_{ij}$, allowing us to define $\chi_{ij} \in \operatorname{End}\mathcal{A}$ by $\chi(aE_{ij}) = \chi_{ij}(a)E_{ij}$. For distinct i, j, p and $a, b \in \mathcal{A}$ we have the multiplication formula

$$abE_{ij} = [[[aE_{ij}, E_{jp}], E_{pi}], bE_{ij}]$$

which implies $\chi_{ij}(ab) = \chi_{ij}(a)b = a\chi_{ij}(b)$, i.e., $\chi_{ij} \in \operatorname{Ctd}(\mathcal{A}) \simeq \mathcal{Z}(\mathcal{A})$ by (1.1.3). Thus, there exists $z_{ij} \in \mathcal{Z}(\mathcal{A})$ such that $\chi_{ij}(a) = z_{ij}a$. From $[aE_{ij}, E_{jp}] = aE_{ip}$ we now obtain $z_{ij} = z_{ip}$ and from $[E_{pi}, aE_{ij}] = aE_{pi}$ we get $z_{ij} = z_{pj}$. Hence the z_{ij} are independent of (ij), say $z_{ij} =: z \in \mathcal{Z}(\mathcal{A})$. Finally $\chi = \zeta_z$ follows. □

2.5. Lemma. *Let ℓ, $\ell' \in \mathbb{N}$ with $\ell, \ell' \geq 2$, and let R be a commutative base ring for which $2 \cdot 1_R$, $\ell \cdot 1_R$ and $\ell' \cdot 1_R$ are invertible in R. Furthermore, let $\mathcal{A}$ and $\mathcal{A}'$ be unital associative R-algebras satisfying*

$$\mathcal{A} = \mathcal{Z}(\mathcal{A}) \oplus [\mathcal{A}, \mathcal{A}] \quad \textit{and} \quad \mathcal{A}' = \mathcal{Z}(\mathcal{A}') \oplus [\mathcal{A}', \mathcal{A}'].$$

(a) *Let $f\colon \mathfrak{sl}_\ell(\mathcal{A}) \to \mathfrak{sl}_{\ell'}(\mathcal{A}')$ be an R-linear isomorphism of Lie algebras. The R-linear algebra isomorphisms* $\operatorname{Ctd}(f)$, *$\zeta_{\mathcal{A}}$ and $\zeta_{\mathcal{A}'}$ of* (1.1.1) *and* (2.4.1) *allow us to define an isomorphism $f_{\mathcal{Z}}\colon \mathcal{Z}(\mathcal{A}) \to \mathcal{Z}(\mathcal{A}')$ by requiring commutativity of the diagram*

$$\begin{array}{ccc} \mathcal{Z}(\mathcal{A}) & \overset{f_{\mathcal{Z}}}{\dashrightarrow} & \mathcal{Z}(\mathcal{A}') \\ \downarrow{\scriptstyle \zeta_{\mathcal{A}}} & & \downarrow{\scriptstyle \zeta_{\mathcal{A}'}} \\ \operatorname{Ctd}\big(\mathfrak{sl}_\ell(\mathcal{A})\big) & \xrightarrow{\operatorname{Ctd}(f)^{-1}} & \operatorname{Ctd}\big(\mathfrak{sl}_{\ell'}(\mathcal{A}')\big) \end{array} \quad . \qquad (2.5.1)$$

For $z \in \mathcal{Z}(\mathcal{A})$ and $X \in \mathfrak{sl}_\ell(\mathcal{A})$ define

$$f_{\mathfrak{gl}}\colon \mathfrak{gl}_\ell(\mathcal{A}) \to \mathfrak{gl}_{\ell'}(\mathcal{A}'), \quad zE_\ell + X \mapsto f_{\mathcal{Z}}(z)E_{\ell'} + f(X).$$

Then $f_{\mathfrak{gl}}$ is an isomorphism of Lie algebras. If $\mathfrak{sl}_\ell(\mathcal{A})$ and $\mathfrak{gl}_\ell(\mathcal{A})$ are viewed as $\mathcal{Z}(\mathcal{A}')$-algebras via the construction of (1.1.1), *then both f and $f_{\mathfrak{gl}}$ are $\mathcal{Z}(\mathcal{A}')$-linear.*

(b) *Let $\varphi\colon \mathrm{M}_\ell(\mathcal{A}) \to \mathrm{M}_{\ell'}(\mathcal{A}')$ be an isomorphism of associative algebras. The induced Lie algebra isomorphism $\varphi_{\mathfrak{sl}}\colon \mathfrak{sl}_\ell(\mathcal{A}) \to \mathfrak{sl}_{\ell'}(\mathcal{A}')$ obtained from φ satisfies $(\varphi_{\mathfrak{sl}})_{\mathfrak{gl}} = \varphi$.*

Proof. (a) is immediate from Lemma 2.3 and Lemma 2.4. For (b) we use that φ maps the centre $\mathcal{Z}(\mathcal{A})E_\ell$ of $\mathrm{M}_\ell(\mathcal{A})$ into the centre $\mathcal{Z}(\mathcal{A}')E_{\ell'}$ of $\mathrm{M}_{\ell'}(\mathcal{A}')$, hence induces an R-linear isomorphism $\psi\colon \mathcal{Z}(\mathcal{A}) \to \mathcal{Z}(\mathcal{A}')$ by $\varphi(zE_\ell) = \psi(z)E_{\ell'}$. Our claim is $(\varphi_{\mathfrak{sl}})_{\mathcal{Z}} = \psi$.

Denoting by L and L' the left multiplication of the associative algebras $\mathrm{M}_\ell(\mathcal{A})$ and $\mathrm{M}_{\ell'}(\mathcal{A}')$ respectively, we have

$$\varphi \circ L_{zE_\ell} \circ \varphi^{-1} = L'_{\varphi(zE_\ell)} = L'_{\psi(z)E_{\ell'}}$$

for all $z \in \mathcal{Z}(\mathcal{A})$. Note that L_{zE_ℓ} stabilizes $\mathfrak{sl}_\ell(\mathcal{A})$. We denote by $(L_{zE_\ell})_{\mathfrak{sl}}$ the restriction of L_{zE_ℓ} to $\mathfrak{sl}_\ell(\mathcal{A})$. Then $\zeta_{\mathcal{A}}(z) = (L_{zE_\ell})_{\mathfrak{sl}}$. Using analogous notation for $\mathrm{M}_{\ell'}(\mathcal{A}')$ and taking the $\mathfrak{sl}$-components of the displayed equation above, we get

$$\begin{aligned}\big(\mathrm{Ctd}(\varphi_{\mathfrak{sl}})^{-1} \circ \zeta_{\mathcal{A}}\big)(z) &= \varphi_{\mathfrak{sl}} \circ \zeta_{\mathcal{A}}(z) \circ \varphi_{\mathfrak{sl}}^{-1} = \varphi_{\mathfrak{sl}} \circ (L_{zE_\ell})_{\mathfrak{sl}} \circ \varphi_{\mathfrak{sl}}^{-1} \\ &= (L'_{\psi(z)E_{\ell'}})_{\mathfrak{sl}} = \zeta_{\mathcal{A}'}\big(\psi(z)\big)\end{aligned}$$

which proves our claim. □

2.6. *Remark.* We will later apply this Lemma in a situation where we are given a Lie algebra isomorphism $f\colon \mathfrak{sl}_\ell(\mathcal{A}) \to \mathfrak{sl}_{\ell'}(\mathcal{A}')$ and

(a) either f extends to an isomorphism $\widehat{f}\colon \mathrm{M}_\ell(\mathcal{A}) \to \mathrm{M}_{\ell'}(\mathcal{A}')$ of associative algebras,

(b) or $f \circ \iota^{\mathrm{op}}\colon \mathfrak{sl}_\ell(\mathcal{A}^{\mathrm{op}}) \to \mathfrak{sl}_{\ell'}(\mathcal{A}')$ extends to an R-linear isomorphism

$$\widehat{f \circ \iota^{\mathrm{op}}}\colon \mathrm{M}_\ell(\mathcal{A}^{\mathrm{op}}) \to \mathrm{M}_{\ell'}(\mathcal{A}')$$

of associative algebras.

In the first case, the Lie algebra isomorphism $f_{\mathfrak{gl}}\colon \mathfrak{gl}_\ell(\mathcal{A}) \to \mathfrak{gl}_{\ell'}(\mathcal{A}')$ of Lemma 2.5 is in fact an isomorphism of associative algebras, namely $\widehat{f} = f_{\mathfrak{gl}}$ (as maps), while in the second case we have $\widehat{f \circ \iota^{\mathrm{op}}} = (f \circ \iota^{\mathrm{op}})_{\mathfrak{gl}}$.

2.7. Definition (AD and MAD subalgebras**).** We now come to the central concept of this paper. Let F be a field. Following [CGP, §6] we call an F-subalgebra $\mathfrak{h}$ of a Lie algebra $\mathcal{L}$ over F an *AD subalgebra* if the adjoint action of each element $x \in \mathfrak{h}$ on $\mathcal{L}$ is F-diagonalizable, i.e., $\mathcal{L}$ admits an F-basis consisting of eigenvectors of $\mathrm{ad}_L(x)$ for all $x \in \mathfrak{h}$.

A maximal AD subalgebra of $\mathcal{L}$, i.e., one which is not properly included in any other AD subalgebra of $\mathcal{L}$, is called a *MAD subalgebra*, or a *MAD* for short.

It is not difficult to show, see, for example, [Hu, Lem. 8.1], that an AD subalgebra is necessarily abelian. Hence, AD can be thought of as an abbreviation for "abelian k-diagonalizable" or "ad k-diagonalizable".

Let Q be a quantum torus over F. Denote by $\mathcal{Z}\big(\mathfrak{gl}_\ell(Q)\big)$ the centre of the Lie algebra $\mathfrak{gl}_\ell(Q)$. Assuming that F is of good characteristic for $\mathfrak{gl}_\ell(Q)$, by Lemma 2.3 we have

$$\mathfrak{gl}_\ell(Q) = \mathcal{Z}\big(\mathfrak{gl}_\ell(Q)\big) \oplus \mathfrak{sl}_\ell(Q), \quad \mathcal{Z}\big(\mathfrak{gl}_\ell(Q)\big) = \mathcal{Z}(Q)E_\ell.$$

It follows that $\mathfrak{h} \subset \mathfrak{sl}_\ell(Q)$ is an AD or a MAD of $\mathfrak{sl}_\ell(Q)$ if and only if $\mathcal{Z}(Q)E_\ell \oplus \mathfrak{h}$ is an AD or a MAD of $\mathfrak{gl}_\ell(Q)$ respectively.

We next give a first example of a MAD.

2.8. Proposition. *Let Q be a quantum torus over a field F of good characteristic for $\mathfrak{sl}_\ell(Q)$, $\ell \ge 2$. The subalgebra $\mathfrak{h}_F$ of* (2.2.3) *is a MAD of the Lie algebra $\mathfrak{sl}_\ell(Q)_{(F)}$.*

We note that in the case that $\mathfrak{sl}_\ell(Q)$ is an fgc Lie torus over a field of characteristic 0, cf. 1.2(f) and (2.4.2), the lemma has been proven in [Al, Cor. 5.5]. The methods of [Al] cannot be applied in our case.

Proof. It is clear that $\mathfrak{h} = \mathfrak{h}_F$ is an AD: the joint eigenspaces of $\mathfrak{h}$ are the subspaces QE_{ij} and $\mathcal{L}_0$ of (2.2.2). To show maximality, let $d \in \mathfrak{sl}_\ell(Q)$ be an ad F-diagonalizable element commuting with $\mathfrak{h}$. It follows that $d \in \mathfrak{sl}_\ell(Q)_0 = \{l \in \mathfrak{sl}_\ell(Q) : [h,l] = 0 \text{ for all } h \in \mathfrak{h}\}$. Thus, $d = \operatorname{diag}(d_1, \dots, d_\ell)$ is a diagonal matrix. For fixed i, $1 \le i \le \ell$, and $q \in [Q,Q]$ we have $qE_{ii} \in \mathfrak{sl}_\ell(Q)_0$ and $[d, qE_{ii}] = [d_i, q]E_{ii}$. Because $Q = \mathcal{Z}(Q) \oplus [Q,Q]$ it follows that $\operatorname{ad} d_i \in \operatorname{End}_F(Q)$ is F-diagonalizable. Hence, by Lemma 1.4(b), all $d_i \in \mathcal{Z}(Q)$. Consider now the action of d on an off-diagonal space QE_{ij}, $i \ne j$. Clearly, $\operatorname{ad} d$ leaves QE_{ij} invariant and acts on qE_{ij} by $(d_i - d_j)qE_{ij}$, Thus, the left multiplication by $d_i - d_j$ is diagonalizable, forcing $d_i - d_j \in F$ by Lemma 1.4(a). Now consider the equation

$$[Q,Q] \ni \textstyle\sum_i d_i = (d_1 - d_2) + 2(d_2 - d_3) + \cdots + (n-1)(d_{\ell-1} - d_\ell) + \ell d_\ell \in \mathcal{Z}(Q).$$

It follows that $\sum_i d_i = 0$ and that $d_\ell \in F$. Analogously, all $d_i \in F$, and $d \in \mathfrak{h}$ follows. □

2.9. Complete orthogonal systems

Let $\mathcal{B}$ be a unital associative R-algebra. A *complete orthogonal system* (of idempotents) in $\mathcal{B}$ is a family $\mathcal{O} = (e_1, \dots, e_m)$ of elements $e_i \in \mathcal{B}$ satisfying

$$e_i e_j = \delta_{ij} e_i \quad \text{for } 1 \le i,j \le m \text{ and } \quad e_1 + \cdots + e_m = 1_{\mathcal{B}}. \tag{2.9.1}$$

In $\mathcal{B} = \mathrm{M}_\ell(\mathcal{A})$, $\mathcal{A}$ unital associative, the *standard orthogonal system* is $\mathcal{O}_{\mathrm{st}} = (E_{11}, E_{22}, \dots, E_{\ell\ell})$, where the E_{ij} are the usual standard matrix units. But also $(E_{11}+E_{22}, E_{33}, \dots, E_{\ell\ell})$ is a complete orthogonal system. Part (a) of the following Lemma 2.10 says that there is a natural bijection between complete orthogonal systems in $\mathrm{M}_\ell(\mathcal{A})$ and decompositions of $V_\mathcal{A}$ as a direct sum of $\mathcal{A}$-modules.

2.10. Lemma. *Let $\mathcal{A}$ be a unital associative R-algebra, and let $V = V_\mathcal{A} = \mathcal{A}^{\oplus \ell}$ be the right $\mathcal{A}$-module of* (2.2.4). *We identify $\mathcal{B} = \mathrm{M}_\ell(\mathcal{A}) \equiv \operatorname{End}_\mathcal{A}(V)$ using* (2.2.7).

(a) *Let $\mathcal{O} = (e_1, \dots, e_m)$ be a complete orthogonal system in $\mathcal{B}$. Define $V_i = e_i(V)$, $1 \le i \le m$. Then V decomposes as $V = V_1 \oplus \cdots \oplus V_m$ where each V_i is a right $\mathcal{A}$-module. Conversely, let $V = V_1 \oplus \cdots \oplus V_m$ be a decomposition of V as a direct sum of $\mathcal{A}$-modules. Define $e_i \in \mathcal{B}$ as the canonical projection of V onto $V_i \subset V$. Then $(e_1, \dots, e_m)$ is a complete orthogonal system in $\mathrm{M}_\ell(\mathcal{A})$.*

The constructions $\mathcal{O} \rightsquigarrow V = V_1 \oplus \cdots \oplus V_m$ and $V = V_1 \oplus \cdots \oplus V_m \rightsquigarrow \mathcal{O}$ defined above are inverses of each other.

(b) *Let $\mathcal{O} = (e_1, \dots, e_m)$ and $\mathcal{O}' = (e'_1, \dots, e'_{m'})$ be complete orthogonal systems in $\mathrm{M}_\ell(\mathcal{A})$, inducing by* (a) *decompositions $V_\mathcal{A} = V_1 \oplus \cdots \oplus V_m = V'_1 \oplus \cdots \oplus V'_{m'}$. Let*

$$\mathcal{D}(\mathcal{O}) = \{f \in \operatorname{End}_\mathcal{A}(V_\mathcal{A}) : f(V_i) \subset V_i, 1 \le i \le m\} = \textstyle\bigoplus_{i=1}^m \operatorname{End}_\mathcal{A}(V_i)$$

and define $\mathcal{D}(\mathcal{O}')$ analogously. Suppose that all $\operatorname{End}_{\mathcal{A}}(V_i)$ *and* $\operatorname{End}_{\mathcal{A}}(V'_j)$ *are connected. Then the following are equivalent for $g \in \mathcal{B}^\times$.*

(i) $g\mathcal{D}(\mathcal{O})g^{-1} = \mathcal{D}(\mathcal{O}')$.

(ii) *$m = m'$ and there exists a permutation $\sigma \in \mathfrak{S}_m$ such that $ge_ig^{-1} = e'_{\sigma(i)}$, $1 \le i \le m$.*

(iii) *$m = m'$ and there exists a permutation $\sigma \in \mathfrak{S}_m$ such that $g(V_i) = V'_{\sigma(i)}$ for $1 \le i \le m$.*

(c) *Let $\mathcal{A} = Q$ be a quantum torus. Let $\mathcal{O}_{\rm st} = (E_{11}, \dots, E_{\ell\ell})$ be the standard orthogonal system in $\mathcal{B} = \mathrm{M}_\ell(Q)$, and let $\mathcal{O} = (e_1, \dots e_\ell)$ be another complete orthogonal system in $\mathcal{B}$ with associated decomposition $V_Q = V_1 \oplus \cdots \oplus V_\ell$, $V_i = e_i(V)$ for $1 \le i \le \ell$. Define $\mathcal{D}(\mathcal{O}_{\rm st})$ and $\mathcal{D}(\mathcal{O})$ as in* (b). *Then the following are equivalent.*

(i) *Each V_i, $1 \le i \le \ell$, is a cyclic Q-module.*

(ii) *Each V_i, $1 \le i \le \ell$, is a free Q-module of rank* 1.

(iii) *There exists $g \in \mathcal{B}^\times$ such that $g\mathcal{D}(\mathcal{O}_{\rm st})g^{-1} = \mathcal{D}(\mathcal{O})$ and each* $\operatorname{End}_Q(V_i)$ *is connected.*

Assuming (c.iii) *holds, let $g \in \mathcal{B}^\times$ be as in* (c.iii) *and let*

$$\mathfrak{h}_{\rm st} = \Big\{ \textstyle\sum_{i=1}^{\ell} s_i E_{ii} : s_i \in F, \sum_i s_i = 0 \Big\}$$

be the standard MAD of $\mathfrak{sl}_\ell(Q)$ and define $\mathfrak{h} \subset \mathcal{D}(\mathcal{O})$ analogously. Then the automorphism $\operatorname{Int}(g)$ *of $\mathfrak{sl}_\ell(Q)$ maps $\mathfrak{h}_{\rm st}$ onto $\mathfrak{h}$. In particular, $\mathfrak{h}$ is also a MAD of $\mathfrak{sl}_\ell(Q)$.*

Proof. (a) That $V = \sum_{i=1}^m V_i$ follows from the second equation in (2.9.1), and that the sum is direct from the first. The converse is equally straightforward.

(b) Assume (b.i). Since $(ge_ig^{-1})_{1 \le i \le m}$ is a complete orthogonal system of $\mathcal{B}$ contained in $\mathcal{D}(\mathcal{O}')$, it follows from our connectedness assumption that each $e_i \in \mathcal{O}$ is a sum of some of the $e'_i \in \mathcal{O}'$, and that distinct idempotents in $\mathcal{O}'$ are used for each e_i. Hence $m \le m'$. By symmetry, $m' \le m$, whence $m = m'$. The remaining part of (b.ii) is now clear. The implications (b.ii) $\Longrightarrow$ (b.iii) $\Longrightarrow$ (b.i) are immediate from the definitions.

(c) A cyclic Q-module is free since Q is a domain. Thus (c.i) $\Longleftrightarrow$ (c.ii). For the proof of (c.ii) $\Longrightarrow$ (c.iii) put $V_{i,\rm st} = E_{ii}(V)$. We know $V_{i,\rm st} \simeq Q$ by (2.2.6). Also, by assumption, there exist Q-linear isomorphisms $g_i \colon V_{i,\rm st} \to V_i$, $1 \le i \le \ell$. Hence $g = g_1 \oplus \cdots \oplus g_\ell$ is an invertible endomorphism of V such that $g(V_{i,\rm st}) = V_i$, $1 \le i \le \ell$. It then follows from (b) that $g\mathcal{D}(\mathcal{O}_{\rm st})g^{-1} = \mathcal{D}(\mathcal{O})$ (note that (b) can be applied since $Q \simeq \operatorname{End}_Q(V_i)$ is connected by 1.2(d). The implication (c.iii) $\Longrightarrow$ (c.ii) also follows from (b). □

3. Isomorphisms between two Lie algebras of type A over rings

3.1. Torsion bijections

We start by reviewing some of the techniques of non-abelian Čech cohomology used later on.

Let **G** be a smooth affine group scheme over a (commutative, unital) ring R. The pointed set of non-abelian Čech cohomology on the étale site of $X = \operatorname{Spec}(R)$

with coefficients in $\mathbf{G}$, is denoted by $H^1_{\text{ét}}(X, \mathbf{G})$. This pointed set measures the isomorphism classes of torsors over X under G (see [Mi, Ch. IV §1] and [DG] for basic definitions and references). Abusing notation a bit we will identify the set of isomorphism classes of $\mathbf{G}$-torsors over X with $H^1_{\text{ét}}(R, \mathbf{G})$.

Recall that any morphism $\mathbf{G} \to \mathbf{H}$ of group schemes induces a natural map $H^1_{\text{ét}}(R, \mathbf{G}) \to H^1_{\text{ét}}(R, \mathbf{H})$. If $[E] \in H^1_{\text{ét}}(R, \mathbf{G})$ we will denote its image in $H^1_{\text{ét}}(R, \mathbf{H})$ by $[E_{\mathbf{H}}]$.

For a $\mathbf{G}$-torsor E we denote by ${}^E\mathbf{G}$ the twisted form of $\mathbf{G}$ by E. This is a smooth affine group scheme over X. Recall that according to [Gi, III.2.6.3.1] there exists a natural bijection

$$\tau_E : H^1_{\text{ét}}(X, {}^E\mathbf{G}) \to H^1_{\text{ét}}(X, \mathbf{G}),$$

called the *torsion bijection*, which takes the class of the trivial torsor under ${}^E\mathbf{G}$ to the class of E.

Let $[E] \in H^1_{\text{ét}}(R, \mathbf{G})$. Any exact sequence

$$1 \to \mathbf{G} \xrightarrow{\psi} \mathbf{H} \to \mathbf{F} \to 1$$

of smooth affine group schemes induces a commutative diagram

$$\begin{array}{ccccccc}
\mathbf{F}(R) & \longrightarrow & H^1_{\text{ét}}(R, {}^E\mathbf{G}) & \xrightarrow{\psi_E} & H^1_{\text{ét}}(R, {}^{E_{\mathbf{H}}}\mathbf{H}) & \longrightarrow & H^1_{\text{ét}}(R, \mathbf{F}) \\
\wr\downarrow & & \downarrow\tau_E & & \downarrow\tau_{E_H} & & \downarrow \\
{}^E\mathbf{F}(R) = \mathbf{F}(R) & \longrightarrow & H^1_{\text{ét}}(R, \mathbf{G}) & \xrightarrow{\psi} & H^1_{\text{ét}}(R, \mathbf{H}) & \longrightarrow & H^1_{\text{ét}}(R, \mathbf{F})
\end{array}.$$

3.2. Lemma. *Using the notation of* 3.1*, the torsion bijection* τ_E *induces a bijection between* $\operatorname{Ker}(\psi_E)$ *and the fiber* $\psi^{-1}(\psi(E)) = \psi^{-1}(E_H)$.

Proof. This follows from an easy diagram chase. □

3.3. Azumaya algebras

Let $\mathcal{A}$ be an Azumaya algebra over a (commutative, associative, unital) ring R. If $\mathcal{A}$ has rank ℓ^2, it is a twisted form of the matrix algebra $\mathrm{M}_\ell(R)$. Since

$$\mathbf{Aut}_R(\mathrm{M}_\ell(R)) \simeq \mathbf{PGL}_{\ell,R}$$

(see [Mi, Chap. IV, Prop. 2.3]), the elements of $H^1_{\text{ét}}(R, \mathbf{PGL}_{\ell,R})$ are in one-to-one correspondence with the isomorphism classes of Azumaya algebras over R of degree ℓ (the bijection is given by twisting). It follows that $\mathcal{A} \simeq {}^\xi\mathrm{M}_\ell(R)$ for some class $[\xi] \in H^1_{\text{ét}}(R, \mathbf{PGL}_{\ell,R})$ and that $\mathbf{Aut}_R(\mathcal{A}) \simeq \mathbf{PGL}_{\mathcal{A}}$.

3.4. Automorphisms of $\mathfrak{sl}_{\mathcal{A}}$

(a) Let $\mathcal{A}$ be an Azumaya algebra over R. Every $\varphi \in \mathbf{Aut}(\mathcal{A})(R)$ leaves the Lie algebra

$$\mathfrak{sl}_{\mathcal{A}} := [\mathcal{A}, \mathcal{A}]$$

invariant, thus induces an automorphism $\varphi_{\mathfrak{sl}}$ of $\mathfrak{sl}_{\mathcal{A}}$. Since the construction $\varphi \mapsto \varphi_{\mathfrak{sl}}$ is functorial, it gives rise to a homomorphism

$$\mathbf{PGL}_{\mathcal{A}} \to \mathbf{Aut}(\mathfrak{sl}_{\mathcal{A}}) \tag{3.4.1}$$

which is injective since $\mathfrak{sl}_{\mathcal{A}}$ generates $\mathcal{A}$ as an associative algebra.

(b) Assume $\mathcal{A}$ has an anti-automorphism κ. Then

$$\kappa_{\mathfrak{sl}} \colon \mathfrak{sl}_{\mathcal{A}} \to \mathfrak{sl}_{\mathcal{A}}, \quad x \mapsto -\kappa(x)$$

is an automorphism of $\mathfrak{sl}_{\mathcal{A}}$. Again by functoriality of the construction, this gives rise to an element of $\mathbf{Aut}(\mathfrak{sl}_{\mathcal{A}})$, also denoted $\kappa_{\mathfrak{sl}}$. In case $\mathcal{A} = \mathrm{M}_\ell(R)$ we use the transpose as anti-automorphism and put

$$\tau \colon \mathfrak{sl}_\ell(R) \to \mathfrak{sl}_\ell(R), \quad x \mapsto -{}^t x. \tag{3.4.2}$$

As before, this gives rise to an automorphism in $\mathbf{Aut}(\mathfrak{sl}_\ell(R))$, also denoted τ.

(c) Putting together the maps in (a) and (b) we have constructed homomorphisms of R-group schemes

$$\mathbf{PGL}_{2,R} \to \mathbf{Aut}\big(\mathfrak{sl}_2(R)\big) \tag{3.4.3}$$

and for $m \geq 3$

$$\mathbf{PGL}_{m,R} \rtimes (\mathbb{Z}/2\mathbb{Z})_R \to \mathbf{Aut}\big(\mathfrak{sl}_m(R)\big); \quad (\varphi, \varepsilon) \mapsto \varphi_{\mathfrak{sl}} \circ \tau^{\varepsilon}. \tag{3.4.4}$$

The reader will easily check that both maps are injective homomorphisms (one needs the assumption $m \geq 3$ to get injectivity in the second case).

3.5. Theorem. *Assume R is a domain of good characteristic for $\mathfrak{sl}_m$ containing a field F. Then the maps* (3.4.3) *and* (3.4.4) *are isomorphisms of R-group schemes.*

Proof. Since all groups involved are obtained from F by base change, and since base change preserves isomorphism, we may assume that $R = F$ is a field of good characteristic. In this case it is shown in [St, 4.7] (or see [Ja, Thm. IX.5] for F of characteristic 0 and [Se, p. 67] for characteristic > 0) that for any field extension E/F the maps (3.4.3) and (3.4.4) evaluated at the E-points are isomorphisms of abstract groups. In particular this holds for the algebraic closure of F. A standard fact in the theory of group schemes (see [KMRT, Prop. 22.5]) then proves the result. □

3.6. Consequences

We will derive some consequences of Theorem 3.5. Let again $\mathcal{A}$ be an Azumaya algebra over R. Assume that R is a domain. Since then $\mathcal{A}$ has constant rank as an R-module, it is a twisted form of $M_\ell(R)$ for some ℓ.

For the remainder of this section we will assume that R contains a field of good characteristic for $\mathfrak{sl}_\ell(R)$. We consider the corresponding Lie R-algebra $\mathfrak{sl}_{\mathcal{A}} =$

$[\mathcal{A},\mathcal{A}]$. By a standard twisting argument, it follows from Theorem 3.5 that we have an R-group scheme isomorphism

$$\mathbf{PGL}_{\mathcal{A}} \simeq \mathbf{Aut}_R(\mathfrak{sl}_{\mathcal{A}})$$

if $\ell = 2$, and the exact sequence of R-group schemes.

$$1 \to \mathbf{PGL}_{\mathcal{A}} \to \mathbf{Aut}_R(\mathfrak{sl}_{\mathcal{A}}) \to (\mathbb{Z}/2\mathbb{Z})_R \to 1$$

if $\ell \geq 3$.

Evaluating at R-points we have

$$1 \to \mathrm{PGL}_{\mathcal{A}} = \mathbf{PGL}_{\mathcal{A}}(R) \hookrightarrow \mathrm{Aut}_R(\mathfrak{sl}_{\mathcal{A}}) = \mathbf{Aut}_R(\mathfrak{sl}_{\mathcal{A}})(R) \to \mathbb{Z}/2\mathbb{Z}.$$

Note that this last map is trivial when $\ell = 2$. We will say that an automorphism $\phi \in \mathrm{Aut}_R(\mathfrak{sl}_{\mathcal{A}})$ is *inner* if it is in the image of $\mathrm{PGL}_{\mathcal{A}}$. Otherwise we say that ϕ is *outer*.

3.7. Theorem. *Let $\phi \in \mathrm{Aut}_R(\mathfrak{sl}_{\mathcal{A}})$. Then the following holds:*

(a) *If ϕ is inner, it is the restriction of a unique automorphism $\psi : \mathcal{A} \to \mathcal{A}$.*
(b) *If ϕ is outer, it is the restriction of the negative of a unique anti-automorphism $\psi : \mathcal{A} \to \mathcal{A}$.*

Proof. The subset $Y = \mathbf{Aut}_R(\mathfrak{sl}_{\mathcal{A}}) \setminus \mathbf{PGL}_{\mathcal{A}}$ is a closed subscheme of $\mathbf{Aut}_R(\mathfrak{sl}_{\mathcal{A}})$ consisting of outer automorphisms of $\mathfrak{sl}_{\mathcal{A}}$. The group $\mathbf{PGL}_{\mathcal{A}}$ acts simply transitively on Y by left multiplication. Thus Y is a $\mathbf{PGL}_{\mathcal{A}}$-torsor. It is trivial if and only if $\mathfrak{sl}_{\mathcal{A}}$ has at least one outer automorphism.

Along the same lines, let X be the scheme of anti-automorphisms of $\mathcal{A}$:

$$X = \{\, \psi : \mathcal{A} \to \mathcal{A} \mid \ \psi \text{ is bijective and } \ \psi(xy) = \psi(y)\psi(x) \text{ for all } \ x, y \in \mathcal{A} \,\}.$$

We observe that if ψ is an anti-automorphism of $\mathcal{A}$ and $\ell \geq 3$ then $-\psi|_{\mathfrak{sl}_{\mathcal{A}}}$ is an outer automorphism of $\mathfrak{sl}_{\mathcal{A}}$. The automorphism group $\mathbf{PGL}_{\mathcal{A}}$ of $\mathcal{A}$ acts simply transitively on X on the left (because the action is simply transitively in the split case). Thus, X is also a $\mathbf{PGL}_{\mathcal{A}}$-torsor. As before, if $\ell \geq 3$ then X is a trivial torsor, i.e., there exists at least one anti-automorphism of $\mathcal{A}$, if and only if $\mathfrak{sl}_{\mathcal{A}}$ has at least one outer automorphism.

Note that the natural restriction map $\lambda : X \to Y$ which takes ψ into $-\psi|_{\mathfrak{sl}_{\mathcal{A}}}$ is an isomorphism of torsors (because it is an isomorphism in the split case).

(a) Let ϕ be inner. Then there exists a unique $g \in \mathbf{PGL}_{\mathcal{A}}(R)$ whose image in $\mathbf{Aut}_R(\mathfrak{sl}_{\mathcal{A}})$ is ϕ. This element g corresponds to the automorphism of $\mathcal{A}$ which we denote by $\psi : \mathcal{A} \to \mathcal{A}$. By construction, the restriction $\psi|_{\mathfrak{sl}_{\mathcal{A}}}$ is ϕ. The uniqueness of such a ψ is immediate, since $\mathfrak{sl}_{\mathcal{A}}$ generates $\mathcal{A}$ as associative algebra.

(b) Now let ϕ be outer. Then since λ is bijective there exists a unique $\psi \in X(R)$ such that $\lambda(\psi) = -\psi|_{\mathfrak{sl}_{\mathcal{A}}}$ equals ϕ. □

3.8. Corollary. *Let $\mathcal{A}$ be an Azumaya algebra over R, thus a twisted form of $\mathrm{M}_\ell(R)$. Assume $\ell \geq 3$. Then $\mathcal{A} \simeq \mathcal{A}^{\mathrm{op}}$ if and only if $\mathfrak{sl}_{\mathcal{A}}$ has an outer automorphism.*

Proof. Assume $\mathcal{A} \simeq \mathcal{A}^{\rm op}$. We have seen in 3.4(b) that an isomorphism $\kappa\colon \mathcal{A} \to \mathcal{A}^{\rm op}$ induces an automorphism $\kappa_{\mathfrak{sl}}$ of $\mathfrak{sl}_{\mathcal{A}}$. We claim that $\kappa_{\mathfrak{sl}}$ is an outer automorphism. It suffices to show this after some base change. Take any base change $R \to S$ which splits $\mathcal{A}$, i.e., $\mathcal{A}_S \simeq \mathrm{M}_\ell(S)$. Then κ is a composite of an inner conjugation, say $\mathrm{int}(g)$, and the transpose τ. It is well known that for $\ell \geq 3$ the restriction of $-\tau$ to $\mathfrak{sl}_\ell$ is an outer automorphism (because it induces the automorphism -1 of the corresponding root system), while the restriction of $\mathrm{int}(g)$ to $\mathfrak{sl}_{\mathcal{A}}$ is an inner automorphism. Therefore $\kappa_{\mathfrak{sl}}$ is indeed outer.

The other direction follows from Theorem 3.7(b). ☐

3.9. Theorem. *Let $\mathcal{A}, \mathcal{A}'$ be Azumaya algebras over our domain R of the same rank l^2. If $\mathfrak{sl}_{\mathcal{A}} \simeq \mathfrak{sl}_{\mathcal{A}'}$ as R-algebras, then*

(a) *$\mathcal{A}' \simeq \mathcal{A}$ if $\ell = 2$;*
(b) *$\mathcal{A}' \simeq \mathcal{A}$ or $\mathcal{A}' \simeq \mathcal{A}^{\rm op}$ if $\ell \geq 3$.*

We note that the converses in (a) and (b) are obvious.

Proof. We abbreviate $H^1 = H^1_{\text{ét}}$ and let $[\xi], [\xi'] \in H^1(R, \mathbf{PGL}_{\ell,R})$ be the classes corresponding to $\mathcal{A}$ and $\mathcal{A}'$. Because $\mathcal{A} = {}^{\xi}\mathrm{M}_\ell(R)$ the Lie algebra $\mathfrak{sl}_{\mathcal{A}}$ is the twisted form of $\mathfrak{sl}_\ell(R)$ by ξ.

Case $\ell = 2$: Recall from Theorem 3.5 that

$$\mathbf{Aut}_R(\mathfrak{sl}_2(R)) = \mathbf{PGL}_{2,R},$$

i.e., the Lie algebra $\mathfrak{sl}_2(R)$ and the Azumaya algebra $\mathrm{M}_2(R)$ have the same automorphism group scheme. Therefore, if the twisted forms $\mathfrak{sl}_{\mathcal{A}}$ and $\mathfrak{sl}_{\mathcal{A}'}$ of the Lie algebra $\mathfrak{sl}_2(R)$ are isomorphic, then $[\xi] = [\xi']$ implies $\mathcal{A} \simeq \mathcal{A}'$ as R-algebras.

Case $\ell \geq 3$: Recall from Theorem 3.5 that

$$\mathbf{Aut}_R(\mathfrak{sl}_\ell(R)) = \mathbf{Aut}_R(\mathbf{PGL}_{\ell,R}) = \mathbf{PGL}_{\ell,R} \rtimes (\mathbb{Z}/2\mathbb{Z})_R.$$

Hence we get the exact sequence of group schemes

$$1 \to \mathbf{PGL}_{\ell,R} \to \mathbf{Aut}_R(\mathfrak{sl}_\ell(R)) \to (\mathbb{Z}/2\mathbb{Z})_R \to 1 \tag{3.9.1}$$

which induces a canonical map

$$H^1(R, \mathbf{PGL}_{\ell,R}) \xrightarrow{\psi} H^1(R, \mathbf{Aut}_R(\mathfrak{sl}_\ell(R)) = H^1(R, \mathbf{Aut}_R(\mathbf{PGL}_{\ell,R}))$$

in cohomology. By assumption, $\psi([\xi]) = \psi([\xi'])$ (because $\mathfrak{sl}_{\mathcal{A}} \overset{R}{\simeq} \mathfrak{sl}_{\mathcal{A}'}$). If $[\xi] = [\xi']$ then the twisted Azumaya algebras $\mathcal{A} = {}^{\xi}\mathrm{M}_\ell(R)$ and $\mathcal{A}' = {}^{\xi'}\mathrm{M}_\ell(R)$ are isomorphic as R-algebras, and we are done.

Assume therefore $[\xi] \neq [\xi']$. Twisting (3.9.1) by ξ we get an exact sequence of group schemes

$$1 \to \mathbf{PGL}_{\mathcal{A}} \to \mathbf{Aut}_R(\mathbf{PGL}_{\mathcal{A}}) \to (\mathbb{Z}/2\mathbb{Z})_R \to 1$$

which in turn induces an exact sequence

$$\begin{aligned}\mathbf{PGL}_{\mathcal{A}}(R) \hookrightarrow \mathbf{Aut}_R(\mathbf{PGL}_{\mathcal{A}})(R) \to \mathbb{Z}/2\mathbb{Z} \\ \to H^1(R, \mathbf{PGL}_{\mathcal{A}}) \xrightarrow{\phi} H^1(R, \mathbf{Aut}_R(\mathbf{PGL}_{\mathcal{A}}))\end{aligned} \tag{3.9.2}$$

of pointed sets. According to Lemma 3.2 there is a natural one-to-one correspondence between the fiber $\psi^{-1}(\psi(\xi))$ and the kernel of ϕ. Since $[\xi] \neq [\xi'] \in \psi^{-1}(\psi(\xi))$, the class $[\xi']$ corresponds to some nontrivial element in $\mathrm{Ker}(\phi)$. On the other hand, by (3.9.2), $\mathrm{Ker}(\phi)$ consists of at most two elements implying $|\mathrm{Ker}(\phi)| = 2$. We then get from (3.9.2) that $\mathbf{PGL}_{\mathcal{A}}(R) = \mathbf{Aut}_R(\mathbf{PGL}_{\mathcal{A}})(R)$, i.e., every automorphism of $\mathrm{PGL}_{\mathcal{A}}$ over R is inner. By Corollary 3.8, this implies $\mathcal{A} \not\simeq \mathcal{A}^{\mathrm{op}}$.

Now we observe that the opposite Azumaya algebra $\mathcal{A}^{\mathrm{op}}$ also corresponds to some element in $H^1(R, \mathbf{PGL}_{\ell,R})$, call it $[\xi^{\mathrm{op}}]$. We have $[\xi] \neq [\xi^{\mathrm{op}}]$ because $\mathcal{A} \not\simeq \mathcal{A}^{\mathrm{op}}$. Furthermore, it is straightforward to verify that the map $\mathbf{GL}_{\mathcal{A}} \to \mathbf{GL}_{\mathcal{A}^{\mathrm{op}}}$, given by $x \to x^{-1}$, is an isomorphism of group schemes which induces in turn an isomorphism $\mathbf{PGL}_{\mathcal{A}} \to \mathbf{PGL}_{\mathcal{A}^{\mathrm{op}}}$. This implies that $\psi([\xi]) = \psi([\xi^{\mathrm{op}}])$, hence the class $[\xi^{\mathrm{op}}]$ also corresponds to the nontrivial element ($\neq [\xi]$) in the fiber $\psi^{-1}(\psi([\xi]))$. Since $|\psi^{-1}(\psi([\xi]))| = |\mathrm{Ker}(\phi)| = 2$, necessarily $[\xi'] = [\xi^{\mathrm{op}}]$ in $H^1(R, \mathbf{PGL}_{\ell,R})$, implying $\mathcal{A}' \simeq \mathcal{A}^{\mathrm{op}}$. □

As a by-product of the proof we obtain that the converse statement in Corollary 3.8 also holds.

3.10. Corollary. *Let $\mathcal{A}$ be an Azumaya algebra over R of rank $\ell^2 \geq 4$, where R has good characteristic for $\mathfrak{sl}_\ell(R)$. Suppose $\mathcal{A} \not\simeq \mathcal{A}^{\mathrm{op}}$. Then every automorphism of the Lie R-algebra $\mathfrak{sl}_{\mathcal{A}}$ is inner. If in addition* $\mathrm{Pic}(R) = 1$ *then every automorphism of $\mathfrak{sl}_{\mathcal{A}}$ is the restriction of the conjugation map by an element $a \in \mathcal{A}^\times$.*

Proof. Without loss of generality we may assume that $\ell \geq 3$. We use the notation of Theorem 3.9. Since $\mathcal{A} \not\simeq \mathcal{A}^{\mathrm{op}}$ we have $[\xi] \neq [\xi^{\mathrm{op}}]$. Therefore, in sequence (3.9.2) the kernel of the canonical map ϕ consists of two elements and this implies $\mathbf{PGL}_{\mathcal{A}}(R) = \mathbf{Aut}_R(\mathbf{PGL}_{\mathcal{A}})(R) = \mathrm{Aut}(\mathcal{A})$. Thus every automorphism of $\mathfrak{sl}_{\mathcal{A}}$ is induced by an automorphism of $\mathcal{A}$. It remains to note that if $\mathrm{Pic}(R) = 1$ the natural map $\mathcal{A}^\times \to \mathbf{PGL}_{\mathcal{A}}(R)$ is surjective. □

In the following sections we apply all these results to the case of fgc quantum tori viewed over their centres.

4. $\mathfrak{sl}_\ell(Q)$ for Q an fgc quantum torus

Our analysis will use the following description of isomorphisms between special linear Lie algebras over rings. The theorem below is an immediate consequence of our Theorem 3.7 and is originally due to Jacobson-Seligman when the Lie algebras in question are defined over fields.

4.1. Theorem. *Let $\mathcal{A}$ and $\mathcal{A}'$ be Azumaya algebras of the same rank over a domain R containing a field of good characteristic. Let $f\colon \mathfrak{sl}_{\mathcal{A}'} \to \mathfrak{sl}_{\mathcal{A}}$ be an R-linear isomorphism of Lie algebras. Then after possibly replacing $\mathcal{A}$ by $\mathcal{A}^{\mathrm{op}}$, the map f uniquely extends to an R-linear isomorphism $\tilde{f}\colon \mathcal{A}' \to \mathcal{A}$ of associative algebras.*

Proof. Since it is clear that $f \circ \iota^{\mathrm{op}}$ is a Lie R-algebra isomorphism, we only need to show that f extends to a map $\tilde{f}\colon \mathcal{A}' \to \mathcal{A}$ which is either an isomorphism or the negative of an anti-isomorphism of associative algebras. By Theorem 3.9, after possibly replacing $\mathcal{A}$ by $\mathcal{A}^{\mathrm{op}}$ we can assume that there exists an isomorphism $g\colon \mathcal{A}' \to \mathcal{A}$ of associative R-algebras. Since the composition $f \circ g^{-1}|_{\mathfrak{sl}_{\mathcal{A}}}$ is an R-automorphism of $\mathfrak{sl}_{\mathcal{A}}$, by Theorem 3.7 it can be extended uniquely to either an automorphism or the negative of an anti-automorphism of $\mathcal{A}$ and the assertion follows. □

4.2. *Remark.* For the special case of central-simple algebras over a field K of characteristic 0, the theorem is proven in [Ja, Thm. IX.5]. The characteristic 0 case can easily be extended to positive characteristic by using the description of $\operatorname{Aut}\mathfrak{sl}_\ell(K)$ in [Se, p. 67].

We will apply this result in the case that $\mathcal{A}$ and $\mathcal{A}'$ are matrix algebras over fgc quantum tori.

4.3. Theorem. *Let $\mathcal{L} = \mathfrak{sl}_\ell(Q)$ and $\mathcal{L}' = \mathfrak{sl}_{\ell'}(Q')$ where Q and Q' are fgc quantum tori over a field F of good characteristic for $\mathcal{L}$ and $\mathcal{L}'$. We denote by $\mathcal{Z} = \mathcal{Z}(Q)$ the centre of Q, and let $f\colon \mathcal{L}' \to \mathcal{L}$ be an F-linear isomorphism of Lie algebras. Then the following hold.*

(a) *After possibly replacing Q by Q^{op}, the map f uniquely extends to a $\mathcal{Z}$-linear isomorphism $\tilde{f}\colon \mathrm{M}_{\ell'}(Q')_{(\mathcal{Z})} \to \mathrm{M}_\ell(Q)_{(\mathcal{Z})}$ of associative algebras.*

(b) *$\ell' = \ell$.*

Proof. (a) By 2.4 the centre $\mathcal{Z}$ of Q is isomorphic to the centroid of $\mathcal{L}$ – we will take this as an identification. Applying 1.1, Example 1, our map f is a $\mathcal{Z}$-linear isomorphism

$$f\colon \mathcal{L}'_{(Z)} = \mathfrak{sl}_{\ell'}(Q'_{(\mathcal{Z})}) \xrightarrow{\sim} \mathcal{L}_{(\mathcal{Z})} = \mathfrak{sl}_\ell(Q_{(\mathcal{Z})}).$$

Observe that the two Azumaya algebras $\mathrm{M}_\ell(Q)$ and $\mathrm{M}_{\ell'}(Q')$ over $\mathcal{Z}$ have the same rank. Indeed, passing to a proper étale base extension we may assume that they are split. Let m^2 and $(m')^2$ be their ranks. Then $\dim \mathcal{L} = m^2 - 1$ and $\dim \mathcal{L}' = (m')^2 - 1$ implying $m = m'$. Now Theorem 4.1 applied to $\mathcal{A} = \mathrm{M}_\ell(Q)_{(\mathcal{Z})}$ and $\mathcal{A}' = \mathrm{M}_{\ell'}(Q')_{(\mathcal{Z})}$ implies the claim.

(b) Let K be the fraction field of $\mathcal{Z}$ (recall from 1.2(b) that $\mathcal{Z}$ is a Laurent polynomial ring over F). Then after possibly replacing Q by Q^{op} we can assume that $\tilde{f}$ uniquely extends to a K-linear isomorphism

$$\tilde{f}_K : M_{\ell'}(Q') \otimes_{\mathcal{Z}} K \xrightarrow{\sim} M_\ell(Q) \otimes_{\mathcal{Z}} K$$

of associative algebras. But $D' = Q'_{(\mathcal{Z})} \otimes_{\mathcal{Z}} K$ and $D = Q_{(\mathcal{Z})} \otimes_{\mathcal{Z}} K$ are central division algebras over K. Now by the theorem of Wedderburn (see [He, Thm. 2.1.6]) we have $D' \simeq D$ and $\ell' = \ell$. □

The spirit of Theorem 4.3, in a sense that will be made precise, will allow us to reduce questions of isomorphisms of Lie algebras to questions of isomorphisms of associative algebras. These are handled in the following result.

4.4. Theorem. *Let Q and Q' be fgc quantum tori over the field F and let $\mathcal{A} = \mathrm{M}_\ell(Q)$ and $\mathcal{A}' = \mathrm{M}_\ell(Q')$. We assume that the characteristic of F is* very good*: it is either 0, or $p > 3$ where p does not divide ℓ or any of the q_{ij}, q'_{ij} occurring in the quantum matrices of Q and Q' respectively with respect to a canonical presentation. We let $\mathcal{Z}$ be the centre of Q, and assume that $f\colon \mathcal{A}' \to \mathcal{A}$ is an F-linear isomorphism of associative algebras. As in* 1.1 *we will view f as a $\mathcal{Z}$-linear isomorphism $\mathcal{A}'_{(\mathcal{Z})} \to \mathcal{A}_{(\mathcal{Z})}$. The isomorphism f induces an isomorphism $\phi\colon G' \to G$ of the associated reductive group schemes $G' = \mathbf{GL}_{A'_{(\mathcal{Z})}}$ and $G = \mathbf{GL}_{A_{(\mathcal{Z})}}$ over $\mathcal{Z}$. Let T' (resp. T) be the split torus in G' (resp. G) corresponding to the diagonal matrices in $A'_{(\mathcal{Z})} = \mathrm{M}_\ell(Q'_{(\mathcal{Z})})$ (resp. in $A_{(\mathcal{Z})} = \mathrm{M}_\ell(Q_{(\mathcal{Z})})$). Then $\phi(T')$ and T are conjugate in G.*

Proof. Clearly, T' and T are generically maximal split tori in the sense of [CGP], i.e., T'_K and T_K are maximal split tori in G_K and G'_K where $K = \tilde{\mathcal{Z}}$ is the fraction field of $\mathcal{Z}$ (because $\widetilde{Q} = Q_{(\mathcal{Z})} \otimes_{\mathcal{Z}} K$ and $\widetilde{Q'}_{(\mathcal{Z})} \otimes_{\mathcal{Z}} K$ are central division algebras over K by 1.2(g)). Note that the centralizer of T (resp. T') in G (resp. in G') is isomorphic to the reductive group scheme $C_G(T) \simeq \mathbf{GL}_{Q_{(\mathcal{Z})}} \times \cdots \times \mathbf{GL}_{Q_{(\mathcal{Z})}}$ over $\mathcal{Z}$ (resp. $C_{G'}(T') \simeq \mathbf{GL}_{Q'_{(\mathcal{Z})}} \times \cdots \times \mathbf{GL}_{Q'_{(\mathcal{Z})}}$).

Since T' and T are generically maximal split tori, the proof of [CGP, Prop. 8.1] shows that an obstacle for conjugacy of $\phi(T')$ and T in G is given by a class

$$[\xi] \in H^1_{\mathrm{Zar}}(\mathcal{Z}, C_G(T)) = H^1_{\mathrm{Zar}}(\mathcal{Z}, \mathbf{GL}_{Q_{(\mathcal{Z})}}) \times \cdots \times H^1_{\mathrm{Zar}}(\mathcal{Z}, \mathbf{GL}_{Q_{(\mathcal{Z})}})$$

and that

$$^{\xi}(C_G(T)) \simeq C_G(\phi(T')) \simeq C_{G'}(T') \simeq \mathbf{GL}_{Q'_{(\mathcal{Z})}} \times \cdots \times \mathbf{GL}_{Q'_{(\mathcal{Z})}}.$$

Let $\xi = (\xi_1, \dots, \xi_\ell)$. Since $^{\xi_i}\mathbf{GL}_{Q_{(\mathcal{Z})}} \simeq \mathbf{GL}_{Q'_{(\mathcal{Z})}}$, for the proof of conjugacy of $\phi(T')$ and T it suffices to show that $\xi_i = 1$.

We now recall some general facts from the theory of reductive group schemes. Let H be any reductive group scheme over $\mathcal{Z}$, $S \subset H$ a maximal torus and $[\xi] \in H^1_{\text{ét}}(\mathcal{Z}, H)$. Let $N = N_H(S)$ and $W = N/S$.

(i) *By* [CGP, Lem. 8.2]*, W is a finite étale $\mathcal{Z}$-group and the canonical map $H^1_{\text{ét}}(\mathcal{Z}, W) \to H^1_{\text{ét}}(K, W)$ is injective; in particular $H^1_{\mathrm{Zar}}(\mathcal{Z}, W) = 1$.*

(ii) *According to* [Gi, Rem. 3.2.5] *if the twisted group scheme $^{\xi}H$ contains a maximal torus then the class $[\xi]$ is in the image of $H^1_{\text{ét}}(\mathcal{Z}, N_H(S)) \to H^1_{\text{ét}}(\mathcal{Z}, H)$.*

(iii) *Let $S' \subset H$ be another maximal torus. Then the transporter $\mathbf{Trans}_H(S, S')$ is an N-torsor over $\mathcal{Z}$, hence it corresponds to a unique class $\lambda \in H^1_{\text{ét}}(\mathcal{Z}, N)$. If S, S' are conjugate over K then λ is in the image of $H^1_{\text{ét}}(\mathcal{Z}, S) \to H^1_{\text{ét}}(\mathcal{Z}, N)$. If in addition $H^1(\mathcal{Z}, S) = 1$ then S and S' are conjugate over $\mathcal{Z}$.*

For the proof of the first statement in (iii) we refer to [CGP, Lem. 8.3]. The second assertion follows from (i) and the commutativity of the following diagram:

$$\begin{array}{ccccc} H^1_{\text{ét}}(\mathcal{Z},S) & \longrightarrow & H^1_{\text{ét}}(\mathcal{Z},N) & \longrightarrow & H^1_{\text{ét}}(\mathcal{Z},W) \\ \downarrow & & \downarrow & & \downarrow \\ H^1_{\text{ét}}(K,S) & \longrightarrow & H^1_{\text{ét}}(K,N) & \longrightarrow & H^1_{\text{ét}}(K,W). \end{array}$$

We will apply these facts in the following two situations. Let m be an integer such that the matrix algebra $\mathrm{M}_m(\mathcal{Z})$ is the split form of $Q_{(\mathcal{Z})}$. Take the pair (E,S) consisting of a maximal commutative étale subalgebra $E \subset Q_{(\mathcal{Z})}$ and the corresponding torus $S = R_{E/\mathcal{Z}}(\mathbf{G}_{m,E})$ constructed in Remark 1.9. Fix any embedding $E \hookrightarrow \mathrm{M}_m(\mathcal{Z})$ (for instance, we can take the regular representation of E). This also gives rise to a closed embedding $S \hookrightarrow H = \mathbf{GL}_{m,\mathcal{Z}}$. Note that the torus S determines uniquely the subalgebra E. Indeed, E is the centralizer of $S(\mathcal{Z})$ in $\mathrm{M}_m(\mathcal{Z})$. When we view E as a subalgebra in $Q_{(\mathcal{Z})}$ we will denote it by E_Q. Similarly the corresponding torus S will be denoted by S_Q. Consider now another embedding $E \to \mathrm{M}_m(\mathcal{Z})$. We denote its image by E' and the corresponding torus by S'.

4.5. Lemma. *The two étale subalgebras $E, E' \subset \mathrm{M}_m(\mathcal{Z})$ are conjugate under the action of $\mathbf{GL}_m(\mathcal{Z})$.*

Proof. It suffices to show the conjugacy of S and S'. The obstacle for conjugacy of S, S' is the N-torsor $\mathbf{Trans}_H(S,S')$ where $N = N_{\mathbf{GL}_{m,\mathcal{Z}}}(S)$. By the Skolem–Noether Theorem (see [KMRT, Thm. 1.4]) the algebras E and E' are conjugate over K. Furthermore, we have

$$H^1_{\text{ét}}(\mathcal{Z},S) = H^1_{\text{ét}}(E,\mathbf{G}_{m,E}) = \mathrm{Pic}(E) = 1,$$

because E is a Laurent polynomial ring and hence has trivial Picard group. The claim now follows from (iii). □

We next pass to adjoint groups.

4.6. Lemma. *We keep the above notation. Let $\overline{H} = \mathbf{PGL}_{m,\mathcal{Z}}$ and let $\overline{S}$ be the image of S under the canonical map $\mathbf{GL}_{m,\mathcal{Z}} \to \mathbf{PGL}_{m,\mathcal{Z}}$. There exists a class $[\theta] \in H^1(\mathcal{Z},\overline{S})$ such that ${}^\theta\mathbf{PGL}_{m,\mathcal{Z}} \simeq \mathbf{PGL}_{Q_{(\mathcal{Z})}}$.*

Proof. Since $\mathbf{PGL}_{Q_{(\mathcal{Z})}}$ has a maximal torus, by (ii) the group scheme $\mathbf{PGL}_{Q_{(\mathcal{Z})}}$ is a twisted form of $\mathbf{PGL}_{m,\mathcal{Z}}$ by some cocycle λ with coefficients in $\overline{N} = N_{\mathbf{PGL}_{m,\mathcal{Z}}}(\overline{S})$. The cohomological class $[\lambda]$ corresponds to the maximal torus $\overline{S}_Q \subset \mathbf{PGL}_{Q_{(\mathcal{Z})}}$ where $\overline{S}_Q$ is the image of $S_Q \subset \mathbf{GL}_{Q_{(\mathcal{Z})}}$ under the canonical map $\mathbf{GL}_{Q_{(\mathcal{Z})}} \to \mathbf{PGL}_{Q_{(\mathcal{Z})}}$. The argument in [Ch, Thm. 3.1] shows that there is another closed embedding $\overline{S} \hookrightarrow \mathbf{PGL}_{m,\mathcal{Z}}$, whose image will be denoted by $\overline{S}'$, such that $[\lambda]$ is equivalent to some class $[\lambda'] \in H^1(\mathcal{Z},\overline{S}')$. Let $S' \subset \mathbf{GL}_{m,\mathcal{Z}}$ be the preimage of $\overline{S}'$. Its centralizer E' in $Q_{(\mathcal{Z})}$ is a maximal commutative étale subalgebra of $Q_{(\mathcal{Z})}$ isomorphic to E. Since by the above lemma E and E' are conjugate over $\mathcal{Z}$, so are the maximal tori S, S'. This in turn implies the conjugacy of $\overline{S}$ and $\overline{S}'$ in $\mathbf{PGL}_{m,\mathcal{Z}}$. Thus the cocycle λ' is equivalent to some cocycle θ with coefficients in $\overline{S}$. □

Assume for a moment that E also admits an embedding into $Q'_{(\mathcal{Z})}$. Then as above the group scheme $\mathbf{PGL}_{Q'_{(\mathcal{Z})}}$ is a twisted from of $\mathbf{PGL}_{m,\mathcal{Z}}$ with some cocycle θ' with coefficients in $\overline{S}$. Consider the exact sequence

$$H^1(\mathcal{Z}, S) = 1 \to H^1(\mathcal{Z}, \overline{S}) \to H^2(\mathcal{Z}, \mathbf{G}_{m,\mathcal{Z}}) = \mathrm{Br}(\mathcal{Z}).$$

The images of $[\theta]$ and $[\theta']$ in $\mathrm{Br}(\mathcal{Z})$ coincide, because the base change morphism $\mathrm{Br}(\mathcal{Z}) \hookrightarrow \mathrm{Br}(K)$ is injective and $Q_{(\mathcal{Z})}$ and $Q'_{(\mathcal{Z})}$ are isomorphic over K. It follows that $[\theta] = [\theta']$ and this implies $Q_{(\mathcal{Z})} \simeq Q'_{(\mathcal{Z})}$; in particular, $\xi_i = 1$ as required.

To sum up: to finish the proof of Theorem 4.4 what is left to show is that E admits an embedding in $Q'_{(\mathcal{Z})}$, i.e., there exist elements $y_1, y_3, \dots, y_{2s-1} \in (Q'_{(\mathcal{Z})})^\times$ which commute and such that

$$y_1^{\ell_1} = t_1,\ y_3^{\ell_3} = t_3, \dots, y_{2s-1}^{\ell_s} = t_{2s-1}.$$

Here the positive integers $\ell_1, \dots, \ell_s$ and variables $t_1, \dots, t_n$ are the same as in Example 1.8 applied to Q.

Fix a presentation $Q' = \bigoplus_{\lambda \in \Lambda'} k x'^\lambda$. As usual it induces a grading of Q'. The isomorphism $f\colon \mathcal{A}'_{(\mathcal{Z})} \to \mathcal{A}_{(\mathcal{Z})}$ induces an isomorphism

$$\mathrm{M}_\ell(Q'_{(\mathcal{Z})}) \otimes_{\mathcal{Z}} K = \mathrm{M}_\ell(Q'_{(\mathcal{Z})} \otimes_{\mathcal{Z}} K) \to \mathrm{M}_\ell(Q_{(\mathcal{Z})}) \otimes_{\mathcal{Z}} K = \mathrm{M}_\ell(Q_{(\mathcal{Z})} \otimes_{\mathcal{Z}} K).$$

From the theory of central simple algebras over fields we know that the last implies that $Q'_{(\mathcal{Z})} \otimes_{\mathcal{Z}} K \simeq Q_{(\mathcal{Z})} \otimes_{\mathcal{Z}} K$. Hence, there exist commuting elements $z_1, z_3, \dots, z_{2s-1} \in Q'_{(\mathcal{Z})} \otimes_{\mathcal{Z}} K$ such that $z_{2i-1}^{l_i} = t_{2i-1}$ for all $i = 1, \dots, s$, where we view $t_i \in Q'_{(\mathcal{Z})}$. Choose an element $r \in \mathcal{Z}$ such that $r z_{2i-1} \in Q'_{(\mathcal{Z})}$ for all i. Then

$$(r z_{2i-1})^{l_i} = r^{l_i} t_{2i-1}. \tag{4.6.1}$$

Recall that $Q'_{(\mathcal{Z})}$ has a natural grading transferred from Q'.

Let u_{2i-1} (resp. a) be the highest homogeneous component of $r z_{2i-1}$ (resp. r) with respect to any order on Λ. Taking the highest components on the left and on the right in (4.6.1) we get $(u_{2i-1})^{l_i} = a^{l_i} t_{2i-1}$. Note that a is some monomial in the centre $\mathcal{Z}^\times$ of $Q'_{(\mathcal{Z})}$, hence it commutes with u_{2i-1}. Then the elements

$$y_{2i-1} = a^{-1} u_{2i-1} \in (Q'_{(\mathcal{Z})})^\times,\ i = 1, 3, \dots, 2s-1,$$

have the required properties. □

4.7. Corollary. *Let $\mathcal{L} = \mathfrak{sl}_\ell(Q)$ and $\mathcal{L}' = \mathfrak{sl}_{\ell'}(Q')$ where Q and Q' are fgc quantum tori over a field of very good characteristic, let $\mathfrak{m} \subset \mathfrak{sl}_\ell(Q)$ (resp. $\mathfrak{m}' \subset \mathfrak{sl}_{\ell'}(Q')$) be the MAD consisting of diagonal matrices with entries in F, and let $f\colon \mathcal{L}' \to \mathcal{L}$ be an F-linear isomorphism of Lie algebras. Then $f(\mathfrak{m}')$ is conjugate to $\mathfrak{m}$.*

Proof. This follows by combining Theorem 4.3 and Theorem 4.4. □

5. Specialization of quantum tori

Our main method of dealing with non-fgc quantum tori is *specialization* which we develop in this section. In this section k denotes a field of characteristic 0.

5.1. Proposition. *Let $\mathcal{F}$ be a finite subset of a field k of characteristic* 0 *consisting of non-zero elements and let $\ell \in \mathbb{N}_+$. Then there exists a finitely generated subring $R \subset k$ and a maximal ideal $\mathfrak{m} \triangleleft R$ such that*

(a) *$\mathcal{F} \subset R \setminus \mathfrak{m}$, and*
(b) *$R/\mathfrak{m}$ is a finite field of characteristic $p > \ell$.*

Proof. We can assume that $f^{-1} \in \mathcal{F}$ for every $f \in \mathcal{F}$. Let $R = \mathbb{Z}[\mathcal{F}] \subset k$ (resp. $C = \mathbb{Q}[\mathcal{F}] \subset k$) be the $\mathbb{Z}$-subalgebra (resp. $\mathbb{Q}$-subalgebra) generated by $\mathcal{F}$. By the Noether Normalization Lemma there exist algebraically independent $u_1, \dots, u_s \in C$ over $\mathbb{Q}$ such that C is integral over its subring $\mathbb{Q}[u_1, \dots, u_s]$ and of finite type as a $\mathbb{Q}[u_1, \dots, u_s]$-module, say with generators $c_1, \dots, c_t$.

Observation. For finitely many polynomials $q_1, \dots, q_m \in \mathbb{Q}[u_1, \dots, u_s]$ there exists $n \in \mathbb{N}_+$ such that $q_1, \dots, q_m \in \mathbb{Z}[1/n][u_1, \dots, u_s]$ (here $\mathbb{Z}[1/n]$ is the localization of $\mathbb{Z}$ in $\{n^a : a \in \mathbb{N}\}$).

Apply this observation to the coefficients of the minimal polynomials of the integral elements $c_1, \dots, c_t$ and to the coefficients appearing in the linear combinations expressing the elements of $\mathcal{F}$ in $\sum_{i=1}^{t} \mathbb{Q}[u_1, \dots, u_s]c_i$. This yields that there exists $n \in \mathbb{N}_+$ such that

(1) all coefficients of the minimal polynomials of the elements $c_1, \dots, c_t$ belong to $E := \mathbb{Z}[1/n][u_1, \dots, u_s]$ and
(2) $R := \mathbb{Z}[\mathcal{F}] \subset E[c_1, \dots, c_t] =: D \subset C = \mathbb{Q}[\mathcal{F}]$.

Because of (1), each c_i is integral over E, whence D/E is an integral extension of finite type. Now choose a prime number p such that $p \nmid n$ and $p > \ell$. The ideal $\mathfrak{p} \triangleleft E$, generated by p and the $u_1, \dots, u_s$, has the property that $E/\mathfrak{p} \simeq \mathbb{Z}[1/n]/\langle p \rangle$, where $\langle p \rangle = p\mathbb{Z}[1/n] \subset \mathbb{Z}[1/n]$ is the ideal generated by p, whence $\langle p \rangle \subset \mathbb{Z}[1/n]$ and therefore also $\mathfrak{p} \subset E$ are maximal ideals. Since D is integral over E, there exists a maximal ideal $\mathfrak{n} \triangleleft D$ lying over $\mathfrak{p} \triangleleft E$. By construction, $D/\mathfrak{n}$ is a finite field of characteristic p. Recall $R \subset D$, and put $\mathfrak{m} = R \cap \mathfrak{n}$. Then $R/\mathfrak{m} \hookrightarrow D/\mathfrak{n}$. So $R/\mathfrak{m}$ is a finite subring of the field $D/\mathfrak{n}$, whence a field itself. It remains to observe that $f \notin \mathfrak{m}$ for every $f \in \mathcal{F}$ because f is a unit in D. □

5.2. Corollary. *Let Q be a quantum torus over a field k of characteristic* 0*, let $q = (q_{ij}) \in \mathrm{M}_n(k)$ be the quantum matrix associated with a coordinatization of Q and write $Q = \bigoplus_{\lambda \in \Lambda} kx^\lambda$ as in* (1.2.1)*. Further, let $a_1, \dots, a_t \in k \setminus \{0\}$, $b_1, \dots, b_m \in Q$ be non-zero elements and let $g_1, \dots, g_p \in \mathrm{M}_\ell(Q)$ be non-zero matrices.*

Then there exists a finitely generated subring $R < k$ and a maximal ideal $\mathfrak{m} \triangleleft R$ with the following properties:

(i) *All a_i and $q_{ij} \in R$, all $b_1, \dots, b_m$ lie in the unital graded subalgebra $\mathcal{A} = \bigoplus_{\lambda \in \Lambda} Rx^\lambda$ of Q, and all $g_1, \dots, g_p \in \mathrm{M}_\ell(\mathcal{A})$.*
(ii) *Denoting by $^-$ the canonical quotient map, we have*

(a) $\bar{R} = R/\mathfrak{m}$ *is a finite field of very good characteristic* $p > 0$ *for* $\mathfrak{sl}_\ell(\bar{\mathcal{A}})$*;*
(b) *all* $\bar{a}_i$ *and* $\bar{q}_{ij} \in \bar{R}$ *are non-zero, hence roots of unity;*
(c) $\bar{\mathcal{A}} = \bigoplus_{\lambda\in\Lambda} \bar{R}x^\lambda$ *is an fgc quantum torus over* $\bar{R}$ *with quantum matrix* $(\bar{q}_{ij})$ *and* $p \not| [\bar{\mathcal{A}} : \mathcal{Z}(\bar{\mathcal{A}})]$*;*
(d) *all* $\bar{b}_i \neq 0$ *in* $\bar{\mathcal{A}}$*;*
(e) *all* $\bar{g}_i$ *are non-zero in* $\mathrm{M}_\ell(\bar{\mathcal{A}})$*.*

Proof. Let $\mathcal{F}$ be the finite subset of k consisting of all a_i, q_{ij}, all non-zero k-coefficients of $b_1, \dots, b_m$ and $g_1, \dots, g_p$ and their inverses (the k-coefficients are taken with respect to a natural k-basis of Q and $\mathrm{M}_\ell(Q)$). Let R and $\mathfrak{m}$ be as in Proposition 5.1. Then all claims follow immediately from $\mathcal{F} \subset R \setminus \mathfrak{m}$. □

5.3. Lemma. *Let* Q *be a quantum torus over a field* k *of characteristic* 0*. Then the Lie algebra* $\mathfrak{sl}_\ell(Q)$ *is finitely generated over* k*.*

Proof. This fact is known for all Lie tori ([Ne1, Thm. 5]). In our concrete case, it can be proven as follows. We fix a parametrization $Q = \bigoplus_{\alpha\in\mathbb{Z}^n} kx^\lambda$; in particular this gives us coordinates x_i, $1 \le i \le n$ corresponding to the standard basis of $\mathbb{Z}^n$. It is straightforward to check that $\mathfrak{sl}_\ell(Q)$ is generated by $\{E_{ij}, x_p^{\pm 1}E_{ij} : 1 \le i \neq j \le \ell\}$. □

We have seen in 1.2(f) that a quantum torus is fgc if for one coordinatization, equivalently for all coordinatizations, the entries of the associated quantum matrix are roots of unity. Hence, in a coordinatization of a non-fgc quantum torus with quantum matrix $q = (q_{ij})$ at least one of the q_{ij} is not a root of unity.

5.4. Theorem. *Let* $Q = \bigoplus_{\lambda\in\Lambda} kx^\lambda$ *and* $Q' = \bigoplus_{\lambda'\in\Lambda'} ky^{\lambda'}$ *be non-fgc quantum tori over a field* k *of characteristic* 0 *with associated quantum matrices* $q = (q_{ij}) \in \mathrm{M}_n(k)$ *and* $q' = (q'_{ij}) \in \mathrm{M}_{n'}(k)$*. We assume that we are given:*

- *non-zero elements* $b_1, \dots, b_m \in Q$ *and non-zero elements* $b'_1, \dots, b'_m \in Q'$*;*
- *non-zero elements* $g_1, \dots, g_s \in \mathfrak{gl}_\ell(Q)$*, and non-zero elements* $g'_1, \dots, g'_{s'} \in \mathfrak{gl}_{\ell'}(Q')$*;*
- *a* k*-linear isomorphism* $f : \mathfrak{sl}_\ell(Q) \to \mathfrak{sl}_{\ell'}(Q')$ *of Lie algebras.*

Then there exists a subring $R < k$ *and a maximal ideal* $\mathfrak{m} \triangleleft R$ *with the following properties:*

(i) *all* $q_{ij} \in R$ *and all* $q'_{ij} \in R$*;*
(ii) *all* $b_1, \dots, b_m$ *lie in the unital graded subalgebra* $\mathcal{A} = \bigoplus_{\lambda\in\Lambda} Rx^\lambda$ *of* Q*, and all* $g_1, \dots, g_s$ *are in* $\mathfrak{gl}_\ell(\mathcal{A})$*;*
(iii) *all* $b'_1, \dots, b'_m$ *lie in the unital graded subalgebra* $\mathcal{A}' = \bigoplus_{\lambda'\in\Lambda'} Ry^{\lambda'}$ *of* Q'*, and all* $g'_1, \dots, g'_{s'} \in \mathfrak{gl}_{\ell'}(\mathcal{A}')$*;*
(iv) $f(\mathfrak{sl}_\ell(\mathcal{A})) = \mathfrak{sl}_{\ell'}(\mathcal{A}')$*.*
(v) *Denoting by* $\bar{\ }$ *the canonical quotient map, we have*
   (a) $\bar{R} = R/\mathfrak{m}$ *is a finite field of very good characteristic for* $\mathfrak{sl}_\ell(\bar{\mathcal{A}})$ *and* $\mathfrak{sl}_{\ell'}(\bar{\mathcal{A}})$*;*
   (b) *all* $\bar{q}_{ij}, \bar{q}'_{ij} \in \bar{R}$ *are non-zero, hence roots of unity;*
   (c) $\bar{\mathcal{A}} = \mathcal{A}/\mathfrak{m}\mathcal{A} = \bigoplus_{\lambda\in\Lambda} \bar{R}x^\lambda$ *and* $\bar{\mathcal{A}}' = \bigoplus_{\lambda'\in\Lambda'} \bar{R}y^{\lambda'}$ *are fgc quantum tori over* $\bar{R}$ *with associated quantum matrices* $(\bar{q}_{ij})$ *and* $(\bar{q}'_{ij})$*;*

(d) $\bar{f} : \mathfrak{sl}_\ell(\bar{\mathcal{A}}) \to \mathfrak{sl}_{\ell'}(\bar{\mathcal{A}}')$ *is an* $\bar{R}$*-isomorphism of the corresponding Lie* $\bar{R}$*-algebras;*
(e) *all* $\bar{b}_i$ *are non-zero in* $\bar{\mathcal{A}}$ *and all* $\bar{b}'_i$ *are non-zero in* $\bar{\mathcal{A}}'$*;*
(f) *all* $\bar{g}_i$ *are non-zero in* $\mathfrak{gl}_\ell(\bar{\mathcal{A}})$ *and all* $\bar{g}'_i$ *are non-zero in* $\mathfrak{gl}_{\ell'}(\bar{\mathcal{A}}')$*.*

Proof. By definition, the coordinates of $0 \neq b \in Q$ are the non-zero $s_\lambda \in k$ when b is written as $b = \sum_{\lambda\in\Lambda} s_\alpha x^\lambda$. The coordinates of $0 \neq g = \sum_{i,j} g_{ij} E_{ij} \in \mathrm{M}_\ell(Q)$ are the coordinates of the non-zero $g_{ij} \in Q$. The coordinates of $0 \neq b' \in Q'$ and $0 \neq g' \in \mathrm{M}_{\ell'}(Q')$ are defined analogously. We choose finite generating systems $S \subset \mathfrak{sl}_\ell(Q)$ and $S' \subset \mathfrak{sl}_{\ell'}(Q')$ as in Lemma 5.3, and put $g = f^{-1}$. We let $\mathcal{F} \subset k$ consist of

— all q_{ij}, q'_{ij}, the coordinates of all b_i, b'_i, g_i, g'_i together with their inverses,
— the elements $(q_{ij} - 1)^{-1}$ and $(q'_{ij} - 1)^{-1}$ whenever q_{ij} or q'_{ij} is not a root of unity, and
— the coefficients of all elements in $f(S)$ and $g(S')$.

We now apply Proposition 5.1 for this $\mathcal{F}$ but with the ℓ there replaced by $\max\{\ell, \ell'\}$. This provides us with $(R, \mathfrak{m})$ as required in the Theorem. Indeed, by construction we have $f(\mathfrak{sl}_\ell(\mathcal{A})) \subset \mathfrak{sl}_{\ell'}(\mathcal{A}')$ and similarly $g(\mathfrak{sl}_{\ell'}(\mathcal{A}')) \subset \mathfrak{sl}_\ell(\mathcal{A})$. It follows that $f(\mathfrak{sl}_\ell(\mathcal{A})) = \mathfrak{sl}_{\ell'}(\mathcal{A}')$ (because g is the inverse for f), so that (iv) holds. The remaining claims follow immediately from $\mathcal{F} \subset R \setminus \mathfrak{m}$. □

6. Some preliminaries for Step 3 of the proof of the main theorem

6.1. Setting I

In this section we use the following setting:

- Q is a quantum torus over a field F with grading group $\Lambda \simeq \mathbb{Z}^n$;
- We fix a basis $\boldsymbol{\varepsilon}$ of Λ; the $\boldsymbol{\varepsilon}$-trace, the corresponding $\mathbb{Z}$-grading, and the $\boldsymbol{\varepsilon}$-degree will all be taken with respect to the fixed $\boldsymbol{\varepsilon}$. We will therefore simply write deg instead of $\deg_{\boldsymbol{\varepsilon}}$. We identify $\Lambda = \mathbb{Z}^n$ via $\boldsymbol{\varepsilon}$, and define $\Lambda^+ = \mathbb{N}^n \subset \mathbb{Z}^n$.
- Corresponding to $\boldsymbol{\varepsilon}$ there exists a coordinatization of Q as $Q = \bigoplus_{\lambda\in\mathbb{Z}^n} Fx^\lambda$. We let $0 \neq x_i \in Q$ denote the element that corresponds to the ith basis vector ε_i in $\boldsymbol{\varepsilon}$, and put $Q^+ = \bigoplus_{\lambda\in\Lambda^+} Q^\lambda$. Note that Q^+ is a unital subring of Q.
- For $\ell \geq 2$ we let V be a free right Q-module of rank ℓ. We fix a basis $e_1, \dots e_n$ of V so that we can write $V = \bigoplus_{i=1}^{\ell} e_i Q$. We put $V^+ = \bigoplus_{i=1}^{\ell} e_i Q^+$.
- We say that x_i *divides* $q \in Q^+$ if every $\lambda \in \mathrm{supp}(q)$ has the form $\lambda = (\lambda_1, \dots, \lambda_n)$ with respect to $\boldsymbol{\varepsilon}$ and $\lambda_i > 0$. In this case $qx_i^{-1} \in Q^+$.
- Any $0 \neq v \in V$ can be uniquely written as $v = \sum_{i=1}^{\ell} e_i q_i$ with $q_i \in Q$. We put
$$\deg(v) = \max\{\deg(q_i) : q_i \neq 0\}.$$
We refer to the q_i as the coordinates of v.

6.2. Lemma. *Let* $0 \neq v \in V$ *and let* $0 \neq q \in Q$*. Then* $vq \neq 0$ *and*
$$\deg(vq) = \deg(v) + \deg(q).$$

Proof. We write $v = \sum_{i=1}^{\ell} e_i q_i$ as above. The first claim is obvious: $q_i q \neq 0 \iff q_i \neq 0$, and at least one coordinate $q_i \neq 0$. For the non-zero q_i we have $\deg(q_i q) = \deg(q_i) + \deg(q)$ by (1.3.3). Hence, if for simpler notation $\deg(v) = \deg(q_1)$ then $\deg(q_i q) \leq \deg(q_1 q)$ for all i with $q_i \neq 0$. □

6.3. Setting II

We continue with the Setting of 6.1. In addition, we fix a non-zero Q-submodule $U \subset V$, and put $U^+ = U \cap V^+$. Observe $U^+ \neq \{0\}$ since $qQ \cap Q^+ \neq \{0\}$ for any $0 \neq q \in Q$. Since $\deg(u) \geq 0$ for every $0 \neq u \in U^+$ there exist *minimal* vectors $u_0 \in U^+$ such that

$$\deg(u_0) \leq \deg(u) \quad \text{for all } 0 \neq u \in U^+.$$

We call $u \in U^+$ *indivisible* if $u \neq 0$ and the coordinates of u do not have a common divisor x_i, $1 \leq i \leq n$, in Q^+. It is obvious that indivisible vectors exist.

Moreover,

every minimal vector $u_0 \in U^+$ is indivisible.

Indeed, write $u_0 = \sum_{j=1}^{\ell} e_j q_j$ with all $q_j \in U^+$. Assume that all q_j are divisible by some x_i. Then $0 \neq u_0 x_i^{-1} = \sum_{j=1}^{\ell} e_j q_j x_i^{-1} \in U^+$ since all $q_j x_i^{-1} \in Q^+$. But $\deg(u_{\min} x_i^{-1}) = \deg(u_0) + \deg(x_i^{-1}) = \deg(u_0) - 1 < \deg(u_0)$, a contradiction.

6.4. Lemma. *Assume Setting* 6.3*, and let $u_0 \in U^+$ be indivisible, and let $q \in Q$. Then*

$$u_0 q \in U^+ \iff q \in Q^+.$$

Proof. We only need to show that the left-hand side implies the right-hand side. We reason by contradiction. Assume $q \notin Q^+$. We write $u_0 = \sum_{j=1}^{\ell} e_j q_j$ with all $q_j \in Q^+$. We choose a minimal $\lambda = (\lambda_1, \ldots, \lambda_n) \in \Lambda^+$ such that $qx^\lambda \in Q^+$. Here "minimal" means that $qx^\lambda x_i^{-1} \notin Q^+$ for all i. Since $q \notin Q^+$, some $\lambda_i > 0$. Let I be the ideal of the ring Q^+ generated by x_i. Note $I = x_i Q^+ = Q^+ x_i$. By definition of an indivisible element, at least one of the coordinate q_j of u_0 is not divisible by x_i in Q^+ (for i with $\lambda_i > 0$). To simplify notation, assume this is q_1, i.e., $q_1 \notin I$. Then the following holds.

(i) $qx^\lambda \notin I$. Otherwise, $qx^\lambda = q' x_i$ for some $q' \in Q^+$, whence $qx^\lambda x_i^{-1} \in Q^+$, and so $qx^\mu \in Q^+$ for $\mu = \lambda - \varepsilon_i \in \Lambda^+$.

(ii) $q_1 q x^\lambda \in Q^+$ because $q_1 \in Q^+$ and $qx^\lambda \in Q^+$. Furthermore, $x^\lambda \in I$ because $\lambda_i > 0$. Hence $q_1 q x^\lambda \in I$. Thus $x_1 | q_1 q x^\lambda$, but $x_1 \not| q_1$ and $x_1 \not| qx^\lambda$. This contradicts (iii) below.

(iii) Q^+/I is a subring of the quantum torus with associated quantum matrix $q' = (q_{ij})$ where $2 \leq i, j \leq n$. It is therefore a domain.

Our assumption $q \notin Q^+$ has thus led to a contradiction. □

6.5. Corollary. *Let $u_0 \in U^+$ be an indivisible vector. Then*

$$U = u_0 Q \iff U^+ = u_0 Q^+.$$

Proof. If $U = u_0 Q$ and $u^+ \in U^+$, then $u^+ = u_0 q$ with $q \in Q$ and Lemma 6.4 shows $q \in Q^+$. Conversely, if $U^+ = u_0 Q^+$ and $u \in U$ is arbitrary, we can take $x^\lambda \in Q$ such that $ux^\lambda \in U^+$, whence $ux^\lambda = u_0 q$ for some $q \in Q^+$. But then $u = u_0 q (x^\lambda)^{-1} \in u_0 Q$. □

6.6. Lemma. *In addition to the Setting* 6.3 *assume that* U *admits a complement:* $V = U \oplus U'$ *for some* Q*-subspace* U'*. Let* $u_0 \in U^+$ *be a minimal vector and put* $Q^{++} = \bigoplus_{\lambda \in \mathbb{N}^n \setminus \{0\}} Q^\lambda$*. Then* $u_0 \notin v(Q^{++})$ *for any* $v \in V^+$*.*

Proof. Assume to the contrary that $u_0 = vq$ for some $q \in Q^{++}$ and $v \in V^+$. Decompose $v = u + u'$ with $u \in U$ and $u' \in U'$. Then $u_0 = uq + u'q$ shows $u'q = 0$. So without loss of generality we can assume $v \in U^+$. Now apply Lemma 6.2 to get

$$\deg(v) + \deg(q) = \deg(vq) = \deg(u_0) \le \deg(v)$$

(because u_0 is minimal) and hence $\deg(q) = 0$, i.e., $q \in F^\times \cdot 1_F$, a contradiction. □

7. Proof of the main theorem

7.1. Setting and plan of the proof

Throughout this section, k is a base field of characteristic 0 and Q and Q' are non-fgc quantum tori. We assume that they are coordinatized as

$$Q = \bigoplus_{\lambda \in \Lambda} kx^\lambda \quad \text{and} \quad Q' = \bigoplus_{\lambda' \in \Lambda'} ky^{\lambda'}$$

for $\Lambda = \mathbb{Z}^n$ and $\Lambda' = \mathbb{Z}^{n'}$ with associated quantum matrices $q = (q_{ij}) \in \mathrm{M}_n(k)$ and $q' = (q'_{ij}) \in \mathrm{M}_{n'}(k)$. We assume that

$$f \colon \mathfrak{sl}_{\ell'}(Q') \to \mathfrak{sl}_\ell(Q)$$

is a k-linear isomorphism. We apply Lemma 2.5 to extend f to an isomorphism

$$f_{\mathfrak{gl}} \colon \mathfrak{gl}_{\ell'}(Q') \to \mathfrak{gl}_\ell(Q).$$

In the first step of the proof (Proposition 7.2) we will show $\ell = \ell'$. Next, it will follow from Proposition 7.3 that we may assume that

$$\phi = f_{\mathfrak{gl}} \colon \mathrm{M}_\ell(Q')) \to \mathrm{M}_\ell(Q)$$

is an isomorphism of associative algebras. In the final step of the proof we will establish that if $\mathfrak{h}' \subset \mathrm{M}_{\ell'}(Q')$ (resp. $\mathfrak{h} \subset \mathrm{M}_\ell(Q)$) is the standard MAD of $\mathrm{M}_{\ell'}(Q')$ (resp. $\mathrm{M}_\ell(Q)$) then $\phi(\mathfrak{h}')$ is conjugate to $\mathfrak{h}$ by an element of $\mathrm{GL}_\ell(Q)$.

7.2. Proposition. *In the setting* 7.1 *we have* $\ell = \ell'$*.*

Proof. According to Theorem 5.4 there exist a subring $R \subset k$ and a maximal ideal $\mathfrak{m} \lhd R$ such that f induces an $\overline{R}$-isomorphism

$$\bar{f} \colon \mathfrak{sl}_{\ell'}(\bar{\mathcal{A}}') \to \mathfrak{sl}_\ell(\bar{\mathcal{A}}).$$

Since $\bar{\mathcal{A}}$ and $\bar{\mathcal{A}}'$ are quantum tori over $\bar{R}$ of fgc type, by Theorem 4.3 we have $\ell = \ell'$. □

7.3. Proposition. *Consider the following two maps:*

(a) $f_{\mathfrak{gl}}\colon \mathfrak{gl}_\ell(Q') \to \mathfrak{gl}_\ell(Q)$,

(b) *the extension of* $f \circ \iota^{\mathrm{op}}\colon \mathfrak{sl}_\ell((Q')^{\mathrm{op}}) \to \mathfrak{sl}_\ell(Q)$ *to a Lie algebra isomorphism*

$$(f \circ \iota^{\mathrm{op}})_{\mathfrak{gl}}\colon \mathfrak{gl}_\ell((Q')^{\mathrm{op}}) \to \mathfrak{gl}_\ell(Q).$$

Then one of these maps is an isomorphism of the underlying associative k-algebras.

Proof. Assume the contrary. Then there exist $g_1, g_2, g_3, g_4 \in \mathrm{M}_\ell(Q')$ such that

$$f_{\mathfrak{gl}}(g_1 g_2) \neq f_{\mathfrak{gl}}(g_1) f_{\mathfrak{gl}}(g_2) \quad \text{and} \quad f_{\mathfrak{gl}}(-g_3 g_4) \neq f_{\mathfrak{gl}}(g_4) f_{\mathfrak{gl}}(g_3).$$

Put

$$g_1' = f_{\mathfrak{gl}}(g_1 g_2),\ g_2' = f_{\mathfrak{gl}}(g_1),\ g_3' = f_{\mathfrak{gl}}(g_2),\ g_4' = g_1' - g_2' g_3'$$

and similarly

$$g_5' = f_{\mathfrak{gl}}(-g_3 g_4),\ g_6' = f_{\mathfrak{gl}}(g_3),\ g_7' = f_{\mathfrak{gl}}(g_4),\ g_8' = g_5' - g_7' g_6'.$$

Recall that by Lemma 2.3 one has the decomposition

$$\mathfrak{gl}_\ell(Q) = \mathcal{Z}(Q) E_\ell \oplus \mathfrak{sl}_\ell(Q)$$

and similarly

$$\mathfrak{gl}_\ell(Q') = \mathcal{Z}(Q') E_\ell \oplus \mathfrak{sl}_\ell(Q').$$

So every element g_i (resp. g_i') can be written as the sum $g_i = q_i E_\ell + \tilde{g}_i$ (resp. $g_i' = q_i' E_\ell + \tilde{g}_i'$) where q_i (resp. q_i') is in the centre of Q (resp. Q') and $\tilde{g}_i$ (resp. $(\tilde{g}_i')$) is a sum of commutators of elements of $\mathfrak{gl}_\ell(Q)$ (resp. $\mathfrak{gl}_{\ell'}(Q')$). We add to our list of elements $g_1, \dots, g_4$ (resp. $g_1', \dots, g_8'$) all their components arising in the above two decompositions (including elements appearing in the writing of $\tilde{g}_i, \tilde{g}_i'$ as sums of commutators).

We now apply Theorem 5.4 with these data. This provides us with a subring $R \subset k$ and a maximal ideal $\mathfrak{m} \lhd R$ satisfying the many conclusions of loc. cit. In particular, denoting by

$$f_{\mathcal{A}}\colon \mathfrak{sl}_\ell(\mathcal{A}') \to \mathfrak{sl}_\ell(\mathcal{A})$$

the isomorphism obtained by restriction of f, we have an isomorphism

$$\bar{f}_{\mathcal{A}}\colon \mathfrak{sl}_\ell(\bar{\mathcal{A}}') \to \mathfrak{sl}_\ell(\bar{\mathcal{A}})$$

where now both $\bar{\mathcal{A}}'$ and $\bar{\mathcal{A}}$ are fgc quantum tori over the finite field $R/\mathfrak{m}$ of very good characteristic. This allows us to apply Theorem 4.3. In view of Remark 2.6 we get that either

$$(\bar{f}_{\mathcal{A}})_{\mathfrak{gl}}\colon \mathfrak{gl}_\ell(\bar{\mathcal{A}}') \to \mathfrak{gl}_\ell(\bar{\mathcal{A}}) \quad \text{or} \quad (\bar{f}_{\mathcal{A}} \circ \iota^{\mathrm{op}})_{\mathfrak{gl}}\colon \mathfrak{gl}_\ell(\bar{\mathcal{A}}'^{\mathrm{op}}) \to \mathfrak{gl}_\ell(\bar{\mathcal{A}})$$

is an isomorphism of the underlying associative algebras.

On the other side, arguing as in Lemma 2.5 we get the following: if $\mathcal{Z}(\mathcal{A})$ (resp. $\mathcal{Z}(\mathcal{A}')$) denotes the centre of $\mathcal{A}$ (resp. $\mathcal{A}'$) our map $f_{\mathcal{A}}$ has a canonical extension to

$$\mathcal{Z}(\mathcal{A}')E_\ell \oplus \mathfrak{sl}_\ell(\mathcal{A}') \to \mathcal{Z}(\mathcal{A})E_\ell \oplus \mathfrak{sl}_\ell(\mathcal{A})$$

which abusing notation we will still denote by $(f_{\mathcal{A}})_{\mathfrak{gl}}$. Note that $(f_{\mathcal{A}})_{\mathfrak{gl}}$ coincides with the restriction of $f_{\mathfrak{gl}}$ to $\mathcal{Z}(\mathcal{A}')E_\ell \oplus \mathfrak{sl}_\ell(\mathcal{A}')$ and that by our construction all matrices q_iE_ℓ, $q_i'E_{\ell'}$, g_i' and g_i live in $\mathcal{Z}(\mathcal{A}')E_\ell \oplus \mathfrak{sl}_\ell(\mathcal{A}')$ and $\mathcal{Z}(\mathcal{A})E_\ell \oplus \mathfrak{sl}_\ell(\mathcal{A})$. Passing to the residues we get an isomorphism

$$\overline{(f_{\mathcal{A}})_{\mathfrak{gl}}} \colon \overline{\mathcal{Z}(\mathcal{A}')} \oplus \mathfrak{sl}_\ell(\bar{\mathcal{A}}') \to \overline{\mathcal{Z}(\mathcal{A})} \oplus \mathfrak{sl}_\ell(\bar{\mathcal{A}}).$$

It is easily seen from the construction that

$$\overline{(f_{\mathcal{A}})_{\mathfrak{gl}}} = ((\bar{f}_{\mathcal{A}})_{\mathfrak{gl}})|_{\overline{\mathcal{Z}(\mathcal{A}')}\oplus\mathfrak{sl}(\bar{\mathcal{A}}')} =: \psi.$$

We now obtain a contradiction: In case $(\bar{f}_{\mathcal{A}})_{\mathfrak{gl}}$ is an isomorphism of the underlying associative algebras we have

$$\overline{g_1'} = \psi(\overline{g_1g_2}) = \psi(\overline{g_1})\psi(\overline{g_2}) = \overline{g_2'g_3'},$$

whence

$$\overline{g_4'} = \overline{g_1' - g_2'g_3'} = \overline{g_1'} - \overline{g_2'g_3'} = 0,$$

contradicting $\overline{g_4'} \neq 0$ by Theorem 5.4. In the other case, one obtains a contradiction in the same way. □

7.4. Final step

As indicated above, from now on we will assume that

$$\phi : \mathrm{M}_\ell(Q') \to \mathrm{M}_\ell(Q)$$

is an isomorphism of associative k-algebras. Let

$$V = Q \oplus \ldots \oplus Q$$

be the free right Q-module of rank ℓ defined in (2.2.4) for $\mathcal{A} = Q$. We know that $\mathrm{M}_\ell(Q)$ acts on V from the left while Q acts from the right. We denote by $\mathsf{B} = \{e_1, \ldots, e_\ell\}$ the standard basis of the Q-module V, defined in (2.2.5). Furthermore, we know that $E_i' = E_{ii}' \in \mathrm{M}_\ell(Q')$, $i = 1, \ldots, \ell$ form a complete orthogonal system of idempotents in $\mathrm{M}_\ell(Q')$. Since ϕ preserves the associative multiplication, the image of the standard orthogonal system $(E_{11}', \ldots, E_{\ell\ell}')$ of $\mathrm{M}_\ell(Q')$ is a complete orthogonal system in $\mathrm{M}_\ell(Q)$. We put

$$\widetilde{E}_i = \phi(E_{ii}') \in \mathrm{M}_\ell(Q), 1 \leq i \leq \ell.$$

We then know from Lemma 2.10 that V decomposes with respect to $(\widetilde{E}_1, \ldots, \widetilde{E}_\ell)$:

$$V = V_1 \oplus \cdots \oplus V_\ell, \quad \text{for} \quad V_i = \widetilde{E}_i(V).$$

As shown in Lemma 2.10(c) conjugacy will follow once we know that all the V_i are cyclic Q-modules. We will prove this using again specialization.

To simplify the notation we let $U = V_i$ for any one of the i, $1 \le i \le \ell$. We can apply the results of §6 and choose a minimal vector $u_0 \in U^+$. For $t \in \mathbb{N}$ define

$$\begin{aligned} P_t &= \{q \in Q^+ : \deg(q) \le t\}, \\ V_t &= \{v \in V^+ : \deg(v) \le t\}, \\ U_t &= V_t \cap U^+. \end{aligned}$$

The spaces P_t, V_t and U_t are finite-dimensional k-vector spaces. We denote by $\mathbb{P}(\cdot)$ the corresponding projective spaces. Since $0 \ne q \implies u_0 q \ne 0$ we have a well-defined regular map

$$\varphi_t \colon \mathbb{P}(P_t) \to \mathbb{P}(U_{t+\deg(u_0)}), \quad [q] \mapsto [u_0 q].$$

Its image is the zero set of a finite set G_t of non-zero homogenous polynomials (in fact linear forms) with coefficients in k:

$$\mathrm{Im}(\varphi_t) = \mathrm{Zero}(G_t).$$

Similarly, for $0 \le s < \deg(u_0)$ we have a regular map

$$\gamma_s \colon \mathbb{P}(V_s) \times \mathbb{P}(P_{\deg(u_0)-s}) \to \mathbb{P}(V_{\deg(u_0)}), \quad ([v],[q]) \to [vq].$$

Since we are dealing with projective spaces, the image of γ_s is a closed subvariety, whence given by a finite set H_s of non-zero homogeneous polynomials with coefficients in k:

$$\mathrm{Im}(\gamma_s) = \mathrm{Zero}(H_s).$$

By Lemma 6.6, $[u_0] \notin \mathrm{Im}(\gamma_s)$ for all $0 \le s < \deg(u_0)$. Hence:

$$h_s(u_0) \ne 0 \quad \text{for some } h_s \in H_s,\ 0 \le s < \deg(u_0). \tag{7.4.1}$$

Recall that our goal is to show $U = u_0 Q$, i.e., in view of Lemma 6.4: $U^+ = u_0 Q^+$. For the purpose of contradiction, assume this is not the case. Thus there exists $v_0 \in U^+ \setminus u_0 Q^+$. Observe

$$d := \deg(v_0) - \deg(u_0) \ge 0.$$

Therefore $[v_0] \notin \mathrm{Im}(\varphi_d)$, i.e.,

$$g_d(v_0) \ne 0 \quad \text{for some } g_d \in G_d. \tag{7.4.2}$$

We now apply Corollary 5.2 to construct a subring $R < k$. The finitely many elements $a_i \in k$, $b_i \in Q$ and $g_i \in \mathrm{M}_\ell(Q)$ of loc. cit. are the following.

- in k: the elements $h_s(u_0)$ and $g_d(v_0)$ of (7.4.1) and (7.4.2) respectively; all q_{ij}; the coefficients of the polynomial g_d and of all polynomials in H_s, $0 \le s < \deg(u_0)$;
- in Q: the (by definition non-zero) coefficients of the vectors u_0 and v_0;
- in $\mathrm{M}_\ell(Q)$: the matrices $\widetilde{E}_i$, $1 \le i \le \ell$.

As in Corollary 5.2 let $\mathcal{A} = \bigoplus_{\lambda\in\Lambda} Rx^\lambda$. We then have objects "over R":

$$V_{\mathcal{A}} = \bigoplus_{i=1}^{\ell} e_i\mathcal{A}, \quad \mathcal{A}^+ = \bigoplus_{\lambda\in\Lambda^+} Rx^\lambda, \quad V_{\mathcal{A}}^+ = \bigoplus_{i=1}^{\ell} e_i\mathcal{A}^+.$$

Since all matrices $\widetilde{E}_i \in \mathrm{M}_\ell(\mathcal{A})$ we get a decomposition

$$V_{\mathcal{A}} = V_{\mathcal{A},1} \oplus \cdots \oplus V_{\mathcal{A},\ell}, \quad V_{\mathcal{A},i} = V_{\mathcal{A}} \cap V_i.$$

In particular, $U_{\mathcal{A}} = U \cap V_{\mathcal{A}}$. We choose the maximal ideal $\mathfrak{m} \lhd R$ as in Corollary 5.2, and denote by $\bar{\ }$ the quotient objects:

$$\begin{aligned} \bar{R} &= R/\mathfrak{m}, \\ \bar{\mathcal{A}} &= \mathcal{A}/\mathfrak{m}\mathcal{A} = \textstyle\bigoplus_{\lambda\in\Lambda} \bar{R}x^\lambda, & \bar{\mathcal{A}}^+ &= \mathcal{A}^+/\mathfrak{m}\mathcal{A}^+ = \textstyle\bigoplus_{\lambda\in\Lambda^+} \bar{R}x^\lambda, \\ \bar{V}_{\mathcal{A}} &= V_{\mathcal{A}}/\mathfrak{m}V_{\mathcal{A}} = \textstyle\bigoplus_{i=1}^{\ell} e_i\bar{\mathcal{A}} \simeq \bar{\mathcal{A}}^\ell, & \bar{V}_{\mathcal{A}}^+ &= V_{\mathcal{A}}^+/\mathfrak{m}V_{\mathcal{A}}^+ = \textstyle\bigoplus_{i=1}^{\ell} e_i\bar{\mathcal{A}}^+ \simeq \bar{\mathcal{A}}^\ell, \\ \bar{U}_{\mathcal{A}} &= U_{\mathcal{A}}/\mathfrak{m}U_{\mathcal{A}}. \end{aligned}$$

By construction, $\bar{\mathcal{A}}$ is an fgc quantum torus over the finite field $\bar{R}$ which has very good characteristic for $\mathfrak{sl}_\ell(\bar{\mathcal{A}})$. Hence, by Corollary 4.7, conjugacy holds in $\mathfrak{sl}_\ell(\bar{\mathcal{A}})$. Thus, by Lemma 2.10, $\bar{U}_{\mathcal{A}}$ is a free $\bar{\mathcal{A}}$-module, say $\bar{U}_{\mathcal{A}} = \bar{c}\cdot\bar{\mathcal{A}}$. We can apply the results of §6: without loss of generality, $\bar{c} \in \bar{U}_{\mathcal{A}}^+ = \bar{U}_{\mathcal{A}} \cap \bar{V}_{\mathcal{A}}^+$. We can even assume that $\bar{c}$ is indivisible. Thus, by Corollary 6.5, $\bar{U}_{\mathcal{A}}^+ = \bar{c}\bar{\mathcal{A}}^+$. Since $\bar{u}_0 \in \bar{U}_{\mathcal{A}}^+$ we get from Lemma 6.4 that

$$\bar{u}_0 = \bar{c}\cdot\bar{a} \quad \text{for some } \bar{a} \in \bar{\mathcal{A}}^+.$$

Our next goal is to show that $\bar{a} \in \bar{R}\cdot 1_{\bar{\mathcal{A}}}$. To this end we use the "bar"-versions of the vector spaces and maps defined above:

$$\begin{gathered} \bar{P}_t = \{q \in \bar{\mathcal{A}}^+ : \deg(\bar{q}) \le t\}, \quad \bar{V}_t = \{\bar{v} \in \bar{V}_{\mathcal{A}}^+ : \deg(\bar{v}) \le t\}, \quad \bar{U}_t = \bar{V}_t \cap \bar{U}^+, \\ \bar{\varphi}_t \colon \mathbb{P}(\bar{P}_t) \to \mathbb{P}(\bar{U}_{t+\deg(u_0)}), \quad [\bar{q}] \mapsto [\bar{u}_0\bar{q}], \\ \bar{\gamma}_s \colon \mathbb{P}(\bar{V}_s) \times \mathbb{P}(\bar{P}_{\deg(u_0)-s}) \to \mathbb{P}(\bar{V}_{\deg(u_0)}), \quad ([\bar{v}],[\bar{q}]) \to [\bar{v}\bar{q}]. \end{gathered}$$

By base change, $\mathrm{Im}(\bar{\varphi}_t)$ is the zero set of the polynomials $\{\bar{g} : g \in G_t\}$. Similarly, $\mathrm{Im}(\bar{\gamma}_s)$ is the zero set of the polynomials $\bar{h}$, $h \in H_s$. From $\bar{u}_0 = \bar{c}\cdot\bar{a}$ we obtain $\deg(\bar{u}_0) = \deg(\bar{c}) + \deg(\bar{a})$. Assuming $\deg(\bar{a}) > 0$, it follows that $\bar{u}_0 \in \mathrm{Im}(\bar{\gamma}_s)$ for $0 \le s = \deg(\bar{c}) < \deg(\bar{u}_0) = \deg(u_0)$. Hence $\bar{h}(\bar{u}_0) = \overline{h(u_0)} = 0$ for all polynomials $h \in H_s$. But this contradicts (7.4.1): $\bar{h}_s(u_0) \ne 0$ by construction of R and $\mathfrak{m}$. Hence $\deg(\bar{a}) = 0$, proving that $\bar{u}_0$ is also a generator of $\bar{U}_{\mathcal{A}}^+$: $\bar{U}_{\mathcal{A}}^+ = \bar{u}_0\bar{\mathcal{A}}^+$.

Recall the element $v_0 \in U^+ \setminus u_0Q^+$. We have $0 \ne \bar{v}_0 \in \bar{U}_{\mathcal{A}}^+ = \bar{u}_0\bar{\mathcal{A}}^+$. Hence $\bar{g}(\bar{v}_0) = \overline{g(v_0)} = 0$ for all $g \in G_d$. But this contradicts (7.4.2): $\overline{g_d(v_0)} \ne 0$ by construction of R and $\mathfrak{m}$. Thus, we have arrived at the final contradiction: There does not exist $v_0 \in U^+ \setminus u_0Q^+$. It follows that U is indeed generated by u_0.

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