

Interacting realization of cosmological singularities with variable vacuum energy

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We examine an interacting dark matter–variable vacuum energy model for a spatially flat Friedmann-Robertson-Walker spacetime, focusing on the appearance of cosmological singularities such as *big rip*, *big brake*, *big freeze*, and *big separation* along with abrupt events (*infinite γ -singularity* and *new w -singularity*) at late times. We introduce a phenomenological interaction which has a nonlinear dependence on the total energy density of the dark sector and its derivative, solve exactly the source equation for the model and find the energy density as function of the scale factor as well as the time dependence of the approximate scale factor in the neighborhood of the singularities. We describe the main characteristics of these singularities by exploring the type of interaction that makes them possible along with behavior of dark components near them. We apply the geometric Tipler and Królak method for determining the fate of time-like geodesic curves around the singularities. We also explore the strength of them by analyzing the leading term in some geometric invariants such as the square Riemann scalar and the Ricci scalar.

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I. INTRODUCTION

In 1998 astrophysical observations coming from distant supernovae–stellar explosions led to the inevitable conclusion that our Universe is actually speeding up rather than slowing down [1] due to the existence of a repulsive agent known as dark energy. Such outstanding finding was promptly confirmed by additional cosmological observations based on the measurements of cosmic microwave background anisotropies, baryon acoustic oscillations, and power spectrum of clustered matter [2, 3]. Despite the plethora of observational evidences, in favor of the present acceleration of the Universe, accumulated over the last years by studying even more remote supernovae and launching new satellites [2], quite little is known about the true nature of dark energy. For instance, there is not a fundamental theory which can account for the origin of dark energy at microscopic level.

In order to shed some light over the dark side of the Universe one could devote some efforts to explore its ultimate fate [4]. An appealing route to follow is to analyze some dark energy models which can explain the current speeding up of the Universe but exhibit the presence of a cosmic singularity in the asymptotic (remote) future [4], [5]; so one may link the present state of the Universe with its drastic final state. A fundamental task to be addressed by these alternative scenarios is the classification of singularities that could emerge at a finite time or abrupt events [4], [5], [6], [7], [8], [9], [10], [11], [12], [13]. Several useful ways to characterize such final doomsdays are based on geometric methods which examine the existence of causal geodesics that cannot be extended to arbitrary values of their proper time (geodesic incompleteness) [14] or the possibility to show that geodesic

curves can be extended beyond cosmic singularity [15], [16]. Another manner consists in taking into account the behavior of curvature invariants near the singularity so the strength of singularities can be determined using the necessary and sufficient conditions obtained by Tipler [17] and Królak [18]. These conditions are considerably useful provided help to classify singularities as strong and weak types, giving some insights on the magnitude of the tidal forces experienced by a co-moving observer toward the singularity. In fact, the Tipler definition requires that any object has its volume crushed to zero as the singularity is approached [17] whereas the Królak definition is weaker than the Tipler version and is related with inquiries on the cosmic censorship conjecture [19].

A serious approach for understanding the nature of cosmological singularities requires to take into account the behavior of dynamical variables which enter into the field equations. Therefore the blow up of the energy density and the divergent behavior of the pressure seem to be of interest along with the behavior of geometric quantities such as the scale factor, the Hubble function and its derivatives. In this way, one must focus on certain physical properties which make these kinds of singularities fairly distinctive among them and how such traits can determine the ultimate fate of the universe [5], [6], [7], [8], [9], [10], [11], [12], [13]. At this point, it would seem crucial to explore a viable cosmological scenario where the aforesaid singularities can appear naturally in order to explore its physical outcome. To do that, we are going to present some interacting dark energy models [20], [21] with the presence of singularities (or singular event) and abrupt events. A natural question that could arise is what kind of phenomenological interactions do imply the existence of a future singularity or abrupt event. We must stress that our approach for studying cosmic sin-

gularities is considerably different from previous works, thus, we focus in an interacting dark energy model with a nonlinear interaction term and then analyze the ultimate fate of the universe. As a consequence of this method, we will firstly study the existence of singularity along with its traits without imposing any particular type of scale factor. However, we will *obtain* the approximate scale factor or the exact one, in the cases where it is possible, to illustrate the model. Since we do not postulate or give the form of scale factor from the very beginning, it is clear that only some singularities will emerge in our model. In this way, we will construct *a posteriori* a scale factor associated with a cosmic singularity for a given interaction. Such expression will be useful for studying then the behavior of the Hubble function, its derivatives in addition to the behavior of the energy density and pressure in terms of the cosmic time.

Our goal is then to consider an interacting dark matter–variable vacuum energy (VVE) framework and explore the new details coming from the appearance of cosmic singularities in the remote future (distant past) or abrupt/singular events. Taking into account that these viable models exhibit a current accelerating phase together with a future doomsday, we are going to describe the physical traits associated with them. Further, we analyze several kinds of singularities, make some comments on the type of interaction supporting them, in particular, we focus on the behavior of both dark component near the singularity. Finally, we examine these singularities with the help of the Tipler and Królak criteria.

II. INTERACTION FRAMEWORK

To study the interacting dark sector model we consider the spatially flat Friedmann-Roberston-Walker (FRW) metric $ds^2 = -dt^2 + a^2(t)dx^i dx_i$, where $a(t)$ is the scale factor and $dx^i dx_i$ is the line element corresponding to hypersurface of constant time. The universe is filled with two perfect fluids, one accommodates as a matter component while the other represents a VVE substratum. Both fluids are described by linear equations of state, having energy densities ρ_m , ρ_x and pressures p_m , p_x , respectively. The total energy density ρ and the conservation equation $\dot{\rho} + 3H(\rho + p) = 0$ for this interacting two-fluid model can be written as

$$\rho = \rho_m + \rho_x, \quad (1)$$

$$\rho' = -\gamma_m \rho_m - \gamma_x \rho_x, \quad (2)$$

where the dot stands for derivative with respect to the cosmic time $\dot{} \equiv d/dt$ being $H = \dot{a}/a$, the prime is the derivative with respect to scale factor $' \equiv d/d\eta = d/3Hdt = d/d\ln(a/a_0)^3$ and a_0 is some value of reference for the scale factor. We have assumed that both interacting components admit an equations of state $p_i = (\gamma_i - 1)\rho_i$ with $i = \{m, x\}$ such that the constant

barotropic indexes γ_m and γ_x satisfy the next condition: $0 < \gamma_x < \gamma_m$. After having solved the algebraic linear system of equations (1)-(2), we are able to get ρ_m and ρ_x as functions of ρ and its η derivative ρ' :

$$\rho_m = -\frac{\gamma_x \rho + \rho'}{\gamma_m - \gamma_x}, \quad (3)$$

$$\rho_x = \frac{\gamma_m \rho + \rho'}{\gamma_m - \gamma_x}. \quad (4)$$

To complete the model we introduce an exchange of energy in the dark sector in terms of a factorized interaction $3HQ(\rho, \rho', \eta)$. Concerning this aim, we split the balance equation (2) as follows:

$$\rho'_m + \gamma_m \rho_m = -Q, \quad (5)$$

$$\rho'_x + \gamma_x \rho_x = Q. \quad (6)$$

From Eqs. (3)-(6), we arrive at a second order differential equation for the energy density

$$\rho'' + (\gamma_m + \gamma_x)\rho' + \gamma_m \gamma_x \rho = Q(\gamma_m - \gamma_x), \quad (7)$$

that we will call “source equation” [20] henceforth.

The uniqueness of the solutions (3)-(4) corresponding to the algebraic system equations (1)-(2) allows us to extract some interesting conclusions of the model. In the procedure outlined above one can find the energy density ρ by solving the source equation (7) for a given interaction term Q , subsequently, the energy densities of the matter and VVE components are reconstructed by means of Eqs. (3)-(4), pointing that this procedure does not rely on the specific cosmological equations that govern the dynamic of an homogeneous isotropic flat universe. Later on, we will investigate a concrete example for a given interaction term and obtain several general features about the existence of initial and final singularities in the interacting dark sector model without knowing of the scale factor.

As a result of this approach, we have been reducing the interacting framework to an effective one-fluid model with energy density $\rho = \rho_m + \rho_x$ and total pressure $p(\rho, \rho') = p_m + p_x = -\rho - \rho'$. Comparing the later equation with the effective equation of state of the dark sector $p = (\gamma - 1)\rho$, we obtain its effective conservation equation

$$\rho' + \gamma\rho = 0, \quad (8)$$

where the effective barotropic index reads

$$\gamma = \frac{(\gamma_m \rho_m + \gamma_x \rho_x)}{\rho}. \quad (9)$$

In calculating the scale factor of the homogeneous and isotropic flat universe, we adopt the Einstein field equations, so that the dynamic of the effective one-fluid model

will be governed by the corresponding Friedmann constraint,

$$3H^2 = \rho. \quad (10)$$

We will investigate a universe which transits from an initial matter-dominated phase into a final era dominated by an unknown component that will be identified with VVE, the former component is associated with an initial singularity and the latter one with a final singularity or a possible doomsday of the universe. In our model the VVE has an equation of state of the form $p_x = -\rho_x$, so its barotropic index vanishes ($\gamma_x = 0$), and therefore the source equation (7), the matter energy density (3), and the VVE density (4) turn to be given by

$$\rho'' + \gamma_m \rho' = \gamma_m Q, \quad (11)$$

$$\rho_m = -\frac{\rho'}{\gamma_m}, \quad \rho_x = \rho + \frac{\rho'}{\gamma_m}. \quad (12)$$

So far we have been speaking of an exchange of energy between matter and VVE fluids without specifying its form. We now propose that the interaction term in the dark sector can be defined via the equation

$$Q = -\frac{df}{d\eta}. \quad (13)$$

Here we suppose that the input function f only depends on ρ, ρ' , and η for simplicity. Replacing this interaction term Q into the source equation (7), it leads us to a nonlinear differential equation

$$\rho'' + \gamma_m \rho' = -\gamma_m f'. \quad (14)$$

Taking into account the first integral of Eq. (14), it allows us to write the equation of state of the mixture $p = -\rho - \rho'$, the effective barotropic index $\gamma = -\rho'/\rho$, the acceleration of the universe \ddot{a} along with the matter and VVE densities (12) as follows

$$\rho' = -\gamma_m(c + \rho + f), \quad (15)$$

$$p = (\gamma_m - 1)\rho + \gamma_m(c + f), \quad (16)$$

$$\gamma = \gamma_m \left[1 + \frac{c}{\rho} + \frac{f}{\rho} \right], \quad (17)$$

$$\frac{\ddot{a}}{a} = -\frac{1}{6}(3\gamma_m - 2)\rho - \frac{\gamma_m}{2}(c + f), \quad (18)$$

$$\rho_m = c + \rho + f, \quad \rho_x = -(c + f). \quad (19)$$

where c is an integration constant. From Eqs. (15)-(19), we observe that the η derivative of the energy density,

the pressure, the effective barotropic index, the acceleration term, the matter and VVE densities, all of them depend linearly with the input function f to describe the whole contribution coming from the energy transfer in the dark sector (13). An interesting point in regard with the role played by the interaction can be understood by analyzing a regime where f gives the largest contribution, neglecting ρ and c . In such regime, the previous dynamical quantities can be easily found as follows

$$\begin{aligned} \rho' &\approx -\gamma_m f, & p &\approx \gamma_m f, & \gamma &\approx \gamma_m f/\rho, \\ \ddot{a} &\approx -\gamma_m a f/2, & \rho_m &\approx -\rho_x \approx -f. \end{aligned} \quad (20)$$

In order to further motivate our results, let us assume that function $f(t) \rightarrow f(t_s)$ when $t \rightarrow t_s$ so that in the limit case $f(t_s) = \pm\infty$, blowing up at the finite value of the cosmic time t_s . Then, we have that the cosmic time derivative of the Hubble variable $\dot{H} = \rho'/2 \approx -\gamma_m f/2 \rightarrow \mp\infty$, the pressure $p \approx \gamma_m f \rightarrow \pm\infty$, the effective barotropic index $\gamma \approx \gamma_m f/\rho \rightarrow \pm\infty$, the acceleration $\ddot{a} \approx -\gamma_m a f/2 \rightarrow \mp\infty$ and the matter and VVE densities $\rho_m \approx f \rightarrow \pm\infty$, $\rho_x \approx -f \rightarrow \mp\infty$, indicating that all these quantities diverge as $t \rightarrow t_s$. This promising approach then opens the possibility of producing an initial or final singularity at t_s within the framework of interacting dark sector, so it will be useful for examining the ultimate fate of a universe. In the next section, we are going to generate Q by selecting the input function f and examine its physical outcome.

III. INTERACTION FRAMEWORK PRODUCES INITIAL AND FINAL SINGULARITIES

A. General properties

In this section, we wish to investigate the initial and final singularities of the universe when the interaction in the dark sector is generated by the function f , written as a power-law of the energy density

$$f = \alpha \rho^{-n}, \quad (21)$$

which yields an exchange of energy associated with the nonlinear interaction term (13)

$$Q = n\alpha \rho^{-n-1} \rho', \quad (22)$$

where α is a coupling constant and n is a non-vanishing real number. Then the first integral (15) becomes

$$\rho' = -\gamma_m(c + \rho + \alpha \rho^{-n}). \quad (23)$$

To simplify our formulation, we slightly redefine the interaction (22) to get a scenario where the integration constant c vanishes and does not contribute to the model. We insert the first integral (23) with $c = 0$ into Eq. (22) and we obtain the final interaction

$$Q = -n\alpha \gamma_m \rho^{-n-1} (\rho + \alpha \rho^{-n}), \quad (24)$$

which can also be written as

$$Q = -n \left(\frac{\rho'^2}{\gamma_m \rho} + \rho' \right). \quad (25)$$

From Eqs. (13), (21) and (25), we obtain the solutions of the source equation (11) in the case of $c = 0$. This kind of interaction is very interesting because the effective equation of state of the effective one-fluid $p = -\rho - \rho'$ for $c = 0$ turns to be that of a Chaplygin or anti-Chaplygin gas depending on the sign of the coupling constant α , as was noticed in Ref.[20]. Summing up, Eqs. (15)-(19) can be recast as

$$\rho' = -\gamma_m(\rho + \alpha\rho^{-n}), \quad (26)$$

$$p = (\gamma_m - 1)\rho + \alpha\gamma_m\rho^{-n}, \quad (27)$$

$$\gamma = \gamma_m [1 + \alpha\rho^{-n-1}], \quad (28)$$

$$\frac{\ddot{a}}{a} = -\frac{1}{6}(3\gamma_m - 2)\rho - \frac{\alpha\gamma_m}{2}\rho^{-n}, \quad (29)$$

$$\rho_m = \rho + \frac{\alpha}{\rho^n}, \quad \rho_x = -\frac{\alpha}{\rho^n}. \quad (30)$$

We are in condition to go a step forward for integrating (26) and obtaining the energy density as a function of the scale factor

$$\rho = \left[-\alpha + b \left(\frac{a_0}{a} \right)^{3\gamma_m(n+1)} \right]^{1/(n+1)}, \quad (31)$$

where a_0 and b are both arbitrary integration constants. A singularity can be achieved if any of the physical quantities, such as the energy density (31), the pressure (27), the barotropic index (28) or the acceleration (29), vanishes or diverges at a finite time t_s . One way to meet this condition is by choosing the integration constant b so that the square bracket in Eq. (31) vanishes as the scale factor reaches the finite value a_s , namely $b = \alpha(a_s/a_0)^{3\gamma_m(n+1)}$. Finally, the energy density (31), its η derivative and the effective barotropic index are given by

$$\rho = \left\{ \alpha \left[-1 + \left(\frac{a_s}{a} \right)^{3\gamma_m(n+1)} \right] \right\}^{1/(n+1)}, \quad (32)$$

$$\rho' = -\alpha\gamma_m \left(\frac{a_s}{a} \right)^{3\gamma_m(n+1)} \rho^{-n}, \quad (33)$$

$$\gamma = \frac{\gamma_m}{1 - \left(\frac{a}{a_s} \right)^{3\gamma_m(n+1)}}. \quad (34)$$

To make contact with previous results we will examine the behavior of several important quantities explicitly.

We start with the case $\alpha > 0$. We achieve that the energy density $\rho \rightarrow 0$ for $a \rightarrow a_s$ with $n > -1$ provided $a \leq a_s$ or in the complementary scenario where $a \rightarrow \infty$ and $n < -1$. The energy density blows up ($\rho \rightarrow \infty$) under two different conditions: i- $a \rightarrow a_s$ with $a \geq a_s$ and $n < -1$ or ii- $a \rightarrow 0$ and $n > -1$. Another branch to study corresponds to a negative coupling constant, $\alpha < 0$. In this case, the energy density $\rho \rightarrow 0$ as $a \rightarrow a_s$ with $a \geq a_s$ and $n > -1$. On the contrary, the energy density becomes divergent ($\rho \rightarrow \infty$) in the $a \rightarrow a_s$ limit if two conditions simultaneously holds: $a \leq a_s$ and $n < -1$. In the last case, , we also have that the energy densities tends to the constant which is indeed an exact solution $\rho \rightarrow \rho_c = (-\alpha)^{1/(n+1)}$ [see Eq. (26)]. It can be obtained in the $a \rightarrow 0$ limit for $n < -1$ or in the case $a \rightarrow \infty$ for $n > -1$ [see Eq. (31)]. This constant solution gives the exact de Sitter scale factor,

$$a_{ds} = \exp \left[\frac{(-\alpha)^{1/(n+1)}}{3} \right]^{1/2} t. \quad (35)$$

The η derivative $\rho' = 2\dot{H}$ is negative or positive definite and therefore the Hubble variable is a decreasing or an increasing function of the time, respectively. An interestingly fact is that the barotropic index always diverges ($\gamma \rightarrow \infty$) in the $a \rightarrow a_s$ limit for any value of n . The remaining quantities, namely the pressure, the acceleration, and the dark matter and VVE densities can be obtained as functions of the scale factor by combining the equations (27), (29), (30) along with (32).

By considering the second term of the first integral (26) as the leading contribution [see Eqs. (20), (21)], it is possible to find the energy density after integrating the approximate conservation equation ($\dot{\rho} \approx -\sqrt{3}\alpha\gamma_m\rho^{-n+1/2}$) to arrive at

$$\rho \approx \left[-\frac{\sqrt{3}\alpha\gamma_m}{2(\nu-1)} \Delta t \right]^{2(\nu-1)}. \quad (36)$$

Now the approximate pressure (27) is given by

$$p \approx \alpha\gamma_m \left[-\frac{\sqrt{3}\alpha\gamma_m}{2(\nu-1)} \Delta t \right]^{\nu-2}, \quad (37)$$

while the approximate scale factors is obtained by integrating the Friedmann equation (10) as

$$a \approx a_s \left\{ 1 - \frac{2(\nu-1)}{3\nu\alpha\gamma_m} \left[-\frac{\sqrt{3}\alpha\gamma_m}{2(\nu-1)} \Delta t \right]^\nu \right\}. \quad (38)$$

We introduce the main parameter

$$\nu = \frac{2(n+1)}{2n+1}, \quad (39)$$

which will be useful for describing the different kinds of singularities in the near future (see Fig. 1). We define $\Delta t = t - t_s$ with t_s being a finite cosmic time. Notice

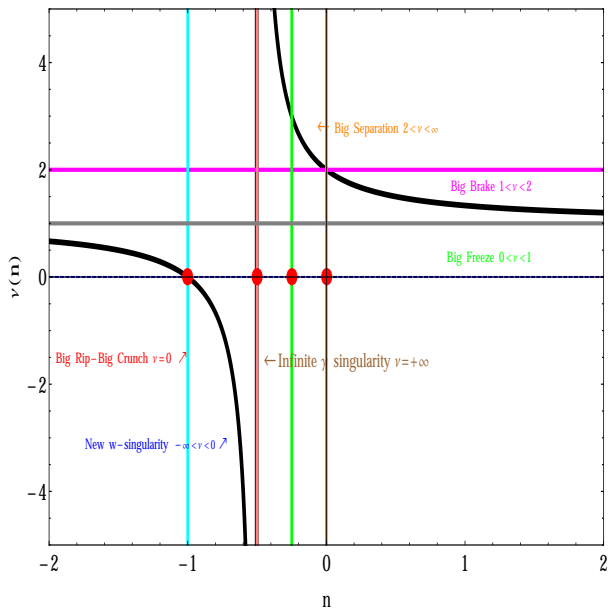


FIG. 1: Plot of the main parameter ν in terms of n indicating the range where the different kinds of singularities or abrupt events are achieved.

that the parameter ν vanishes for $n = -1$ or diverges at $n = -1/2$, hence, the expansions (36)-(38) are not well defined in those cases and we are going to deal with them separately. In fact, we will be able to solve the whole dynamics of the model and give the *exact* scale factor for both values of ν .

Combining the energy density (36), (24), and (26)-(30), we can rewrite the interaction term, the approximate Hubble variable, its first, its second time derivatives, the acceleration term, the barotropic index, and the matter and VVE densities, under the assumption that the $\alpha\rho^{-n}$ -term in Eq. (26) becomes dominant:

$$Q \approx \frac{2(\nu-2)(\nu-1)}{3\gamma_m \Delta t^2}, \quad (40)$$

$$H \approx -\frac{1}{\sqrt{3}} \left[-\frac{\sqrt{3}\alpha\gamma_m}{2(\nu-1)} \Delta t \right]^{\nu-1}, \quad (41)$$

$$\dot{H} \approx -\frac{\alpha\gamma_m}{2} \left[-\frac{\sqrt{3}\alpha\gamma_m}{2(\nu-1)} \Delta t \right]^{\nu-2}, \quad (42)$$

$$\ddot{H} \approx \frac{\sqrt{3}\alpha^2\gamma_m^2(\nu-2)}{4(\nu-1)} \left[-\frac{\sqrt{3}\alpha\gamma_m}{2(\nu-1)} \Delta t \right]^{\nu-3}, \quad (43)$$

$$\frac{\ddot{a}}{a} \approx \dot{H}, \quad (44)$$

$$\gamma \approx \frac{\alpha\gamma_m}{\left[-\frac{\sqrt{3}\alpha\gamma_m}{2(\nu-1)} \Delta t \right]^\nu}, \quad (45)$$

$$\rho_m \approx \frac{-2\dot{H}}{\gamma_m}, \quad \rho_x \approx \frac{2\dot{H}}{\gamma_m}. \quad (46)$$

We have carried out a detailed calculation of the dynamical quantities (densities, pressures, interaction term, acceleration, etc) that we will employ for addressing the issue of cosmic singularities. The next step is to classify the singularities using the behavior of the above quantities near them.

B. Classification of singularities

After having discussed the main traits of an interacting cosmology with two components in the presence of an initial or a final singularity, we must classify the different kinds of singularities that will emerge within this interacting framework. In order to do that, we mainly use the approximate energy density (36), the pressure (37), the scale factor (38) and remaining quantities (40)-(46). We classify the singularities according to the values taken by the main parameter, ν . Here, we will exclude the non interacting cases corresponding to $\nu = 2$ and $\nu = 1$.

1. ($1 < \nu < 2$)-case **Big Brake singularity**

If we now consider the case of a positive coupling constant ($\alpha > 0$) and the time approaching to the finite value t_s from the left $t \rightarrow t_s$ (thus $\Delta t < 0$) then the scale factor of the universe (38) reaches a finite value $a \rightarrow a_s$, the time derivative of the scale factor vanishes $\dot{a} \rightarrow 0$ but the second and third derivatives both diverge at the singularity ($\ddot{a}_s = -\infty$, $\ddot{a}_s = +\infty$). However, the energy density (36) along with the Hubble variable (41) are zero near the singularity, namely $\rho(t_s) = 0$ and $H(t_s) = 0$. The time derivative of the Hubble variable and their subsequent ones diverge as well as the acceleration of the universe $\ddot{a} \approx a_s \dot{H} \rightarrow -\infty$, see Eq. (42), while the pressure (37) positively grows without limit ($p \rightarrow +\infty$). Consequently, the interacting dark sector model has a late-time singularity at the finite cosmic time t_s for $n > 0$ and $\alpha > 0$ characterized by a finite scale factor, a vanishing time derivative, energy density and Hubble variable as well. But the acceleration and the pressure both diverge while the effective fluid fulfills anti-chaplygin gas equation of state (27). These results show us that this kind of behavior corresponds to a big Brake singularity, see [7]. Interestingly enough, it turned out the energy transfer from the VVE to the matter diverges as $t \rightarrow t_s$ ($Q \rightarrow -\infty$), $\rho_m \rightarrow +\infty$, and $\rho_x \rightarrow -\infty$ [see Eq. (40)]. It is important to stress that the physical set up associated with a big brake singularity can be described in terms of a tachyonic scalar field [7]. A weaker extension of the big brake singularity is obtained when a dust component is included

in the Friedmann equation provided the Hubble function does not vanish any longer at the singularity [8].

We end this case with the analysis of a complementary branch where the scale factor is defined in the region $t > t_s$ and the coupling constant is negative ($\alpha < 0$). The former branch which corresponds to $\alpha > 0$ and $t < t_s$ matches with the latter one at $t = t_s$ up to its first time derivative whereas their higher order derivatives are all divergent [8]. Near the singularity the effects introduced by the interaction (40) on the matter and VVE components (30) are the same of the $\alpha > 0$ case, thus, their limits are coincident.

2. ($2 < \nu < \infty$)-case **Big Separation singularity**

In the interval of $\nu \in (2, \infty)$ under the assumption of positive coupling constant ($\alpha > 0$), we find that the scale factor (38) reaches a constant value a_s as $t \rightarrow t_s$ from the left while the energy density (36) and the pressure (37) both approach to zero [6], however, the barotropic index (45) becomes divergent as $t \rightarrow t_s$. The first time derivative of the scale factor $\dot{a} \rightarrow 0$, its second derivative $\ddot{a} \rightarrow 0$, the Hubble variable $H \rightarrow 0$ and its first time derivative $\dot{H} \rightarrow 0$. Taking into account that $\rho' = 2\dot{H} \rightarrow 0$ as $t \rightarrow t_s$ and $\rho'' = 2\ddot{H}/3H$, the source equation (11) then becomes

$$\ddot{H} \approx \frac{3\gamma_m}{2} H Q. \quad (47)$$

This equation explicitly shows that the second and subsequent time derivatives of the Hubble variable are strongly dependent on the interaction term. In fact, from Eqs. (43) and (47), we find that H , \dot{H} , and their subsequently time derivatives $H^{(k)}$ vanish up to order $k \leq [[\nu - 1]]$ while the remaining ones diverge for $k \geq [[\nu - 1]]$ if $t \rightarrow t_s$, where $[[x]]$ stands for the integer part of its argument. From Eqs. (40) and (46), we also have that $Q \rightarrow \infty$, $\rho_m \rightarrow 0$ and $\rho_x \rightarrow 0$ as $t \rightarrow t_s$. In conclusion, the interacting dark sector model exhibits a big separation singularity, sometimes considered as a more softer singularity.

3. ($\nu \rightarrow +\infty$)-case **Exact solution**

We will discuss in detail the mysterious case associated with $n \rightarrow -1/2$. In fact, we are going to show that the field equations can be solved exactly and therefore we can obtain a complete picture of the interacting model. To this end, we introduce a new variable $s = (a/a_s)^{3\gamma_m/2}$ so that the Friedmann equation (10) with the energy density (32) reduces to the linear differential equation

$$\dot{s} = \pm \sqrt{3} \gamma_m \alpha (1 - s). \quad (48)$$

Having solved this equation we obtain four different types of solutions which need to be examined. Our starting point is to consider the expanding a_1 -solution and its effective barotropic index ($\gamma = -2\dot{H}/3H^2$),

$$a_1 = a_s \left[1 + e^{\frac{\sqrt{3}}{2} \alpha \gamma_m \Delta t} \right]^{\frac{2}{3\gamma_m}}, \quad (49)$$

$$\gamma_1 = -\gamma_m e^{-\frac{\sqrt{3}}{2} \alpha \gamma_m \Delta t}, \quad (50)$$

where $H_1 = \dot{a}_1/a_1 > 0$ and $\gamma_1 < 0$. The contracting a_3 -solution, obtained from the time reversal symmetry of the former one, leads to $a_3 = a_1(-t)$ with $H_3 = \dot{a}_3/a_3 < 0$ and $\gamma_3 = \gamma_1(-t)$. We name the another expanding solution as a_4 and is given by

$$a_4 = a_s \left[1 - e^{-\frac{\sqrt{3}}{2} \alpha \gamma_m \Delta t} \right]^{\frac{2}{3\gamma_m}}, \quad (51)$$

$$\gamma_4 = \gamma_m e^{\frac{\sqrt{3}}{2} \alpha \gamma_m \Delta t}. \quad (52)$$

Here $H_4 = \dot{a}_4/a_4 > 0$ and the contracting case is defined as $a_2 = a_4(-t)$ with $H_2 = \dot{a}_2/a_2 < 0$ and $\gamma_2 = \gamma_4(-t)$. There we have assumed a positive coupling constant, $\alpha > 0$. The branches a_1 and a_3 are defined in the interval $-\infty < t < \infty$ while for the branches a_2 and a_4 are specified in the intervals $t \leq 0$ and $t \geq 0$, respectively. We have fixed the integration constants so that the solution $a_1 \rightarrow a_s$ and $a_3 \rightarrow a_s$ in the limits $t \rightarrow -\infty$ and $t \rightarrow +\infty$, respectively. We have fixed the final big crunch or the initial big bang singularity in a_2 or in a_4 at $t = 0$.

The solution a_1 (a_3) describes a universe which expands (contracts) from a finite scale factor a_s , free of an initial singularity, (from an infinite scale factor) at $t = -\infty$ and a final de Sitter stage (a finite scale factor). However, the solution a_4 (a_2) represents a universe which evolves from the big bang singularity (a finite scale factor a_s) at $t = 0$ (in the limit $t = -\infty$) and increases (decreases) monotonically until reaches the finite scale factor a_s (the big crunch singularity) in the limit $t \rightarrow \infty$ (at $t = 0$). The last two scale factors a_4 and a_2 behave as a power-law solution $a \approx a_s [\sqrt{3} \alpha \gamma_m |t|/2]^{2/3\gamma_m}$ in the limit $|t| \rightarrow 0$ when the universe begins from a big bang singularity or ends in a big crunch singularity.

The scale factors a_3 and a_4 (a_1 and a_2) all of them go to a_s as $\Delta t \rightarrow \infty$ ($\Delta t \rightarrow -\infty$) while its first and higher order time derivatives vanish in those limits. Furthermore, we find that the Hubble variables can be written in terms of the barotropic index as

$$H_i = \frac{\alpha}{\sqrt{3}} \frac{1}{\left(1 - \frac{\gamma_i}{\gamma_m}\right)}, \quad (53)$$

where the index i runs from 1 to 4. Their subsequent time derivatives $H_i^{(k)} \rightarrow 0$ as $a \rightarrow a_s$.

The energy density of the mix, its pressure and the interaction term read

$$\rho_i = \alpha^2 \left[-1 + \left(\frac{a_s}{a_i} \right)^{\frac{3\gamma_m}{2}} \right]^2, \quad (54)$$

$$p_i = (\gamma_m - 1)\rho_i + \alpha \gamma_m \rho_i^{1/2}, \quad (55)$$

$$Q_i = \frac{\alpha \gamma_m}{2} \left[\alpha + \rho_i^{1/2} \right]. \quad (56)$$

At this point, we highlight that the time dependence of these quantities can be obtained by replacing each one of the exact scale factors found before [see Eqs. (49) and (49)]. From (54) and (55), we have that the energy density $\rho(a) \rightarrow 0$ and the pressure $p \rightarrow 0$ in the limit $a \rightarrow a_s$. Although the energy density and the pressure vanish in this limit, however, the effective barotropic index $\gamma = (\rho + p)/\rho = \gamma_m[1 + \alpha\rho^{-1/2}]$ diverges. Most importantly, the higher order time derivatives of γ also diverge as $a \rightarrow a_s$ so we name this behavior as “infinite γ -singularity”.

Consequently, the universe presents an infinite γ -singularity in the distant past or the remote future ($t = \pm\infty$) where the scale factor reaches a finite values a_s , but the first and subsequent time derivatives of the scale factor vanish, further, the Hubble variable along with their higher order time derivatives also vanish near a_s . We remark this “new singularity” differs from the one reported in [23] provided it could not be reached at finite time. Nevertheless, this singularity is characterized by a non-vanishing interaction term (56) at a_s , $Q \rightarrow \alpha^2\gamma_m/2$, while the matter component $\rho_m \rightarrow 0$ and the VEE $\rho_x \rightarrow 0$. Even though all physical quantities (p, ρ, ρ_m, ρ_x) fades away quickly as one approaches to the singularity, in the far remote future or distant asymptotic past, the interaction cannot be turned off neither the barotropic index.

4. ($-\infty < \nu < 0$)-case **New w -singularity**

In addition to the spacetime singularities at finite time mentioned above there are another kinds of abrupt events [22] related with the ultimate fate of the universe which indeed requires some mentioning for sake of completeness. The abrupt events were discovered within the framework of dark energy model and attracted some attention mostly because are less dangerous than singularity. Concerning this aim, we are going to analyze a new kind of abrupt event that is related with the so called w -singularity found in [23] and re-examined in [24].

For these values of the main parameter ν , the scale factor (38), the energy density (36) and the pressure (37) have the limits $a \rightarrow a_s$, $\rho \rightarrow 0$ and $p \rightarrow 0$ as the time variable $t \rightarrow \infty$ while the first and higher order time derivatives of these three quantities vanish in the remote future ($t \rightarrow \infty$). Due to that the effective barotropic index (45) and their subsequent time derivative $\gamma^{(k)}$ diverges up to order $k \leq \lceil -\nu \rceil$ but vanishes for all the following ones in the limit $t \rightarrow \infty$; as a result the universe ends in an abrupt γ -singularity in the remote future, again, this kind of singular event differs from the case explored in [23]. The approximate interaction term (40) vanishes in the abrupt event whereas Eq. (46) shows us that $\rho_x \rightarrow 0$ and $\rho_m \rightarrow 0$ as $a \rightarrow a_s$. This singular event differs from the “infinite γ -singularity” because the interaction vanishes in the latter case while the former one leads to $Q \rightarrow \alpha^2\gamma_m/2$.

5. ($\nu = 0$)-case **Big Rip/Crunch Singularity**

For this particular value of the main parameter ν , the constant $n = -1$ and the interaction term (22) becomes a linear function of the η -derivative of the energy density, $Q = -\alpha\rho'$, and therefore the first integral (23) turns to be

$$\rho' + \gamma_m(1 + \alpha)\rho = c\gamma_m. \quad (57)$$

We will solve this equation for the energy density and below show the general exact solution of the Friedmann equation for any value of the integration constant c . This includes the corresponding power law scale factor for $c = 0$:

$$a = a_s t^{2/3\gamma_m(1+\alpha)}, \quad c = 0, \quad (58)$$

$$\rho = \frac{4}{3\gamma_m^2(1+\alpha)^2} \left(\frac{a_s}{a}\right)^{3\gamma_m(1+\alpha)}, \quad c = 0, \quad (59)$$

and two families of solutions associated with $c \neq 0$,

$$a = a_s \left[\frac{1 - \cos \omega t}{2} \right]^{1/3\gamma_m(1+\alpha)}, \quad (60)$$

$$\rho = \frac{3c^2\gamma_m^2}{\omega^2} \left[\frac{\sin \omega t}{1 - \cos \omega t} \right]^2, \quad (61)$$

where $\omega^2 = 3c\gamma_m^2(1+\alpha) > 0$ and

$$a = a_s \left[\frac{1 + \cosh \omega t}{2} \right]^{1/3\gamma_m(1+\alpha)}, \quad (62)$$

$$\rho = \frac{3c^2\gamma_m^2}{\omega^2} \left[\frac{\sinh \omega t}{1 + \cosh \omega t} \right]^2, \quad (63)$$

where $\omega^2 = -3c\gamma_m^2(1+\alpha) > 0$ while the pressure and the interaction term are

$$p = c\gamma_m + [\gamma_m(1 + \alpha) - 1]\rho, \quad (64)$$

$$Q = \alpha\gamma_m[c + (1 + \alpha)\rho]. \quad (65)$$

For $\omega^2 = 3c\gamma_m^2(1+\alpha) > 0$ and $\alpha > -1$, we have that the integration constant $c > 0$ and the solution (60) represents a universe with a finite time span that begins with a big bang at $t = 0$, then passes by a maximum at $t_{\max} = \pi/\omega$ and ends in a big crunch at $t_{bc} = 2\pi/\omega$ [6]. However, for $\alpha < -1$ and $c < 0$ there is a significant difference in the behavior of the solution (60) provided the universe has a finite time span but begins with a contracting phase at $t = 0$ associated with an infinite scale factor, then bounces at $t_{\text{bounce}} = \pi/\omega$ and ends with an infinite scale factor in a big rip singularity at $t_{br} = 2\pi/\omega$ [6]. Near the big bang, big crunch and big rip singularities, the energy density (61), the pressure (64), interaction term (65) and the matter and the VVE

densities (46) diverge, namely $\rho \rightarrow \infty$, $p \rightarrow -\infty$, $Q \rightarrow +\infty$, $\rho_m \rightarrow -\infty$, and $\rho_x \rightarrow +\infty$. At the extrema of the scale factor (60) $t_{\text{max}/\text{bounce}} = \pi/\omega$, we have that ρ vanishes, both p and Q become constants $p = c\gamma_m$, $Q = \alpha p$ while $\rho_m = \rho_x = 0$.

For $\omega^2 = -3c\gamma_m^2(1 + \alpha) > 0$ and $\alpha > -1$, we have that the integration constant $c < 0$ and the solution (62) describes a universe with an initial contracting de Sitter phase in the remote past, then bounces at $t = 0$ and ends in an expanding de Sitter stage in the remote future. For $\alpha < -1$ and $c > 0$, the universe evolves from a zero radius (vanishing scale factor) at the distant past, then passes by a maximum at $t = 0$ and ends in the remote future, again, with a vanishing scale factor. The energy density (63), pressure (64), the interaction term (65) and the matter and VVE densities (30) approach to the constants $\rho \rightarrow 3c^2\gamma_m^2/\omega^2$, $p \rightarrow -3c^2\gamma_m^2/\omega^2 = -\rho$, $Q \rightarrow \alpha\gamma_m(1 - c)$, $\rho_m \rightarrow -c$ and $\rho_x \rightarrow c\alpha/(1 + \alpha)$ as time $t \rightarrow \pm\infty$. At $t = 0$ the scale factor exhibits an extreme where $\rho = 0$, $p = c\gamma_m$, $Q = \alpha p$ and $\rho_m = \rho_x = 0$.

6. ($0 < \nu < 1$)-case **Big Freeze**

The big freeze singularity was firstly encountered in the literature by Bouhmadi-López *et al.* within the context of a phantom generalized Chaplygin gas [11]. Further, they proved that this singularity at a finite scale factor arises in a Randall-Sundrum I brane-world scenario if the brane is filled with the dual version of the generalized phantom Chaplygin gas [11]. Besides, the avoidance of this singularity within the context of quantum cosmology was explored with the help of the Wheeler-de Witt equation by mimicking a (phantom) generalized Chaplygin gas with a scalar field [12].

The scale factor (38) approaches to a finite value $a \rightarrow a_s$ in the limit $t \rightarrow t_s$. However, its time derivative \dot{a} and their subsequent ones $a^{(k)}$ with $k > 1$ diverge for $t \rightarrow t_s$, indicating the existence of the big freeze singularity [10]. Concerning the Hubble variable (41), the energy density (36), the pressure (37), the barotropic index (45) and the interaction term (40), these and their higher order time derivatives diverge in the limit $t \rightarrow t_s$. Then, we have a divergent behavior of the dark matter and VVE density as $t \rightarrow t_s$.

C. Krolak and Tipler criteria

In what follows, we give a reinterpretation of the above classification making a description of the singularities and/or abrupt events from a geometric point of view based on a method developed by Tipler [17] and Królak [18]. A spacetime is Tipler strong [17] iff as the proper time $t \rightarrow t_s$, the integral

$$\mathcal{T}(t) = \int_0^t dt' \int_0^{t'} |\mathcal{R}_{ab}u^a u^b| dt'' \rightarrow \infty. \quad (66)$$

In same manner, a spacetime is Królak strong [18] iff as the proper time $t \rightarrow t_s$, the integral

$$\mathcal{K}(t) = \int_0^t |\mathcal{R}_{ab}u^a u^b| dt \rightarrow \infty, \quad (67)$$

where the component of Ricci tensor are understood to be written in a parallel transported frame along the geodesic curves. Notice that a singularity can be strong by Królak criteria but weak according to Tipler's criteria, however, the reverse situation always holds. Because weak singularities can be extended beyond them, the method developed by Tipler and Królak are useful tools for determining the fate of the universe in terms of the fate of geodesic curves near potential strong singular point.

Let us consider time-like geodesic curves, $x^i = c$ with i spatial index and c a constant [25], associated with co-moving observer, i.e, we take into account a co-moving world-line congruence with velocity $u^a = (\partial_t)^a = (1, 0, 0, 0)$ so that the proper time and the coordinate time are the same. Moreover, the components of the Ricci tensor measured by an observer along this congruence lead to $\mathcal{R}_{ab}u^a u^b = -3\ddot{a}(t)/a(t)$. Using the latter fact together (38) and (42) is not difficult to see that a big brake singularity is T-weak and K-weak provided both integrals (66)-(67) vanish for $\nu \in (1, 2)$, namely $\mathcal{T} \propto \Delta\tau^\nu$ and $\mathcal{K} \propto \Delta\tau^{\nu-1}$ as $\Delta\tau \rightarrow 0$. Nevertheless, the leading term in the square Riemann $\mathcal{R}^{abcd}\mathcal{R}_{abcd} \propto (\ddot{a}/a)^2$, given by $\Delta\tau^{2(\nu-2)}$, diverges as $\Delta\tau \rightarrow 0$ [26]. Also the Ricci scalar, $\mathcal{R} \propto \ddot{a}/a \propto \Delta\tau^{\nu-2}$, blows up at the singularity [26]. A big separation singularity can be considered as a weaker event provided is not only T-weak and K-weak but also $\mathcal{R}^{abcd}\mathcal{R}_{abcd}$ and \mathcal{R} both vanish at the singularity. The "new w -singularity" is T/K-weak and avoids divergences in geometric invariant as $\mathcal{R}^{abcd}\mathcal{R}_{abcd}$ in the asymptotic past or remote future, indeed a similar situation occurs for the infinite γ -singularity. On the other hand, the behavior of scale factor near a big rip singularity is $a(t) \propto (t - t_{\text{br}})^{-p}$ with $p > 0$, so we classify it as a K/T-strong singularity, and all scalar invariants blow up at t_{br} . Besides, one could expect that both big bang and big crunch singularity exhibit a similar behavior regarding the K/T criteria or the blow up of $\mathcal{R}^{abcd}\mathcal{R}_{abcd}$. In the case of a big freeze ultimate fate, the leading term in Krolak strength gives $\mathcal{K} \propto \Delta\tau^{\nu-1}$ so it diverges as $t \rightarrow t_s$, however, Tipler measure involves a second integration and yields $\mathcal{T} \propto (t - t_s)^\nu$, being totally regular as $t \rightarrow t_s$ provided $\nu \in (0, 1)$. Notice that the squared Riemann and the Ricci scalar both diverge for the big freeze event, as can be seen from (38) and (42). It is important to stress that the analysis performed with casual geodesic which meets a T/K -strong singularity gave the same kind of finding because involves the integral of component Riemann tensor parallel transported, $\mathcal{R}^a_{bcd}u^b u^d$, for which the non-vanishing components turned to be the same ($\mathcal{R}^a_{tct} = -\ddot{a}/a$) also [15]. We end this section by studying the violation or not of the energy conditions for the aforesaid singularities [27]. Interestingly enough, we obtained that all the energy conditions (WEC, NEC,

SEC) are fulfilled near the singularities except for a few cases, namely the big rip and the infinite γ singularity (associated with the expanding solution a_1) both violate SEC [27].

An open question related with the present work is what kind of fundamental theory can support these singularities? In this direction, Barrow and Graham have recently shown that a new ultra-weak generalized sudden singularity can be supported by a scalar field with a simple power-law potential [28]. We hope to return to this issue in the near future.

IV. CONCLUSIONS

We have investigated the evolution of an interacting dark sector model in a spatially flat FRW space-time, solved exactly the source equation when the energy exchange is produced by a nonlinear interaction term and found that the equation of state of the effective dark fluid turned to be that of a Chaplygin or anti-Chaplygin gas according to the sign of the coupling constant. We highlighted that this interacting dark matter-VVE model, generated by a nonlinear interaction term, harbored future-like singularities related with the ultimate fate of the universe and two abrupt events called infinite γ singularity and new w -singularity.

We have obtained the approximate scale factor in the vicinity of the singular events and used it to study physically viable models describing the current along with final

state of the universe, in particular, we have examined the behavior of the interaction and the dark component energy densities near the singularities. We have classified entirely the singularities of the model for any value of the main parameter ν while for certain value of ν , we have shown that the two-fluid system is fully integrable, so the exact form of the scale factor and the remaining representative quantities of the model were found.

Using the aforesaid scale factor and the geometric method developed by Tipler and Królak for the case of time-like geodesic curves, associated with co-moving observer, we have obtained a regular behavior of Tipler measure in the case of big separation, big brake, big freeze, new w -singularity, and the infinite γ -singularity, making them traversable in the sense that a time-like geodesic curves can be extended beyond such singular events. However, big freeze singularities are K-strong due to Królak measure becomes divergent near them. On the other side, we have shown that big rip singularity cannot be smoothed within interacting scenarios and therefore Tipler and Królak measures diverges as geodesics meet such final event.

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