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High-order time-splitting methods for irreversible equations

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In this work, high-order splitting methods of integration without negative steps are shown which can be used in irreversible problems, like reaction–diffusion or complex Ginzburg–Landau equations. These methods consist of suitable affine combinations of Lie–Tortter schemes with different positive steps. The number of basic steps for these methods grows quadratically with the order, while for symplectic methods, the growth is exponential. Furthermore, the calculations can be performed in parallel, so that the computation time can be significantly reduced using multiple processors. Convergence results of these methods are proved for a large range of semilinear problems, which includes reaction–diffusion systems and dissipative perturbation of Hamiltonian systems.

Keywords: splitting methods; irreversible dynamics; high-order method.

1. Introduction

The goal of the present article is to derive arbitrary-order splitting integrators for irreversible problems. We are mainly interested in dissipative pseudo-differentiable problems which cannot be solved either by the methods of lines or by the usual splitting integrators with negative steps. In order to avoid negative steps, symplectic methods with complex steps are proposed in the literature, but in this case analytic properties on the operators are required. These assumptions on the operators restrict the application of this kind of method to reaction–diffusion-type problems.

In this article we obtain integrators that, at the same time, avoid the use of negative steps and do not require special assumptions on the operator, as well as exploiting the simplicity of the decomposition of the original problem. These methods can also be applied to problems with nonlocal nonlinearities as shown below. It is possible to build arbitrary high-order integrators for which the number of basic steps is lower than previous symplectic methods. Moreover, these methods can naturally be parallelized. In this work, we present a rigorous proof of the convergence of the proposed methods, and we also test their performance in several examples of interest.

We study the initial value problem

$$\begin{cases} \partial_t u = A_0 u + A_1(u), \\ u(0) = u_0, \end{cases} \tag{1.1}$$

where A_0 is a linear closed operator densely defined in $D(A_0) \subset \mathsf{X}$, X is a Banach space, which generates a strongly continuous semigroup of operators. We assume that the nonlinear term $A_1 : \mathsf{X} \to \mathsf{X}$ is a smooth mapping with $A_1(0) = 0$. In many problems of interest, the partial differential equations (PDEs)

$$\partial_t u = A_0 u, \tag{1.2a}$$

$$\partial_t u = A_1(u), \tag{1.2b}$$

can be easily solved either analytically or numerically, enabling approximated solutions of problem (1.1) to be found, applying in turn the flows ϕ_0 and ϕ_1 associated with the partial problems (1.2a) and (1.2b), respectively.

There exist many numerical integration methods for (1.1) based on splitting methods; the best known are the Lie–Trotter and Strang methods defined by

$$\Phi_{\mathrm{Lie}}(h,u) = \phi_1(h, \phi_0(h,u)),$$

$$\Phi_{\mathrm{Strang}}(h,u) = \phi_0(h/2, \phi_1(h, \phi_0(h/2,u))),$$

where h is the time step of the numerical integration. It can be proved that Φ_{Lie} has order 1 and Φ_{Strang} has order 2, where the order q represents the greatest natural number such that the truncation error between the real flow ϕ of equation (1.1) and the numerical method Φ satisfies

$$\|\phi(h,u) - \Phi(h,u)\|_{\mathsf{X}} \leqslant M(u) h^{q+1}$$

for $0 < h < h^*$.

A well known example of problem (1.1) is the nonlinear Schrödinger equation (NLS)

$$\partial_t u = \mathrm{i}\Delta u + \mathrm{i}|u|^2 u, \tag{1.3}$$

where the partial flows associated with each term of the equation are given by

$$\phi_0(t,u) = \exp(\mathrm{i}t\Delta)u,$$

$$\phi_1(t,u) = \exp(\mathrm{i}t|u|^2)u,$$

which represent the evolution of a free particle and self-phase modulation, respectively. This is not exactly the problem we are interested in solving since A_0 generates a strongly continuous group of operators, that is, we are in the presence of a reversible system. In Ruth (1983), Neri (1987) and Yoshida (1990), the authors present numerical integrators for Hamiltonian systems of order $q = 3, 4, 2n$, respectively, which are known as symplectic integrators. The general form of these methods is

$$\Phi_{\mathrm{Sym}}(h) = \phi_1(b_m h) \circ \phi_0(a_m h) \circ \cdots \circ \phi_1(b_1 h) \circ \phi_0(a_1 h), \tag{1.4}$$

with $a_1 + \cdots + a_m = b_1 + \cdots + b_m = 1$. In the pioneering work Ruth (1983), a symplectic operator Φ_{Sym} of order 3 is presented, taking $a_1 = 7/24$, $a_2 = 3/4$, $a_3 = -1/24$ and $b_1 = 2/3$, $b_2 = -2/3$, $b_3 = 1$.

In Neri (1987), a symplectic operator of order 4 is considered, where

$$a_1 = a_4 = \frac{1}{2(2-2^{1/3})}, \quad a_2 = a_3 = -\frac{2^{1/3}-1}{2(2-2^{1/3})},$$
$$b_1 = b_3 = \frac{1}{2-2^{1/3}}, \quad b_2 = -\frac{2^{1/3}}{2-2^{1/3}}, \quad b_4 = 0.$$

In Yoshida (1990), a systematic way to obtain integrators of arbitrary even order, based on the Baker–Campbell–Hausdorff formula, is presented. These integrators can be set inductively,

$$\Phi_{\mathrm{Sym},2n+2}(h) = \Phi_{\mathrm{Sym},2n}(z_1 h) \circ \Phi_{\mathrm{Sym},2n}(z_0 h) \circ \Phi_{\mathrm{Sym},2n}(z_1 h),$$

with $z_0 + 2z_1 = 1$ and $z_0^{2n+1} + z_1^{2n+1} = 0$. The total number of steps of the method of order $q = 2n$ is $S_T = 3^n$. Nevertheless, for order $q = 6, 8$ there can be shown to be symplectic integrators with 8 and 16 steps, respectively.

In recent years, many authors have started the rigorous study of the convergence of symplectic methods applied to Hamiltonian systems of infinite dimension. In Besse *et al.* (2002) the NLS problem given by (1.3) in dimension 2 is considered and the convergence of the Lie–Trotter and Strang methods in $L^2(\mathbb{R}^2)$ with order 1 and 2, respectively, is proved (see also Descombes & Thalhammer, 2010, 2013). In Lubich (2008) and Gauckler (2011) similar results are proved for the Gross–Pitaevskii equation given by

$$\mathrm{i}\partial_t u = -\Delta u + |x|^2 u + |u|^2 u.$$

In both cases, the solutions are required to be differentiable with respect to time, and therefore initial data in $D(A_0^k)$ is considered, where A_0 is the corresponding differential operator.

The symplectic methods with order $q > 2$ require some steps to be negative (see Goldman & Kaper, 1996), inhibiting their application to irreversible problems. In Castella *et al.* (2009), the authors develop splitting methods for irreversible problems, that use complex time steps having positive real part: going to the complex plane allows the accuracy to be considerably increased, while keeping small time steps. The total number of steps using the so-called triple jump method of order $q = 2n$ is $S_T = 3^{n-1}$ for order not greater than 8 and for the quadruple jump method is $S_T = 4 \times 3^{n-2}$ for order not greater than 12. Finally, we recall that the rigorous approach given in this article is based upon the results for linear operators given in Hansen & Ostermann (2009) while the nonlinear problem is only formally discussed.

Since our interest is focused on irreversible pseudo-differential problems, the paradigmatic example we have in mind is the regularized cubic Schrödinger equation

$$\partial_t u = \mathrm{i}\Delta u - (-\Delta)^\beta u + \mathrm{i}|u|^2 u, \tag{1.5}$$

where $0 < \beta < 1$. It is natural to split the problem into the linear equation $\partial_t u = \mathrm{i}\Delta u - (-\Delta)^\beta u$ and the ordinary differential equation (ODE) system given by $\dot{u} = \mathrm{i}|u|^2 u$, where the linear problem is ill posed for negative times. Note that the same procedure can be applied to nonlocal nonlinearities like convolution potentials as in Example 5.3 (see also Borgna *et al.*, 2015, example 4.1). Since $\mathrm{i}\Delta - (-\Delta)^\beta$ is a pseudo-differential operator, it cannot be discretized in space in order to use a method of lines, as Runge–Kutta schemes. Observe that the strongly continuous semigroup generated by the linear part of equation (1.5) cannot be extended to an open sector $\{z \in \mathbb{C} : |\arg(z)| < \theta\}$ since its spectrum is $\{-\mathrm{i}\lambda - \lambda^\beta : \lambda \geqslant 0\} \not\subseteq \{\lambda \in \mathbb{C} : \arg|\lambda - \omega| \geqslant \pi/2 + \theta\}$ for any $\omega \in \mathbb{R}$, contrary to the Hille–Yosida–Phillips theorem (see Reed & Simon, 1975, theorem X.47b). Therefore, the splitting methods with complex times described in Castella et al. (2009) cannot be used. The case $\beta = 1$ corresponds to the complex Ginzburg–Landau equation (see Aranson & Kramer, 2002 and references there)

$$\partial_t u = a\Delta u + b|u|^2 u, \tag{1.6}$$

where $a, b \in \mathbb{C}$ with $\mathrm{Re}(a) > 0$. The spectrum of the operator $a\Delta$ is $\{-a\lambda : \lambda \geqslant 0\}$ and generates a strongly continuous semigroup on the open sector $\{z \in \mathbb{C} : |\arg(z)| < \pi/2 - |\arg(a)|\}$. In Castella et al. (2009), it is shown that the arguments of the complex steps grow with the order of the method, exceeding the value $\pi/2 - |\arg(a)|$ for order high enough. Therefore, among integrators proposed in Castella et al. (2009), only the low-order methods can be used.

In this work, we present a family of splitting-type methods for arbitrary order with positive time step, that exploit the simplicity of the partial flows in nonreversible problems. Here we describe the methods proposed: given the associated flows ϕ_0, ϕ_1 of the partial problems (1.2a) and (1.2b), respectively, we define the maps $\Phi^+(h) = \phi_1(h) \circ \phi_0(h)$, $\Phi^-(h) = \phi_0(h) \circ \phi_1(h)$ and $\Phi^{\pm}_m(h) = \Phi^{\pm}(h) \circ \Phi^{\pm}_{m-1}(h)$ with $\Phi^{\pm}_1 = \Phi^{\pm}$, and consider the following methods:

$$\Phi(h) = \sum_{m=1}^{s} \gamma_m \Phi^{\pm}_m(h/m) \quad \text{(asymmetric)}, \tag{1.7a}$$

$$\Phi(h) = \sum_{m=1}^{s} \gamma_m (\Phi^{+}_m(h/m) + \Phi^{-}_m(h/m)) \quad \text{(symmetric)}. \tag{1.7b}$$

We will show below that under appropriate assumptions, the integrators given by (1.7a) and (1.7b) are convergent with order q, if $\gamma = (\gamma_1, \ldots, \gamma_s)$ satisfies the following conditions:

$$\begin{aligned} 1 &= \gamma_1 + \gamma_2 + \cdots + \gamma_s, \\ 0 &= \gamma_1 + 2^{-k}\gamma_2 + \cdots + s^{-k}\gamma_s, \quad 1 \leqslant k \leqslant q-1, \end{aligned} \tag{1.8a}$$

$$\begin{aligned} \frac{1}{2} &= \gamma_1 + \gamma_2 + \cdots + \gamma_s, \\ 0 &= \gamma_1 + 2^{-2k}\gamma_2 + \cdots + s^{-2k}\gamma_s, \quad 1 \leqslant k \leqslant n-1, \end{aligned} \tag{1.8b}$$

respectively, where $2n = q$. The first method (1.7a) is the h-extrapolation of the first-order Lie–Trotter splitting method and the second method (1.7b) is the h^2-extrapolation of the symmetrization of this method. The general extrapolation technique is described in Hairer et al. (1993) and an application of these techniques applied to classical Hamiltonian systems is shown in Chin (2010).

The possibility of computing $\Phi^{\pm}_m$ simultaneously, allows the total time of computation using multiple processors to be reduce significantly. The total number of steps for (1.7a) is given by $S_T = 2\sum_{\gamma_m \neq 0} m$ and $S_T = 4\sum_{\gamma_m \neq 0} m$ for (1.7b). Neglecting the communication time between the processors, the total time of computation, working in parallel, turns out to be proportional to $S_P = 2\max_{\gamma_m \neq 0} m$ in both cases. System (1.8) has a solution for $s \geqslant q$, and hence there exist methods of arbitrary order q with $S_P = 2q$ and $S_T = q(q+1)$. On the other hand, system (1.8b) has a solution for $s \geqslant n$, which shows that there exist integrators of arbitrary even order $q = 2n$ with $S_P = q$ and $S_T = q(q/2+1)$, using double the number of processors. As can be seen, the minimum number of steps working in parallel for the symmetric method is smaller than the corresponding one for the asymmetric method. Also, in the examples considered below, the symmetric method presents less error than the asymmetric method. These two issues justify the choice of the symmetric method over the asymmetric one. Even using one single processor, the total number of steps grows quadratically with the order, while both methods presented in Yoshida (1990) and Castella *et al.* (2009) have exponential growth.

Since the methods (1.7) are an affine combination of Lie–Trotter integrators, they do not preserve the structure for Hamiltonian systems. Nevertheless, there is numerical evidence, which is beyond the scope of this article and therefore not presented here, that shows that the performance is comparable with symplectic methods.

The paper is organized as follows: in Section 2 we give basic definitions and preliminary results. Following the ideas of Besse *et al.* (2002), Lubich (2008) and Gauckler (2011), we consider a decreasing sequence of dense subspaces where the flows are repeatedly differentiable. In Section 3 we prove consistency and stability results for the methods (1.7), from where we deduce convergence in the standard way. In Section 4 we study the full discretization of methods (1.7). In Section 5 we give several examples of the application of the methods to initial value problems for ODEs and irreversible PDEs.

2. Notation and preliminary results

From now on, we denote by ϕ the flow of equation (1.1), and by ϕ_0 and ϕ_1 the flows associated with the respective partial problems (1.2a) and (1.2b). We write as $\Phi^{\pm}$ the maps defined by $\Phi^{+}(h) = \phi_1(h) \circ \phi_0(h)$, $\Phi^{-}(h) = \phi_0(h) \circ \phi_1(h)$ and $\Phi^{\pm}_m(h) = \Phi^{\pm}(h) \circ \Phi^{\pm}_{m-1}(h)$ with $\Phi^{\pm}_1 = \Phi^{\pm}$. Finally, we will use the letter Φ for the numerical integrators given by (1.7a) and (1.7b).

In the next subsections we will give some preliminary results which will be used in Section 3. Section 2.1 provides combinatorial results necessary to prove the consistency in Section 3.1. In order to prove Theorems 3.1 and 3.2 we establish the concept of compatible flows given in Section 2.2.

2.1 *Combinatorial results*

For a multiindex $\beta = (\beta_1, \dots, \beta_r) \in \mathbb{N}^r$, we define $\beta! = \beta_1! \cdots \beta_r!$ and $I_{r,k} = \{\beta \in \mathbb{N}^r : \beta_1 + \cdots + \beta_r = k\}$ which satisfy $\mathbb{N}^r = \bigcup_{k=1}^{\infty} I_{r,k}$.

REMARK 2.1 It holds that $I_{r,k} = \emptyset$ if $r > k$, $I_{k,k} = \{(1, \dots, 1)\}$ and for $r + s \leqslant k$, $I_{r+s,k} = \bigcup_{j=s}^{k-r} I_{r,k-j} \times I_{s,j}$.

We will need the following lemmas. We will give an outline of the proof of the first lemma and skip the proof of the second one.

LEMMA 2.2 Let $q \in \mathbb{N}$; if $\gamma = (\gamma_1, \dots, \gamma_s)$ satisfies conditions (1.8), then for $1 \leqslant k \leqslant q$, it holds that

$$\sum_{m=r}^{s} \binom{m}{r} m^{-k} \gamma_m = 0, \qquad r = 1, \dots, k-1,$$

$$\sum_{m=k}^{s} \binom{m}{k} m^{-k} \gamma_m = \frac{1}{k!}.$$

Proof. We consider the falling factorial $(x)_k = x(x-1)\cdots(x-k+1)$, which is a monic polynomial of degree k such that $(x)_k = \sum_{j=0}^{k} S(k,j)x^j$. Then, for any natural number m satisfying $0 \leqslant m \leqslant k-1$, we have $(m)_k = 0$ and therefore $\sum_{j=0}^{k} S(k,j)m^j = 0$. For the second equality we use that for $1 \leqslant r \leqslant k-1$,

$$\sum_{m=r}^{s} \binom{m}{r} m^{-k} \gamma_m = \frac{1}{r!} \sum_{m=r}^{s} (m)_r m^{-k} \gamma_m = -\frac{1}{r!} \sum_{m=1}^{r-1} \sum_{j=0}^{r} S(r,j) m^j \frac{\gamma_m}{m^k} = 0,$$

where we have used the hypothesis on the second equality. Analogously for the first equality we have

$$k!\sum_{m=k}^{s}\binom{m}{k}m^{-k}\gamma_m=\sum_{m=k}^{s}(m)_k m^{-k}\gamma_m=1-\sum_{m=1}^{k-1}\left(\frac{\sum_{j=0}^{k}S(k,j)m^j}{m^k}\right)\gamma_m=1,$$

where we have used the hypothesis on the second equality. □

LEMMA 2.3 Let $n\in\mathbb{N}$; if $\gamma=(\gamma_1,\ldots,\gamma_s)$ satisfies conditions (1.8b), then for $1\leqslant k\leqslant q=2n$, it holds that

$$\sum_{m=1}^{s}\left[\binom{m}{r}+(-1)^{k+r}\binom{m+r-1}{m-1}\right]m^{-k}\gamma_m=0,\quad r=1,\ldots,k-1,$$

$$\sum_{m=1}^{s}\left[\binom{m}{k}+\binom{m+k-1}{m-1}\right]m^{-k}\gamma_m=\frac{1}{k!}.$$

Proof. The proof is similar to the previous lemma. □

2.2 *Compatible flows*

We denote by $\mathcal{L}_n(\mathsf{X},\mathsf{Y})$ the Banach space consisting of the set of continuous multilinear operators from $\mathsf{X}^n=\prod_{i=1}^n\mathsf{X}$ to Y and by D^nf the n Fréchet derivative of f.

Let X,Y be Banach spaces, $I\subset\mathbb{R}^n$ an n-cube and $0\leqslant k_1,\ldots,k_n,q$. We define $C_{\mathrm{st}}^{k_1,\ldots,k_n,q}(I\times X,Y)$ to be the linear subspace of functions $C(I\times X,Y)$ verifying

(1) for any $t\in I$, $\varphi(t,\cdot)\in C^q(X,Y)$ in the sense of Fréchet;

(2) for $0\leqslant m\leqslant q$ and $u,v_1,\ldots,v_m\in X$, the map $t\mapsto D^m\varphi(t,u)(v_1,\ldots,v_m)$ belongs to $C^{k_1,\ldots,k_n}(I,Y)$;

(3) for $0\leqslant j_i\leqslant k_i$, the map given by $(t,u,v_1,\ldots,v_m)\mapsto(\partial_{t_1}^{j_1}\cdots\partial_{t_n}^{j_n}D^m\varphi)(t,u)(v_1,\ldots,v_m)$ is continuous, i.e., $\partial_{t_1}^{j_1}\cdots\partial_{t_n}^{j_n}D^m\varphi$ is strong continuous.

REMARK 2.4 Let X_0,X_1,X_2 be Banach spaces, $I_1,I_2\subset\mathbb{R}$ intervals. For $\varphi\in C_{\mathrm{st}}^{k_1,q-k_1}(I_1\times X_1,X_0)$ and $\psi\in C_{\mathrm{st}}^{k_2,q-k_2}(I_2\times X_2,X_1)$ with $0\leqslant k_1+k_2\leqslant q$, it is easy to see that $\theta\in C(I_1\times I_2\times X_2,X_0)$ defined by $\theta(t_1,t_2,u)=\varphi(t_1,\psi(t_2,u))$ satisfies $\theta\in C_{\mathrm{st}}^{k_1,k_2,q-k_1-k_2}(I_1\times I_2\times X_2,X_0)$.

Let $\{\mathsf{X}_k\}_{0\leqslant k\leqslant q}$ be a sequence of nested Banach spaces, i.e., $\mathsf{X}_{k+1}\hookrightarrow\mathsf{X}_k$ and $I\subset\mathbb{R}$ an interval. Given $\varphi\in C(I\times\mathsf{X}_0,\mathsf{X}_0)$, we say that φ is compatible with $\{\mathsf{X}_k\}_{0\leqslant k\leqslant q}$ if and only if for $0\leqslant k\leqslant j\leqslant q$, it holds that $\varphi\in C_{\mathrm{st}}^{k,q-k}(I\times\mathsf{X}_j,\mathsf{X}_{j-k})$.

EXAMPLE 2.5 Let $A_0:D(A_0)\to\mathsf{X}$ be an infinitesimal generator of a strongly continuous semigroup ϕ_0, $\mathsf{X}_0=\mathsf{X}$ and $\mathsf{X}_k=D(A_0^k)$ with the graph norm $\|u\|_{\mathsf{X}_k}=\sum_{j=0}^{k}\|A_0^ju\|_{\mathsf{X}}$. We can see that ϕ_0 is compatible with the sequence of nested Banach spaces $\{\mathsf{X}_k\}_{0\leqslant k\leqslant q}$ for any $q\geqslant 0$, since $\partial_t^k\phi_0(t,u)=\phi_0(t)A_0^ku$, $\partial_t^kD\phi_0(t,u)v=\phi_0(t)A_0^kv$, $\partial_t^kD^m\phi_0(t,u)(v_1,\ldots,v_m)=0$, for $m\geqslant 2$. Note that the map $t\mapsto\phi_0(t)A_0^k$ from $[0,\infty)$ on $\mathcal{L}_1(\mathsf{X}_j,\mathsf{X}_{j-k})$ is not continuous in the uniform topology for A_0 unbounded, but it is strongly continuous.

PROPOSITION 2.6 If φ and ψ are compatible with $\{\mathsf{X}_k\}_{0\leqslant k\leqslant q}$, then $(\varphi\circ\psi)(t,u)=\varphi(t,\psi(t,u))$ is also compatible with $\{\mathsf{X}_k\}_{0\leqslant k\leqslant q}$.

Proof. Let $\theta(t_1,t_2,u)=\varphi(t_1,\psi(t_2,u))$; from Remark 2.4 we can see that $\theta\in C^{i,k-i,q-k}_{\rm st}(I\times I\times X_l,X_{l-k})$ for $0\leqslant i\leqslant k\leqslant l\leqslant q$. Using that

$$\partial_t^j D^m(\varphi\circ\psi)(t,u)(v_1,\ldots,v_m)=\sum_{i=0}^{j}\binom{j}{i}\,\partial_{t_1}^i\partial_{t_2}^{j-i}D^m\theta(t_1,t_2,u)(v_1,\ldots,v_m)\Big|_{t_1=t_2=t}$$

for $0\leqslant j\leqslant k$ and $0\leqslant m\leqslant q-k$, the result follows. □

Now, we discuss the compatibility of the nonlinear flow ϕ associated with equation (1.1). Note that for $u_0\in\mathsf{X}_q$ the time of existence T^* of the solution in X_k could depend on u_0 and k. Taking $T<T_q^*(u_0)=\min_{0\leqslant k\leqslant q}\{T_k^*(u_0)\}$, the nonlinear flow ϕ is defined in $[0,T]$ for any initial data close to u_0. Since consistency is a local problem, this is sufficient for our purposes. Moreover, in many applications T does not depend on k and it could be taken the same for any $u_0\in X_q$ with $\|u_0\|_{\mathsf{X}_0}\leqslant R$.

LEMMA 2.7 Let X be a Banach space, ϕ_0 a continuous semigroup on X and $A_1\in C^q(\mathsf{X},\mathsf{X})$ such that for $0\leqslant m\leqslant q$, $D^mA_1:\mathsf{X}\to\mathcal{L}_m(\mathsf{X},\mathsf{X})$ is a locally Lipschitz continuous map. If $\phi\in C([0,T]\times\mathsf{X},\mathsf{X})$ is the flow associated with (1.1), i.e.,

$$\phi(t,u_0)=\phi_0(t)u_0+\int_0^t\phi_0(t-t')A_1(\phi(t',u_0))\,{\rm d}t',$$

then $\phi\in C^{0,q}_{\rm st}([0,T]\times\mathsf{X},\mathsf{X})$.

Proof. The proof is by induction on q; the statement is true for $q=0$. Let $\overline{\mathsf{X}}=\mathsf{X}\times\mathsf{X}$, $\overline{\phi}_0$ the semigroup given by $\overline{\phi}_0(t)(u,v)=(\phi_0(t)u,\phi_0(t)v)$, and $\overline{A}_1$ the map defined by $\overline{A}_1(u,v)=(A_1(u),DA_1(u)v)$. Since $\overline{A}_1$ verifies the hypothesis on $\overline{\mathsf{X}}$ for $0\leqslant m\leqslant q-1$, applying the inductive hypothesis to $\overline{\phi}$, the flow associated with the integral equation

$$(u(t),v(t))=\overline{\phi}_0(t)(u_0,v_0)+\int_0^t\overline{\phi}_0(t-t')\overline{A}_1(u(t'),v(t'))\,{\rm d}t',$$

we have $\overline{\phi}\in C^{0,q-1}_{\rm st}([0,T]\times\overline{\mathsf{X}},\overline{\mathsf{X}})$. We can see that $(u(t),v(t))=\overline{\phi}(t,(u_0,v_0))=(\phi(t,u_0),D\phi(t,u_0)v_0)$, and a straightforward computation shows that

$$D^m\overline{\phi}(t,(u_0,v_0))((v_1,0),\ldots,(v_m,0))=(D^m\phi(t,u)(v_1,\ldots,v_m),D^{m+1}\phi(t,u)(v_0,v_1,\ldots,v_m))$$

for $0\leqslant m\leqslant q-1$, and the result follows. □

PROPOSITION 2.8 Let $A_0:D(A_0)\to\mathsf{X}$ be an infinitesimal generator of a strongly continuous semigroup ϕ_0 and $\mathsf{X}_k=D(A_0^k)$ with the graph norm $\|u\|_{\mathsf{X}_k}=\sum_{j=0}^k\|A_0^ju\|_{\mathsf{X}}$ and let $A_1\in C^q(\mathsf{X}_k,\mathsf{X}_k)$ for $0\leqslant k\leqslant q$ such that $D^mA_1:\mathsf{X}_k\to\mathcal{L}_m(\mathsf{X}_k,\mathsf{X}_k)$ is a locally Lipschitz continuous map for $0\leqslant m\leqslant q$. If ϕ,ϕ_0,ϕ_1 are the flows associated with (1.1), (1.2a), (1.2b), respectively, then ϕ,ϕ_0,ϕ_1 are compatible with $\{\mathsf{X}_k\}_{0\leqslant k\leqslant q}$.

Proof. For the compatibility of ϕ_0 see Example 2.5. From Lemma 2.7, it holds that $\phi\in C^{0,q}_{\rm st}(I\times\mathsf{X}_k,\mathsf{X}_k)$. Since $A_0\in C^\infty(\mathsf{X}_k,\mathsf{X}_{k-1})$ and $A_1\in C^q(\mathsf{X}_k,\mathsf{X}_k)$, we can prove that $A_0\phi\in C^{0,q}_{\rm st}(I\times\mathsf{X}_k,\mathsf{X}_{k-1})$ and $A_1(\phi)\in C^{0,q}_{\rm st}(I\times\mathsf{X}_k,\mathsf{X}_k)$. Therefore $\partial_t\phi=A_0\phi+A_1(\phi)\in C^{0,q}_{\rm st}(I\times\mathsf{X}_k,\mathsf{X}_{k-1})$ and then $\phi\in C^{1,q-1}_{\rm st}(I\times\mathsf{X}_k,\mathsf{X}_{k-1})$. A recursive argument shows that $\phi\in C^{l,q-l}_{\rm st}(I\times\mathsf{X}_k,\mathsf{X}_{k-l})$. Taking $A_0=0$, we obtain that ϕ_1, the partial flow associated with (1.2b), is also compatible. □

REMARK 2.9 Let $\mu \in \mathcal{L}_m(\mathsf{X}_k, \mathsf{X}_k)$ and define the map $A_1(u) = \mu(u, \ldots, u)$. Since $A_1(u) = \mu_s(u, \ldots, u)$, where

$$\mu_s(v_1, \ldots, v_m) = \frac{1}{m!} \sum_{\sigma \in S_m} \mu(v_{\sigma_1}, \ldots, v_{\sigma_m}),$$

without loss of generality we can assume that μ is symmetric. Then it holds that $A_1 \in C^\infty(\mathsf{X}_k, \mathsf{X}_k)$ and $D^n A_1$ is a locally Lipschitz continuous map from X_k on $\mathcal{L}_n(\mathsf{X}_k, \mathsf{X}_k)$. To see this, observe that

$$D^n A_1(u)(v_1, \ldots, v_n) = \frac{m!}{(m-n)!} \mu_s(u, \ldots, u, v_1, \ldots, v_n), \quad 1 \leqslant n \leqslant m,$$

and $D^n A_1 = 0$ if $n > m$. Let $u, \tilde{u} \in \mathsf{X}_k$ with $\|u\|_{\mathsf{X}_k}, \|\tilde{u}\|_{\mathsf{X}_k} \leqslant R$ and $n \leqslant m$, it follows that

$$\|D^n A_1(u) - D^n A_1(\tilde{u})\|_{\mathcal{L}_n(\mathsf{X}_k, \mathsf{X}_k)} \leqslant \frac{m!(m-n)}{(m-n)!} \|\mu\|_{\mathcal{L}_m(\mathsf{X}_k, \mathsf{X}_k)} R^{m-n-1} \|u - \tilde{u}\|_{\mathsf{X}_k}.$$

As an example, consider $\mathsf{X}_0 = H^s(\mathbb{R}^d)$ with $s > d/2$ as a real vectorial space and $A_0 = \mathrm{i}\Delta - (-\Delta)^\beta$ with $0 < \beta < 1$, then $\mathsf{X}_k = H^{s+2k}(\mathbb{R}^d)$. Since X_k is a real Banach algebra with the pointwise product, the multilinear map $\mu \in \mathcal{L}_3(\mathsf{X}_k, \mathsf{X}_k)$ given by $\mu(u, v, w) = \lambda u v w^*$ defines $A_1(u) = \mu(u, u, u) = \lambda |u|^2 u$, which satisfies the hypothesis of Proposition 2.8. On the other hand, let $g \in L^1(\mathbb{R}^d)$; the map given by $Gu = g * u$ is a bounded linear operator in X_k. Consider $\mu \in \mathcal{L}_3(\mathsf{X}_k, \mathsf{X}_k)$ defined by

$$\mu(u, v, w) = (g * (uv^*))w;$$

we obtain $A_1(u) = (g * |u|^2)u$ which satisfies the hypothesis of Proposition 2.8.

In Section 3 we will need to compute the successive derivatives of the method proposed (see Theorem 3.1 and Theorem 3.2). Observe that these derivatives are linear combinations of the derivatives of the composition of the partial flows, that is

$$\partial_t^k \Phi(0, u) = \sum_{m=1}^{s} \gamma_m \partial_t^k \Phi_m^{\pm}(0, u) = \sum_{m=1}^{s} \gamma_m \partial_t^k (\cdots \circ \phi_0(t/m) \circ \phi_1(t/m) \circ \phi_0(t/m) \circ \cdots)(0, u).$$

In order to do this, we consider mixed Lie derivatives with respect to the flows ϕ_0 and ϕ_1. For $0 \leqslant k \leqslant q$, define the linear spaces

$$\mathcal{D}_{q,k} = \{f \in C^{q-k}(\mathsf{X}_k, \mathsf{X}_0) : f|_{\mathsf{X}_j} \in C^{q-k}(\mathsf{X}_j, \mathsf{X}_{j-k}),\ k \leqslant j \leqslant q\}.$$

We can see that if $f \in \mathcal{D}_{q,k}$ and $g \in \mathcal{D}_{q,l}$ with $k + l \leqslant q$, then $f \circ g \in \mathcal{D}_{q,k+l}$. If φ is compatible with $\{\mathsf{X}_k\}_{0 \leqslant k \leqslant q}$, then $\partial_t^k \varphi(t, \cdot) \in \mathcal{D}_{q,k}$ and, for $f \in \mathcal{D}_{q,l}$, it holds that $f \circ \varphi \in C_{\mathrm{st}}^{k,q-k-l}([0,T] \times \mathsf{X}_j, \mathsf{X}_{j-k-l})$, with $k + l \leqslant j \leqslant q$. We define the linear operator $L_k[\varphi] : \mathcal{D}_{q,l} \to \mathcal{D}_{q,k+l}$ as

$$(L_k[\varphi]f)(u) = \partial_t^k f(\varphi(t, u))|_{t=0}.$$

Note that if $f \in \mathcal{L}_1(\mathsf{X}_j, \mathsf{X}_{j-l})$ for $l \leqslant j \leqslant q$, then $f \in \mathcal{D}_{q,l}$ and $(L_k[\varphi]f)(u) = f(\partial_t^k \varphi(0,u))$. In particular, for $f = \mathrm{id}$, we obtain that $L_k[\varphi]\,\mathrm{id} = \partial_t^k \varphi(0,\cdot)$ and for a linear combination $\varphi = \sum_{m=1}^{s} \gamma_m \varphi_m$, we have $L_k[\varphi]\,\mathrm{id} = \sum_{m=1}^{s} \gamma_m L_k[\varphi_m]\,\mathrm{id}$.

LEMMA 2.10 If φ, ψ are compatible with $\{\mathsf{X}_k\}_{0 \leqslant k \leqslant q}$, then

$$L_k[\varphi \circ \psi] = \sum_{j=0}^{k} \binom{k}{j} L_{k-j}[\psi] L_j[\varphi].$$

Proof. From Proposition 2.6, $\varphi \circ \psi$ is compatible and from Remark 2.4, $\theta(t_1, t_2, u) = \varphi(t_1, \psi(t_2, u))$ verifies $\theta \in C_{\mathrm{st}}^{j,k-j,q-k}([0,T] \times [0,T] \times \mathsf{X}_{l+k}, \mathsf{X}_l)$. Given $f \in \mathcal{D}_{q,l}$, for any $u \in \mathsf{X}_{l+k}$,

$$\begin{aligned}(L_k[\varphi \circ \psi]f)(u) &= \sum_{j=0}^{k} \binom{k}{j} \partial_{t_1}^{j} \partial_{t_2}^{k-j} f(\theta(t_1, t_2, u))|_{(t_1,t_2)=(0,0)} \\ &= \sum_{j=0}^{k} \binom{k}{j} \partial_{t_2}^{k-j} (L_j[\varphi]f)(\psi(t_2, u))|_{t_2=0} = \sum_{j=0}^{k} \binom{k}{j} (L_{k-j}[\psi] L_j[\varphi] f)(u)\end{aligned}$$

is satisfied. Since f and u are arbitrary, the proposition follows. □

PROPOSITION 2.11 Let φ be a compatible map with $\{\mathsf{X}_k\}_{0 \leqslant k \leqslant q}$. Let $\varphi_1 = \varphi$ and $\varphi_{m+1} = \varphi \circ \varphi_m$; then

$$L_k[\varphi_m] = \sum_{r=1}^{k} \binom{m}{r} \sum_{\beta \in I_{r,k}} \frac{k!}{\beta!} L_{\beta_1}[\varphi] \cdots L_{\beta_r}[\varphi].$$

Proof. Using Lemma 2.10, we get

$$L_k[\varphi_{m+1}] = L_k[\varphi_m] + L_k[\varphi] + \sum_{j=1}^{k-1} \binom{k}{j} L_{k-j}[\varphi_m] L_j[\varphi];$$

applying induction and using Remark 2.1, we obtain the result. □

LEMMA 2.12 If φ is a flow, compatible with $\{\mathsf{X}_k\}_{0 \leqslant k \leqslant q}$, then $L_k[\varphi] = (L_1[\varphi])^k$.

Proof. The proof is by induction: suppose the result holds for $1 \leqslant j \leqslant k-1$; using the lemma above we obtain

$$\begin{aligned}L_k[\varphi \circ \varphi] &= 2L_k[\varphi] + \sum_{j=1}^{k-1} \binom{k}{j} L_{k-j}[\varphi] L_j[\varphi] \\ &= 2L_k[\varphi] + \sum_{j=1}^{k-1} \binom{k}{j} (L_1[\varphi])^{k-j} (L_1[\varphi])^j = 2L_k[\varphi] + (2^k - 2)(L_1[\varphi])^k.\end{aligned}$$

Since φ is a flow, $\varphi(t) \circ \varphi(t) = \varphi(2t)$ and therefore $L_k[\varphi \circ \varphi] = 2^k L_k[\varphi]$, which implies the result for $j = k$. □

3. Convergence

3.1 *Consistency*

The next two theorems ensures consistency results for the schemes given by (1.7a) and (1.7b), when the coefficients of the affine combination that defines the methods Φ satisfy the algebraic conditions (1.8a) and (1.8b), respectively.

In the present section, we take on the assumptions on the operators A_0 and A_1 made precise in Proposition 2.8 which implies that the flows ϕ_0, ϕ_1 and ϕ are compatible with $\{\mathsf{X}_k\}_{0\leqslant k\leqslant q}$. We have the following consistency results.

THEOREM 3.1 (Asymmetric case) For any $q \in \mathbb{N}$, $\gamma = (\gamma_1, \dots, \gamma_s)$ satisfying (1.8a) and $u \in \mathsf{X}_q$, the method Φ given by (1.7a) satisfies

$$\partial_t^k \Phi(0, u) = \partial_t^k \phi(0, u), \quad \text{for } k = 0, \dots, q.$$

THEOREM 3.2 (Symmetric case) For any $n \in \mathbb{N}$, $\gamma = (\gamma_1, \dots, \gamma_s)$ satisfying (1.8b) and $u \in \mathsf{X}_q$ with $q = 2n$, the method Φ given by (1.7b) satisfies

$$\partial_t^k \Phi(0, u) = \partial_t^k \phi(0, u), \quad \text{for } k = 0, \dots, q.$$

3.1.1 *Asymmetric case* We prove the consistency of method (1.7a) using Lemmas 2.10 and 2.2.

PROPOSITION 3.3 For any $q \in \mathbb{N}$ and $\gamma = (\gamma_1, \dots, \gamma_s)$ satisfying (1.8a), the method Φ given by (1.7a) satisfies $L_k[\Phi]\,\mathrm{id} = (L_1[\Phi^\pm])^k\,\mathrm{id}$, for $k = 1, \dots, q$.

Proof. Since $L_k[\Phi]\,\mathrm{id} = \sum_{m=1}^s m^{-k}\gamma_m L_k[\Phi_m^\pm]\,\mathrm{id}$, using Proposition 2.11 we can see that

$$L_k[\Phi]\,\mathrm{id} = \sum_{r=1}^k \left(\sum_{m=1}^s \binom{m}{r} m^{-k}\gamma_m \right) \sum_{\beta\in I_{r,k}} \frac{k!}{\beta!} L_{\beta_1}[\Phi^\pm]\cdots L_{\beta_r}[\Phi^\pm]\,\mathrm{id};$$

from Lemma 2.2, we get $L_k[\Phi]\,\mathrm{id} = \sum_{\beta\in I_{k,k}} (1/\beta!) L_{\beta_1}[\Phi^\pm]\cdots L_{\beta_k}[\Phi^\pm]\,\mathrm{id} = (L_1[\Phi^\pm])^k\,\mathrm{id}$. □

Proof of Theorem 3.1. Since $\Phi^+ = \phi_1 \circ \phi_0$, from Lemma 2.10, $L_1[\Phi^+] = L_1[\phi_0] + L_1[\phi_1] = L_1[\phi]$. In the same way it follows that $L_1[\Phi^-] = L_1[\phi]$. Using Proposition 3.3 we obtain

$$\partial_t^k \Phi(0, u) = ((L_1[\Phi^\pm])^k\,\mathrm{id})(u) = ((L_1[\phi])^k\,\mathrm{id})(u)$$

and the theorem follows from Lemma 2.12. □

3.1.2 *Symmetric case* If ϕ_0, ϕ_1 were reversible flows, then it would hold that $\Phi^-(t) \circ \Phi^+(-t) = I$ and using Lemma 2.10 we would obtain that M_k, defined below by (3.1), is identically zero. We get the same result for irreversible flows.

LEMMA 3.4 Let $M_k : \mathcal{D}_{q,0} \to \mathcal{D}_{q,k}$ be the operator given by

$$M_k = \sum_{j=0}^{k} (-1)^j \binom{k}{j} L_j[\Phi^+] L_{k-j}[\Phi^-]; \tag{3.1}$$

then $M_k = 0$.

Proof. Using Lemma 2.10 for $\Phi^\pm$ and Lemma 2.12,

$$M_k = \sum_{j=0}^{k} \sum_{i=0}^{j} \sum_{l=0}^{k-j} (-1)^j \binom{k}{j} \binom{j}{i} \binom{k-j}{l} L_1[\phi_0]^{j-i} L_1[\phi_1]^{k+i-j-l} L_1[\phi_0]^l.$$

Interchanging the order of summation, considering $n = j - i$ and using the identity

$$\binom{k}{n+i} \binom{n+i}{i} \binom{k-n-i}{l} = \binom{k-n-l}{i} \frac{k!}{n!\,l!\,(k-n-l)!},$$

we can write M_k as

$$M_k = \sum_{n=0}^{k} (-1)^n \sum_{l=0}^{k-n-1} \left(\sum_{i=0}^{k-n-l} (-1)^i \binom{k-n-l}{i} \right) \frac{k!}{n!\,l!\,(k-n-l)!}$$
$$\times L_1[\phi_0]^n L_1[\phi_1]^{k-n-l} L_1[\phi_0]^l + \sum_{n=0}^{k} (-1)^n \binom{k}{k-n} L_1[\phi_0]^k.$$

Since $\sum_{i=0}^{k-n-l} (-1)^i \binom{k-n-l}{i} = 0$, we have the result. □

PROPOSITION 3.5 For $m \geqslant 1$ it holds that

$$L_k[\Phi_m^-] = (-1)^k \sum_{r=1}^{k} C_{m,r} \sum_{\beta \in I_{r,k}} \frac{k!}{\beta!} L_{\beta_1}[\Phi^+] \cdots L_{\beta_r}[\Phi^+],$$

where $C_{m,r} = (-1)^r \binom{m+r-1}{r}$.

Proof. We proceed by induction in m and in k: for $m = 1$, eliminating $L_k[\Phi^-]$ from (3.1) we have

$$L_k[\Phi^-] = -\sum_{j=1}^{k} (-1)^j \binom{k}{j} L_j[\Phi^+] L_{k-j}[\Phi^-];$$

by inductive hypothesis for $k - j < k$ and using Remark 2.1 we obtain the case $m = 1$. Applying Lemma 2.10 to $\Phi_{m+1}^- = \Phi^- \circ \Phi_m^-$ and using $C_{m+1,r} = \sum_{s=0}^{r} C_{m,s} C_{1,r-s}$, we have the result. □

PROPOSITION 3.6 If γ satisfies conditions (1.8b), then the method Φ defined by (1.7b) satisfies for $k = 0, \ldots, 2n$, $L_k[\Phi]\,\mathrm{id} = (L_1[\Phi^+])^k\,\mathrm{id}$.

Proof. Applying Proposition 2.11 to Φ^+, using Proposition 3.5 and Lemma 2.3 the result may be concluded. □

Proof of Theorem 3.2. From Proposition 3.6 we have

$$\partial_t^k \Phi(0,u) = ((L_1[\Phi^+])^k \,\mathrm{id})(u) = ((L_1[\phi])^k \,\mathrm{id})(u),$$

and the theorem follows from Lemma 2.12. □

3.2 *Stability*

If ϕ_0 is the semigroup associated with A_0, we can assume that ϕ_0 is quasi-contractive with respect to some appropriate equivalent norm on X_0, i.e., $\|\phi_0(t)\|_{\mathcal{L}_1(X_0,X_0)} \leqslant e^{\kappa t}$. Since $A_0^k\phi_0(t) = \phi_0(t)A_0^k$, it holds that $\|\phi_0(t)u\|_{\mathsf{X}_k} \leqslant e^{\kappa t}\|u\|_{\mathsf{X}_k}$.

To prove stability, we proceed as in Borgna *et al.* (2015). We define the 1-periodic function α given by

$$\alpha(t) = \begin{cases} 1 & \text{if } 0 \leqslant t < 1/2, \\ -1 & \text{if } 1/2 \leqslant t < 1, \end{cases}$$

$\alpha_h(t) = \alpha(t/h)$, $\alpha_h^{\pm}(t) = 1 \pm \alpha_h(t)$ and $\eta_h^{\pm}(t_1,t_0) = \int_{t_0}^{t_1} \alpha_h^{\pm}(t)\,\mathrm{d}t$. Observe that α_h is h-periodic with mean zero. We can see that, for any $h > 0$ and $0 \leqslant t_0 \leqslant t_1$ and $n \in \mathbb{N}$, it is verified that

- $0 \leqslant \eta_h^{\pm}(t_1,t_0) \leqslant 2(t_1 - t_0)$;
- $|(t_1 - t_0) - \eta_h^{\pm}(t_1,t_0)| \leqslant h/2$;
- $\eta_h^{\pm}(t_1 + nh, t_0 + nh) = \eta_h^{\pm}(t_1,t_0)$;
- $\eta_{h/m}^{\pm}(t_0 + nh, t_0) = nh$.

If $\phi_{0,h}^{\pm}(t,t') = \phi_0(\eta_h^{\pm}(t,t'))$, it holds that $\|\phi_{0,h}^{\pm}(t,t')\|_{\mathcal{L}_1(\mathsf{X}_k,\mathsf{X}_k)} \leqslant e^{2\kappa(t-t')}$ and the solution of the integral equation

$$u_{h,m}^{\pm}(t) = \phi_{0,h/m}^{\pm}(t,0)u_0 + \int_0^t \alpha_{h/m}^{\mp}\left(t'\right)\phi_{0,h/m}^{\pm}\left(t,t'\right)A_1\left(u_{h,m}^{\pm}(t')\right)\,\mathrm{d}t' \tag{3.2}$$

verifies $u_{h,m}^{\pm}(h) = \Phi_m^{\pm}(h/m,u_0)$ (see Borgna *et al.*, 2015).

PROPOSITION 3.7 Given $R > 0$ and $0 \leqslant k \leqslant q$, there exists $h^* > 0$ such that for any $h \leqslant h^*$ and $u_0 \in \mathsf{X}_k$ with $\|u_0\|_{\mathsf{X}_k} \leqslant R$, the solution $u_{h,m}^{\pm}$ of (3.2) is defined on $[0,h]$ for any $m \in \mathbb{N}$. Moreover, if $u_0, \tilde{u}_0 \in \mathsf{X}_k$ and $\|u_0\|_{\mathsf{X}_k}, \|\tilde{u}_0\|_{\mathsf{X}_k} \leqslant R$, then

$$\max_{0\leqslant t\leqslant h} \|u_{h,m}^{\pm}(t) - \tilde{u}_{h,m}^{\pm}(t)\|_{\mathsf{X}_k} \leqslant C\|u_0 - \tilde{u}_0\|_{\mathsf{X}_k},$$

where $u_{h,m}^{\pm}, \tilde{u}_{h,m}^{\pm}$ are the solutions of (3.2) with initial data $u_0, \tilde{u}_0 \in \mathsf{X}_k$, respectively, $C = e^{(2\kappa + 2\Lambda\, e^{2\kappa h^*})h^*}$ and Λ is a Lipschitz constant of A_1 in $B_{2R}(0) \subset \mathsf{X}_k$.

Proof. The existence of the solutions can be proved by applying a fixed point argument, that is, for any $u_0 \in B_R(0)$ the solution $u_{h,m}^{\pm}$ of (3.2) is defined on $[0,h]$ and $\|u_{h,m}^{\pm}(t)\|_{\mathsf{X}_k} \leqslant 2R$ for $0 \leqslant t \leqslant h$.

Let $u_0, \tilde{u}_0 \in B_R(0)$; we have

$$u^{\pm}_{h,m}(t) - \tilde{u}^{\pm}_{h,m}(t) = \phi^{\pm}_{0,h/m}(t,0)\,(u_0 - \tilde{u}_0) + \int_0^t \alpha^{\mp}_{h/m}(t')\,\phi^{\pm}_{0,h/m}(t,t')\left(A_1\left(u^{\pm}_{h,m}(t')\right) - A_1\left(\tilde{u}^{\pm}_{h,m}(t')\right)\right)\mathrm{d}t'.$$

Then, we can estimate

$$\left\|u^{\pm}_{h,m}(t) - \tilde{u}^{\pm}_{h,m}(t)\right\|_{\mathsf{X}_k} \leqslant e^{2\kappa h}\,\|u_0 - \tilde{u}_0\|_{\mathsf{X}_k} + 2\Lambda\, e^{2\kappa h}\int_0^t \left\|u^{\pm}_{h,m}(t') - \tilde{u}^{\pm}_{h,m}(t')\right\|_{\mathsf{X}_k}\,\mathrm{d}t'.$$

From Gronwall's lemma, we obtain $\|u^{\pm}_{h,m}(t) - \tilde{u}^{\pm}_{h,m}(t)\|_{\mathsf{X}_k} \leqslant C\|u_0 - \tilde{u}_0\|_{\mathsf{X}_k}$. □

The next theorem is the stability result. Observe that the stability constant depends strongly on the Lipschitz constant of A_1.

THEOREM 3.8 Given $R > 0$ and $0 \leqslant k \leqslant q$, there exists $h^* > 0$ such that for any $h \leqslant h^*$, the method $\Phi(h,\cdot)$ given by (1.7) (both cases (1.7a) and (1.7b)) is defined on $B_R(0)$ and is Lipschitz continuous with constant e^{Kh}, where $K = K(\kappa, \Lambda, \sum_{m=1}^{s}|\gamma_m|, h^*)$.

Proof. We give the proof only for the asymmetric case (1.7a); the symmetric case is completely similar. From the proposition above, $\Phi^{\pm}_m$ is defined on $[0,h]$ and then Φ is also defined. We can write

$$\begin{aligned}\Phi(h,u_0) - \Phi(h,\tilde{u}_0) &= \sum_{m=1}^{s}\gamma_m\left(\Phi^{\pm}_m(h/m,u_0) - \Phi^{\pm}_m(h/m,\tilde{u}_0)\right) = \sum_{m=1}^{s}\gamma_m\left(u^{\pm}_{h,m}(h) - \tilde{u}^{\pm}_{h,m}(h)\right)\\ &= \sum_{m=1}^{s}\gamma_m\Bigg(\phi^{\pm}_{0,h/m}(h,0)\,(u_0 - \tilde{u}_0)\\ &\quad + \int_0^h \alpha^{\mp}_{h/m}(t')\,\phi^{\pm}_{0,h/m}(h,t')\left(A_1\left(u^{\pm}_{h,m}(t')\right) - A_1\left(\tilde{u}^{\pm}_{h,m}(t')\right)\right)\mathrm{d}t'\Bigg).\end{aligned}$$

Since $\phi^{\pm}_{0,h/m}(h,0) = \phi_0(h)$ and $\sum_{m=1}^{s}\gamma_m = 1$ we deduce that

$$\begin{aligned}\Phi(h,u_0) - \Phi(h,\tilde{u}_0) &= \phi_0(h)\,(u_0 - \tilde{u}_0)\\ &\quad + \sum_{m=1}^{s}\gamma_m\left(\int_0^h \alpha^{\mp}_{h/m}(t')\,\phi^{\pm}_{0,h/m}(h,t')\left(A_1\left(u^{\pm}_{h,m}(t')\right) - A_1\left(\tilde{u}^{\pm}_{h,m}(t')\right)\right)\mathrm{d}t'\right).\end{aligned}$$

Therefore

$$\begin{aligned}&\|\Phi(h,u_0) - \Phi(h,\tilde{u}_0)\|_{\mathsf{X}_k}\\ &\quad\leqslant e^{\kappa h}\|u_0 - \tilde{u}_0\|_{\mathsf{X}_k} + 2e^{2\kappa h}\sum_{m=1}^{s}|\gamma_m|\int_0^h\left\|A_1\left(u^{\pm}_{h,m}(t')\right) - A_1\left(\tilde{u}^{\pm}_{h,m}(t')\right)\right\|_{\mathsf{X}_k}\mathrm{d}t'\\ &\quad\leqslant\left(e^{\kappa h} + 2e^{2\kappa h}\Lambda C\sum_{m=1}^{s}|\gamma_m|h\right)\|u_0 - \tilde{u}_0\|_{\mathsf{X}_k} \leqslant e^{Kh}\|u_0 - \tilde{u}_0\|_{\mathsf{X}_k},\end{aligned}$$

where $K = 2\kappa + 2\Lambda C\sum_{m=1}^{s}|\gamma_m|$. Then, we have the result. □

3.3 *Convergence results*

To prove the convergence result in X_0 with order q (see Theorem 3.11) we begin by proving the following convergence result in each X_k for any $0 \leqslant k \leqslant q$.

THEOREM 3.9 Let $u \in C([0,T^*), \mathsf{X}_k)$ be the solution of (1.1) $(0 \leqslant k \leqslant q)$, $T < T^*(u_0)$, $\varepsilon > 0$ and $\Phi(h,\cdot)$ the method given by (1.7) (both cases (1.7a) and (1.7b)). There exist $\delta > 0$ and $h^* > 0$ such that if $\|u_0 - U_0\|_{\mathsf{X}_k} < \delta$, $h \in (0,h^*]$, then the sequence $U_n = \Phi(h, U_{n-1})$ is defined and verifies $\|u_n - U_n\|_{\mathsf{X}_k} < \varepsilon$ for $n \leqslant [T/h]$, where $u_n = u(nh)$.

In order to do this we will first prove the following results.

LEMMA 3.10 Let $\Omega_T = \{(t_1,t_0) \in \mathbb{R}^2 : 0 \leqslant t_0 \leqslant t_1 \leqslant T\}$, $f \in C([0,T], \mathsf{X}_k)$ and

$$I_1(h,t_1,t_0) = \left(\phi_0(t_1-t_0) - \phi_{0,h}^{\pm}(t_1,t_0)\right) f(t_0).$$

Given $\delta > 0$, there exists $h^* > 0$ such that if $0 < h \leqslant h^*$, then $\sup_{(t_1,t_0)\in\Omega_T} \|I_1(h,t_1,t_0)\|_{\mathsf{X}_k} < \delta$.

Proof. Let $g \in C([0,T], \mathsf{X}_{k+1})$ such that $\|f(t) - g(t)\|_{\mathsf{X}_k} < e^{-2\kappa T}\delta/4$, for $0 \leqslant t \leqslant T$; then

$$\sup_{(t_1,t_0)\in\Omega_T} \left\|\left(\phi_0(t_1-t_0) - \phi_{0,h}^{\pm}(t_1,t_0)\right)(f(t_0) - g(t_0))\right\|_{\mathsf{X}_k} < \delta/2.$$

Using that $\partial_t \phi_0(t) g(t_0) = \phi_0(t) A_0 g(t_0)$, we get

$$\left(\phi_0(t_1-t_0) - \phi_{0,h}^{\pm}(t_1,t_0)\right) g(t_0) = \int_0^{t_1-t_0} \phi_0(t) A_0 g(t_0)\,\mathrm{d}t - \int_0^{\eta_h^{\pm}(t_1,t_0)} \phi_0(t) A_0 g(t_0)\,\mathrm{d}t$$

and then

$$\begin{aligned}\left\|\left(\phi_0(t_1-t_0) - \phi_{0,h}^{\pm}(t_1,t_0)\right) g(t_0)\right\|_{\mathsf{X}_k} &\leqslant e^{\kappa T} |(t_1-t_0) - \eta_h^{\pm}(t_1,t_0)| \|A_0 g(t_0)\|_{\mathsf{X}_k} \\ &\leqslant e^{\kappa T} \max_{0\leqslant t\leqslant T} \|g(t)\|_{\mathsf{X}_{k+1}} \frac{h}{2}.\end{aligned}$$

Taking h^* small enough, we have

$$\sup_{(t_1,t_0)\in\Omega_T} \left\|\left(\phi_0(t_1-t_0) - \phi_{0,h}^{\pm}(t_1,t_0)\right) g(t_0)\right\|_{\mathsf{X}_k} < \delta/2$$

and the result follows. □

COROLLARY 3.11 Let $f \in C([0,T], \mathsf{X}_k)$ and

$$I_2(h,t_1,t_0) = \int_{t_0}^{t_1} \alpha_h^{\mp}(t) \left(\phi_0(t_1-t) - \phi_{0,h}^{\pm}(t_1,t)\right) f(t)\,\mathrm{d}t.$$

Given $\delta > 0$, there exists $h^* > 0$ such that if $0 < h \leqslant h^*$, then it is verified that $\|I_2(h,t_1,t_0)\|_{\mathsf{X}_k} < \delta(t_1 - t_0)$.

LEMMA 3.12 Let $F \in C(\Omega_T, \mathsf{X}_k)$ and

$$I_3(h, t_1, t_0) = \int_{t_0}^{t_1} \alpha_h(t) F(t_1, t)\, \mathrm{d}t.$$

Given $\delta > 0$, there exists $h^* > 0$ such that if $0 < h \leqslant h^*$, $h \leqslant t_1 \leqslant T$ and $m \in \mathbb{N}$, then

$$\|I_3(h/m, t_1, t_1 - h)\|_{\mathsf{X}_k} \leqslant \delta h.$$

Proof. Since F is uniformly continuous, for $\delta > 0$, there exists $h^* > 0$ such that if $0 \leqslant t_1 - t \leqslant h^*$, then $\|F(t_1, t) - F(t_1, t_1)\|_{\mathsf{X}_k} < \delta$. Using that $\alpha_{h/m}$ is h-periodic with mean zero, we have

$$\|I_3(h/m, t_1, t_1 - h)\|_{\mathsf{X}_k} \leqslant \left\| \int_{t_1 - h}^{t_1} \alpha_{h/m}(t)(F(t_1, t) - F(t_1, t_1))\, \mathrm{d}t \right\|_{\mathsf{X}_k} < \delta h. \qquad \square$$

Proof of Theorem 3.9. We give the proof only for the asymmetric case (1.7a); the symmetric case is similar. Let $R = \max_{0 \leqslant t \leqslant T} \|u(t)\|_{\mathsf{X}_k} + \varepsilon$ and Λ the Lipschitz constant of A_1 in $B_R(0) \subset \mathsf{X}_k$. From Proposition 3.7, there exists $h^* > 0$ such that $\Phi_m^{\pm}(h/m, u)$ is defined for $0 < h \leqslant h^*$, $u \in B_R(0)$. Let $v(t) = u(t + nh - h) = \phi(t, u_{n-1})$ and $v_{h,m}^{\pm}$ be the solution of (3.2) with $v_{h,m}^{\pm}(0) = U_{n-1}$. First, we prove that given $\delta > 0$, there exist $C, h^* > 0$ such that $\max_{0 \leqslant t \leqslant h} \|v(t) - v_{h,m}^{\pm}(t)\|_{\mathsf{X}_k} \leqslant C(\|u_{n-1} - U_{n-1}\|_{\mathsf{X}_k} + \delta)$ for $0 < h \leqslant h^*$ and $m \in \mathbb{N}$. We can write

$$\begin{aligned} v(t) - v_{h,m}^{\pm}(t) = {}& \phi_{0,h/m}^{\pm}(t, 0)\,(u_{n-1} - U_{n-1}) + I_1(t) + I_2(t) \pm I_3(t) \\ & + \int_0^t \alpha_{h/m}^{\mp}(t')\, \phi_{0,h/m}^{\pm}(t, t') \left(A_1\left(v(t')\right) - A_1\left(v_{h,m}^{\pm}(t')\right)\right) \mathrm{d}t', \end{aligned}$$

where

$$\begin{aligned} I_1(t) &= \left(\phi_0(t) - \phi_{0,h/m}^{\pm}(t, 0)\right) u_{n-1}, \\ I_2(t) &= \int_0^t \alpha_{h/m}^{\mp}(t') \left(\phi_0(t - t') - \phi_{0,h/m}^{\pm}(t, t')\right) A_1\left(v(t')\right) \mathrm{d}t', \\ I_3(t) &= \int_0^t \alpha_{h/m}(t')\, \phi_0(t - t')\, A_1\left(v(t')\right) \mathrm{d}t'. \end{aligned}$$

Note that $I_1(t)$ corresponds to $I_1(h/m, t + nh - h, nh - h)$ with $f = u$ from Lemma 3.10; then we get $\|I_1(t)\|_{\mathsf{X}_k} < \delta/2$ for $0 \leqslant t \leqslant h \leqslant h_1^*$ and $1 \leqslant n \leqslant [T/h]$. Taking h_2^* small enough, we obtain

$$\|I_2(t)\|_{\mathsf{X}_k} + \|I_3(t)\|_{\mathsf{X}_k} \leqslant 5e^{2\kappa h} \max_{0 \leqslant t \leqslant T} \|A_1(u(t))\|_{\mathsf{X}_k} h < \delta/2,$$

for $0 < h \leqslant h_2^*$. Therefore, for $0 < h \leqslant h^* = \min\{h_1^*, h_2^*\}$, we have

$$\begin{aligned}\left\|v(t) - v_{h,m}^{\pm}(t)\right\|_{\mathsf{X}_k} &\leqslant \left\|\phi_{0,h/m}^{\pm}(t,0)\,(u_{n-1} - U_{n-1})\right\|_{\mathsf{X}_k} + \|I_1(t)\|_{\mathsf{X}_k} + \|I_2(t)\|_{\mathsf{X}_k} + \|I_3(t)\|_{\mathsf{X}_k} \\ &\quad + \int_0^t \left\|\alpha_{h/m}^{\mp}(t')\,\phi_{0,h/m}^{\pm}(t,t')\left(A_1(v(t')) - A_1\left(v_{h,m}^{\pm}(t')\right)\right)\right\|_{\mathsf{X}_k} \mathrm{d}t' \\ &\leqslant e^{2\kappa h}\,\|u_{n-1} - U_{n-1}\|_{\mathsf{X}_k} + \delta + 2\Lambda e^{2\kappa h}\int_0^t \left\|v(t') - v_{h,m}^{\pm}(t')\right\|_{\mathsf{X}_k} \mathrm{d}t',\end{aligned}$$

and using Gronwall's lemma, $\|v(t) - v_{h,m}^{\pm}(t)\|_{\mathsf{X}_k} \leqslant C(\|u_{n-1} - U_{n-1}\|_{\mathsf{X}_k} + \delta)$ with $C = e^{(2\kappa + 2\Lambda\, e^{2\kappa h^*})h^*}$.

If we define $U_{n-1}(t) = \sum_{m=1}^{s} \gamma_m v_{h,m}^{\pm}(t)$, we can see that $U_{n-1}(h) = U_n$ and then

$$\begin{aligned}U_n &= \sum_{m=1}^{s} \gamma_m \phi_{0,h/m}^{\pm}(h,0)\,U_{n-1} + \sum_{m=1}^{s} \gamma_m \int_0^h \alpha_{h/m}^{\mp}(t')\,\phi_{0,h/m}^{\pm}(h,t')\,A_1\left(v_{h,m}^{\pm}(t')\right)\,\mathrm{d}t' \\ &= \phi_0(h)\,U_{n-1} + \sum_{m=1}^{s} \gamma_m \int_0^h \alpha_{h/m}^{\mp}(t')\,\phi_{0,h/m}^{\pm}(h,t')\,A_1\left(v_{h,m}^{\pm}(t')\right)\,\mathrm{d}t';\end{aligned}$$

writing $u_n = \phi_0(h)u_{n-1} + \sum_{m=1}^{s} \gamma_m \int_0^h \phi_0(h - t')A_1(v(t'))\,\mathrm{d}t'$ we obtain

$$\begin{aligned}u_n - U_n &= \phi_0(h)\,(u_{n-1} - U_{n-1}) + \sum_{m=1}^{s} \gamma_m\,(I_2(h) \pm I_3(h)) \\ &\quad + \sum_{m=1}^{s} \gamma_m \int_0^h \alpha_{h/m}^{\mp}(t')\,\phi_{0,h/m}^{\pm}(t,t')\left(A_1(v(t')) - A_1\left(v_{h,m}^{\pm}(t')\right)\right)\,\mathrm{d}t'.\end{aligned}$$

Using that $I_2(h)$ corresponds to $I_2(h/m, nh, nh - h)$ with $f = A_1(u)$ from Corollary 3.11 and $I_3(h)$ corresponds to $I_3(h/m, nh, nh - h)$ with $F(t,t') = \phi_0(t - t')A_1(u(t'))$ from Lemma 3.12, we can see that $\|I_2(h)\|_{\mathsf{X}_k} + \|I_3(h)\|_{\mathsf{X}_k} < \delta h$, for $h \leqslant h^*$. Then, we have

$$\begin{aligned}\|u_n - U_n\|_{\mathsf{X}_k} &\leqslant e^{\kappa h}\|u_{n-1} - U_{n-1}\|_{\mathsf{X}_k} + \sum_{m=1}^{s} |\gamma_m|\delta h \\ &\quad + 2\Lambda C\, e^{2\kappa h}\sum_{m=1}^{s} |\gamma_m| h(\|u_{n-1} - U_{n-1}\|_{\mathsf{X}_k} + \delta) \leqslant e^{Kh}\|u_{n-1} - U_{n-1}\|_{\mathsf{X}_k} + C'\delta h,\end{aligned}$$

where $K = 2\kappa + 2\Lambda C \sum_{m=1}^{s} |\gamma_m|$ and $C' = (1 + 2\Lambda C\, e^{2\kappa h^*}) \sum_{m=1}^{s} |\gamma_m|$. Then, we obtain

$$\|u_n - U_n\|_{\mathsf{X}_k} \leqslant e^{Knh}\|u_0 - U_0\|_{\mathsf{X}_k} + \frac{C'}{K}\left(e^{Knh} - 1\right)\delta \leqslant (1 + C'/K)\,e^{KT}\delta < \varepsilon,$$

for δ small enough, which proves the theorem. □

The next result shows that for initial data in X_{q+1}, the method converges in X_0 with order q. The proof of convergence falls naturally from consistency and stability in the usual way (see Hairer *et al.*, 1993).

THEOREM 3.13 Let $A_0 : D(A_0) \to \mathsf{X}$ be an infinitesimal generator of a strongly continuous semigroup ϕ_0 and $\mathsf{X}_k = D(A_0^k)$ with the graph norm $\|u\|_{\mathsf{X}_k} = \sum_{j=0}^{k} \|A_0^j u\|_{\mathsf{X}}$ and let $A_1 \in C^{q+1}(\mathsf{X}_k, \mathsf{X}_k)$ for $0 \leqslant k \leqslant q+1$ such that $D^m A_1 : \mathsf{X}_k \to \mathcal{L}_m(\mathsf{X}_k, \mathsf{X}_k)$ is a locally Lipschitz continuous map for $0 \leqslant m \leqslant q+1$. Let ϕ, ϕ_0, ϕ_1 be the flows associated with (1.1), (1.2a) and (1.2b), respectively, and let Φ be the method defined by (1.7a) or (1.7b) with $\gamma = (\gamma_1, \ldots, \gamma_s)$ satisfying (1.8a) or (1.8b), respectively. Then, given $u_0 \in \mathsf{X}_{q+1}$ and $u(t) = \phi(t, u_0)$ the maximal solution of (1.1) defined on $[0, T^*)$, for any $T \in (0, T^*)$ there exist h^*, δ, K, C such that if $U_0 \in \mathsf{X}_{q+1}$ satisfies $\|u_0 - U_0\| < \delta$ and $0 < h \leqslant h^*$, then the sequence $U_n = \Phi(h, U_{n-1})$ is defined for $n \leqslant [T/h]$ and satisfies

$$\|\phi(nh, u_0) - U_n\|_{\mathsf{X}} \leqslant e^{Knh}\|u_0 - U_0\|_{\mathsf{X}} + C\frac{e^{Knh} - 1}{K}h^q.$$

Proof. From Theorem 3.9, there exist $\delta, h^* > 0$ such that for any $h \in [0, h^*]$ and $\|u_0 - U_0\|_{\mathsf{X}_{q+1}} < \delta$, the sequence $U_n = \Phi(h, U_{n-1})$ is defined for $n \leqslant [T/h]$ and $\|u_n - U_n\|_{\mathsf{X}_{q+1}} = o(1)$, when $h \to 0$. From Propositions 2.8 and 2.6, we can see that ϕ and Φ are compatible with $\{\mathsf{X}_k\}_{0 \leqslant k \leqslant q+1}$; therefore ϕ, $\Phi \in C_{\mathrm{st}}^{q+1,0}([0, h^*] \times \mathsf{X}_{q+1}, \mathsf{X}_0)$. Using the Taylor formula, we have

$$\phi(h, u_{n-1}) = \sum_{k=0}^{q} \frac{1}{k!}\partial_t^k \phi(0, u_{n-1})h^k + \frac{1}{q!}\int_0^h (h-t)^q \partial_t^{q+1}\phi(t, u_{n-1})\,\mathrm{d}t,$$

$$\Phi(h, u_{n-1}) = \sum_{k=0}^{q} \frac{1}{k!}\partial_t^k \Phi(0, u_{n-1})h^k + \frac{1}{q!}\int_0^h (h-t)^q \partial_t^{q+1}\Phi(t, u_{n-1})\,\mathrm{d}t;$$

from Theorem 3.1 (or Theorem 3.2) we obtain the local error estimation $\|\phi(h, u_{n-1}) - \Phi(h, u_{n-1})\|_{\mathsf{X}_0} \leqslant M(u)h^{q+1}$, where

$$M(u) = \frac{1}{(q+1)!} \max_{\substack{0 \leqslant t_0 \leqslant T \\ 0 \leqslant t \leqslant h^*}} \left(\left\| \partial_t^{q+1}\phi(t, u(t_0)) \right\|_{\mathsf{X}_0} + \left\| \partial_t^{q+1}\Phi(t, u(t_0)) \right\|_{\mathsf{X}_0} \right)$$

and using Theorem 3.8 the result follows. □

4. Full discretization

In actual problems the computation of Φ requires the partial problems to be solved exactly. Apart from some simple cases of ODEs, this is not possible. In what follows we will show how the method defined by (1.7) can be used to define integration methods of order q using suitable approximations of the partial flows ϕ_0 and ϕ_1. In order to gain some insight we briefly discuss the simplest case given by the spectral projections for linear flows.

Let X be a Hilbert space, and let $\{u_\nu\}_{\nu \in \mathbb{N}}$ be an orthonormal basis of eigenfunctions of A_0, i.e., $A_0 u_\nu = \lambda_\nu u_\nu$. Assume that $\mathrm{Re}(\lambda_\nu) \leqslant \kappa$, define $\phi_0(t)u = \sum_{\nu \in \mathbb{N}} e^{\lambda_\nu t}\langle u_\nu, u\rangle u_\nu$ and the spaces

$$\mathsf{X}_k = \left\{ u \in \mathsf{X} : \sum_{\nu \in \mathbb{N}} |\lambda_\nu|^{2k} |\langle u_\nu, u\rangle|^2 < \infty \right\};$$

then a straightforward computation shows that ϕ_0 is compatible with $\{\mathsf{X}_k\}_{k \geqslant 0}$ and satisfies $\|\phi_0(t)u\|_{\mathsf{X}_k} \leqslant e^{\kappa t}\|u\|_{\mathsf{X}_k}$. In addition, the related orthogonal projection P^σ (onto the subspace spanned by $\{u_1, \ldots, u_\sigma\}$)

given by

$$P^{\sigma}u = \sum_{1\leqslant\nu\leqslant\sigma} \langle u_\nu, u\rangle u_\nu,$$

satisfies the estimate $\|u - P^{\sigma}u\|_{X_l} \leqslant (\inf_{\nu>\sigma}|\lambda_\nu|)^{l-k}\|u\|_{X_k}$.

Following these ideas we consider a family of operators $\{P^{\sigma}\}_{\sigma\in\mathbb{N}} \subset \mathcal{L}_1(X_k, X_k)$ verifying

- $P^{\sigma}P^{\sigma} = P^{\sigma}$;
- $\|P^{\sigma}\|_{\mathcal{L}_1(X_k,X_k)} \leqslant C$;
- $P^{\sigma}\phi_0(t)P^{\sigma} = \phi_0(t)P^{\sigma}$ for $t \geqslant 0$ and $\sigma \in \mathbb{N}$;
- $\lim_{\sigma\to\infty}\|u - P^{\sigma}u\|_{X_k} = 0$ for $u \in X_k$;
- $\lim_{\sigma\to\infty}\|\,\mathrm{id} - P^{\sigma}\|_{\mathcal{L}(X_k,X_l)} = 0$ for $0 \leqslant l < k$.

Note that the members of the family need not be orthogonal projections as suggested in the example. Actually, nonorthogonal projectors are considered in many applications due to their lower computational cost (see Remark 4.2 and the examples in Sections 5.2 and 5.3).

Accordingly, we set $X_k^{\sigma} = P^{\sigma}X_k$ as a workspace, the linear flow $\phi_0^{\sigma}(t) = \phi_0(t)P^{\sigma}$, and ϕ_1^{σ} the flow associated with $A_1^{\sigma} = P^{\sigma}A_1$, and define the method Φ^{σ} by (1.7) with ϕ_0^{σ} and ϕ_1^{σ}. Under the same hypothesis as Theorem 3.13, we have the following result.

THEOREM 4.1 Given u and T as in Theorem 3.13 and $\varepsilon > 0$, there exist $\delta > 0$, $\sigma^* \in \mathbb{N}$ and $h^* > 0$ such that if $\sigma \geqslant \sigma^*$, $U_0 \in X_{q+1}^{\sigma}$ with $\|u_0 - U_0\|_{X_{q+1}} < \delta$ and $0 < h \leqslant h^*$, then $\|u_n - U_n^{\sigma}\|_{X_{q+1}} < \varepsilon$ for $n \leqslant [T/h]$, where $U_0^{\sigma} = U_0$ and $U_n^{\sigma} = \Phi^{\sigma}(h, U_{n-1}^{\sigma})$. Furthermore, there exists $C, K > 0$ such that for any $0 < h \leqslant h^*$ and $\sigma \geqslant \sigma^*$,

$$\left\|u_n - U_n^{\sigma}\right\|_{X_0} \leqslant e^{Knh}\|u_0 - U_0\|_{X_0} + C\frac{e^{Knh}-1}{K}\left(\|\mathrm{id} - P^{\sigma}\|_{\mathcal{L}(X_{q+1},X_0)} + h^q\right).$$

Sketch of the proof. We prove the asymmetric case (1.7a); the symmetric case is completely similar. The proof very closely follows the one given for Theorem 3.9. Let $v(t) = u(t - nh + h)$ and $v_{h,m}^{\sigma\pm}$ the solution of integral equation

$$v_{h,m}^{\sigma\pm}(t) = \phi_{0,h/m}^{\pm}(t,0)\,U_{n-1}^{\sigma} + \int_0^t \alpha_{h/m}^{\mp}(t')\,\phi_{0,h/m}^{\pm}(t,t')\,P^{\sigma}A_1\left(v_{h,m}^{\sigma\pm}(t')\right)\,\mathrm{d}t';$$

we can write

$$\begin{aligned} v(t) - v_{h,m}^{\sigma\pm}(t) &= \phi_{0,h/m}^{\pm}(t,0)\left(u_{n-1} - U_{n-1}^{\sigma}\right) + I_1(t) + I_2(t) \pm I_3(t) + I_4(t) \\ &\quad + \int_0^t \alpha_{h/m}^{\mp}(t')\,\phi_{0,h/m}^{\pm}(t,t')\,P^{\sigma}\left(A_1\left(v(t')\right) - A_1\left(v_{h,m}^{\sigma\pm}(t')\right)\right)\,\mathrm{d}t', \end{aligned}$$

where I_1, I_2, I_3 are the same as in the proof of Theorem 3.9 and

$$I_4(t) = \int_0^t \alpha_{h/m}^{\mp}(t')\,\phi_{0,h/m}^{\pm}(t,t')\,(\mathrm{id} - P^{\sigma})\,A_1\left(v(t')\right)\,\mathrm{d}t'.$$

Since $u \in C([0,T], \mathsf{X}_{q+1})$, it holds that $A_1(u)$ is uniformly continuous and then

$$\lim_{\sigma\to\infty} \max_{0\leqslant t\leqslant T} \|(\mathrm{id}-P^\sigma) A_1(u(t))\|_{\mathsf{X}_{q+1}} = 0.$$

The first part of the result follows as in Theorem 3.9.

For the second part, we can see that

$$v_{h,m}^{\pm}(t) - v_{h,m}^{\sigma\pm}(t) = \phi_{0,h/m}^{\pm}(t,0)\left(U_{n-1}-U_{n-1}^{\sigma}\right) + \int_0^t \alpha_{h/m}^{\mp}(t')\,\phi_{0,h/m}^{\pm}(t,t')\,(\mathrm{id}-P^\sigma)A_1\left(v_{h,m}^{\pm}(t')\right)\mathrm{d}t'$$
$$- \int_0^t \alpha_{h/m}^{\mp}(t')\,\phi_{0,h/m}^{\pm}(t,t')\,P^\sigma\left(A_1\left(v_{h,m}^{\pm}(t')\right) - A_1\left(v_{h,m}^{\sigma\pm}(t')\right)\right)\mathrm{d}t',$$

from where we obtain the estimate

$$\left\|v_{h,m}^{\pm}(t) - v_{h,m}^{\sigma\pm}(t)\right\|_{\mathsf{X}_0} \leqslant C\left(\|U_{n-1}-U_{n-1}^{\sigma}\|_{\mathsf{X}_0} + \|\mathrm{id}-P^\sigma\|_{\mathcal{L}_1(\mathsf{X}_{q+1},\mathsf{X}_0)}\right).$$

Writing $U_{n-1}(t) = \sum_{m=1}^{s}\gamma_m v_{h,m}^{\pm}(t)$ and $U_{n-1}^{\sigma}(t) = \sum_{m=1}^{s}\gamma_m v_{h,m}^{\sigma\pm}(t)$, in the same manner as in Theorem 3.9, we can show that

$$\|U_n - U_n^\sigma\|_{\mathsf{X}_0} \leqslant e^{Kh}\|U_{n-1}-U_{n-1}^{\sigma}\|_{\mathsf{X}_0} + Ch\|\mathrm{id}-P^\sigma\|_{\mathcal{L}_1(\mathsf{X}_{q+1},\mathsf{X}_0)}$$

and using that $U_0^\sigma = U_0$, we get $\|U_n - U_n^\sigma\|_{\mathsf{X}_0} \leqslant CK^{-1}(e^{Knh}-1)\|\mathrm{id}-P^\sigma\|_{\mathcal{L}_1(\mathsf{X}_{q+1},\mathsf{X}_0)}$. Finally, from Theorem 3.13 and the triangle inequality the proof is completed. □

REMARK 4.2 If $\{u_1,\ldots,u_\sigma\}$ is a basis of X_k^σ and $\mu_1,\ldots,\mu_\sigma \in \mathsf{X}_k^*$ are Hahn–Banach extensions of the dual basis, then P^σ, defined by $P^\sigma u = \sum_{\nu=1}^{\sigma}\langle\mu_\nu,u\rangle u_\nu$, is a projection onto X_k^σ and the flow ϕ_1^σ is given by $\phi_1^\sigma(t,u) = \sum_{\nu=1}^{\sigma}\hat{U}_\nu(t)u_\nu$, where $(\hat{U}_1,\ldots,\hat{U}_\sigma)$ is the solution of the ODE system

$$\begin{cases} \mathrm{d}\hat{U}_\nu/\mathrm{d}t = \left\langle \mu_\nu, A_1\left(\sum_{j=1}^{\sigma}\hat{U}_j(t)u_j\right)\right\rangle, \\ \hat{U}_\nu(0) = \langle\mu_\nu,u\rangle. \end{cases}$$

See the examples in Sections 5.2 and 5.3.

5. Numerical examples

We present several examples which illustrate the performance of the proposed methods.

5.1 *Ordinary differential system*

We begin by considering an elementary example which is simple for the proposed methods, but would be more expensive to solve with symplectic methods. The bidimensional system

$$\begin{cases} \dot{u}_1 = 4u_2 - \tan(u_1), \\ \dot{u}_2 = -4u_1 - \tan(u_2) \end{cases} \tag{5.1}$$

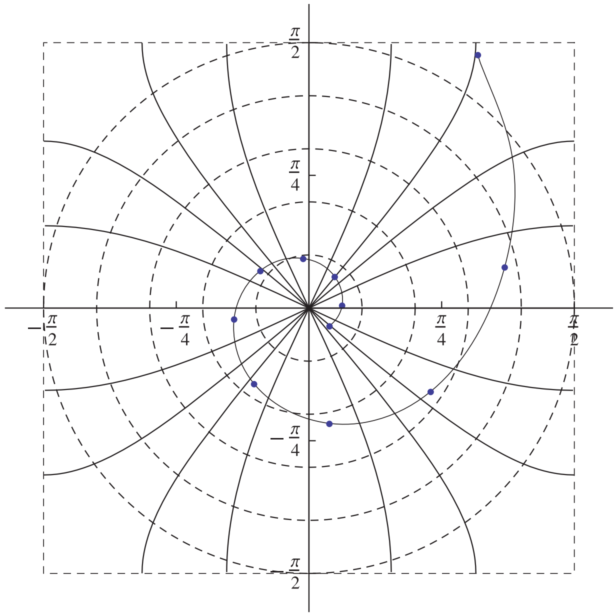

FIG. 1. Flows ϕ_0, ϕ_1 and solution of (5.1) obtained with Φ of fourth order.

can be split into a linear system and a decoupled system. The linear flow is a clockwise rotation; orbits are showed in Fig. 1 for concentric circles. Lines that go through the origin are the orbits of the system $\dot{u}_j = -\tan(u_j)$, whose solution is $u_j(t) = \arcsin(e^{-t}\sin(u_{j,0}))$. Note that solutions are not defined for $t < \ln|\sin(u_{j,0})| \leqslant 0$, which implies that h should be small for symplectic methods (with negative steps). For initial data $(1, 3/2)$, the solution computed with Runge–Kutta with very small h is shown in Fig. 1; the points are the solution obtained with the symmetric method Φ of fourth order with $s = 2$, $\gamma_1 = -1/6$, $\gamma_2 = 2/3$ and $h = 0.2$. It can be seen numerically that for this step, $h = 0.2$, the symplectic method proposed in Neri (1987) cannot be used.

5.2 *Oscillatory reaction–diffusion system*

In this example, we consider a reaction–diffusion system, as the ones analyzed in Kopell & Howard (1973). We study the performance of the methods for this system. Since this system is an irreversible problem, symplectic methods with negative steps cannot be used. We consider the system

$$\begin{aligned} \partial_t v &= \Delta v + \left(1 - r^2\right) v - \left(\omega_0 - \omega_1 r^2\right) w, \\ \partial_t w &= \Delta w + \left(\omega_0 - \omega_1 r^2\right) v + \left(1 - r^2\right) w, \end{aligned} \tag{5.2}$$

where $r^2 = v^2 + w^2$. If $u = v + \mathrm{i}w$, equation (5.2) reads as follows:

$$\partial_t u = \Delta u + \left(1 - |u|^2\right) u + \mathrm{i}\left(\omega_0 - \omega_1 |u|^2\right) u.$$

The right-hand term can be written as $A_0 u + A_1(u)$, where $A_0 u = \Delta u$ and

$$A_1(u) = \left(1 - |u|^2\right) u + \mathrm{i}\left(\omega_0 - \omega_1 |u|^2\right) u.$$

The flow ϕ_1 is given by

$$\phi_1(h,u) = ue^h\left(1+\left(e^{2h}-1\right)|u|^2\right)^{-1/2} e^{\mathrm{i}\left(\omega_0 h - \omega_1/2\ln\left(1+\left(e^{2h}-1\right)|u|^2\right)\right)}.$$

We will restrict our discussion to L-periodic solutions; flow ϕ_0 can be computed approximately by using the discrete Fourier transform (DFT). Let σ be an odd integer, $\sigma = 2l+1$ with $l \in \mathbb{N}$; consider

$$(P^\sigma u)(x) = \sum_{\nu=-l}^{l} \hat{U}_\nu\, e^{\mathrm{i}a\nu x},$$

where $a = 2\pi/L$ and $\hat{U}_\nu$ is the DFT coefficient given by

$$\hat{U}_\nu = \frac{1}{\sigma}\sum_{r=0}^{\sigma-1} U_r\, e^{-\mathrm{i}2\pi r\nu/\sigma} = \frac{1}{\sigma}\sum_{r=0}^{\sigma-1} u\left(Lr/\sigma\right)\, e^{-\mathrm{i}2\pi r\nu/\sigma}.$$

Since $e^{-\mathrm{i}2\pi r\nu/\sigma} = e^{-\mathrm{i}2\pi r(\nu\pm\sigma)/\sigma}$, it holds that $\hat{U}_\nu = \hat{U}_{\nu\pm\sigma}$. From Tadmor (1986, Lemma 2.2.), for $u \in \mathrm{H}^s(\mathbb{T})$ with $s > 1/2$ we have $\|u - P^\sigma u\|_{\mathrm{H}^r(\mathbb{T})} \leqslant C_{L,s,r}\sigma^{r-s}\|u\|_{\mathrm{H}^s(\mathbb{T})}$. Since $\phi_0^\sigma(t) = \phi_0(t)P^\sigma$ and using that $\hat{U}_\nu = \hat{U}_{\nu\pm\sigma}$, we get

$$\begin{aligned}\left(\phi_0^\sigma(t)u\right)(Lr/\sigma) &= \sum_{\nu=-l}^{l} \hat{U}_\nu\, e^{-a^2\nu^2 t} e^{\mathrm{i}2\pi r\nu/\sigma}\\ &= \sum_{\nu=l+1}^{\sigma-1} \hat{U}_\nu\, e^{-a^2(\sigma-\nu)^2 t}\, e^{\mathrm{i}2\pi r\nu/\sigma} + \sum_{\nu=0}^{l} \hat{U}_\nu e^{-a^2\nu^2 t} e^{\mathrm{i}2\pi r\nu/\sigma}\\ &= \sum_{\nu=0}^{\sigma-1} \hat{U}_\nu\, e^{-a^2\lambda_\nu t}\, e^{\mathrm{i}2\pi r\nu/\sigma},\end{aligned}$$

where $\lambda_\nu = \sigma^2 g(\nu/\sigma)$ for $0 \leqslant \nu \leqslant \sigma - 1$ and $g(\xi) = \xi^2 - 2(\xi - 1/2)_+$.

In Kopell & Howard (1973) the stability of the planar waves

$$\begin{aligned}v(x,t) &= r^*\cos\left(\theta_0 \pm ax + \left(\omega_0 - \omega_1 r^{*2}\right)t\right),\\ w(x,t) &= r^*\sin\left(\theta_0 \pm ax + \left(\omega_0 - \omega_1 r^{*2}\right)t\right)\end{aligned}$$

is proved, if $L > 2\pi(3+2\omega_1^2)^{1/2}$, where $r^* = L^{-1}(L^2 - 4\pi^2)^{1/2}$ and θ_0 is an arbitrary constant (see also Sherratt, 2003). Taking $L = 4\pi$, $\omega_0 = 1$, $\omega_1 = 1/2$ and $u_0 = r^* e^{\mathrm{i}ax}$, we compare methods given by (1.7b) of order $q = 4, 6, 8$ with $\sigma = 63$. The fourth-order method used is the same as the previous example; for the sixth-order method we take $s = 3$, $\gamma_1 = 1/48$, $\gamma_2 = -8/15$ and $\gamma_3 = 81/80$; for the eighth-order method we take $s = 4$, $\gamma_1 = -1/720$, $\gamma_2 = 8/45$, $\gamma_3 = -729/560$ and $\gamma_4 = 512/315$. In Fig. 2, global errors for $T = 10$ are shown. We note that the slopes coincide with the expected order up to the point where the rounding error dominates the total error.

In order to show the stability of the planar waves, we consider the initial data $\tilde{u}_0(x) = 0.8u_0(x) + 0.1 + 2.5e^{\mathrm{i}2ax} - 0.8\mathrm{i}e^{\mathrm{i}3ax}$. In Fig. 3, we can see the evolution of the fourth-order method $\Phi^\sigma(t, \tilde{u}_0)$ for $t \in [0, 50]$, calculated with $\sigma = 63$ and $h = 0.1$ and $\phi(t, u_0)$ is showed as a dashed line.

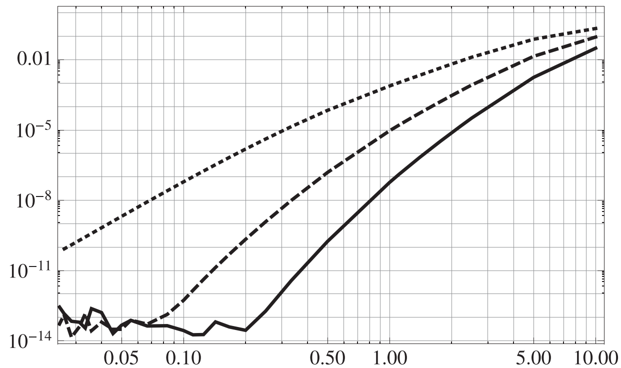

FIG. 2. Global error of Φ^σ vs. h for $q = 4, 6, 8$.

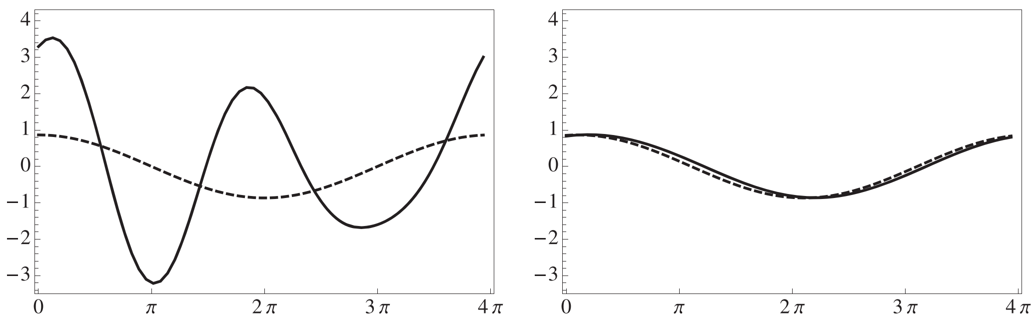

FIG. 3. $\mathrm{Re}(\Phi^\sigma(t, \tilde{u}_0))$ for $t = 0$ (left) and $t = 50$ (right).

5.3 *Regularized Schrödinger–Poisson equation*

In this example, we study the 2π-periodic solutions of the regularized Schrödinger–Poisson equation

$$\begin{cases} \partial_t u = \mathrm{i}\partial_x^2 u - \left(-\partial_x^2\right)^\beta u + \mathrm{i}|u|^2 u + \mathrm{i}\left(g * |u|^2\right) u, \\ u(0) = u_0, \end{cases} \tag{5.3}$$

where $0 < \beta < 1$ and g is a real kernel. Similar equations are considered in Aloui (2008a,b) and Aloui *et al.* (2013), on bounded domains of $\mathbb{R}^n$ as well as on compact manifolds. In order to apply the methods given by (1.7b), we consider the flow ϕ_0 generated by the linear operator $L = \mathrm{i}\partial_x^2 - (-\partial_x^2)^\beta$, and the flow $\phi_1(h, u) = \exp(\mathrm{i}h(|u|^2 + g * |u|^2))u$ associated with $\partial_t u = \mathrm{i}(|u|^2 + g * |u|^2)u$. If $\rho = |u|^2$ and $\rho(x, t) = \sum_{\nu \in \mathbb{Z}} \hat{\rho}_\nu(t)\, e^{\mathrm{i}\nu x}$, we have

$$(g * |u|^2)(x, t) = \sum_{\nu \in \mathbb{Z}} \hat{g}_\nu \hat{\rho}_\nu(t)\, e^{\mathrm{i}\nu x}.$$

Both ϕ_0, ϕ_1 can be numerically solved using DFT as in the example above. Using fast Fourier transform, the computational cost of each evaluation is $\mathcal{O}(\sigma \log \sigma)$, where σ is the number of points in the spatial discretization.

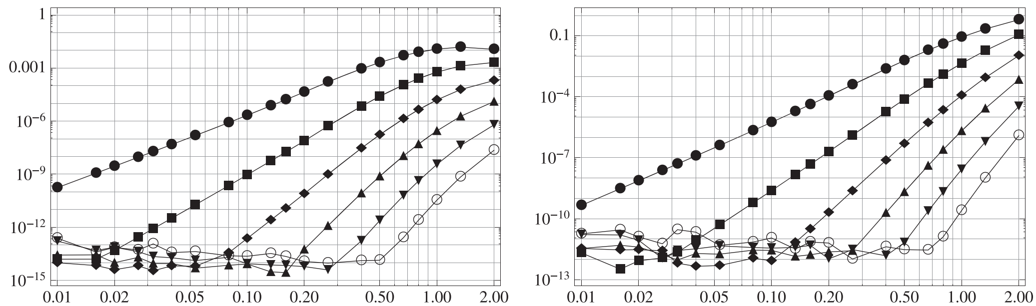

FIG. 4. Global error vs. h for $q = 4, 6, \ldots, 14$, absolute error (left) and relative error (right).

In order to analyse the performance of the integrators proposed, we consider the exact solutions $u(x,t) = r(t)\, e^{\mathrm{i}(\nu_0 x + \theta(t))}$, with $r(t) = r_0\, e^{-|\nu_0|^{2\beta} t}$ and

$$\theta(t) = -\nu_0^2 t + \tfrac{1}{2}(1+\hat{g}_0) r_0^2 |\nu_0|^{-2\beta} \left(1 - e^{-2|\nu_0|^{2\beta} t}\right) + \theta_0.$$

Note that $u(\cdot,t)$ has only one oscillation mode, and taking ν_0 as the momentum of the wave as usual, we can say that u is a monokinetic wave. As an example, we consider the Poisson kernel given by

$$g(x) = \frac{\sinh(\lambda)}{\cosh(\lambda) - \cos(x)};$$

then $\hat{g}_\nu = e^{-\lambda|\nu|}$. In Fig. 4, absolute global error and relative global error, defined by

$$\mathcal{E}_{\mathrm{abs}} = \max_{0 \leqslant n \leqslant [T/h]} \|u_n - U_n\|_{L^2}, \quad \mathcal{E}_{\mathrm{rel}} = \max_{0 \leqslant n \leqslant [T/h]} \frac{\|u_n - U_n\|_{L^2}}{\|u_n\|_{L^2}},$$

are shown, with $\beta = 1/4$, $T = 4$, $\lambda = 1$, initial condition $u_0 = e^{\mathrm{i}4x}$ and methods varying from fourth to fourteenth order. The number of points in the spatial discretization is $\sigma = 31$ and temporal steps h range from 0.01 to 2. As in the example above, the slopes coincide with the expected order up to the point where the rounding error dominates the total error.

For $\nu_0 = 0$, it holds that $u(x,t) = r_0\, e^{\mathrm{i}2|r_0|^2 t + \mathrm{i}\theta_0}$, which are time-periodic solutions. Multiplying (5.3) by $\bar{u}$ and integrating by parts, we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\|u\|_{L^2}^2 = -2\left\| \left(-\partial_x^2\right)^{\beta/2} u \right\|_{L^2}^2 = -2 \sum_{\substack{\nu \in \mathbb{Z} \\ \nu \neq 0}} |\nu|^{2\beta} |\hat{u}_\nu|^2 \leqslant -2\|Pu\|_{L^2}^2,$$

where $Pu = \sum_{\nu \neq 0} \hat{u}_\nu\, e^{\mathrm{i}\nu x}$ and therefore the monokinetic solution with $\nu_0 = 0$ is the only time-periodic solution.

It is easy to see that the flow ϕ of equation (5.3) preserves parity; then for any odd initial data u_0, $u(t)$ is an odd function and $u(t) = Pu(t)$ for $t > 0$. Therefore, it holds that $\mathrm{d}\|u\|_{L^2}^2/\mathrm{d}t \leqslant -2\|u\|_{L^2}^2$ and $\|u\|_{L^2} \leqslant e^{-t}\|u_0\|_{L^2}$. We will test the numerical methods by verifying these properties. Consider the odd initial data $u_0(x) = e^{\cos(2x) + \mathrm{i}\pi/6}\sin(5x)$; in Fig. 5 we show the numerical solution obtained with the eighth symmetric integrator with $\sigma = 255$ and $h = 0.1$. Since the higher the frequencies are, the stronger is the damping, u asymptotically behaves like $a\, e^{-t-\mathrm{i}t}\sin(x)$. Figure 6a shows the evolution of

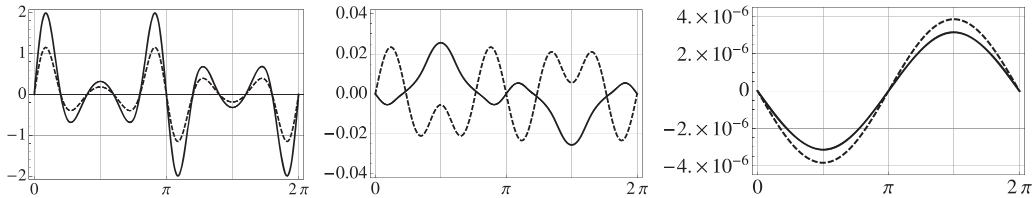

FIG. 5. $\mathrm{Re}(\Phi^\sigma(t,u_0))$ and $\mathrm{Im}(\Phi^\sigma(t,u_0))$ for $t=0$ (left), $t=2$ (centre) and $t=10$ (right).

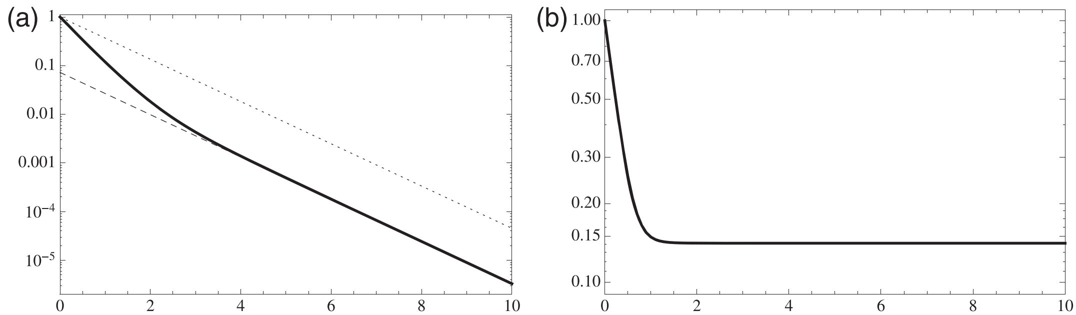

FIG. 6. Evolution of $\|u\|_{L^2}/\|u_0\|_{L^2}$ vs. time. (a) Odd solution and (b) even solution.

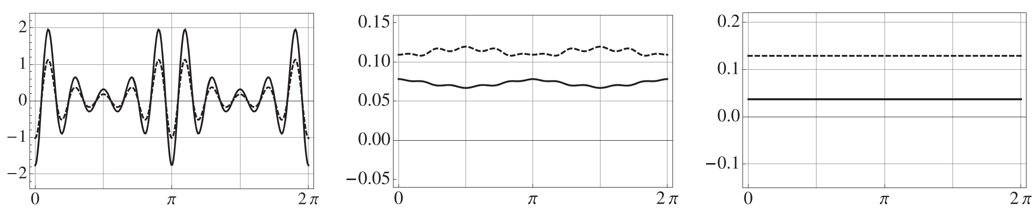

FIG. 7. $\mathrm{Re}(\Phi^\sigma(t,u_0))$ and $\mathrm{Im}(\Phi^\sigma(t,u_0))$ for $t=0$ (left), $t=2$ (centre) and $t=10$ (right).

$\|u(\cdot,t)\|_{L^2}/\|u_0\|_{L^2}$ as a continuous line, the function e^{-t} as a dotted line and the asymptotic behaviour as a dashed line.

We also consider a numerical computation with $u_0(x)=e^{\cos(2x)+\mathrm{i}\pi/6}(1-1.75\cos^2(5x))$, which is even initial data. Using the same integrator as in the odd case, we see that the solution converges to the periodic solution $u(x,t)\sim ae^{\mathrm{i}2|a|^2t}$ as seen in Fig. 7. In Fig. 6b, fast stabilization of the norm can be observed. This suggests that the periodic solutions are limit cycles of the dynamic given by equation (5.3).

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