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Newton's method and a mesh-independence principle for certain semilinear boundary-value problems

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ABSTRACT

We exhibit an algorithm which computes an ϵ-approximation of the positive solutions of a family of boundary-value problems with Neumann boundary conditions. Such solutions arise as the stationary solutions of a family of semilinear parabolic equations with Neumann boundary conditions. The algorithm is based on a finite-dimensional Newton iteration associated with a suitable discretized version of the problem under consideration. To determine the behavior of such a discrete iteration we establish an explicit mesh-independence principle. We apply a homotopy-continuation algorithm to compute a starting point of the discrete Newton iteration, and the discrete Newton iteration until an ϵ-approximation of the stationary solution is obtained. The algorithm performs roughly $\mathcal{O}((1/\epsilon)^{1/2})$ flops and function evaluations.

1. Introduction

Consider the following boundary-value problem with Neumann boundary conditions:

$$\begin{cases} \boldsymbol{u}''(x) = g(\boldsymbol{u}(x)) & \text{in } (0,\ell), \\ \boldsymbol{u}'(\ell) = f\big(\boldsymbol{u}(\ell)\big), \\ \boldsymbol{u}'(0) = 0, \end{cases} \tag{1}$$

where $f, g : \mathbb{R}_{\geq 0} \to \mathbb{R}$ are nonnegative nondecreasing functions of class C^2 and ℓ is a positive real. We are interested in the positive solutions of (1). Such solutions arise as the stationary solutions of the semilinear heat equation with Neumann boundary conditions. In particular, the dynamics of the latter are usually described in terms of the solutions of the former (see, e.g., [1–4]).

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The term $f\big(\boldsymbol{u}(\ell)\big)$ corresponds to a "boundary flux" term $f\big(\boldsymbol{u}(\ell,t)\big)$ in the semilinear heat equation, while the term $g(\boldsymbol{u})$ arises as a "reaction" term in the semilinear heat equation. When any of these two terms are absent, conditions on f and g are known which imply either the global existence and boundedness of solutions of the semilinear heat equation or blow-up in finite time. On the other hand, in presence of both terms, the corresponding solutions admit an interesting asymptotic behavior which strongly depends on f and g and is described in terms of the stationary solutions (see, e.g., [2,5,1,6] and the references therein).

The usual numerical approach to the solutions of (1) consists of considering a second-order finite-difference discretization, with a uniform mesh. For a given mesh size, the solutions of the discretization of (1) were studied in [7,3,4,8,9], for different conditions concerning g and f. In this paper we consider the behavior of the discretization of the solutions (1) as the mesh size tends to zero, that is, we aim to approximate the solutions of (1). As a case study, we analyze the case where f is constant, for which global existence and boundedness of solutions is known.

In the process of approximation of discrete and continuous solutions of (1) for constant f we shall use the Newton method, on a certain closed convex subset $\mathcal{X}$ of a Banach space and on suitable closed convex subsets $\mathcal{X}_j$ of finite-dimensional vector spaces. More precisely, the set $\mathcal{X}$ we consider is that of the twice continuously differentiable functions which satisfy the boundary conditions in (1), endowed with a suitable norm. On the other hand, the sets $\mathcal{X}_j$ are formed by complete cubic splines satisfying the boundary conditions.

In order to keep track of the relation between continuous and discrete Newton iterations we establish an explicit mesh-independence principle for (1). Generally speaking, a mesh-independence principle asserts that, when the Newton method is applied to a nonlinear equation between Banach spaces, as well as to some finite-dimensional discretization of that equation, the behavior of the corresponding continuous and discrete Newton iterations is essentially the same, provided that the discretization is sufficiently fine (see, e.g., [10–13] or [14, Section 8.1]). Such mesh-independence principles are usually stated in terms of certain Lipschitz constants associated with the behavior of the corresponding Newton operator.

In this paper we establish explicit values for the Lipschitz constants mentioned before in terms of the parameters defining the family of problems (1) under consideration. As a consequence, we determine an explicit mesh size h^* such that for $h \leq h^*$ the discrete Newton iterations associated with (1) and mesh size h differ from the continuous ones by a factor which is determined by the precision of the mesh (Theorem 29). For this purpose, we rely on a general framework on mesh-independence principles developed in [13] (see also [14, Section 8.1]), which is based on an invariant version of the Newton–Mysovskikh theorem. In Theorem 13 we obtain an explicit version of the latter for the convergence of the Newton iteration to the positive solution of (1).

Then we consider the computation of a starting point for the discrete Newton iteration with mesh size h^*. Combining an algorithm of [4] or [8] for the approximation of the discrete solutions of (1) with mesh size h^* and estimates provided by our mesh-independence principle we obtain an algorithm which computes a starting point (Theorem 36). Using this starting point and a discrete Newton iteration we obtain an ϵ-approximation of the positive solution of (1) with $\mathcal{O}((1/\epsilon)^{1/2}\log_2\log_2(1/\epsilon))$ flops and function evaluations (Theorem 37).

There is a well-established framework for the analysis of the ϵ-complexity (that is, the optimal complexity of finding an ϵ-approximation) of the solutions of linear boundary-value problems or initial-value problems for differential equations (see, e.g., [15–17]). On the other hand, the ϵ-complexity of nonlinear boundary-value problems is far from been understood. To the best of our knowledge, only mildly nonlinear boundary-value problems for ordinary differential equations with Dirichlet conditions have been considered so far (see, e.g., [18–20]). Furthermore, global boundedness of the function g of (1) is usually assumed. The paradigm arising from these papers is that optimal ϵ-complexity should be of order $(1/\epsilon)^{1/r}$ for problems defined by functions of class C^r. We contribute to this stream of work with the analysis of the ϵ-complexity of a family of boundary-value problems with Neumann conditions which matches this optimal ϵ-complexity paradigm. We remark that, unlike these previous works, no requirements of global boundedness of the function g of (1) have been imposed. Besides, our algorithm is stable, in the sense that the $\mathcal{O}$-constant in our complexity estimate behaves well for well-conditioned input instances (see Remark 4).

The paper is organized as follows. In Section 2 we show that the instance of (1) under consideration has a unique positive solution $\boldsymbol{x}^*$, and provide upper and lower bounds for it. In Section 3 we obtain an explicit version of the Newton–Mysovskikh theorem for the convergence of the Newton iteration to $\boldsymbol{x}^*$. Section 4 is devoted to the mesh-independence principle. Finally, in Section 5 we discuss the computation of the starting point for the discrete Newton iteration with mesh size h^* and the computation of an ϵ-approximation of $\boldsymbol{x}^*$.

2. Existence and uniqueness of the problem under consideration

As expressed in the Introduction, we consider the boundary-value problem

$$\begin{cases} \boldsymbol{u}''(x) = g\big(\boldsymbol{u}(x)\big), & x \in (0,\ell) \\ \boldsymbol{u}'(0) = 0, \\ \boldsymbol{u}'(\ell) = \alpha > 0, \end{cases} \tag{2}$$

where $g : \mathbb{R}_{\geq 0} \to \mathbb{R}$ is an increasing convex C^2 function with $g(0) = g'(0) = g''(0) = 0$. We shall assume further that g'' is Lipschitz continuous on any compact interval of $\mathbb{R}_{\geq 0}$. In particular, any power-law nonlinearity $g(x) := x^p$ with $p \geq 3$ may be considered as a prototype for our model.

For the sake of completeness, in this section we show that there exists a unique positive solution of (2) and obtain lower and upper bounds for such a solution.

Denote by $G : \mathbb{R}_{\geq 0} \to \mathbb{R}$ the primitive of g with $G(0) = 0$. Let $\boldsymbol{u}$ be a positive solution of (2). Integrating $\boldsymbol{u}''$ in the interval $[0, x]$ for any $x \in [0, \ell]$, we obtain

$$\boldsymbol{u}'(x) = \sqrt{2\big(G(\boldsymbol{u}(x)) - G(\boldsymbol{u}(0))\big)}.$$

Substituting ℓ for x yields

$$\alpha = \sqrt{2\big(G(\boldsymbol{u}(\ell)) - G(\boldsymbol{u}(0))\big)}. \tag{3}$$

On the other hand, from the previous expression for $\boldsymbol{u}'(x)$ we deduce that

$$\sqrt{2} = \frac{1}{\ell}\int_0^{\ell} \frac{\boldsymbol{u}'(x)\,dx}{\sqrt{G(\boldsymbol{u}(x)) - G(\boldsymbol{u}(0))}}.$$

Applying the change of variables $t(x) := G(\boldsymbol{u}(x)) - G(\boldsymbol{u}(0))$ to the integral on the right-hand side of this identity and taking into account (3), we see that

$$B(s) = \sqrt{2} \tag{4}$$

where $s := G(u(0))$ and $B : [0, +\infty) \to \mathbb{R}$ is the following function:

$$B(s) := \frac{1}{\ell}\int_0^{\frac{\alpha^2}{2}} \frac{dt}{\sqrt{t}\,(g \circ G^{-1})(t+s)}.$$

Observe that $t \mapsto (g \circ G^{-1})(t+s)$ is an increasing function on $\mathbb{R}_{\geq 0}$. Therefore,

$$\frac{\sqrt{2}\,\alpha}{\ell\,(g \circ G^{-1})(s + \alpha^2/2)} \leq B(s) \leq \frac{\sqrt{2}\,\alpha}{\ell\,(g \circ G^{-1})(s)}.$$

We conclude that $B(s_0) \leq \sqrt{2} \leq B(s_1)$, where $s_0 := (G \circ g^{-1})(\alpha/\ell)$ and $s_1 := (G \circ g^{-1})(\alpha/\ell) - \alpha^2/2$. As B is a decreasing function, there exists a unique $s_1 < s^* < s_0$ satisfying (4). As a consequence, we have the following result.

Lemma 1. *For any $\ell > 0$, there exists a unique positive solution of (2).*

By the definition of s^* and (3) it follows that

$$\boldsymbol{u}(0) = G^{-1}(s^*), \qquad \boldsymbol{u}(\ell) = G^{-1}(s^* + \alpha^2/2). \tag{5}$$

In particular, (5) shows that $\boldsymbol{u}$ is the solution of the following initial-value problem:

$$\begin{cases} \boldsymbol{u}''(x) = g(\boldsymbol{u})(x), & x \in (0, \ell) \\ \boldsymbol{u}(0) = G^{-1}(s^*), \\ \boldsymbol{u}'(0) = 0. \end{cases}$$

In order to obtain upper and lower bounds for $\boldsymbol{u}$, combining (5) with the definition of s_0 and s_1, we obtain

$$\boldsymbol{u}(0) \geq m := G^{-1}(s_1) = G^{-1}\big((G \circ g^{-1})(\alpha/\ell) - \alpha^2/2\big),$$
$$\boldsymbol{u}(\ell) \leq M := G^{-1}(s_0 + \alpha^2/2) = G^{-1}\big((G \circ g^{-1})(\alpha/\ell) + \alpha^2/2\big).$$

Since $\boldsymbol{u}$ is an increasing function, m and M are a lower and an upper bound for $\boldsymbol{u}$ in the interval $[0, \ell]$. Our results will be expressed in terms of these bounds.

3. On the convergence of Newton's method

In this section we obtain conditions which imply the convergence of Newton's method applied to (2). For this purpose, we shall use the "invariant" version of the Newton–Mysovskikh theorem of [13], which we now describe.

Let X, Y be Banach spaces. For $\boldsymbol{x} \in X$ and $\rho > 0$, we denote by $S(\boldsymbol{x}, \rho)$ the open ball with center $\boldsymbol{x}$ and radius ρ and by $\overline{S}(\boldsymbol{x}, \rho)$ its closure. Let $D \subset X$ be a convex domain and $F : D \to Y$ a nonlinear operator of class C^1. Suppose that the equation $F(\boldsymbol{x}) = 0$ has a unique solution $\boldsymbol{x}^* \in D$.

The Newton method consists of the iteration

$$\boldsymbol{x}^{k+1} = \boldsymbol{x}^k + \Delta\boldsymbol{x}^k, \qquad F'(\boldsymbol{x}^k)\Delta\boldsymbol{x}^k = -F(\boldsymbol{x}^k) \quad (k \geq 0), \tag{6}$$

assuming that the derivatives $F'(\boldsymbol{x}^k)$ are invertible. We have the following convergence result.

Theorem 2 (*[13, Theorem 1.1]*). *Let $\|\cdot\|$ be the norm of X. Suppose that:*

(1) *$F'(\boldsymbol{x})$ is invertible for each $\boldsymbol{x} \in D$.*
(2) *For collinear $\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z} \in D$, the following affine invariant Lipschitz condition is satisfied:*

$$\left\|F'(\boldsymbol{z})^{-1}\big(F'(\boldsymbol{y}) - F'(\boldsymbol{x})\big)\boldsymbol{v}\right\| \le \omega\|\boldsymbol{y} - \boldsymbol{x}\|\,\|\boldsymbol{v}\|. \tag{7}$$

Let $\boldsymbol{x}^0 \in D$ be such that

$$h_0 := \omega\|\Delta\boldsymbol{x}^0\| < 2 \quad \text{and} \quad \overline{S}(\boldsymbol{x}^0, \rho) \subset D, \quad \text{where } \rho := \frac{\|\Delta\boldsymbol{x}^0\|}{1 - h_0/2}.$$

Then the Newton sequence $(\boldsymbol{x}^k)_{k\ge 0}$ of (6) remains in $S(\boldsymbol{x}^0, \rho)$ and converges to the unique solution $\boldsymbol{x}^* \in \overline{S}(\boldsymbol{x}^0, \rho)$. Furthermore, we have

$$\|\boldsymbol{x}^{k+1} - \boldsymbol{x}^k\| \le \frac{1}{2}\omega\|\boldsymbol{x}^k - \boldsymbol{x}^{k-1}\|^2.$$

3.1. Well-definedness of Newton's method applied to (2)

For the analysis of convergence of the Newton method applied to (2), we consider the following closed and convex subset of the Banach space $C^2([0,\ell])$ of functions of class C^2 on the interval $[0,\ell]$ with the norm $\|\boldsymbol{x}\| := \sum_{i=0}^{2}\|\boldsymbol{x}^{(i)}\|_\infty$:

$$\mathcal{X} := \{\boldsymbol{x} \in C^2([0,\ell]) : \boldsymbol{x} \ge 0,\ \boldsymbol{x}'(0) = \boldsymbol{x}'(\ell) - \alpha = 0\}.$$

Recall that $m > 0$ and $M > 0$ are lower and upper bounds for the unique positive solution $\boldsymbol{u}$ of (2). Our results will be expressed in terms of the following quantities:

$$\tilde{m} := \frac{m}{2}, \qquad \hat{\ell} := \max\left\{\frac{\tilde{m}}{\ell g(\tilde{m})}, \ell\right\},$$
$$\tilde{M} := \max\left\{\frac{3M}{2}, 4\hat{\ell}\alpha\right\}, \qquad A := \min\left\{\frac{1}{8\ell^2\hat{\ell}^2 g''(\tilde{M})}, \frac{\tilde{m}}{\hat{\ell}\ell}\right\}, \qquad \omega := 2\hat{\ell}\ell g''(\tilde{M}).$$

Remark 3. As g is an increasing convex C^2 function with $g(0) = g'(0) = 0$, it follows that $g(\tilde{m})/\tilde{m} \le g'(\tilde{m})$. This implies the f ollowing inequality, which shall be frequently used in the next lemmas:

$$\frac{1}{\ell g'(\tilde{m})} \le \frac{\tilde{m}}{\ell g(\tilde{m})} \le \hat{\ell}.$$

Remark 4. The number $2\hat{\ell}\ell$ may be considered as a condition number of the boundary-value problem (2). Indeed, consider the following perturbation of (2):

$$\begin{cases} \boldsymbol{v}''(x) = (g + \Delta g)\big(\boldsymbol{v}(x)\big), & x \in (0,\ell) \\ \boldsymbol{v}'(0) = 0, \\ \boldsymbol{v}'(\ell) = \alpha + \Delta\alpha > 0, \end{cases} \tag{8}$$

where $g + \Delta g : \mathbb{R}_{\ge 0} \to \mathbb{R}$ is an increasing convex C^2 function with $(g + \Delta g)(0) = (g + \Delta g)'(0) = (g + \Delta g)''(0) = 0$. Existence and uniqueness of solutions of (8) is proved as in Section 2. Denote by $\boldsymbol{w} := \boldsymbol{v} - \boldsymbol{u}$ the difference between the unique solutions of (2) and (8) respectively. Arguing as in Lemmas 6 and 7, it can be shown that, for Δg and $\Delta\alpha$ sufficiently small,

$$\|\boldsymbol{w}\|_\infty \le 2\hat{\ell}(\ell\|\Delta g\|_\infty + |\Delta\alpha|).$$

This proves that $2\hat{\ell}\ell$ is an upper bound for the condition number of (2).

Denote by $\mathcal{D}$ the following (non-convex) subset of $\mathcal{X}$:

$$\mathcal{D} := \{\boldsymbol{z} \in \mathcal{X} : \tilde{m} < \boldsymbol{z} < \tilde{M},\ \|\boldsymbol{z}'' - g(\boldsymbol{z})\|_\infty < A\},$$

where the inequality $\boldsymbol{u} > \beta$ or $\boldsymbol{u} < \gamma$ for $\boldsymbol{u} \in \mathcal{X}$ and $\beta, \gamma \in \mathbb{R}$ means $\boldsymbol{u}(x) > \beta$ or $\boldsymbol{u}(x) < \gamma$ for any $x \in [0,\ell]$. For $\boldsymbol{z} \in \mathcal{D}$, the Newton operator N associated with (2) is defined in the following way:

$$\begin{aligned} N : \mathcal{D} \times \{0\} \times \{0\} &\to \mathcal{C}([0,\ell]) \times \{0\} \times \{0\}, \\ \boldsymbol{z} &\mapsto \boldsymbol{y} := \boldsymbol{z} + \boldsymbol{w}, \end{aligned} \tag{9}$$

where $\boldsymbol{w}$ is the solution of the problem $F'(\boldsymbol{z})\boldsymbol{w} = -F(\boldsymbol{z})$, with

$$F(\boldsymbol{z}) := (F_1, F_2, F_3)^t(\boldsymbol{z}) := \begin{pmatrix} \boldsymbol{z}'' - g(\boldsymbol{z}) \\ \boldsymbol{z}'(0) \\ \boldsymbol{z}'(\ell) - \alpha \end{pmatrix}, \qquad F'(\boldsymbol{z}) := \begin{pmatrix} D^2 - g'(\boldsymbol{z})I \\ D|_{x=0} \\ D|_{x=\ell} \end{pmatrix}.$$

According to these definitions, the term $\boldsymbol{w} := -F'(\boldsymbol{z})^{-1}F(\boldsymbol{z})$ is defined by

$$\begin{cases} (\boldsymbol{w}'' - g'(\boldsymbol{z})\boldsymbol{w})(x) = (-\boldsymbol{z}'' + g(\boldsymbol{z}))(x), & x \in (0, \ell) \\ \boldsymbol{w}'(0) = 0, \\ \boldsymbol{w}'(\ell) = 0. \end{cases} \tag{10}$$

As a consequence, $\boldsymbol{y} := N(\boldsymbol{z})$ is the solution of the problem

$$\begin{cases} \boldsymbol{y}''(x) = (g'(\boldsymbol{z})\boldsymbol{y} + g(\boldsymbol{z}) - g'(\boldsymbol{z})\boldsymbol{z})(x), & x \in (0, \ell) \\ \boldsymbol{y}'(0) = 0, \\ \boldsymbol{y}'(\ell) = \alpha. \end{cases} \tag{11}$$

It is clear that Theorem 2 cannot be applied to show the convergence of the Newton sequence (9), because the hypothesis of convexity of the domain D of the statement of Theorem 2 is not satisfied by the set $\mathcal{D} \subset \mathcal{X}$ defined above. We shall nevertheless obtain a variant of Theorem 2 which proves that, starting at an arbitrary element $\boldsymbol{z} \in \mathcal{D}$, the Newton sequence (9) converges.

We start showing that the Newton operator N is well-defined on $\mathcal{D}$.

Lemma 5. *For any $\boldsymbol{z} \in \mathcal{D}$, the operator $F'(\boldsymbol{z})$ is invertible.*

Proof. Let $\boldsymbol{z} \in \mathcal{D}$. We have to show existence and uniqueness of solutions of the boundary-value problem

$$F_1'(\boldsymbol{z})\,\boldsymbol{v} := \boldsymbol{v}'' - g'(\boldsymbol{z})\boldsymbol{v} = \boldsymbol{w}, \quad \boldsymbol{v}'(0) = 0,\ \boldsymbol{v}'(\ell) = 0, \tag{12}$$

for $\boldsymbol{w} \in \mathcal{C}([0, \ell])$. Associated to (12), we have the Sturm–Liouville problem

$$\boldsymbol{v}'' - g'(\boldsymbol{z})\boldsymbol{v} = 0, \quad \boldsymbol{v}'(0) = 0, \qquad \boldsymbol{v}'(\ell) = 0. \tag{13}$$

According to, e.g., [21, Chapter XI, Theorem 4.1], if (13) has no nontrivial solutions, then (12) has a unique solution for any function $\boldsymbol{w}$ which is integrable in $[0, \ell]$. Hence, it suffices to prove that (13) has a unique solution, $\boldsymbol{v} = 0$.

If $\boldsymbol{v}$ is a solution of (13), then

$$\int_0^\ell (\boldsymbol{v}\boldsymbol{v}'')(x)dx - \int_0^\ell (g'(\boldsymbol{z})\boldsymbol{v}^2)(x)dx = 0.$$

Integrating by parts, we deduce that

$$\int_0^\ell \left((\boldsymbol{v}')^2 + g'(\boldsymbol{z})\boldsymbol{v}^2\right)(x)dx = 0.$$

Observe that $\boldsymbol{z} \geq 0$, because $\boldsymbol{z} \in \mathcal{D}$, and then $(\boldsymbol{v}')^2 + g'(\boldsymbol{z})\boldsymbol{v}^2 \geq 0$. This implies $(\boldsymbol{v}')^2 + g'(\boldsymbol{z})\boldsymbol{v}^2 = 0$, which shows that $\boldsymbol{v} = 0$. □

By Lemma 5 it follows that, if $\boldsymbol{z} \in \mathcal{D}$, then the Newton iteration $N(\boldsymbol{z})$ is well-defined. Next we show that $N(\boldsymbol{z}) \in \mathcal{D}$. For this purpose, we first obtain a simple estimate which will be frequently used in the sequel.

Lemma 6. *Let $\boldsymbol{w} \in C^2([0, \ell])$ be such that $\boldsymbol{w}'(0) = 0$ and let $x_0 \in [0, \ell]$ be such that $|\boldsymbol{w}(x_0)| = \min\{|\boldsymbol{w}(x)| : x \in [0, \ell]\}$. Then*

$$\|\boldsymbol{w}\|_\infty \leq \sqrt{2}\left(\boldsymbol{w}(x_0)^2 + \hat{\ell}\,\boldsymbol{w}(\ell)\boldsymbol{w}'(\ell) - \hat{\ell}\int_0^\ell (\boldsymbol{w}\boldsymbol{w}'')(x)dx\right)^{\frac{1}{2}}.$$

Proof. Let $x_1 \in [0, \ell]$ be such that $|\boldsymbol{w}(x_1)| = \max\{|\boldsymbol{w}(x)| : x \in [0, \ell]\}$. Then

$$|\boldsymbol{w}(x_1) - \boldsymbol{w}(x_0)| = \left|\int_{x_0}^{x_1} \boldsymbol{w}'(x)dx\right|.$$

We deduce that

$$\begin{aligned}\|\boldsymbol{w}\|_\infty &= |\boldsymbol{w}(x_1)| \le |\boldsymbol{w}(x_0)| + \left|\int_{x_0}^{x_1} \boldsymbol{w}'(x)dx\right| \le |\boldsymbol{w}(x_0)| + |x_1 - x_0|^{\frac{1}{2}} \left|\int_{x_0}^{x_1} (\boldsymbol{w}')^2(x)dx\right|^{\frac{1}{2}} \\ &\le |\boldsymbol{w}(x_0)| + \ell^{\frac{1}{2}}\left(\int_0^\ell (\boldsymbol{w}')^2(x)dx\right)^{\frac{1}{2}}.\end{aligned}$$

Applying integration by parts on the right-hand side, we obtain

$$\begin{aligned}\|\boldsymbol{w}\|_\infty &\le |\boldsymbol{w}(x_0)| + \hat{\ell}^{\frac{1}{2}}\left(\boldsymbol{w}(\ell)\boldsymbol{w}'(\ell) - \int_0^\ell (\boldsymbol{w}\boldsymbol{w}'')(x)dx\right)^{\frac{1}{2}} \\ &\le \sqrt{2}\left(\boldsymbol{w}(x_0)^2 + \hat{\ell}\,\boldsymbol{w}(\ell)\boldsymbol{w}'(\ell) - \hat{\ell}\int_0^\ell (\boldsymbol{w}\boldsymbol{w}'')(x)dx\right)^{\frac{1}{2}}.\end{aligned}$$

This finishes the proof of the lemma. □

For $\boldsymbol{z} \in \mathcal{D}$, denote $\boldsymbol{y} := N(\boldsymbol{z})$ and $\boldsymbol{w} := \boldsymbol{y} - \boldsymbol{z}$. Now we obtain an upper bound on the infinity norm of the "update" of a Newton iteration.

Lemma 7. *If $\boldsymbol{w} := N(\boldsymbol{z}) - \boldsymbol{z}$ is the solution of (10) for a given $\boldsymbol{z} \in \mathcal{D}$, then*

$$\|\boldsymbol{w}\|_\infty < 2\hat{\ell}\ell A.$$

Proof. Let $x_0 \in [0, \ell]$ be such that $|\boldsymbol{w}(x_0)| = \min\{|\boldsymbol{w}(x)| : x \in [0, \ell]\}$. By Lemma 6,

$$\|\boldsymbol{w}\|_\infty \le \sqrt{2}\left(\boldsymbol{w}(x_0)^2 - \hat{\ell}\int_0^\ell (\boldsymbol{w}\boldsymbol{w}'')(x)dx\right)^{\frac{1}{2}}.$$

According to (10) we have $\boldsymbol{w}'' = g'(\boldsymbol{z})\boldsymbol{w} - \boldsymbol{z}'' + g(\boldsymbol{z})$. Therefore,

$$\begin{aligned}\|\boldsymbol{w}\|_\infty &\le \sqrt{2}\left(\boldsymbol{w}(x_0)^2 + \hat{\ell}\int_0^\ell \big(\boldsymbol{w}\boldsymbol{z}'' - \boldsymbol{w}g(\boldsymbol{z}) - \boldsymbol{w}^2 g'(\boldsymbol{z})\big)(x)dx\right)^{\frac{1}{2}} \\ &\le \sqrt{2}\left(\frac{1}{\ell g'(\tilde{m})}\int_0^\ell \big(\boldsymbol{w}^2 g'(\boldsymbol{z})\big)(x)dx + \hat{\ell}\int_0^\ell \big(\boldsymbol{w}\boldsymbol{z}'' - \boldsymbol{w}g(\boldsymbol{z}) - \boldsymbol{w}^2 g'(\boldsymbol{z})\big)(x)dx\right)^{\frac{1}{2}} \\ &\le \sqrt{2}\left(\hat{\ell}\int_0^\ell \big(\boldsymbol{w}\boldsymbol{z}'' - \boldsymbol{w}g(\boldsymbol{z})\big)(x)dx\right)^{\frac{1}{2}} \le (2\hat{\ell}\ell)^{\frac{1}{2}}\|\boldsymbol{z}'' - g(\boldsymbol{z})\|_\infty^{\frac{1}{2}}\|\boldsymbol{w}\|_\infty^{\frac{1}{2}}.\end{aligned}$$

We conclude that $\|\boldsymbol{w}\|_\infty < 2\hat{\ell}\ell A$, finishing thus the proof of the lemma. □

The aim of the next four lemmas is to show that $\boldsymbol{y} := N(\boldsymbol{z})$ belongs to $\mathcal{D}$. For this purpose, we first prove that $\boldsymbol{y}$ satisfies the upper bound in the definition of $\mathcal{D}$.

Lemma 8. *If $\boldsymbol{y} := N(\boldsymbol{z})$ is the solution of (11) for a given $\boldsymbol{z} \in \mathcal{D}$, then*

$$\|\boldsymbol{y}\|_\infty < 3\tilde{M}/4 < \tilde{M}.$$

Proof. Let $x_0 \in [0, \ell]$ be such that $|\boldsymbol{y}(x_0)| = \min\{|\boldsymbol{y}(x)| : x \in [0, \ell]\}$. By Lemma 6,

$$\|\boldsymbol{y}\|_\infty \le \sqrt{2}\left(\boldsymbol{y}(x_0)^2 + \hat{\ell}\,\boldsymbol{y}(\ell)\,\alpha - \hat{\ell}\int_0^\ell (\boldsymbol{y}\boldsymbol{y}'')(x)dx\right)^{\frac{1}{2}}.$$

By (11) it follows that $\boldsymbol{y}'' = \big(g'(\boldsymbol{z}) - \frac{g(\boldsymbol{z})}{\boldsymbol{z}}\big)\boldsymbol{w} + \frac{g(\boldsymbol{z})}{\boldsymbol{z}}\boldsymbol{y}$. As a consequence,

$$\begin{aligned}\|\boldsymbol{y}\|_\infty &\le \sqrt{2}\left(\boldsymbol{y}(x_0)^2 + \hat{\ell}\,\boldsymbol{y}(\ell)\,\alpha - \hat{\ell}\int_0^\ell \left(\Big(g'(\boldsymbol{z}) - \frac{g(\boldsymbol{z})}{\boldsymbol{z}}\Big)\boldsymbol{w}\boldsymbol{y} + \frac{g(\boldsymbol{z})}{\boldsymbol{z}}\boldsymbol{y}^2\right)(x)dx\right)^{\frac{1}{2}} \\ &\le \sqrt{2}\left(\Big(\frac{\tilde{m}}{\ell g(\tilde{m})} - \hat{\ell}\Big)\int_0^\ell \Big(\frac{g(\boldsymbol{z})}{\boldsymbol{z}}\boldsymbol{y}^2\Big)(x)dx + \hat{\ell}\,\boldsymbol{y}(\ell)\,\alpha - \hat{\ell}\int_0^\ell \left(\Big(g'(\boldsymbol{z}) - \frac{g(\boldsymbol{z})}{\boldsymbol{z}}\Big)\boldsymbol{w}\boldsymbol{y}\right)(x)dx\right)^{\frac{1}{2}}\end{aligned}$$

$$\leq \sqrt{2}\Big(\hat{\ell}\,\boldsymbol{y}(\ell)\,\alpha - \hat{\ell}\int_0^{\ell}\Big(\Big(g'(\boldsymbol{z}) - \frac{g(\boldsymbol{z})}{\boldsymbol{z}}\Big)\boldsymbol{w}\boldsymbol{y}\Big)(x)dx\Big)^{\frac{1}{2}}$$

$$\leq \Big(2\hat{\ell}\|\boldsymbol{y}\|_\infty\,\alpha + 2\hat{\ell}\ell\|\boldsymbol{y}\|_\infty\Big\|g'(\boldsymbol{z}) - \frac{g(\boldsymbol{z})}{\boldsymbol{z}}\Big\|_\infty\|\boldsymbol{w}\|_\infty\Big)^{\frac{1}{2}}.$$

By the Taylor theorem, considering the function g in a neighborhood of the point $\boldsymbol{z}(x)$, evaluated at 0, we deduce that there exists $\xi(x)$ in the interval $[0, \boldsymbol{z}(x)]$ such that $\big(g'(\boldsymbol{z}) - g(\boldsymbol{z})/\boldsymbol{z}\big)(x) = -g''(\xi(x))\boldsymbol{z}/2$. Therefore, Lemma 7 implies

$$\begin{aligned}\|\boldsymbol{y}\|_\infty &\leq 2\hat{\ell}\alpha + 2\hat{\ell}\ell\|g''(\xi)\,\boldsymbol{z}/2\|_\infty\|\boldsymbol{w}\|_\infty\\ &\leq 2\hat{\ell}\alpha + \hat{\ell}\ell g''(\tilde{M})\|\boldsymbol{z}\|_\infty\|\boldsymbol{w}\|_\infty < 2\hat{\ell}\alpha + 2\hat{\ell}^2\ell^2 g''(\tilde{M})\tilde{M}A.\end{aligned}$$

Taking into account that

$$2\hat{\ell}\alpha + 2\hat{\ell}^2\ell^2 g''(\tilde{M})\tilde{M}A \leq \big(1/2 + 2\hat{\ell}^2\ell^2 g''(\tilde{M})A\big)\tilde{M} \leq 3\tilde{M}/4,$$

where the last inequality is a consequence of the definition of A, we deduce the statement of the lemma. □

Next we prove that the differential operator $\boldsymbol{y} \mapsto \boldsymbol{y}'' - g(\boldsymbol{y})$ maps elements of $\mathcal{D}$ to elements of small infinity norm.

Lemma 9. *If $\boldsymbol{y} := N(\boldsymbol{z})$ is the solution of (11) for a given $\boldsymbol{z} \in \mathcal{D}$, then*

$$\|\boldsymbol{y}'' - g(\boldsymbol{y})\|_\infty < A/4 < A.$$

Proof. According to (11), we have

$$\boldsymbol{y}'' - g(\boldsymbol{y}) = g'(\boldsymbol{z})(\boldsymbol{y} - \boldsymbol{z}) + g(\boldsymbol{z}) - g(\boldsymbol{y}).$$

By the Taylor theorem, considering the function g in a neighborhood of the point $\boldsymbol{z}(x)$, evaluated at $\boldsymbol{y}(x)$, we see that there exists $\xi(x)$ in the real interval defined by $\boldsymbol{y}(x)$ and $\boldsymbol{z}(x)$ such that

$$\big(\boldsymbol{y}'' - g(\boldsymbol{y})\big)(x) = -\frac{1}{2}g''(\xi(x))\big((\boldsymbol{y} - \boldsymbol{z})(x)\big)^2 = -\frac{1}{2}g''(\xi(x))\boldsymbol{w}(x)^2. \tag{14}$$

By the definition of $\mathcal{D}$ and Lemma 8 we conclude that $|\xi(x)| < \tilde{M}$. As a consequence, by Lemma 7 it follows that

$$\|\boldsymbol{y}'' - g(\boldsymbol{y})\|_\infty < 2g''(\tilde{M})(\hat{\ell}\ell A)^2.$$

Since $8\,\ell^2\hat{\ell}^2 g''(\tilde{M})A \leq 1$, we readily deduce the statement of the lemma. □

Our next result asserts that $\mathcal{D}$ is contained in a ball of small radius in the infinity norm whose center is the positive solution of (2).

Lemma 10. *If $\boldsymbol{z} \in \mathcal{D}$ and $\boldsymbol{x}^*$ is the positive solution of (2), then*

$$\|\boldsymbol{z} - \boldsymbol{x}^*\|_\infty < 2\hat{\ell}\ell A.$$

Proof. By (2), the function $\boldsymbol{v} := \boldsymbol{z} - \boldsymbol{x}^*$ satisfies the following conditions:

$$\begin{cases}\boldsymbol{v}''(x) = \big(\boldsymbol{z}'' - g(\boldsymbol{x}^*)\big)(x), & x \in (0, \ell)\\ \boldsymbol{v}'(0) = 0,\\ \boldsymbol{v}'(\ell) = 0.\end{cases} \tag{15}$$

Let $x_0 \in [0, \ell]$ be such that $|\boldsymbol{v}(x_0)| = \min\{|\boldsymbol{v}(x)| : x \in [0, \ell]\}$. Lemma 6 shows that

$$\|\boldsymbol{v}\|_\infty \leq \sqrt{2}\Big(\boldsymbol{v}(x_0)^2 - \hat{\ell}\int_0^{\ell}(\boldsymbol{v}\boldsymbol{v}'')(x)dx\Big)^{\frac{1}{2}}.$$

Since $\boldsymbol{v}'' = \boldsymbol{z}'' - g(\boldsymbol{x}^*)$, we deduce that

$$\|\boldsymbol{v}\|_\infty \leq \sqrt{2}\Big(\boldsymbol{v}(x_0)^2 - \hat{\ell}\int_0^{\ell}\Big(\boldsymbol{v}\,\big(\boldsymbol{z}'' - g(\boldsymbol{z})\big) + \boldsymbol{v}\,\big(\boldsymbol{z}^p - g(\boldsymbol{x}^*)\big)\Big)(x)dx\Big)^{\frac{1}{2}}.$$

Fix $x \in [0, \ell]$. By the Mean Value theorem there exists $\xi(x)$ in the real interval defined by $\boldsymbol{z}(x)$ and $\boldsymbol{x}^*(x)$ such that $\big(g(\boldsymbol{z}) - g(\boldsymbol{x}^*)\big)(x) = g'(\xi(x))\boldsymbol{v}(x)$. Furthermore, the bounds $\boldsymbol{x}^*(x) > m > \tilde{m}$ and $\boldsymbol{z}(x) > \tilde{m}$ imply $g'(\xi(x)) > g'(\tilde{m})$. Therefore,

$$\begin{aligned}\|\boldsymbol{v}\|_\infty &\leq \sqrt{2}\bigg(|\boldsymbol{v}(x_0)|^2 - \hat{\ell}\int_0^\ell \Big(\boldsymbol{v}\left(\boldsymbol{z}'' - g(\boldsymbol{z})\right)\Big)(x)dx - \hat{\ell}\int_0^\ell \left(\boldsymbol{v}^2 g'(\xi)\right)(x)dx\bigg)^{\frac{1}{2}} \\ &\leq \sqrt{2}\bigg(\Big(\frac{1}{\ell g'(\tilde{m})} - \hat{\ell}\Big)\int_0^\ell \left(\boldsymbol{v}^2 g'(\xi)\right)(x)dx - \hat{\ell}\int_0^\ell \Big(\boldsymbol{v}\left(\boldsymbol{z}'' - g(\boldsymbol{z})\right)\Big)(x)dx\bigg)^{\frac{1}{2}} \\ &\leq \sqrt{2}\bigg(-\hat{\ell}\int_0^\ell \Big(\boldsymbol{v}\left(\boldsymbol{z}'' - g(\boldsymbol{z})\right)\Big)(x)dx\bigg)^{\frac{1}{2}} \leq (2\hat{\ell}\ell)^{\frac{1}{2}}\|\boldsymbol{z}'' - g(\boldsymbol{z})\|_\infty^{\frac{1}{2}}\|\boldsymbol{v}\|_\infty^{\frac{1}{2}}.\end{aligned}$$

Hence $\|\boldsymbol{z} - \boldsymbol{x}^*\|_\infty < 2\hat{\ell}\ell A$, which shows the lemma. □

Now we obtain a lower bound for a Newton iteration, showing thus that the Newton operator N of (9) maps $\mathcal{D}$ to itself.

Lemma 11. *If $\boldsymbol{y} := N(\boldsymbol{z})$ is the solution of (11) for $\boldsymbol{z} \in \mathcal{D}$, then*

$$\boldsymbol{y} > \frac{3}{2}\tilde{m}.$$

Proof. By (2) and (11), the function $\boldsymbol{v} := \boldsymbol{y} - \boldsymbol{x}^*$ satisfies the following equalities:

$$\begin{cases}\boldsymbol{v}''(x) = \big(g'(\boldsymbol{z})\boldsymbol{v} + g'(\boldsymbol{z})\boldsymbol{x}^* + g(\boldsymbol{z}) - g'(\boldsymbol{z})\boldsymbol{z} - g(\boldsymbol{x}^*)\big)(x), \quad x \in (0, \ell) \\ \boldsymbol{v}'(0) = 0, \\ \boldsymbol{v}'(\ell) = 0.\end{cases} \tag{16}$$

Let $x_0 \in [0, \ell]$ be such that $|\boldsymbol{v}(x_0)| = \min\{|\boldsymbol{v}(x)| : x \in [0, \ell]\}$. By Lemma 6,

$$\|\boldsymbol{v}\|_\infty \leq \sqrt{2}\bigg(\boldsymbol{v}(x_0)^2 - \hat{\ell}\int_0^\ell (\boldsymbol{v}\boldsymbol{v}'')(x)dx\bigg)^{\frac{1}{2}}.$$

From (16) we see that $\boldsymbol{v}'' = g'(\boldsymbol{z})\boldsymbol{v} + g'(\boldsymbol{z})\boldsymbol{x}^* + g(\boldsymbol{z}) - g'(\boldsymbol{z})\boldsymbol{z} - g(\boldsymbol{x}^*)$. Arguing as in the proof of Lemma 7, we conclude that

$$|\boldsymbol{v}(x_1)| \leq (2\hat{\ell})^{\frac{1}{2}}\bigg(\int_0^\ell \Big(\big(g'(\boldsymbol{z})\boldsymbol{z} - g'(\boldsymbol{z})\boldsymbol{x}^* - g(\boldsymbol{z}) + g(\boldsymbol{x}^*)\big)\boldsymbol{v}\Big)(x)dx\bigg)^{\frac{1}{2}}.$$

Fix $x \in [0, \ell]$. Applying the Taylor theorem to the function g in a neighborhood of $\boldsymbol{x}^*(x)$, evaluated at $\boldsymbol{z}(x)$, it follows that there exists $\xi(x)$ in the real interval defined by $\boldsymbol{z}(x)$ and $\boldsymbol{x}^*(x)$ such that

$$\big(g'(\boldsymbol{z})\boldsymbol{z} - g'(\boldsymbol{z})\boldsymbol{x}^* - g(\boldsymbol{z}) + g(\boldsymbol{x}^*)\big)(x) = \frac{1}{2}g''(\xi(x))\big(\boldsymbol{z}(x) - \boldsymbol{x}^*(x)\big)^2.$$

By the definition of $\tilde{M}$ and $\mathcal{D}$ we deduce that $|\xi(x)| < \tilde{M}$. As a consequence,

$$\|\boldsymbol{v}\|_\infty < \hat{\ell}\ell g''(\tilde{M})\|\boldsymbol{z} - \boldsymbol{x}^*\|_\infty^2.$$

Lemma 10 implies

$$\|\boldsymbol{v}\|_\infty < \hat{\ell}\ell g''(\tilde{M})(2\hat{\ell}\ell A)^2 \leq \hat{\ell}\ell A/2 \leq \tilde{m}/2.$$

We conclude that $\boldsymbol{y}(x) - \boldsymbol{x}^*(x) \geq -\tilde{m}/2$. Therefore, $\boldsymbol{y}(x) > \boldsymbol{x}^*(x) - \tilde{m}/2 \geq m - \tilde{m}/2 = 3\tilde{m}/2$, finishing thus the proof of the lemma. □

Let $\boldsymbol{z} \in \mathcal{D}$. Lemma 5 shows that the Newton iteration $N(\boldsymbol{z})$ is well-defined. Furthermore, Lemmas 8, 9 and 11 prove that $N(\boldsymbol{z}) \in \mathcal{D}$. In other words, the Newton sequence defined by N starting at $\boldsymbol{z} \in \mathcal{D}$ is well-defined.

3.2. *Convergence of Newton's method to the solution of (2)*

To establish the convergence of the Newton sequence associated with (2), we show that our hypotheses imply that an affine invariant Lipschitz condition as in (7) is satisfied.

Lemma 12. *For $\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z} \in \mathcal{D}$, the following affine invariant Lipschitz condition is satisfied:*

$$\|F'(\boldsymbol{x})^{-1}(F'(\boldsymbol{y}+s(\boldsymbol{x}-\boldsymbol{y}))-F'(\boldsymbol{z}))\boldsymbol{v}\|_\infty \leq \omega\|\boldsymbol{y}-\boldsymbol{z}+s(\boldsymbol{x}-\boldsymbol{y})\|_\infty\|\boldsymbol{v}\|_\infty,$$

where $0 \leq s \leq 1$ *and* $\boldsymbol{v} \in C^2([0,\ell])$.

Proof. Let $\boldsymbol{w} := F'(\boldsymbol{x})^{-1}(F'(\boldsymbol{y}+s(\boldsymbol{x}-\boldsymbol{y}))-F'(\boldsymbol{z}))\boldsymbol{v}$, where $0 \leq s \leq 1$. We have that $\boldsymbol{w}$ satisfies the following conditions:

$$\begin{cases} \boldsymbol{w}''(x) = -\Big(\big(g'(\boldsymbol{y}+s(\boldsymbol{x}-\boldsymbol{y}))-g'(\boldsymbol{z})\big)\boldsymbol{v} - g'(\boldsymbol{x})\boldsymbol{w}\Big)(x), \quad x \in (0,\ell) \\ \boldsymbol{w}'(0) = 0, \\ \boldsymbol{w}'(\ell) = 0. \end{cases} \tag{17}$$

Let $x_0 \in [0,\ell]$ be such that $|\boldsymbol{w}(x_0)| = \min\{|\boldsymbol{w}(x)| : x \in [0,\ell]\}$. By Lemma 6,

$$\|\boldsymbol{w}\|_\infty \leq \sqrt{2}\left(\boldsymbol{w}(x_0)^2 - \hat{\ell}\int_0^\ell (\boldsymbol{w}\boldsymbol{w}'')(x)dx\right)^{\frac{1}{2}}.$$

Expressing $\boldsymbol{w}''$ as in the first equation of (17), we obtain

$$\begin{aligned} \|\boldsymbol{w}\|_\infty &\leq \sqrt{2}\left(\boldsymbol{w}(x_0)^2 + \hat{\ell}\int_0^\ell \Big(\big(g'(\boldsymbol{y}+s(\boldsymbol{x}-\boldsymbol{y}))-g'(\boldsymbol{z})\big)\boldsymbol{v}\boldsymbol{w} - g'(\boldsymbol{x})\boldsymbol{w}^2\Big)(x)dx\right)^{\frac{1}{2}} \\ &\leq \sqrt{2}\left(\hat{\ell}\int_0^\ell \Big(\big(g'(\boldsymbol{y}+s(\boldsymbol{x}-\boldsymbol{y}))-g'(\boldsymbol{z})\big)\boldsymbol{v}\boldsymbol{w}\Big)(x)dx\right)^{\frac{1}{2}}. \end{aligned}$$

Observe that

$$\big\|g'(\boldsymbol{y}+s(\boldsymbol{x}-\boldsymbol{y}))-g'(\boldsymbol{z})\big\|_\infty \leq g''(\tilde{M})\|\boldsymbol{y}-\boldsymbol{z}+s(\boldsymbol{x}-\boldsymbol{y})\|_\infty.$$

It follows that

$$\begin{aligned} \|\boldsymbol{w}\|_\infty &\leq (2\hat{\ell}\ell g''(\tilde{M})\|\boldsymbol{y}-\boldsymbol{z}+s(\boldsymbol{x}-\boldsymbol{y})\|_\infty\|\boldsymbol{v}\|_\infty\|\boldsymbol{w}\|_\infty)^{\frac{1}{2}} \\ &= (\omega\|\boldsymbol{y}-\boldsymbol{z}+s(\boldsymbol{x}-\boldsymbol{y})\|_\infty\|\boldsymbol{v}\|_\infty\|\boldsymbol{w}\|_\infty)^{\frac{1}{2}}. \end{aligned}$$

This shows that $\|\boldsymbol{w}\|_\infty \leq \omega\|\boldsymbol{y}-\boldsymbol{z}+s(\boldsymbol{x}-\boldsymbol{y})\|_\infty\|\boldsymbol{v}\|_\infty$ and finishes the proof. □

Now we state the main result of this section, namely a version of the Newton–Mysovskikh theorem for (2). Unlike Theorem 2, where convexity of the domain D is required, our result is valid for a domain $\mathcal{D}$ which is not convex. This is essentially due to the fact that $\mathcal{X}$ is convex and F' is defined everywhere in $\mathcal{X}$.

Theorem 13. *If $\boldsymbol{x}^0 \in \mathcal{D}$, then the sequence $(\boldsymbol{x}^k)_{k\geq 0}$ determined by the Newton iteration (6) is well-defined, remains in $\mathcal{D}$ and converges to the solution $\boldsymbol{x}^* \in \overline{\mathcal{D}}$ of (2). Furthermore, we have the following estimates for any $k \geq 1$:*

$$\|\boldsymbol{x}^{k+1}-\boldsymbol{x}^k\|_\infty \leq \frac{\omega}{2}\|\boldsymbol{x}^k-\boldsymbol{x}^{k-1}\|_\infty^2, \tag{18}$$

$$\|\boldsymbol{x}^k-\boldsymbol{x}^*\|_\infty \leq \frac{\|\boldsymbol{x}^k-\boldsymbol{x}^{k+1}\|_\infty}{1-\frac{\omega}{2}\|\boldsymbol{x}^k-\boldsymbol{x}^{k+1}\|_\infty}. \tag{19}$$

Proof. Combining Lemmas 5, 8, 9 and 11 we conclude that the sequence $(\boldsymbol{x}^k)_{k\geq 0}$, starting at $\boldsymbol{x}^0 \in \mathcal{D}$, is well-defined and remains in $\mathcal{D}$.

Next we analyze the convergence of $(\boldsymbol{x}^k)_{k\geq 0}$. Let $\Delta\boldsymbol{x}^k := -F'(\boldsymbol{x}^k)^{-1}F(\boldsymbol{x}^k)$ and $h_k := \omega\|\Delta\boldsymbol{x}^k\|_\infty$ for any $k \geq 0$. By the definition of $\Delta\boldsymbol{x}^{k-1}$ and $\Delta\boldsymbol{x}^k$, we infer that

$$\Delta\boldsymbol{x}^k = -F'(\boldsymbol{x}^k)^{-1}\Big(F(\boldsymbol{x}^k) - \big(F(\boldsymbol{x}^{k-1}) + F'(\boldsymbol{x}^{k-1})\Delta\boldsymbol{x}^{k-1}\big)\Big).$$

Now we use the affine invariant Lipschitz condition of Lemma 12 to estimate the norm of $\Delta\boldsymbol{x}^k$. For this purpose, observe that

$$F(\boldsymbol{x}^k) - F(\boldsymbol{x}^{k-1}) - F'(\boldsymbol{x}^{k-1})\Delta\boldsymbol{x}^{k-1} = \int_0^1 \big(F'(\boldsymbol{x}^{k-1}+s(\boldsymbol{x}^k-\boldsymbol{x}^{k-1})) - F'(\boldsymbol{x}^{k-1})\big)(\boldsymbol{x}^k-\boldsymbol{x}^{k-1})ds.$$

As a consequence, since $\boldsymbol{x}^k$ and $\boldsymbol{x}^{k-1}$ belong to $\mathcal{D}$, by Lemma 12 we have

$$\left\| \int_0^1 F'(\boldsymbol{x}^k)^{-1}\big(F'(\boldsymbol{x}^{k-1}+s(\boldsymbol{x}^k-\boldsymbol{x}^{k-1}))-F'(\boldsymbol{x}^{k-1})\big)(\boldsymbol{x}^k-\boldsymbol{x}^{k-1})ds \right\|_\infty$$
$$\le \int_0^1 s\,\omega\|\boldsymbol{x}^k-\boldsymbol{x}^{k-1}\|_\infty^2 ds = \frac{\omega}{2}\|\boldsymbol{x}^k-\boldsymbol{x}^{k-1}\|_\infty^2.$$

We conclude that

$$\|\Delta\boldsymbol{x}^k\|_\infty \le \frac{\omega}{2}\|\Delta\boldsymbol{x}^{k-1}\|_\infty^2,$$

which is (18). Multiplying (18) by ω we see that $h_k \le h_{k-1}^2/2$ for any k. Since $h_0 < 2$ by Lemma 7, $(h_k)_{k\ge 0}$ is a decreasing sequence and

$$0 \le \lim_{k\to\infty} h_k \le \lim_{k\to\infty} 2(h_0/2)^{2^k} = 0.$$

A simple inductive argument proves that

$$\|\Delta\boldsymbol{x}^{l+k}\|_\infty \le \left(\frac{h_k}{2}\right)^l \|\Delta\boldsymbol{x}^k\|_\infty.$$

Then

$$\|\boldsymbol{x}^{l+k+1}-\boldsymbol{x}^k\|_\infty \le \sum_{j=k}^{k+l}\|\Delta\boldsymbol{x}^j\|_\infty \le \|\Delta\boldsymbol{x}^k\|_\infty \sum_{j=0}^{\infty}\left(\frac{h_k}{2}\right)^j = \frac{h_k}{\omega\left(1-\frac{1}{2}h_k\right)}. \tag{20}$$

As $\lim_{k\to\infty} h_k = 0$, the Newton sequence $(\boldsymbol{x}^k)_{k\ge 0}$ is a Cauchy sequence of $C([0,\ell])$, with respect to the infinite norm, and therefore converges in $C([0,\ell])$.

To see that $(\boldsymbol{x}^k)_{k\ge 0}$ converges in $\mathcal{X}$, we show that $((\boldsymbol{x}^k)')_{k\ge 0}$ and $((\boldsymbol{x}^k)'')_{k\ge 0}$ are Cauchy sequences of $C([0,\ell])$. First we observe that it suffices to prove that $((\boldsymbol{x}^k)'')_{k\ge 0}$ is a Cauchy sequence of $C([0,\ell])$. Indeed, assuming that this is the case, taking into account that $(\boldsymbol{x}^k)'(0) = 0$ for each $k \ge 0$ we easily conclude that $((\boldsymbol{x}^k)')_{k\ge 0}$ is also a Cauchy sequence of $C([0,\ell])$.

Next we show that $((\boldsymbol{x}^k)'')_{k\ge 0}$ is a Cauchy sequence of $C([0,\ell])$. By the definition of $\Delta\boldsymbol{x}^k$ we have $(\Delta\boldsymbol{x}^k)'' = g'(\boldsymbol{x}^k)\Delta\boldsymbol{x}^k - (\boldsymbol{x}^k)'' + g(\boldsymbol{x}^k)$. By (14),

$$\|(\boldsymbol{x}^k)'' - g(\boldsymbol{x}^k)\|_\infty \le \frac{1}{2}g''(\tilde{M})\|\Delta\boldsymbol{x}^{k-1}\|_\infty^2.$$

Therefore,

$$\|(\Delta\boldsymbol{x}^k)''\|_\infty = g'(\tilde{M})\|\Delta\boldsymbol{x}^k\|_\infty + \frac{1}{2}g''(\tilde{M})\|\Delta\boldsymbol{x}^{k-1}\|_\infty^2$$
$$\le g'(\tilde{M})\frac{\omega}{2}\|\Delta\boldsymbol{x}^{k-1}\|_\infty^2 + \frac{\omega}{4\ell\hat{\ell}}\|\Delta\boldsymbol{x}^{k-1}\|_\infty^2 = C\omega\|\Delta\boldsymbol{x}^{k-1}\|_\infty^2,$$

where $C := g'(\tilde{M})/2 + 1/4\ell\hat{\ell}$. Now we argue as in (20):

$$\|(\Delta\boldsymbol{x}^{l+k+1})'' - (\Delta\boldsymbol{x}^k)''\|_\infty \le \sum_{j=k}^{k+l}\|(\Delta\boldsymbol{x}^j)''\|_\infty \le C\omega\sum_{j=k}^{k+l}\|\Delta\boldsymbol{x}^{j-1}\|_\infty^2 \le \frac{Ch_{k-1}^2}{\omega}\sum_{j=0}^{\infty}\left(\frac{h_{k-1}}{2}\right)^{2j} = \frac{Ch_{k-1}^2}{\omega\left(1-\frac{1}{4}h_{k-1}^2\right)}.$$

We conclude that $((\boldsymbol{x}^k)'')_{k\ge 0}$ is a Cauchy sequence, as claimed.

It follows that $(\boldsymbol{x}^k)_{k\ge 0}$ converges in $\mathcal{X}$. Denoting by $\boldsymbol{x}^*$ the limit of this sequence, we see that

$$0 = \lim_{k\to\infty} F'(\boldsymbol{x}^k)\Delta\boldsymbol{x}^k = -\lim_{k\to\infty} F(\boldsymbol{x}^k) = -F(\boldsymbol{x}^*),$$

which proves that $\boldsymbol{x}^* \in \overline{\mathcal{D}}$ is the solution of (2). Finally, taking limits as l tends to infinity in (20) we obtain (19). $\square$

4. On the mesh-independence principle

In this section we obtain an explicit mesh-independence principle for (2). As in Section 3, let X, Y be Banach spaces, $D \subset X$ a convex domain and $F : D \to Y$ a nonlinear C^1 operator such that the equation $F(\boldsymbol{x}) = 0$ has a unique solution $\boldsymbol{x}^* \in D$. Recall that the Newton sequence applied to $F(\boldsymbol{x}) = 0$ is defined as

$$\boldsymbol{x}^{k+1} = \boldsymbol{x}^k + \Delta\boldsymbol{x}^k, \qquad F'(\boldsymbol{x}^k)\Delta\boldsymbol{x}^k = -F(\boldsymbol{x}^k) \quad (k \ge 0),$$

assuming that the derivatives $F'(\boldsymbol{x}^k)$ are invertible.

Under the hypotheses of Theorem 2, suppose that for each $j \geq 0$ there exist finite-dimensional vector spaces $X_j \subset X$ and $Y_j \subset Y$, a convex domain $D_j \subset X_j$ and a C^1 morphism $F_j : D_j \to Y_j$. According to Theorem 2, if:

(1) F_j' is invertible for each $\boldsymbol{x}_j \in D_j$; and
(2) there exists ω_j such that, for collinear $\boldsymbol{x}_j, \boldsymbol{y}_j, \boldsymbol{z}_j \in D_j$,

$$\left\| F_j'(\boldsymbol{z}_j)^{-1}\big(F_j'(\boldsymbol{y}_j) - F_j'(\boldsymbol{x}_j)\big)\boldsymbol{v}_j \right\| \leq \omega_j \|\boldsymbol{y}_j - \boldsymbol{x}_j\| \|\boldsymbol{v}_j\|;$$

then for each $j \geq 0$ we can apply the Newton method in X_j to solve the equation

$$F_j(\boldsymbol{x}_j) = 0,$$

which has a unique solution $\boldsymbol{x}_j^*$ in a suitable neighborhood. The corresponding Newton sequence is the following:

$$\boldsymbol{x}_j^{k+1} = \boldsymbol{x}_j^k + \Delta\boldsymbol{x}_j^k, \qquad F_j'(\boldsymbol{x}_j^k)\Delta\boldsymbol{x}_j^k = -F_j(\boldsymbol{x}_j^k) \quad (k \geq 0),$$

and converges to $\boldsymbol{x}_j^*$ starting sufficiently close. It is to be expected that the discretization method implies that

$$\lim_{j\to\infty} \boldsymbol{x}_j^* = \boldsymbol{x}^*.$$

We have the following convergence result.

Theorem 14 ([13, Theorem 2.2]). *Let $\boldsymbol{x}^0 \in \bigcap X_j \subset X$ be such that*

$$h_0 := \omega\|\Delta\boldsymbol{x}^0\| < 2 \quad \textit{and} \quad \overline{S}(\boldsymbol{x}^0, \rho) \subset D, \quad \textit{where } \rho := \frac{\|\Delta\boldsymbol{x}^0\|}{1 - h_0/2}.$$

For each $j \geq 0$ and each $\boldsymbol{x}_j \in S\big(\boldsymbol{x}^0, \rho + \frac{2}{\omega}\big) \cap X_j$, we define

$$F_j'(\boldsymbol{x}_j)\Delta\boldsymbol{x}_j = -F_j(\boldsymbol{x}_j), \qquad F'(\boldsymbol{x}_j)\Delta\boldsymbol{x} = -F(\boldsymbol{x}_j).$$

Assume that the discretization is sufficiently fine so that

$$\|\Delta\boldsymbol{x}_j - \Delta\boldsymbol{x}\| \leq \delta_j \leq \frac{\min\{1, 2 - h_0\}}{2\omega}$$

(uniformly for $\boldsymbol{x}_j \in D_j$). Suppose further that $\overline{S}(\boldsymbol{x}^0, \rho_j) \cap X_j \subset D_j$ for

$$\rho_j := \frac{\|\Delta\boldsymbol{x}^0\|}{1 - h_0/2} + \frac{2\delta_j}{\min\{1, 2 - h_0\}}.$$

Then the discrete Newton sequences $(\boldsymbol{x}_j^k)_{k\geq 0}$ remain in $S(\boldsymbol{x}_0, \rho_j) \cap X_j$ and we have the following error estimates:

$$\|\boldsymbol{x}_j^k - \boldsymbol{x}^k\| \leq \frac{2\delta_j}{\min\{1, 2 - h_0\}} \leq \frac{1}{\omega} \quad (k \geq 0), \qquad \limsup_{k\to\infty} \|\boldsymbol{x}_j^k - \boldsymbol{x}^k\| \leq 2\delta_j.$$

4.1. Discrete Newton iterations associated with (2)

Our aim is to obtain a mesh-independence principle for (2). For this purpose, for $j \geq 0$ we consider a uniform mesh $0 =: x_0 < \cdots < x_j := \ell$, and the space of *complete cubic splines* on $[0, \ell]$ with boundary conditions at 0 and ℓ, that is,

$$\mathcal{X}_j := \{\boldsymbol{x}_j \in C^2([0, \ell]) : \boldsymbol{x}_j|_{[x_{i-1}, x_i]} \text{ is a cubic } (1 \leq i \leq j), \boldsymbol{x}_j'(0) = 0,\ \boldsymbol{x}_j'(\ell) = \alpha\}.$$

Any choice of values $\boldsymbol{x}_{j,i}$ $(0 \leq i \leq j)$ for the nodes x_i $(0 \leq i \leq j)$ determines a unique element of $\mathcal{X}_j$ taking such values. More precisely, denote $\boldsymbol{x}_{j,i} := \boldsymbol{x}_j(x_i)$ and $\boldsymbol{x}_{j,i}'' := \boldsymbol{x}_j''(x_i)$ for $0 \leq i \leq j$. Then the unique element $\boldsymbol{x}_j$ of $\mathcal{X}_j$ satisfying these conditions can be expressed in the following way:

$$\boldsymbol{x}_j|_{[x_i, x_{i+1}]}(t) = \frac{\boldsymbol{x}_{j,i}''}{6h}(x_{i+1} - t)^3 + \frac{\boldsymbol{x}_{j,i+1}''}{6h}(t - x_i)^3 + \left(\frac{\boldsymbol{x}_{j,i+1}}{h} - \frac{\boldsymbol{x}_{j,i+1}''h}{6}\right)(t - x_i) + \left(\frac{\boldsymbol{x}_{j,i}}{h} - \frac{\boldsymbol{x}_{j,i}''h}{6}\right)(x_{i+1} - t). \tag{21}$$

Here, the values $\boldsymbol{x}_{j,i}''$ are uniquely determined in terms of the values $\boldsymbol{x}_{j,i}$ according to the following identity:

$$\begin{pmatrix} 2 & 1 & & & \\ 1 & 4 & \ddots & & \\ & \ddots & \ddots & 1 & \\ & & 1 & 4 & 1 \\ & & & 1 & 2 \end{pmatrix} \begin{pmatrix} \boldsymbol{x}_{j,0}'' \\ \boldsymbol{x}_{j,1}'' \\ \vdots \\ \boldsymbol{x}_{j,j-1}'' \\ \boldsymbol{x}_{j,j}'' \end{pmatrix} = \frac{6}{h^2} \begin{pmatrix} \boldsymbol{x}_{j,1} - \boldsymbol{x}_{j,0} \\ \boldsymbol{x}_{j,2} - 2\boldsymbol{x}_{j,1} + \boldsymbol{x}_{j,0} \\ \vdots \\ \boldsymbol{x}_{j,j} - 2\boldsymbol{x}_{j,j-1} + \boldsymbol{x}_{j,j-2} \\ h\alpha - \boldsymbol{x}_{j,j} + \boldsymbol{x}_{j,j-1} \end{pmatrix}. \tag{22}$$

In what follows, unless otherwise stated, by $\|\cdot\|$ we shall refer to the infinite norm of $C([0,\ell])$. We shall use the following estimates.

Theorem 15 *([22, p. 210]). Let $f \in C^3([0,\ell])$ be such that $f'(0)=0$ and $f'(\ell)=\alpha$, and let $\boldsymbol{x}_j \in \mathcal{X}_j$ be the complete cubic spline interpolating f. Then*

$$\|f^{(r)} - \boldsymbol{x}_j^{(r)}\| \le 5\Omega(f^{(3)}, h)h^{3-r} \quad (0 \le r \le 3),$$

where $h := \ell/j$ and $\Omega(f^{(3)},\cdot)$ is the modulus of continuity of $f^{(3)}$.

In order to introduce a discrete Newton iteration associated with (2), we consider the open subset $\mathcal{D}_j$ of $\mathcal{X}_j$ defined as follows:

$$\mathcal{D}_j := \mathcal{D} \cap \mathcal{X}_j = \{\boldsymbol{z} \in \mathcal{X}_j : \tilde{m} < \boldsymbol{z} < \tilde{M},\ \|\boldsymbol{z}'' - g(\boldsymbol{z})\|_\infty < A\}.$$

Further, for $\boldsymbol{z}_j \in \mathcal{D}_j$, define

$$F_j(\boldsymbol{z}_j) := \frac{1}{h^2}\begin{pmatrix} -2 & 2 & & & \\ 1 & -2 & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & 1 & -2 & 1 \\ & & & 2 & -2 \end{pmatrix}\begin{pmatrix} \boldsymbol{z}_{j,0} \\ \boldsymbol{z}_{j,1} \\ \vdots \\ \boldsymbol{z}_{j,j-1} \\ \boldsymbol{z}_{j,j} \end{pmatrix} - \begin{pmatrix} g(\boldsymbol{z}_{j,0}) \\ g(\boldsymbol{z}_{j,1}) \\ \vdots \\ g(\boldsymbol{z}_{j,j-1}) \\ g(\boldsymbol{z}_{j,j}) - \dfrac{2\alpha}{h} \end{pmatrix},$$

and denote by $F_j'(\boldsymbol{z}_j)$ the Jacobian matrix of F_j with respect to $\boldsymbol{z}_{j,0},\dots,\boldsymbol{z}_{j,j}$. Then a discrete Newton operator N_j is defined in the following way:

$$\begin{aligned} N_j :\ \mathcal{D}_j &\to \mathcal{X}_j, \\ \boldsymbol{z}_j &\mapsto \boldsymbol{y}_j := \boldsymbol{z}_j + \boldsymbol{w}_j. \end{aligned} \tag{23}$$

Here $\boldsymbol{w}_j$ is the complete cubic spline taking the values $\boldsymbol{w}_{j,i}$ $(0 \le i \le j)$ at nodes x_i $(0 \le i \le j)$ and satisfying the conditions $\boldsymbol{w}_j'(0) = \boldsymbol{w}_j'(\ell) = 0$, where

$$\begin{pmatrix} \boldsymbol{w}_{j,0} \\ \vdots \\ \boldsymbol{w}_{j,j} \end{pmatrix} := -F_j'(\boldsymbol{z}_j)^{-1}F_j(\boldsymbol{z}_j).$$

4.2. Estimates for the mesh-independence principle

We need upper bounds on the infinite norm of certain functions associated with the discrete and continuous Newton iterations defined by the operators N and N_j of (9) and (23). For this purpose, we start with the following remark.

Remark 16. Let $\boldsymbol{w} \in C^2([0,\ell])$ be such that $\boldsymbol{w}'(0)=0$. For any $x \in [0,\ell]$, we have $|\boldsymbol{w}'(x)| = \left|\int_0^x \boldsymbol{w}''(t)dt\right| \le \ell\|\boldsymbol{w}''\|$. As a consequence,

$$\|\boldsymbol{w}'\| \le \ell\|\boldsymbol{w}''\|.$$

Next we obtain an upper bound on the third derivative of a continuous Newton iteration and its Lipschitz constant in $[0,\ell]$. We observe that, although not explicitly stated, our bounds can be easily expressed as functions of mild growth in terms of the constants $\hat{\ell}$, $\tilde{M}$, ω and A.

Lemma 17. *Let $\boldsymbol{z} \in \mathcal{D}$ and $\boldsymbol{y} := N(\boldsymbol{z})$. Then $\boldsymbol{y}$ is a C^3 function with $y^{(3)}$ Lipschitz continuous in $[0,\ell]$, and there exist constants $\lambda_3 > 0$ and $\lambda_4 > 0$, independent of $\boldsymbol{y}$, such that $\|\boldsymbol{y}^{(3)}\| \le \lambda_3$ and $\mathrm{Lip}(\boldsymbol{y}^{(3)}) \le \lambda_4$, where $\mathrm{Lip}(\boldsymbol{y}^{(3)})$ denotes the best Lipschitz constant for $\boldsymbol{y}^{(3)}$ in the interval $[0,\ell]$.*

Proof. According to (11), we have $\boldsymbol{y}'' = g'(\boldsymbol{z})\boldsymbol{y} - g'(\boldsymbol{z})\boldsymbol{z} + g(\boldsymbol{z}) = g'(\boldsymbol{z})\boldsymbol{w} + g(\boldsymbol{z})$, where $\boldsymbol{w}$ is defined as in (10), i.e., $\boldsymbol{w} := -F'(\boldsymbol{z})^{-1}F(\boldsymbol{z})$. Therefore,

$$\begin{aligned} \boldsymbol{y}^{(3)} &= \big(g'(\boldsymbol{z})\boldsymbol{w} + g(\boldsymbol{z})\big)' = g''(\boldsymbol{z})\boldsymbol{z}'\boldsymbol{w} + g'(\boldsymbol{z})\boldsymbol{w}' + g(\boldsymbol{z})\boldsymbol{z}' \\ &= g''(\boldsymbol{z})\boldsymbol{z}'\boldsymbol{w} + g'(\boldsymbol{z})\boldsymbol{y}'. \end{aligned} \tag{24}$$

By Remark 16 we deduce that $\|\boldsymbol{z}'\| \le \ell\|\boldsymbol{z}''\|$ and $\|\boldsymbol{y}'\| \le \ell\|\boldsymbol{y}''\|$. Hence,

$$\begin{aligned} \|\boldsymbol{y}^{(3)}\| &\le \ell g''(\|\boldsymbol{z}\|)\|\boldsymbol{z}''\|\|\boldsymbol{w}\| + \ell g'(\|\boldsymbol{z}\|)\|\boldsymbol{y}''\| \\ &\le \ell g''(\|\boldsymbol{z}\|)\big(\|\boldsymbol{z}'' - g(\boldsymbol{z})\| + g(\|\boldsymbol{z}\|)\big)\|\boldsymbol{w}\| + \ell g'(\|\boldsymbol{z}\|)\big(\|\boldsymbol{y}'' - g(\boldsymbol{y})\| + g(\|\boldsymbol{y}\|)\big). \end{aligned}$$

By the definition of $\mathcal{D}$ and Lemma 7 it follows that $\|\boldsymbol{z}''-g(\boldsymbol{z})\|<A$, and $\|\boldsymbol{z}\|$ and $\|\boldsymbol{w}\|$ are uniformly bounded. On the other hand, Lemma 9 shows that $\|\boldsymbol{y}''-g(\boldsymbol{y})\|<A$. We conclude that $\boldsymbol{y}^{(3)}$ is also uniformly bounded.

Concerning the second assertion, we prove the Lipschitz continuity of each function arising in the expression of $\boldsymbol{y}^{(3)}$ in (24). Recall that $\boldsymbol{z}$ is uniformly bounded by definition. Since $\|\boldsymbol{z}''-g(\boldsymbol{z})\|<A$, we readily conclude that $\boldsymbol{z}''$ is uniformly bounded in $[0,\ell]$. This implies that $\boldsymbol{z}'$ is Lipschitz continuous in $[0,\ell]$, and the inequality $\|\boldsymbol{z}'\|\le\ell\|\boldsymbol{z}''\|$ proves that $\boldsymbol{z}'$ is also uniformly bounded in $[0,\ell]$. It follows that $\boldsymbol{z}$ is Lipschitz continuous in $[0,\ell]$. A similar argument shows that $\boldsymbol{y}'$ is Lipschitz continuous in $[0,\ell]$.

To show that $\boldsymbol{w}$ is Lipschitz continuous, it suffices to see that $\|\boldsymbol{w}'\|$ is uniformly bounded in $[0,\ell]$. Combining Remark 16 and the identity $\boldsymbol{w}''=g(\boldsymbol{z})-\boldsymbol{z}''+g'(\boldsymbol{z})\boldsymbol{w}$, we find that

$$\|\boldsymbol{w}'\|\le\ell\|\boldsymbol{w}''\|\le\ell\|\boldsymbol{z}''-g(\boldsymbol{z})\|+\ell\|g'(\boldsymbol{z})\|\|\boldsymbol{w}\|,$$

which proves the claim. Summarizing, (24) shows that $\boldsymbol{y}^{(3)}$ is a sum of products of Lipschitz continuous function in $[0,\ell]$, from which the second assertion follows. $\square$

As the unique positive solution $\boldsymbol{x}^*\in\mathcal{D}$ of (2) is a fixed point of the continuous Newton operator N, the conclusions of the lemma are valid for $\boldsymbol{x}^*$.

In what follows we shall need to obtain quantitative information concerning the discretization of second derivative of a continuous Newton iteration. We have the following result.

Lemma 18. *Let $\boldsymbol{z}\in\mathcal{D}$ and let $\boldsymbol{y}:=N(\boldsymbol{z})$. We have*

$$\begin{aligned}
\big(g'(\boldsymbol{z})(\boldsymbol{y}-\boldsymbol{z})+g(\boldsymbol{z})\big)(x_0)&=\frac{2\big(\boldsymbol{y}(x_1)-\boldsymbol{y}(x_0)\big)}{h^2}-C_0h,\\
\big(g'(\boldsymbol{z})(\boldsymbol{y}-\boldsymbol{z})+g(\boldsymbol{z})\big)(x_i)&=\frac{\boldsymbol{y}(x_{i-1})-2\boldsymbol{y}(x_i)+\boldsymbol{y}(x_{i+1})}{h^2}-C_ih^2,\quad(1\le i\le j-1)\\
\big(g'(\boldsymbol{z})(\boldsymbol{y}-\boldsymbol{z})+g(\boldsymbol{z})\big)(x_j)&=\frac{2\big(\boldsymbol{y}(x_{j-1})-\boldsymbol{y}(x_j)\big)}{h^2}+\frac{2\alpha}{h}-C_jh,
\end{aligned}$$

where each C_i is uniformly bounded, independently of j.

Proof. We extend the definition of $\boldsymbol{y}$ to the interval $[-h,\ell+h]$ by considering the third-order Taylor polynomial of $\boldsymbol{y}$ around 0 and ℓ. Considering suitable Taylor expansions of $\boldsymbol{y}$ at the nodes $x_0,\dots,x_j$, we see that there exist $\xi_{3,k}\in(x_k-h,x_k)$ and $\xi_{4,k}\in(x_k,x_k+h)$ for $k=0,j$ and $\xi_{1,i}\in(x_i-h,x_i)$ and $\xi_{2,i}\in(x_i,x_i+h)$ for $0\le i\le j$ such that

$$\begin{aligned}
\boldsymbol{y}''(x_0)&=\frac{2\,(\boldsymbol{y}(x_1)-\boldsymbol{y}(x_0))}{h^2}+\frac{\boldsymbol{y}'''(\xi_{3,0})+\boldsymbol{y}'''(\xi_{4,0})}{3!}h+\frac{\boldsymbol{y}'''(\xi_{1,0})-\boldsymbol{y}'''(\xi_{2,0})}{3!}h,\\
\boldsymbol{y}''(x_i)&=\frac{\boldsymbol{y}(x_{i-1})-2\boldsymbol{y}(x_i)+\boldsymbol{y}(x_{i+1})}{h^2}+\frac{\boldsymbol{y}'''(\xi_{1,i})-\boldsymbol{y}'''(\xi_{2,i})}{3!}h,\quad(1\le i\le j-1)\\
\boldsymbol{y}''(x_j)&=\frac{2\,\big(\boldsymbol{y}(x_{j-1})-\boldsymbol{y}(x_j)\big)}{h^2}+\frac{2\alpha}{h}+\frac{\boldsymbol{y}'''(\xi_{3,j})+\boldsymbol{y}'''(\xi_{4,j})}{3!}h+\frac{\boldsymbol{y}'''(\xi_{1,j})-\boldsymbol{y}'''(\xi_{2,j})}{3!}h.
\end{aligned}$$

By definition we have $\boldsymbol{y}''=g'(\boldsymbol{z})(\boldsymbol{y}-\boldsymbol{z})+g(\boldsymbol{z})$. Hence, we obtain

$$\begin{aligned}
\big(g'(\boldsymbol{z})(\boldsymbol{y}-\boldsymbol{z})+g(\boldsymbol{z})\big)(x_0)&=\frac{2\big(\boldsymbol{y}(x_1)-\boldsymbol{y}(x_0)\big)}{h^2}+C_0h,\\
\big(g'(\boldsymbol{z})(\boldsymbol{y}-\boldsymbol{z})+g(\boldsymbol{z})\big)(x_i)&=\frac{\boldsymbol{y}(x_{i-1})-2\boldsymbol{y}(x_i)+\boldsymbol{y}(x_{i+1})}{h^2}+C_ih,\quad(1\le i\le j-1)\\
\big(g'(\boldsymbol{z})(\boldsymbol{y}-\boldsymbol{z})+g(\boldsymbol{z})\big)(x_j)&=\frac{2\big(\boldsymbol{y}(x_{j-1})-\boldsymbol{y}(x_j)\big)}{h^2}+\frac{2\alpha}{h}+C_jh,
\end{aligned}$$

where the constants C_i are defined in the following way:

$$\begin{aligned}
C_0&:=\frac{\boldsymbol{y}'''(\xi_{3,0})+\boldsymbol{y}'''(\xi_{4,0})}{3!}+\frac{\boldsymbol{y}'''(\xi_{1,0})-\boldsymbol{y}'''(\xi_{2,0})}{3!},\\
C_i&:=\frac{\boldsymbol{y}'''(\xi_{1,i})-\boldsymbol{y}'''(\xi_{2,i})}{3!},\quad(1\le i\le j-1),\\
C_j&:=\frac{\boldsymbol{y}'''(\xi_{3,j})+\boldsymbol{y}'''(\xi_{4,j})}{3!}+\frac{\boldsymbol{y}'''(\xi_{1,j})-\boldsymbol{y}'''(\xi_{2,j})}{3!}.
\end{aligned}$$

By Lemma 17 we deduce that

$$
\begin{aligned}
|C_0| &\le \frac{|\boldsymbol{y}'''(0)| + |\boldsymbol{y}'''(\xi_{4,0})|}{3!} + \frac{|\boldsymbol{y}'''(\xi_{1,0}) - \boldsymbol{y}'''(\xi_{2,0})|}{3!} \le \frac{\|\boldsymbol{y}'''\|}{3} + \frac{\mathrm{Lip}(\boldsymbol{y}''')}{3}h \le \frac{\lambda_3}{3} + \frac{\lambda_4}{3}h,\\
|C_i| &\le \frac{|\boldsymbol{y}'''(\xi_{1,i}) - \boldsymbol{y}'''(\xi_{2,i})|}{3!} \le \frac{\mathrm{Lip}(y''')}{3}h \le \frac{\lambda_4}{3}h, \quad (1 \le i \le j-1)\\
|C_j| &\le \frac{|\boldsymbol{y}'''(\xi_{3,j})| + |\boldsymbol{y}'''(\ell)|}{3!} + \frac{|\boldsymbol{y}'''(\xi_{1,j}) - \boldsymbol{y}'''(\xi_{2,j})|}{3!} \le \frac{\|\boldsymbol{y}'''\|}{3} + \frac{\mathrm{Lip}(\boldsymbol{y}''')}{3}h \le \frac{\lambda_3}{3} + \frac{\lambda_4}{3}h.
\end{aligned}
$$

This finishes the proof of the lemma. □

Our next estimate is concerned with a comparison between discrete and continuous Newton iterations. Given $\boldsymbol{z}_j \in \mathcal{D}_j$, there are two elements of $\mathcal{D}_j$ associated with $\boldsymbol{z}_j$. On one hand, we have the discrete Newton iteration $\boldsymbol{y}_j := N_j(\boldsymbol{z}_j)$. On the other hand, we may consider a continuous Newton iteration $\boldsymbol{y} := N(\boldsymbol{z}_j)$ and its corresponding discretization $\boldsymbol{y}_{sp}$, namely the complete cubic spline of $\mathcal{D}_j$ interpolating $\boldsymbol{y}$. Our next result enables us to compare these functions.

Proposition 19. *For $\boldsymbol{z}_j \in \mathcal{D}_j$, let $\boldsymbol{y} := N(\boldsymbol{z}_j)$ and $\boldsymbol{y}_j := N_j(\boldsymbol{z}_j)$ be the corresponding continuous and discrete Newton iterations. Let $\boldsymbol{y}_{sp}$ be the complete cubic spline of $\mathcal{D}_j$ interpolating $\boldsymbol{y}$. Then there exists $\lambda_2 > 0$, independent of j, such that*

$$\|\boldsymbol{y}_j'' - \boldsymbol{y}_{sp}''\| \le 3g'(\tilde{M})\|\boldsymbol{w}_j - \boldsymbol{w}\| + \lambda_2 h,$$

where $\boldsymbol{w}_j := \boldsymbol{y}_j - \boldsymbol{z}_j$ and $\boldsymbol{w} := \boldsymbol{y} - \boldsymbol{z}_j$.

Proof. Observe that $\boldsymbol{y}_j - \boldsymbol{y}_{sp}$ is the complete cubic spline which interpolates $\boldsymbol{y}_j - \boldsymbol{y}$ at $x_0, \dots, x_j$ and satisfies the conditions $(\boldsymbol{y}_j - \boldsymbol{y}_{sp})'(0) = (\boldsymbol{y}_j - \boldsymbol{y}_{sp})'(\ell) = 0$. Hence,

$$\|\boldsymbol{y}_j'' - \boldsymbol{y}_{sp}''\| = \max_{1\le i\le j} \left\{ \left|(\boldsymbol{y}_j - \boldsymbol{y}_{sp})''(x_i)\right| \right\}.$$

Let $\boldsymbol{x}_{j,i}'' := (\boldsymbol{y}_j - \boldsymbol{y}_{sp})''(x_i)$ for $0 \le i \le j$. According to (22), we may express the $\boldsymbol{x}_{j,i}''$ in terms of $\boldsymbol{x}_{j,i} := (\boldsymbol{y}_j - \boldsymbol{y})(x_i)$ $(0 \le i \le j)$ in the following way:

$$
\begin{pmatrix}
2 & 1 & & & \\
1 & 4 & \ddots & & \\
 & \ddots & \ddots & 1 & \\
 & & 1 & 4 & 1\\
 & & & 1 & 2
\end{pmatrix}
\begin{pmatrix}
\boldsymbol{x}_{j,0}''\\ \boldsymbol{x}_{j,1}''\\ \vdots \\ \boldsymbol{x}_{j,j-1}''\\ \boldsymbol{x}_{j,j}''
\end{pmatrix}
= \frac{6}{h^2}
\begin{pmatrix}
\boldsymbol{x}_{j,1} - \boldsymbol{x}_{j,0}\\
\boldsymbol{x}_{j,2} - 2\boldsymbol{x}_{j,1} + \boldsymbol{x}_{j,0}\\
\vdots\\
\boldsymbol{x}_{j,j} - 2\boldsymbol{x}_{j,j-1} + \boldsymbol{x}_{j,j-2}\\
-\boldsymbol{x}_{j,j} + \boldsymbol{x}_{j,j-1}
\end{pmatrix}. \tag{25}
$$

By Lemma 18,

$$
\begin{aligned}
\frac{1}{2}\big(g'(\boldsymbol{z}_j)(\boldsymbol{y} - \boldsymbol{z}_j) + g(\boldsymbol{z}_j)\big)(x_0) &= \frac{\boldsymbol{y}(x_1) - \boldsymbol{y}(x_0)}{h^2} - \frac{C_0}{2}h,\\
\big(g'(\boldsymbol{z}_j)(\boldsymbol{y} - \boldsymbol{z}_j) + g(\boldsymbol{z}_j)\big)(x_i) &= \frac{\boldsymbol{y}(x_{i-1}) - 2\boldsymbol{y}(x_i) + \boldsymbol{y}(x_{i+1})}{h^2} - C_i h^2, \quad (1 \le i \le j-1)\\
\frac{1}{2}\big(g'(\boldsymbol{z}_j)(\boldsymbol{y} - \boldsymbol{z}_j) + g(\boldsymbol{z}_j)\big)(x_j) &= \frac{\boldsymbol{y}(x_{j-1}) - \boldsymbol{y}(x_j)}{h^2} + \frac{\alpha}{h} - \frac{C_j}{2}h,
\end{aligned} \tag{26}
$$

where each C_i is uniformly bounded, independently of j.

For $\boldsymbol{u} \in C^2([0,\ell])$, denote $\mathrm{Ev}_j(\boldsymbol{u}) := \big(\boldsymbol{u}(x_0), \dots, \boldsymbol{u}(x_j)\big)^t$. Since $\boldsymbol{y}_j = N_j(\boldsymbol{z}_j)$, we have $F_j'(\boldsymbol{z}_j)\mathrm{Ev}_j(\boldsymbol{y}_j) = F_j'(\boldsymbol{z}_j)\mathrm{Ev}_j(\boldsymbol{z}_j) - F_j(\boldsymbol{z}_j) = \mathrm{Ev}_j\big(g(\boldsymbol{z}_j)\big) - \mathrm{Ev}_j\big(g'(\boldsymbol{z}_j)\boldsymbol{z}_j\big) - \frac{2\alpha}{h}\mathbf{e}_j$, where $\mathbf{e}_j := (0, \dots, 0, 1)^t \in \mathbb{R}^j$. As a consequence,

$$
\begin{aligned}
\frac{1}{2}\big(g'(\boldsymbol{z}_j)(\boldsymbol{y}_j - \boldsymbol{z}_j) + g(\boldsymbol{z}_j)\big)(x_0) &= \frac{\boldsymbol{y}_j(x_1) - \boldsymbol{y}_j(x_0)}{h^2},\\
\big(g'(\boldsymbol{z}_j)(\boldsymbol{y}_j - \boldsymbol{z}_j) + g(\boldsymbol{z}_j)\big)(x_i) &= \frac{\boldsymbol{y}_j(x_{i-1}) - 2\boldsymbol{y}_j(x_i) + \boldsymbol{y}_j(x_{i+1})}{h^2}, \quad (1 \le i \le j-1),\\
\frac{1}{2}\big(g'(\boldsymbol{z}_j)(\boldsymbol{y}_j - \boldsymbol{z}_j) + g(\boldsymbol{z}_j)\big)(x_j) &= \frac{\boldsymbol{y}_j(x_{j-1}) - \boldsymbol{y}_j(x_j)}{h^2} + \frac{\alpha}{h}.
\end{aligned} \tag{27}
$$

We subtract (26) in (27) and combine the resulting identities with (25) to obtain

$$\frac{1}{6}\begin{pmatrix} 2 & 1 & & & \\ 1 & 4 & \ddots & & \\ & \ddots & \ddots & 1 & \\ & & 1 & 4 & 1 \\ & & & 1 & 2 \end{pmatrix}\begin{pmatrix} \boldsymbol{x}''_{j,0} \\ \boldsymbol{x}''_{j,1} \\ \vdots \\ \boldsymbol{x}''_{j,j-1} \\ \boldsymbol{x}''_{j,j} \end{pmatrix} = \begin{pmatrix} \frac{1}{2}g'(\boldsymbol{z}_{j,0})\boldsymbol{x}_{j,0} \\ g'(\boldsymbol{z}_{j,1})\boldsymbol{x}_{j,1} \\ \vdots \\ g'(\boldsymbol{z}_{j,j-1})\boldsymbol{x}_{j,j-1} \\ \frac{1}{2}g'(\boldsymbol{z}_{j,j})\boldsymbol{x}_{j,j} \end{pmatrix} - \begin{pmatrix} \frac{1}{2}C_0 h \\ C_1 h^2 \\ \vdots \\ C_{j-1}h^2 \\ \frac{1}{2}C_j h \end{pmatrix}.$$

Let

$$B := \frac{1}{6}\begin{pmatrix} 4 & 2 & & & \\ 1 & 4 & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & 1 & 4 & 1 \\ & & & 2 & 4 \end{pmatrix}, \qquad \boldsymbol{C}_h := \begin{pmatrix} C_0 h \\ C_1 h^2 \\ \vdots \\ C_{j-1}h^2 \\ C_j h \end{pmatrix}. \tag{28}$$

We conclude that

$$\|\boldsymbol{y}''_j - \boldsymbol{y}''_{sp}\| = \max_{0\le i\le j} |\boldsymbol{x}''_{j,i}| \le \|B^{-1}\| \left\| \mathrm{Ev}_j(g'(\boldsymbol{z}_j)\boldsymbol{x}_j) + \boldsymbol{C}_h \right\|.$$

As B is diagonally dominant, [23, Theorem 1] shows that $\|B^{-1}\| \le 3$. Further, since $|\boldsymbol{x}_{j,i}| = |(\boldsymbol{y}_j - \boldsymbol{y})(x_i)| \le \|\boldsymbol{y}_j - \boldsymbol{y}\| = \|\boldsymbol{w}_j - \boldsymbol{w}\|$ for $0 \le i \le j$, we see that

$$\|\boldsymbol{y}''_j - \boldsymbol{y}''_{sp}\| \le 3\big(g'(\|\boldsymbol{z}_j\|)\|\boldsymbol{w}_j - \boldsymbol{w}\| + \|\boldsymbol{C}_h\|\big) \le 3g'(\tilde{M})\|\boldsymbol{w}_j - \boldsymbol{w}\| + 3\|\boldsymbol{C}_h\|. \tag{29}$$

Lemma 18 shows that there exists a constant $\lambda_2 > 0$, independent of j, such that $\|\boldsymbol{C}_h\| \le \lambda_2 h$. Then the proposition follows. □

We finish this section with another estimate similar to that of Proposition 19, as it concerns a further comparison of continuous and discrete solutions of (2).

Proposition 20. *Let $\boldsymbol{x}^* \in \mathcal{D}$ be the positive solution of (2) and $\boldsymbol{x}^*_j \in \mathcal{X}_j$ the positive solution of $F_j = 0$. Let $\boldsymbol{x}^*_{sp}$ be the complete cubic spline of $\mathcal{X}_j$ interpolating $\boldsymbol{x}^*$. Then there exists a constant $\lambda^*_2 > 0$, independent of j, such that*

$$\|\boldsymbol{x}^*_j - \boldsymbol{x}^*_{sp}\| \le \lambda^*_2 h^2.$$

Proof. Observe that $\boldsymbol{x}^*_j - \boldsymbol{x}^*_{sp}$ is the complete cubic spline which interpolates $\boldsymbol{x}^*_j - \boldsymbol{x}^*$ at the nodes $x_0, \ldots, x_j$ and satisfies the boundary conditions $(\boldsymbol{x}^*_j - \boldsymbol{x}^*_{sp})'(0) = (\boldsymbol{x}^*_j - \boldsymbol{x}^*_{sp})'(\ell) = 0$. Hence, by (21) we have

$$\|\boldsymbol{x}^*_j - \boldsymbol{x}^*_{sp}\| \le \frac{2h^2}{3}\left\|\mathrm{E}''_j\right\| + 2\left\|\mathrm{E}_j\right\|,$$

with $\mathrm{E}_j := \mathrm{Ev}_j(\boldsymbol{x}^*_j - \boldsymbol{x}^*_{sp})$ and $\mathrm{E}''_j := \mathrm{Ev}_j\big((\boldsymbol{x}^*_j - \boldsymbol{x}^*_{sp})''\big)$. According to (22), we may express the E''_j in terms of E_j in the following way:

$$\frac{h^2}{6}\mathrm{E}''_j = \begin{pmatrix} 2 & 1 & & & \\ 1 & 4 & \ddots & & \\ & \ddots & \ddots & 1 & \\ & & 1 & 4 & 1 \\ & & & 1 & 2 \end{pmatrix}^{-1}\begin{pmatrix} -1 & 1 & & & \\ 1 & -2 & \ddots & & \\ & \ddots & \ddots & 1 & \\ & & 1 & -2 & 1 \\ & & & 1 & -1 \end{pmatrix}\mathrm{E}_j.$$

Taking into account the upper bound in [23, Theorem 1] for the infinity norm of inverses of diagonally dominant matrices, we obtain

$$\|\boldsymbol{x}^*_j - \boldsymbol{x}^*_{sp}\| \le \frac{2h^2}{3}\left\|\mathrm{E}''_j\right\| + 2\left\|\mathrm{E}_j\right\| \le 18\|\mathrm{E}_j\|. \tag{30}$$

As $\|\mathrm{E}_j\|$ is the global error of the second-order finite difference approximation of $\boldsymbol{x}^*$, it is well-known that $\|\mathrm{E}_j\| = \mathcal{O}(h^2)$ (see, e.g., [24, Chapter 8, Problem 7.7]). As a consequence, there exists $\lambda^*_2 > 0$, independent of j, such that

$$\|\boldsymbol{x}^*_j - \boldsymbol{x}^*_{sp}\| \le 18\|\mathrm{E}_j\| \le \lambda^*_2 h^2.$$

This shows the statement of the proposition. □

The following result is an easy consequence of Proposition 20.

Corollary 21. *Let $\boldsymbol{x}^* \in \mathcal{D}$ be the positive solution of (2) and $\boldsymbol{x}_j^* \in \mathcal{X}_j$ the complete cubic spline interpolating the positive solution of $F_j = 0$. Then there exists a constant $\lambda_2^{**} > 0$, independent of j, such that*

$$\|\boldsymbol{x}_j^* - \boldsymbol{x}^*\| \le \lambda_2^{**} h^2.$$

Proof. Let $\boldsymbol{x}_{sp}^*$ be the complete cubic spline of $\mathcal{X}_j$ which interpolates $\boldsymbol{x}^*$ at $x_0, \ldots, x_j$. We have

$$\|\boldsymbol{x}_j^* - \boldsymbol{x}^*\| \le \|\boldsymbol{x}_j^* - \boldsymbol{x}_{sp}^*\| + \|\boldsymbol{x}_{sp}^* - \boldsymbol{x}^*\|.$$

By Proposition 20 there exists $\lambda_2^* > 0$, independent of j, such that $\|\boldsymbol{x}_j^* - \boldsymbol{x}_{sp}^*\| \le \lambda_2^* h^2$. On the other hand, as $\boldsymbol{x}_{sp}^*$ is a complete cubic spline which interpolates $\boldsymbol{x}^*$, Theorem 15 shows that $\|\boldsymbol{x}_{sp}^* - \boldsymbol{x}^*\| \le 5\,\Omega((\boldsymbol{x}^*)^{(3)}, h)\, h^3 \le 5\,\mathrm{Lip}((\boldsymbol{x}^*)^{(3)})h^4$. Furthermore, according to Lemma 17, there exists $\lambda_4 > 0$, independent of j, such that $\mathrm{Lip}((\boldsymbol{x}^*)^{(3)}) \le \lambda_4$ holds. As a consequence, we obtain

$$\|\boldsymbol{x}_j^* - \boldsymbol{x}^*\| \le \lambda_2^* h^2 + 5\lambda_4 h^4.$$

Setting $\lambda_2^{**} := \lambda_2^* + 5\lambda_4 \ell^2$ finishes the proof of the corollary. □

4.3. Well-definedness of discrete Newton iterations

Let $\boldsymbol{z}_j \in \mathcal{D}_j$ and denote as before $\boldsymbol{y} := N(\boldsymbol{z}_j)$ and $\boldsymbol{y}_j := N_j(\boldsymbol{y}_j)$. Further, denote $\boldsymbol{w} := \boldsymbol{y} - \boldsymbol{z}_j$ and $\boldsymbol{w}_j := \boldsymbol{y}_j - \boldsymbol{z}_j$. In this section we show that, if $\|\boldsymbol{w}_j - \boldsymbol{w}\|$ is sufficiently small, then the discrete Newton iteration associated with N_j is well-defined.

For this purpose, in the next few results we obtain conditions on $\|\boldsymbol{w}_j - \boldsymbol{w}\|$ which imply that $\boldsymbol{y}_j := N_j(\boldsymbol{z}_j)$ belongs to $\mathcal{D}_j$.

Lemma 22. *If*

$$\|\boldsymbol{w} - \boldsymbol{w}_j\| < \lambda_M := \tilde{M}/4, \tag{31}$$

then $\|\boldsymbol{y}_j\| < \tilde{M}$.

Proof. Since $\boldsymbol{y} = \boldsymbol{z}_j + \boldsymbol{w}$, we have $\|\boldsymbol{y}_j\| = \|\boldsymbol{z}_j + \boldsymbol{w} + \boldsymbol{w}_j - \boldsymbol{w}\| \le \|\boldsymbol{y}\| + \|\boldsymbol{w}_j - \boldsymbol{w}\|$. By Lemma 8 it follows that

$$\|\boldsymbol{y}\| < 3\tilde{M}/4.$$

This immediately implies the statement of the lemma. □

Next we show that for j sufficiently large and $\|\boldsymbol{w} - \boldsymbol{w}_j\|$ sufficiently small, $\boldsymbol{y}_j'' - g(\boldsymbol{y}_j)$ has small norm for a discrete Newton iteration $\boldsymbol{y}_j = N_j(\boldsymbol{z}_j)$ with $\boldsymbol{z}_j \in \mathcal{D}_j$.

Proposition 23. *Let λ_4 and λ_2 be the constants of Lemma 17 and Proposition 19 respectively. If (31) holds and additionally*

$$j > j_0 := \lceil 4\ell(\lambda_2 + 5\ell\lambda_4)/3A \rceil, \tag{32}$$

$$\|\boldsymbol{w}_j - \boldsymbol{w}\| < \lambda_A := \big(3A/4 - (\lambda_2 + 5\ell\lambda_4)h\big)/4g'(\tilde{M}), \tag{33}$$

then $\|\boldsymbol{y}_j'' - g(\boldsymbol{y}_j)\| < A$.

Proof. Let $\boldsymbol{y} := \boldsymbol{z}_j + \boldsymbol{w}$ and let $\boldsymbol{y}_{sp} \in \mathcal{X}_j$ be the complete cubic spline interpolating $\boldsymbol{y}$ at $x_0, \ldots, x_j$. We have

$$\|\boldsymbol{y}_j'' - g(\boldsymbol{y}_j)\| \le \|\boldsymbol{y}_j'' - \boldsymbol{y}_{sp}''\| + \|\boldsymbol{y}_{sp}'' - \boldsymbol{y}''\| + \|\boldsymbol{y}'' - g(\boldsymbol{y})\| + \|g(\boldsymbol{y}) - g(\boldsymbol{y}_j)\|. \tag{34}$$

We bound each term on the right-hand side of (34). Concerning the first term, Proposition 19 shows that there exists $\lambda_2 > 0$, independent of j, such that

$$\|\boldsymbol{y}_j'' - \boldsymbol{y}_{sp}''\| \le 3g'(\tilde{M})\|\boldsymbol{w}_j - \boldsymbol{w}\| + \lambda_2 h. \tag{35}$$

Next we consider the second term in (34). According to Theorem 15,

$$\|\boldsymbol{y}_{sp}'' - \boldsymbol{y}''\| \le 5\,\Omega(\boldsymbol{y}^{(3)}, h)h \le 5\,\mathrm{Lip}(\boldsymbol{y}^{(3)})h^2.$$

By Lemma 17, there exists $\lambda_4 > 0$, independent of j, such that $\mathrm{Lip}(\boldsymbol{y}^{(3)}) \le \lambda_4$. As a consequence, we obtain

$$\|\boldsymbol{y}_{sp}'' - \boldsymbol{y}''\| \le 5\lambda_4 h^2. \tag{36}$$

In order to bound the third term in (34), since $\boldsymbol{y} = N(\boldsymbol{z}_j)$, Lemma 9 implies

$$\|\boldsymbol{y}'' - g(\boldsymbol{y})\| \le A/4. \tag{37}$$

Finally, regarding the fourth term in (34), by the Mean Value Theorem we have $\|g(\boldsymbol{y})-g(\boldsymbol{y}_j)\| \le g'(\max\{\|\boldsymbol{y}\|, \|\boldsymbol{y}_j\|\})\|\boldsymbol{y}-\boldsymbol{y}_j\| = g'(\max\{\|\boldsymbol{y}\|, \|\boldsymbol{y}_j\|\})\|\boldsymbol{w}-\boldsymbol{w}_j\|$. By Lemmas 8 and 22 we see that $\max\{\|\boldsymbol{y}\|, \|\boldsymbol{y}_j\|\} \le \tilde{M}$. Therefore,

$$\|g(\boldsymbol{y})-g(\boldsymbol{y}_j)\| \le g'(\tilde{M})\|\boldsymbol{w}-\boldsymbol{w}_j\|. \tag{38}$$

Summarizing, from (35)–(38) it follows that

$$\|\boldsymbol{y}_j''-g(\boldsymbol{y}_j)\| \le 4g'(\tilde{M})\|\boldsymbol{w}-\boldsymbol{w}_j\| + A/4 + (\lambda_2+5\ell\lambda_4)h.$$

Hence, if

$$\|\boldsymbol{w}-\boldsymbol{w}_j\| < \big(3A/4-(\lambda_2+5\ell\lambda_4)h\big)/4g'(\tilde{M}),$$

then $\|\boldsymbol{y}_j''-g(\boldsymbol{y}_j)\| < A$. In order to obtain a feasible condition on $\|\boldsymbol{w}-\boldsymbol{w}_j\|$, we need that the right-hand side in the previous expression is a strictly positive number. This is the case provided that

$$j > 4\ell(\lambda_2+5\ell\lambda_4)/3A.$$

This finishes the proof of the proposition. □

It remains to consider a further condition on $\|\boldsymbol{w}-\boldsymbol{w}_j\|$ in order to assure that $\boldsymbol{y}_j := N_j(\boldsymbol{z}_j) \in \mathcal{D}_j$ for any $\boldsymbol{z}_j$.

Lemma 24. *If*

$$\|\boldsymbol{w}-\boldsymbol{w}_j\| < \lambda_m := \tilde{m}/2, \tag{39}$$

then $\boldsymbol{y}_j(x) > \tilde{m}$ *for any* $x \in [0,\ell]$.

Proof. Let $\boldsymbol{y} := \boldsymbol{z}_j + \boldsymbol{w}$. By Lemma 11 we have

$$\boldsymbol{y}_j(x) = \boldsymbol{y}_j(x) - \boldsymbol{y}(x) + \boldsymbol{y}(x) = \boldsymbol{w}_j(x) - \boldsymbol{w}(x) + \boldsymbol{y}(x) > -\tilde{m}/2 + 3\tilde{m}/2 = \tilde{m}$$

for any $x \in [0,\ell]$, which shows the statement of the lemma. □

Let j_0 be as in (32) and let λ_M, λ_A and λ_m be the constants of (31), (33) and (39) respectively. If $j > j_0$ and $\|\boldsymbol{w}_j-\boldsymbol{w}\| < \min\{\lambda_M, \lambda_A, \lambda_m\}$, then Lemmas 22 and 24 and Proposition 23 show that $\boldsymbol{y}_j := N_j(\boldsymbol{z}_j)$ remains in $\mathcal{D}_j$, namely the discrete Newton iteration defined by N_j is well-defined in $\mathcal{D}_j$.

Next we show that the condition $\|\boldsymbol{w}_j-\boldsymbol{w}\| < \min\{\lambda_M, \lambda_A, \lambda_m\}$ is satisfied if the mesh under consideration is sufficiently fine. For this purpose, we obtain an upper bound for $\|\boldsymbol{w}_j-\boldsymbol{w}\|$ in terms of j.

Proposition 25. *There exists a universal constant* $\lambda > 0$ *such that*

$$\|\boldsymbol{w}_j-\boldsymbol{w}\| \le \lambda/j^2. \tag{40}$$

Proof. Let $\boldsymbol{y}_j := \boldsymbol{z}_j + \boldsymbol{w}_j$, $\boldsymbol{y} := \boldsymbol{z}_j + \boldsymbol{w}$, and let $\boldsymbol{y}_{sp} \in \mathcal{D}_j$ be the complete cubic spline interpolating $\boldsymbol{y}$ at $x_0, \dots, x_j$. We have

$$\|\boldsymbol{w}_j-\boldsymbol{w}\| = \|\boldsymbol{y}_j-\boldsymbol{y}\| \le \|\boldsymbol{y}_j-\boldsymbol{y}_{sp}\| + \|\boldsymbol{y}_{sp}-\boldsymbol{y}\|. \tag{41}$$

We first bound the second term on the right-hand side of (41). By Theorem 15,

$$\|\boldsymbol{y}_{sp}-\boldsymbol{y}\| \le 5\,\Omega(\boldsymbol{y}^{(3)}, h)h^3 \le 5\,\mathrm{Lip}(\boldsymbol{y}^{(3)})h^4.$$

According to Lemma 17, there exists $\lambda_4 > 0$, independent of j, such that $\mathrm{Lip}(\boldsymbol{y}^{(3)}) \le \lambda_4$. This implies

$$\|\boldsymbol{y}_{sp}-\boldsymbol{y}\| \le 5\lambda_4 h^4. \tag{42}$$

Next we bound the first term on the right-hand side of (41). Observe that $\boldsymbol{y}_j - \boldsymbol{y}_{sp}$ is the complete cubic spline which interpolates $\boldsymbol{y}_j - \boldsymbol{y}$ at $x_0, \dots, x_j$ and satisfies the conditions $(\boldsymbol{y}_j-\boldsymbol{y}_{sp})'(0) = 0$ and $(\boldsymbol{y}_j-\boldsymbol{y}_{sp})'(\ell) = 0$. Arguing as in (30), we have

$$\|\boldsymbol{y}_j-\boldsymbol{y}_{sp}\| \le \frac{2h^2}{3}\left\|\mathrm{E}_j''\right\| + 2\left\|\mathrm{E}_j\right\| \le 18\|\mathrm{E}_j\|,$$

with $\mathrm{E}_j := \mathrm{Ev}_j(\boldsymbol{y}_j-\boldsymbol{y}_{sp})$ and $\mathrm{E}_j'' := \mathrm{Ev}_j\big((\boldsymbol{y}_j-\boldsymbol{y}_{sp})''\big)$. Since the discrete Newton iteration defined by N_j is the discretization of the continuous Newton iteration defined by N, $\|\mathrm{E}_j\|$ is the global error of the approximation of (11). Therefore, there exists a constant $\widehat{\lambda} > 0$ independent of j such that

$$\|\boldsymbol{y}_j-\boldsymbol{y}_{sp}\| \le 18\|\mathrm{E}_j\| \le 18\widehat{\lambda}h^2.$$

Combining this inequality with (41) and (42), the proposition follows. □

Next we combine the results above to obtain a condition on j which implies that the discrete Newton iteration corresponding to the mesh $x_0, \dots, x_j$ is well-defined on $\mathcal{D}_j$.

Corollary 26. *Let j_0 be as in (32) and let $\lambda_M, \lambda_A, \lambda_m$ and λ be the constants of (31), (33), (39) and (40) respectively. Let $\boldsymbol{z}_j \in \mathcal{D}_j$ and $\boldsymbol{y}_j := N_j(\boldsymbol{z}_j)$. If*

$$j > \max\{j_0, (\lambda/\lambda_M)^{1/2}, (\lambda/\lambda_A)^{1/2}, (\lambda/\lambda_m)^{1/2}\}, \tag{43}$$

then $\boldsymbol{y}_j \in \mathcal{D}_j$.

Proof. Let $\boldsymbol{y} := N(\boldsymbol{z}_j)$, $\boldsymbol{w}_j := \boldsymbol{y}_j - \boldsymbol{z}_j$ and $\boldsymbol{w} := \boldsymbol{y} - \boldsymbol{z}_j$. Proposition 25 shows that $\|\boldsymbol{w}_j - \boldsymbol{w}\| \leq \lambda/j^2$. Therefore, if (43) holds, then $j > j_0$ and

$$\|\boldsymbol{w}_j - \boldsymbol{w}\| < \min\{\lambda_M, \lambda_A, \lambda_m\}.$$

By Lemmas 22 and 24 and Proposition 23 it follows that $\boldsymbol{y}_j \in \mathcal{D}_j$. $\square$

4.4. A mesh-independence principle

In this section we establish an explicit version of the mesh-independence principle for (2). For this purpose, we are going to require that the norm of the difference of the discrete and continuous Newton corrections satisfies a further condition, which implies the quadratic convergence of the discrete Newton iteration.

Lemma 27. *Let $\boldsymbol{z}_j \in \mathcal{D}_j$ and $h_0 := \omega\|\boldsymbol{w}\|$. If*

$$j > j_1^{1/2} := \big(2\omega\lambda\big)^{1/2}, \tag{44}$$

then $\|\boldsymbol{w}_j - \boldsymbol{w}\| < \min\{1, 2 - h_0\}/2\omega = 1/2\omega$.

Proof. By Proposition 25 there exists $\lambda > 0$ such that $\|\boldsymbol{w}_j - \boldsymbol{w}\| \leq \lambda/j^2$. On other hand, Lemma 7 proves that $2 - h_0 > 2 - 2\hat{\ell}\ell\omega A \geq 3/2$. This shows that the condition in the statement of the lemma is satisfied if

$$\frac{\lambda}{j^2} < \frac{1}{2\omega}.$$

We easily deduce the statement of the lemma. $\square$

To establish our mesh-independence principle we need the following perturbation lemma, which is an adaptation of [13, Lemma 2.1] to our context.

Lemma 28. *Let $(\boldsymbol{x}^k)_{k\geq 0}$, $(\boldsymbol{y}^k)_{k\geq 0}$ be two continuous Newton sequences of $\mathcal{D}$. Then*

$$\|\boldsymbol{x}^{k+1} - \boldsymbol{y}^{k+1}\| \leq \omega\left(\frac{1}{2}\|\boldsymbol{y}^k - \boldsymbol{x}^k\| + \|\Delta\boldsymbol{y}^k\|\right)\|\boldsymbol{y}^k - \boldsymbol{x}^k\|,$$

where $\Delta\boldsymbol{y}^k := \boldsymbol{y}^{k+1} - \boldsymbol{y}^k$ and ω is the constant of Lemma 12.

Proof. Following the proof of [13, Lemma 2.1], we obtain

$$\boldsymbol{x}^{k+1} - \boldsymbol{y}^{k+1} = F'(\boldsymbol{x}^k)^{-1}\left(F'(\boldsymbol{x}^k)(\boldsymbol{x}^k - \boldsymbol{y}^k) - \int_0^1 F'\big(\boldsymbol{y}^k + t(\boldsymbol{x}^k - \boldsymbol{y}^k)\big)(\boldsymbol{x}^k - \boldsymbol{y}^k)dt\right) + F'(\boldsymbol{x}^k)^{-1}\big(F'(\boldsymbol{x}^k) - F'(\boldsymbol{y}^k)\big)\Delta\boldsymbol{y}^k.$$

Taking norms in the previous identity, by the affine invariant Lipschitz condition of Lemma 12 we obtain

$$\begin{aligned}
\|\boldsymbol{x}^{k+1} - \boldsymbol{y}^{k+1}\| &\leq \int_0^1 \left\|F'(\boldsymbol{x}^k)^{-1}\Big(F'(\boldsymbol{x}^k) - F'\big(\boldsymbol{y}^k + t(\boldsymbol{x}^k - \boldsymbol{y}^k)\big)\Big)(\boldsymbol{x}^k - \boldsymbol{y}^k)\right\|dt + \|F'(\boldsymbol{x}^k)^{-1}\big(F'(\boldsymbol{x}^k) - F'(\boldsymbol{y}^k)\big)\Delta\boldsymbol{y}^k\| \\
&\leq \omega\left(\|\boldsymbol{y}^k - \boldsymbol{x}^k\|\int_0^1(1-t)dt + \|\Delta\boldsymbol{y}^k\|\right)\|\boldsymbol{y}^k - \boldsymbol{x}^k\| \\
&\leq \omega\left(\frac{1}{2}\|\boldsymbol{y}^k - \boldsymbol{x}^k\| + \|\Delta\boldsymbol{y}^k\|\right)\|\boldsymbol{y}^k - \boldsymbol{x}^k\|.
\end{aligned}$$

This finishes the proof of the lemma. $\square$

Now we are able to obtain a mesh-independence principle for (2). This result combines our previous estimates in a way which follows the general line of argumentation of the proof of [13, Theorem 2.2].

Theorem 29. *Let j_0 and j_1 be as in (32) and (44), and let λ_M, λ_A, λ_m and λ be the constants of (31), (33), (39) and (40) respectively. Assume that*

$$j > \max\{j_0, j_1^{1/2}, (\lambda/\lambda_M)^{1/2}, (\lambda/\lambda_A)^{1/2}, (\lambda/\lambda_m)^{1/2}\}. \tag{45}$$

Fix $\boldsymbol{x}^0 := \boldsymbol{x}_j^0 \in \mathcal{D}_j$ and let $h_0 := \omega\|\Delta\boldsymbol{x}_0\|$. For any $\boldsymbol{x}_j \in \mathcal{D}_j$, define

$$\Delta\boldsymbol{x}_j := N_j(\boldsymbol{x}_j) - \boldsymbol{x}_j, \qquad \Delta\boldsymbol{x} := N(\boldsymbol{x}_j) - \boldsymbol{x}_j.$$

Then the discrete Newton iteration $(\boldsymbol{x}_j^k)_{k\geq 0}$ remains in $\mathcal{D}_j$ and we have the following error estimate:

$$\|\boldsymbol{x}_j^k - \boldsymbol{x}^k\| \leq 2\delta_j \leq \frac{1}{\omega} \quad (k \geq 0),$$

where $\delta_j := \sup\{\|\Delta\boldsymbol{x}_j - \Delta\boldsymbol{x}\| : \boldsymbol{x}_j \in \mathcal{D}_j\} \leq \lambda/j^2$.

Proof. Since (45) implies (43), by Corollary 26 we have that the discrete Newton iteration $(\boldsymbol{x}^{k,k})_{k\geq 0} := (\boldsymbol{x}_j^k)_{k\geq 0}$ starting at $\boldsymbol{x}_j^0$ is well-defined and remains in $\mathcal{D}_j$. For any $k \geq 0$, we denote by $(\boldsymbol{x}^{i,k})_{i\geq k}$ the continuous Newton iteration starting at $\boldsymbol{x}^{k,k}$. By Theorem 13 we have that $(\boldsymbol{x}^{i,k})_{i\geq k}$ remains in $\mathcal{D}$ and converges for any $k \geq 0$. In particular, if $k = 0$, then the sequence $(\boldsymbol{x}^{i,0})_{i\geq 0} = (\boldsymbol{x}^k)_{k\geq 0}$ agrees with the continuous Newton iteration starting at $\boldsymbol{x}^0 := \boldsymbol{x}_j^0$.

Furthermore, since (45) implies (44), Lemma 27 proves that the following condition holds uniformly for $\boldsymbol{x}_j \in \mathcal{D}_j$:

$$\|\Delta\boldsymbol{x}_j - \Delta\boldsymbol{x}\| \leq \delta_j \leq \frac{1}{2\omega}.$$

Now we follow the general line of argumentation of the proof of [13, Theorem 2.2]. In order to control the distance between the corresponding terms of $(\boldsymbol{x}_j^k)_{k\geq 0}$ and $(\boldsymbol{x}^k)_{k\geq 0}$, let h_k and ϵ_k be upper bounds for $\omega\|\Delta\boldsymbol{x}^k\|$ and $\|\boldsymbol{x}_j^k - \boldsymbol{x}^k\|$, i.e.,

$$\omega\|\Delta\boldsymbol{x}^k\| \leq h_k, \qquad \|\boldsymbol{x}_j^k - \boldsymbol{x}^k\| \leq \epsilon_k \quad (k \geq 0).$$

From the proof of Theorem 13 we conclude that the following is an admissible definition for $(h_k)_{k\geq 0}$:

$$h_{k+1} := \frac{1}{2}h_k^2, \qquad h_0 := \omega\|\Delta\boldsymbol{x}_0\|.$$

Next we obtain an admissible explicit definition for $(\epsilon_k)_{k\geq 0}$. To this end, we have

$$\begin{aligned}\|\boldsymbol{x}_j^{k+1} - \boldsymbol{x}^{k+1}\| &= \|\boldsymbol{x}^{k+1,\,k+1} - \boldsymbol{x}^{k+1,\,0}\| \\ &\leq \|\boldsymbol{x}^{k+1,\,k+1} - \boldsymbol{x}^{k+1,\,k}\| + \|\boldsymbol{x}^{k+1,\,k} - \boldsymbol{x}^{k+1,\,0}\|.\end{aligned}$$

By assumption,

$$\|\boldsymbol{x}^{k+1,\,k+1} - \boldsymbol{x}^{k+1,\,k}\| = \|\boldsymbol{x}_j^k + \Delta\boldsymbol{x}_j^k - \boldsymbol{x}^{k,\,k} - \Delta\boldsymbol{x}^{k,\,k}\| = \|\Delta\boldsymbol{x}_j^k - \Delta\boldsymbol{x}^{k,\,k}\| \leq \delta_j.$$

On the other hand, the perturbation lemma (Lemma 28) shows that

$$\|\boldsymbol{x}^{k+1,\,k} - \boldsymbol{x}^{k+1,\,0}\| \leq \omega\Big(\frac{1}{2}\|\boldsymbol{x}^{k,\,k} - \boldsymbol{x}^{k,\,0}\| + \|\Delta\boldsymbol{x}^{k,\,0}\|\Big)\|\boldsymbol{x}^{k,\,k} - \boldsymbol{x}^{k,\,0}\|.$$

Combining the previous bounds we deduce that

$$\|\boldsymbol{x}_j^{k+1} - \boldsymbol{x}^{k+1}\| = \|\boldsymbol{x}^{k+1,\,k+1} - \boldsymbol{x}^{k+1,\,0}\| \leq \delta_j + \frac{\omega}{2}\epsilon_k^2 + h_k\epsilon_k,$$

which yields an admissible recursive definition $(\epsilon_k)_{k\geq 0}$. More precisely, we have the following admissible recursive definitions for $(h_k)_{k\geq 0}$ and $(\epsilon_k)_{k\geq 0}$:

$$\begin{aligned}&h_{k+1} := \frac{1}{2}h_k^2, \qquad h_0 := \omega\|\Delta\boldsymbol{x}_0\|, \\ &\epsilon_{k+1} := \delta_j + \frac{\omega}{2}\epsilon_k^2 + h_k\epsilon_k, \quad \epsilon_0 := 0.\end{aligned}$$

Now, a majorizing sequence for $(\epsilon_k)_{k\geq 0}$ is obtained by following *mutatis mutandis* the proof of [13, Theorem 2.2]. Since $h_0 \leq 1$, we deduce that $\epsilon_k \leq 2\delta_j$ for any $k \in \mathbb{N}$. This readily implies the statement of the theorem. $\square$

5. Computing an ϵ-approximation of the solution of (2)

Let $j \in \mathbb{N}$ satisfy (45) and let $\boldsymbol{x}^0 := \boldsymbol{x}_j^0$ be any point of $\mathcal{D}_j$. Denote by $(\boldsymbol{x}^k)_{k\geq 0}$ and $(\boldsymbol{x}_j^k)_{k\geq 0}$ the continuous and discrete Newton iterations starting at $\boldsymbol{x}^0$. According to Theorems 13 and 29, the continuous Newton iteration $(\boldsymbol{x}^k)_{k\geq 0}$ converges to the solution $\boldsymbol{x}^*$ of (2) and for any $k \geq 0$ we have

$$\|\boldsymbol{x}_j^k - \boldsymbol{x}^k\| \leq \frac{2\lambda}{j^2}.$$

This shows that, by means of a discrete Newton iteration, the quantity $\|\boldsymbol{x}_j^k - \boldsymbol{x}^*\|$ can be made arbitrarily small for j sufficiently large, provided that a starting point $\boldsymbol{x}^0 := \boldsymbol{x}_j^0 \in \mathcal{D}_j$ is obtained.

In [7,3,4,8] we exhibited an algorithm which, for a given $j \in \mathbb{N}$ and $\epsilon' > 0$, computes an ϵ'-approximation of the discrete system $F_j = \mathbf{0}$, i.e., a point $\boldsymbol{x}_j \in \mathbb{R}^n_{>0}$ with $\|\boldsymbol{x}_j - \boldsymbol{x}_j^*\| < \epsilon'$, where $\boldsymbol{x}_j^* \in \mathbb{R}^n_{>0}$ is the unique positive solution of the system $F_j = \mathbf{0}$. The algorithm performs $\mathcal{O}(j \log_2 \log_2(1/\epsilon'))$ flops and function evaluations. In this section we discuss how we can use this algorithm to obtain a starting point $\boldsymbol{x}^0 \in \mathcal{D}_j$ for our discrete Newton iteration. Then we shall compute discrete Newton iterations, starting at $\boldsymbol{x}^0 \in \mathcal{D}_j$, until an ϵ-approximation of the solution of (2) is obtained, for a given $\epsilon > 0$.

5.1. On the starting point for the discrete Newton iteration

Assume that we are given $\epsilon > 0$ and $\boldsymbol{x}_j \in \mathcal{X}_j$ with

$$\hat{m} := \frac{3\tilde{m}}{2} \leq \boldsymbol{x}_j(x_i) \leq \hat{M} := \frac{5\tilde{M}}{6}$$

for any $0 \leq i \leq j$ such that $\|F_j(\boldsymbol{x}_j)\| < \epsilon$. We shall obtain a sufficient condition on j which implies that the complete cubic spline in $\mathcal{X}_j$ interpolating $\boldsymbol{x}_j$ belongs to $\mathcal{D}_j$, and thus yields a starting point for the discrete Newton iteration.

In the sequel, if $0 =: x_0 \leq \cdots \leq x_j := \ell$ denotes the uniform mesh which we take as the interpolation nodes for the space of complete cubic splines $\mathcal{X}_j$, we shall frequently refer to the mapping $\mathrm{Ev}_j : \mathcal{X}_j \to \mathbb{R}^j$ defined by $\mathrm{Ev}_j(\boldsymbol{x}_j) := (\boldsymbol{x}_j(x_0), \ldots, \boldsymbol{x}_j(x_j))^t$. We start with the following technical lemma.

Lemma 30. *Given $\epsilon > 0$ and $\boldsymbol{x}_j \in \mathcal{X}_j$ with $\boldsymbol{x}_j(x_i) \in [\hat{m}, \hat{M}]$ for $0 \leq i \leq j$ and $\|F_j(\boldsymbol{x}_j)\| < \epsilon$, we have*

$$\|\mathrm{Ev}_j(\boldsymbol{x}_j'')\| < 3(\epsilon + g(\hat{M})).$$

Proof. We may rewrite (22) in the following way:

$$\mathrm{Ev}_j(\boldsymbol{x}_j'') = B^{-1}\big(F_j(\boldsymbol{x}_j) + \mathrm{Ev}_j(g(\boldsymbol{x}_j))\big), \tag{46}$$

where the matrix B is defined as in (28). In the proof of Proposition 19 we show that $\|B^{-1}\| \leq 3$, which readily implies the statement of the lemma. □

Our next result yields a sufficient condition on j which implies that the complete cubic spline defined by an ϵ-approximation of the discrete system $F_j = \mathbf{0}$ meets the expected upper and lower bounds.

Lemma 31. *Given $\epsilon > 0$ and $\boldsymbol{x}_j \in \mathcal{X}_j$ with $\boldsymbol{x}_{j,i} := \boldsymbol{x}_j(x_i) \in [\hat{m}, \hat{M}]$ for $0 \leq i \leq j$ and $\|F_j(\boldsymbol{x}_j)\| < \epsilon$, if*

$$j > \lambda_{1,\epsilon} := \ell(\epsilon + \hat{M}^p)^{1/2} \max\{4/\tilde{m}, 12\tilde{M}\}^{1/2},$$

then $\tilde{m} < \boldsymbol{x}_j(x) < \tilde{M}$ for any $x \in [0, \ell]$.

Proof. Fix i with $0 \leq i \leq j-1$ and $t \in [x_i, x_{i+1}]$. By (21), we have

$$\left|\boldsymbol{x}_j(t) - \left(\frac{\boldsymbol{x}_{j,i+1}}{h}(t - x_i) + \frac{\boldsymbol{x}_{j,i}}{h}(x_{i+1} - t)\right)\right| \leq \frac{|\boldsymbol{x}_{j,i}''|}{3}h^2 + \frac{|\boldsymbol{x}_{j,i+1}''|}{3}h^2 \leq \frac{2}{3}\|\mathrm{Ev}_j(\boldsymbol{x}_j'')\|h^2.$$

On the other hand,

$$\hat{m} \leq \min\{\boldsymbol{x}_{j,i}, \boldsymbol{x}_{j,i+1}\} \leq \frac{\boldsymbol{x}_{j,i+1}}{h}(t - x_i) + \frac{\boldsymbol{x}_{j,i}}{h}(x_{i+1} - t) \leq \max\{\boldsymbol{x}_{j,i}, \boldsymbol{x}_{j,i+1}\} \leq \hat{M}.$$

Combining both estimates and Lemma 30 we deduce that

$$\hat{m} - 2(\epsilon + g(\hat{M}))h^2 \leq \frac{3\tilde{m}}{2} - \frac{2}{3}\|\mathrm{Ev}_j(\boldsymbol{x}_j'')\|h^2 \leq \boldsymbol{x}_j(t) \leq \hat{M} + \frac{2}{3}\|\mathrm{Ev}_j(\boldsymbol{x}_j'')\|h^2 \leq \hat{M} + 2(\epsilon + g(\hat{M}))h^2.$$

This readily implies the statement of the lemma. □

Next we obtain a sufficient condition on j which implies that the second derivative of the complete cubic spline defined by an ϵ-approximation of the discrete system $F_j = \mathbf{0}$ meets the expected requirements. To this end, we need the following result.

Lemma 32. *Given $\epsilon > 0$ and $\boldsymbol{x}_j \in \mathcal{X}_j$ with $\boldsymbol{x}_{j,i} := \boldsymbol{x}_j(x_i) \in [\hat{m}, \hat{M}]$ for $0 \le i \le j$ and $\|F_j(\boldsymbol{x}_j)\| < \epsilon$, the following estimate holds for $0 \le i \le j$:*

$$|\boldsymbol{x}''_{j,i} - g(\boldsymbol{x}_{j,i})| \le g'(\hat{M}) \max_{1 \le s \le j} |\boldsymbol{x}_{j,s} - \boldsymbol{x}_{j,s-1}| + 3\epsilon.$$

Proof. Observe that (46) may be rewritten in the following way:

$$\mathrm{Ev}_j(\boldsymbol{x}''_j) - \mathrm{Ev}_j(g(\boldsymbol{x}_j)) + (B - I)\mathrm{Ev}_j(\boldsymbol{x}''_j) = F_j(\boldsymbol{x}),$$

where I denotes the identity matrix of size $(j+1) \times (j+1)$. A critical remark is that the sum of the elements of each row in $\frac{1}{6}B - I$ is equal to zero, which allows us to express the vector $(B - I)\mathrm{Ev}_j(\boldsymbol{x}''_j)$ in terms of the differences $\boldsymbol{x}''_{j,s} - \boldsymbol{x}''_{j,s+1}$ for $0 \le s \le j-1$. More precisely, we have

$$\begin{aligned}
&\boldsymbol{x}''_{j,0} - g(\boldsymbol{x}_{j,0}) + \frac{2}{6}(\boldsymbol{x}''_{j,1} - \boldsymbol{x}''_{j,0}) = F_{j,0}(\boldsymbol{x})\\
&\boldsymbol{x}''_{j,1} - g(\boldsymbol{x}_{j,1}) + \frac{1}{6}(\boldsymbol{x}''_{j,2} - \boldsymbol{x}''_{j,1}) - \frac{1}{6}(\boldsymbol{x}''_{j,1} - \boldsymbol{x}''_{j,0}) = F_{j,1}(\boldsymbol{x})\\
&\vdots\\
&\boldsymbol{x}''_{j,j-1} - g(\boldsymbol{x}_{j,j-1}) + \frac{1}{6}(\boldsymbol{x}''_{j,j} - \boldsymbol{x}''_{j,j-1}) - \frac{1}{6}(\boldsymbol{x}''_{j,j-1} - \boldsymbol{x}''_{j,j-2}) = F_{j,j-1}(\boldsymbol{x})\\
&\boldsymbol{x}''_{j,j} - g(\boldsymbol{x}_{j,j}) - \frac{2}{6}(\boldsymbol{x}''_{j,j} - \boldsymbol{x}''_{j,j-1}) = F_{j,j}(\boldsymbol{x}).
\end{aligned}$$

As a consequence, we deduce the following inequality for $0 \le i \le j$:

$$|\boldsymbol{x}''_{j,i} - g(\boldsymbol{x}_{j,i})| \le \frac{1}{3} \max_{1 \le s \le j}\{|\boldsymbol{x}''_{j,s} - \boldsymbol{x}''_{j,s-1}|\} + \epsilon. \tag{47}$$

Let

$$R := \begin{pmatrix} 1 & -1 & & \\ & \ddots & \ddots & \\ & & 1 & -1 \\ & & & 1 \end{pmatrix} \quad \text{and} \quad R^{-1} = \begin{pmatrix} 1 & \dots & \dots & 1 \\ & \ddots & & \vdots \\ & & 1 & 1 \\ & & & 1 \end{pmatrix}.$$

From (46) we see that $R(6B)\,\mathrm{Ev}_j(\boldsymbol{x}''_j) = 6R\,\mathrm{Ev}_j(g(\boldsymbol{x}_j)) + 6RF_j(\boldsymbol{x}_j)$. Expressing this identity in matrix form we obtain

$$\begin{pmatrix} 3 & 1 & & & & \\ 1 & 4 & 1 & & & \\ & \ddots & \ddots & \ddots & & \\ & & 1 & 4 & 1 & \\ & & & 1 & 3 & \\ & & & & 2 & 6 \end{pmatrix} \begin{pmatrix} \boldsymbol{x}''_{j,0} - \boldsymbol{x}''_{j,1} \\ \vdots \\ \boldsymbol{x}''_{j,j-1} - \boldsymbol{x}''_{j,j} \\ \boldsymbol{x}''_{j,j} \end{pmatrix} = 6 \begin{pmatrix} g(\boldsymbol{x}_{j,0}) - g(\boldsymbol{x}_{j,1}) \\ \vdots \\ g(\boldsymbol{x}_{j,j-1}) - g(\boldsymbol{x}_{j,j}) \\ g(\boldsymbol{x}_{j,j}) \end{pmatrix} + 6RF_j(\boldsymbol{x}_j).$$

Denote by C the $(j \times j)$-submatrix of the matrix of the left-hand side consisting of the first j rows and j columns of this matrix. By considering the first j rows of the previous inequality we conclude that

$$\max_{0 \le i \le j-1} |\boldsymbol{x}''_{j,i} - \boldsymbol{x}''_{j,i+1}| \le 6\|C^{-1}\| \Big(\max_{0 \le i \le j-1} |g(\boldsymbol{x}_{j,i}) - g(\boldsymbol{x}_{j,i+1})| + \|R\| \|F_j(\boldsymbol{x}_j)\| \Big).$$

By [23, Theorem 1], it follows that $\|C^{-1}\| \le 1/2$. Therefore,

$$\max_{0 \le i \le j-1} |\boldsymbol{x}''_{j,i} - \boldsymbol{x}''_{j,i+1}| \le 3 \max_{0 \le i \le j-1} |g(\boldsymbol{x}_{j,i}) - g(\boldsymbol{x}_{j,i+1})| + 6\epsilon \le 3\, g'(\hat{M}) \max_{0 \le i \le j-1} |\boldsymbol{x}_{j,i} - \boldsymbol{x}_{j,i+1}| + 6\epsilon.$$

Combining this inequality with (47) finishes the proof of the lemma. □

Now we are able to obtain the sufficient condition on j that we are looking for.

Proposition 33. *Given $\epsilon > 0$ and $\boldsymbol{x}_j \in \mathcal{X}_j$ with $\boldsymbol{x}_{j,i} := \boldsymbol{x}_j(x_i) \in [\hat{m}, \hat{M}]$ for $0 \le i \le j$ and $\|F_j(\boldsymbol{x}_j)\| < \epsilon$, let*

$$\lambda_{2,\epsilon} := \Big(9g''(\tilde{M})\big(\epsilon + g(\hat{M})\big)\ell^2 + 3g'(\tilde{M}) + 8g'(\hat{M})/\ell\Big)\big(\epsilon + g(\hat{M})\big)\ell^2/4A.$$

If $\epsilon \le A/6$ and $j > \max\big\{\lambda_{1,\epsilon}, \lambda_{2,\epsilon}\big\}$, then $\|\boldsymbol{x}_j'' - g(\boldsymbol{x}_j)\| < A$.

Proof. Fix i with $0 \le i \le j-1$ and $t \in [x_i, x_{i+1}]$. From (21) we see that

$$\big|\boldsymbol{x}_j''(t) - g(\boldsymbol{x}_j)(t)\big| \le \left|\frac{\boldsymbol{x}_{j,i+1}'' - g(\boldsymbol{x}_{j,i+1})}{h}(t - x_i) + \frac{\boldsymbol{x}_{j,i}'' - g(\boldsymbol{x}_{j,i})}{h}(x_{i+1} - t)\right| + \left|g(\boldsymbol{x}_j)(t) - \left(\frac{g(\boldsymbol{x}_{j,i+1})}{h}(t - x_i) + \frac{g(\boldsymbol{x}_{j,i})}{h}(x_{i+1} - t)\right)\right|. \tag{48}$$

We first bound the second term on the right-hand side of (48). Since $g(\boldsymbol{x}_j)(x_i) = g(\boldsymbol{x}_{j,i})$ for $0 \le i \le j$, this term agrees with the error of Lagrange interpolation of $g(\boldsymbol{x}_j)$ at x_i, x_{i+1}. This implies that there exists $\xi_t \in [x_i, x_{i+1}]$ such that

$$\begin{aligned}\mathcal{T} &:= \left|g(\boldsymbol{x}_j)(t) - \left(\frac{g(\boldsymbol{x}_{j,i+1})}{h}(t - x_i) + \frac{g(\boldsymbol{x}_{j,i})}{h}(x_{i+1} - t)\right)\right| \\ &\le \frac{\big|(g(\boldsymbol{x}_j))''(\xi_t)\big|}{2}(t - x_i)(x_{i+1} - t) \le \|(g(\boldsymbol{x}_j))''\|\frac{h^2}{8}.\end{aligned}$$

Observe that $(g(\boldsymbol{x}_j))'' = g''(\boldsymbol{x}_j)(\boldsymbol{x}_j')^2 + g'(\boldsymbol{x}_j)\boldsymbol{x}_j''$. Furthermore, for any $s \in [0, \ell]$,

$$|\boldsymbol{x}_j'(s)| \le \frac{|\boldsymbol{x}_{j,i+1} - \boldsymbol{x}_{j,i}|}{h} + \frac{|\boldsymbol{x}_{j,i}'' - \boldsymbol{x}_{j,i+1}''|}{6}h + \frac{|\boldsymbol{x}_{j,i+1}''|}{2h}(s - x_i)^2 + \frac{|\boldsymbol{x}_{j,i}''|}{2h}(x_{i+1} - s)^2 \le \frac{|\boldsymbol{x}_{j,i+1} - \boldsymbol{x}_{j,i}|}{h} + \frac{4}{3}\|\mathrm{Ev}_j(\boldsymbol{x}_j'')\|h.$$

Since $\|F_j(\boldsymbol{x}_j)\| < \epsilon$, by, e.g., [3, Remark 1] it follows that

$$\frac{|\boldsymbol{x}_{j,i+1} - \boldsymbol{x}_{j,i}|}{h} < \left(\left(i + \frac{1}{2}\right)\epsilon + \frac{1}{2}g(\boldsymbol{x}_{j,0}) + g(\boldsymbol{x}_{j,1}) + \cdots + g(\boldsymbol{x}_{j,i})\right)h < (\epsilon + g(\hat{M}))\ell \tag{49}$$

for $0 \le i \le j-1$. We conclude that

$$\|\boldsymbol{x}_j'\| \le \left(\epsilon + g(\hat{M}) + \frac{4}{3j}\|\mathrm{Ev}_j(\boldsymbol{x}_j'')\|\right)\ell.$$

As a consequence,

$$\|(g(\boldsymbol{x}_j))''\| \le g''(\|\boldsymbol{x}_j\|)\left(\epsilon + g(\hat{M}) + \frac{4}{3j}\|\mathrm{Ev}_j(\boldsymbol{x}_j'')\|\right)^2\ell^2 + g'(\|\boldsymbol{x}_j\|)\|\mathrm{Ev}_j(\boldsymbol{x}_j'')\|.$$

Since $j > 1$, combining this inequality with Lemmas 30 and 31, we see that

$$\begin{aligned}\|(g(\boldsymbol{x}_j))''\| &\le 9g''(\|\boldsymbol{x}_j\|)(\epsilon + g(\hat{M}))^2\ell^2 + 3g'(\|\boldsymbol{x}_j\|)(\epsilon + g(\hat{M})) \\ &\le 9g''(\tilde{M})(\epsilon + g(\hat{M}))^2\ell^2 + 3g'(\tilde{M})(\epsilon + g(\hat{M})).\end{aligned}$$

This enables us to bound the second term on the right-hand side of (48):

$$\mathcal{T} \le \Big(9g''(\tilde{M})\big(\epsilon + g(\hat{M})\big)\ell^2 + 3g'(\tilde{M})\Big)\big(\epsilon + g(\hat{M})\big)h^2/8. \tag{50}$$

Next we consider the first term on the right-hand side of (48). By Lemma 32 and (49) we easily see that

$$|\boldsymbol{x}_{j,i}'' - g(\boldsymbol{x}_{j,i})| \le g'(\hat{M})\big(\epsilon + g(\hat{M})\big)\ell h + 3\epsilon. \tag{51}$$

Now we are ready to establish an upper bound for (48) in terms of ϵ and h. By (50) and (51) we conclude that

$$\|\boldsymbol{x}_j'' - g(\boldsymbol{x}_j)\| \le 3\epsilon + g'(\hat{M})\big(\epsilon + g(\hat{M})\big)\ell h + \Big(9g''(\tilde{M})\big(\epsilon + g(\hat{M})\big)\ell^2 + 3g'(\tilde{M})\Big)\big(\epsilon + g(\hat{M})\big)h^2/8.$$

From this inequality and the hypotheses $\epsilon \le A/6$ and $j > \lambda_{2,\epsilon}$ we readily deduce the statement of the lemma. □

We summarize Lemma 31 and Proposition 33 in the following statement.

Corollary 34. *Given $\epsilon > 0$ and $\boldsymbol{x}_j \in \mathcal{X}_j$ with $\boldsymbol{x}_{j,i} := \boldsymbol{x}_j(x_i) \in [\hat{m}, \hat{M}]$ for $0 \le i \le j$ and $\|F_j(\boldsymbol{x}_j)\| < \epsilon$, let $\lambda_{1,\epsilon}$ and $\lambda_{2,\epsilon}$ be defined as in Lemma 31 and Proposition 33 respectively. If $\epsilon \le A/6$ and $j > \max\{\lambda_{1,\epsilon}, \lambda_{2,\epsilon}\}$, then $\boldsymbol{x}_j \in \mathcal{D}_j$.*

Next we show that, for j large enough, a sufficiently good approximation $\boldsymbol{x}_j$ to the positive solution $\boldsymbol{x}_j^*$ of the discrete system $F_j = 0$ satisfies the conditions in the statement of Corollary 34.

Proposition 35. *Given $\epsilon > 0$ and $\boldsymbol{x}_j \in \mathcal{X}_j$ with $\|\mathrm{Ev}_j(\boldsymbol{x}_j - \boldsymbol{x}_j^*)\| < \epsilon h^2/5$, where $\boldsymbol{x}_j^* \in \mathcal{X}_j$ is the positive solution of $F_j = 0$, if*

$$j > \lambda_{3,\epsilon} := \max\left\{\frac{2\ell^2}{\tilde{m}}\left(\frac{\epsilon}{5} + \lambda_2^{**}\right), \ell^2 g'(\hat{M})\right\}^{1/2},$$

then $\|F_j(\boldsymbol{x}_j)\| < \epsilon$ and $\boldsymbol{x}_j(x_i) \in [\hat{m}, \hat{M}]$ for $0 \le i \le j$.

Proof. Let $\boldsymbol{x}^* \in \mathcal{D}$ be the positive solution of (2). We have

$$\|\boldsymbol{x}_j - \boldsymbol{x}^*\| \le \|\boldsymbol{x}_j - \boldsymbol{x}_j^*\| + \|\boldsymbol{x}_j^* - \boldsymbol{x}^*\|.$$

By hypothesis, $\|\mathrm{Ev}_j(\boldsymbol{x}_j - \boldsymbol{x}_j^*)\| \le \epsilon h^2/5$. On the other hand, Corollary 21 proves that there exists $\lambda_2^{**} > 0$, independent of j, such that $\|\boldsymbol{x}_j^* - \boldsymbol{x}^*\| \le \lambda_2^{**} h^2$. Hence,

$$\|\mathrm{Ev}_j(\boldsymbol{x}_j - \boldsymbol{x}^*)\| \le \left(\frac{\epsilon}{5} + \lambda_2^{**}\right) h^2.$$

As $m \le \boldsymbol{x}^* \le M$, we deduce that, if

$$j^2 \ge \frac{\left(\frac{\epsilon}{5} + \lambda_2^{**}\right)\ell^2}{\min\{m - \hat{m}, \hat{M} - M\}} = \frac{2\left(\frac{\epsilon}{5} + \lambda_2^{**}\right)\ell^2}{\tilde{m}}, \tag{52}$$

then $\boldsymbol{x}_j(x_i) \in [\hat{m}, \hat{M}]$ for $0 \le i \le j$.

It remains to find a condition on j which implies $\|F_j(\boldsymbol{x}_j)\| < \epsilon$. Since $F_j(\boldsymbol{x}_j^*) = 0$, we have $F_j(\boldsymbol{x}_j) = F_j(\boldsymbol{x}_j) - F_j(\boldsymbol{x}_j^*)$, and thus

$$F_j(\boldsymbol{x}_j) = \frac{1}{h^2}\begin{pmatrix} -2 & 2 & & & \\ 1 & -2 & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & 1 & -2 & 1 \\ & & & 2 & -2 \end{pmatrix}\begin{pmatrix} \boldsymbol{x}_{j,0} - \boldsymbol{x}_{j,0}^* \\ \vdots \\ \boldsymbol{x}_{j,j} - \boldsymbol{x}_{j,j}^* \end{pmatrix} - \begin{pmatrix} g(\boldsymbol{x}_{j,0}) - g(\boldsymbol{x}_{j,0}^*) \\ \vdots \\ g(\boldsymbol{x}_{j,j}) - g(\boldsymbol{x}_{j,j}^*) \end{pmatrix},$$

where $\boldsymbol{x}_{j,i} := \boldsymbol{x}_j(x_i)$ and $\boldsymbol{x}_{j,i}^* := \boldsymbol{x}_j^*(x_i)$ for $0 \le i \le j$. By the Mean Value theorem it follows that there exists ξ_i^* in the real interval defined by $\boldsymbol{x}_{j,i}$ and $\boldsymbol{x}_{j,i}^*$ such that $g(\boldsymbol{x}_{j,i}) - g(\boldsymbol{x}_{j,i}^*) = g'(\xi_i)(\boldsymbol{x}_{j,i} - \boldsymbol{x}_{j,i}^*)$ for $0 \le i \le j$. Denote $\gamma_i := h^2 g'(\xi_i)$ for $0 \le i \le j$. Then

$$F_j(\boldsymbol{x}_j) = \frac{-1}{h^2}\begin{pmatrix} 2+\gamma_0 & -2 & & & \\ -1 & 2+\gamma_1 & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & 2+\gamma_{j-1} & -1 \\ & & & -2 & 2+\gamma_j \end{pmatrix}\begin{pmatrix} \boldsymbol{x}_{j,0} - \boldsymbol{x}_{j,0}^* \\ \vdots \\ \boldsymbol{x}_{j,j} - \boldsymbol{x}_{j,j}^* \end{pmatrix}.$$

Taking into account the upper bound $|\boldsymbol{x}_{j,i}| \le \hat{M}$ and the hypothesis $\|\mathrm{Ev}_j(\boldsymbol{x}_j - \boldsymbol{x}_j^*)\| < \epsilon h^2/5$, we deduce that

$$\|F_j(\boldsymbol{x}_j)\| \le \frac{1}{h^2}(4 + \max\{\gamma_i : 0 \le i \le j\})\|\mathrm{Ev}_j(\boldsymbol{x}_j - \boldsymbol{x}_j^*)\| < (4 + h^2 g'(\hat{M}))\frac{\epsilon}{5}.$$

We see that, if

$$j^2 > \ell^2\, g'(\hat{M}), \tag{53}$$

then $\|F_j(\boldsymbol{x}_j)\| < \epsilon$. Combining (52) and (53) the proposition follows. □

We can now prove the main result of this section, namely there is an explicitly computable positive integer j^* such that for $j \ge j^*$ we are able to obtain a starting point for the discrete Newton iteration defined by N_j, and this iteration yields good approximations of the positive solution of (2). In fact, we have the following result.

Theorem 36. *There is an explicitly computable positive integer j^* with the following properties:*

(1) *we can compute $\boldsymbol{x}_{j^*} \in \mathcal{D}_{j^*}$ with $\mathcal{O}(j^* \log_2 \log_2(j^*))$ flops and function evaluations;*

(2) *$\boldsymbol{x}_{j^*} \in \mathcal{D}_j$ for each integer multiple j of j^*.*

Proof. Let $\epsilon^* := A/6$ and let $\lambda^* := \max\{\lambda_{1,\epsilon^*}, \lambda_{2,\epsilon^*}, \lambda_{3,\epsilon^*}\}$, where $\lambda_{1,\epsilon^*}, \lambda_{2,\epsilon^*}$ and λ_{3,ϵ^*} are defined as in the statements of Lemma 31 and Propositions 33 and 35 respectively. Let j^* be the least positive integer with $j^* > \lambda^*$. We claim that j^* satisfies the conditions of the statement of the theorem.

Let $0 := x_0 < \cdots < x_{j^*} := \ell$ be the uniform mesh of $[0, \ell]$ with $j^* + 1$ elements and let $h^* := \ell/j^*$. Applying the algorithm of [4] or [8] we compute the values at the nodes $x_0, \ldots, x_{j^*}$ of an element $\boldsymbol{x}_{j^*} \in \mathcal{X}_{j^*}$ such that

$$\|\mathrm{Ev}_j(\boldsymbol{x}_{j^*} - \boldsymbol{x}^*_{j^*})\| < \epsilon^* {h^*}^2/5,$$

where $\boldsymbol{x}_{j^*}$ is the positive solution of $F_{j^*} = 0$. The algorithm performs $\mathcal{O}(j^* \log_2 \log_2(j^*))$ flops and function evaluations, showing thus the first assertion.

As $j^* > \lambda_{3,\epsilon^*}$, by Proposition 35 it follows that $\|F_{j^*}(\boldsymbol{x}_{j^*})\| < \epsilon$ and $\boldsymbol{x}_{j^*}(x_i) \in [\hat{m}, \hat{M}]$ for $0 \le i \le j^*$. As $\epsilon^* \le A/6$ and $j^* > \max\{\lambda_{1,\epsilon^*}, \lambda_{2,\epsilon^*}\}$, from Corollary 34 we deduce the second assertion, finishing the proof of the theorem. $\square$

5.2. The cost of computing an ϵ-approximation

Theorem 36 asserts that we can obtain a starting point $\boldsymbol{x}_j^0 \in \mathcal{D}_j$ for the discrete Newton operator N_j for j sufficiently large with $\mathcal{O}(1)$ flops and function evaluations. Given $\epsilon > 0$, we aim to compute an ϵ-approximation of the positive solution $\boldsymbol{x}^*$ of (2). To this end, we determine a value of j and a positive integer k such that the kth iteration $\boldsymbol{x}_j^k$ of N_j, starting at $\boldsymbol{x}_j^0$, is an ϵ-approximation of $\boldsymbol{x}^*$, namely

$$\|\boldsymbol{x}_j^k - \boldsymbol{x}^*\| < \epsilon.$$

According to Theorems 13 and 29,

$$\|\boldsymbol{x}_j^k - \boldsymbol{x}^k\| \le \frac{2\lambda}{j^2} \quad \text{and} \quad \|\boldsymbol{x}^k - \boldsymbol{x}^*\| \le \frac{\frac{2}{\omega}\left(\frac{h_0}{2}\right)^{2^k}}{1 - \left(\frac{h_0}{2}\right)^{2^k}} \le \frac{\frac{2}{\omega}\left(\frac{1}{4}\right)^{2^k}}{1 - \left(\frac{1}{4}\right)^{2^k}}.$$

As a consequence, for

$$j > \left(\frac{4\lambda}{\epsilon}\right)^{1/2} \quad \text{and} \quad k > \log_2 \log_2\big((4 + \epsilon\omega)/\epsilon\omega\big),$$

we obtain $\|\boldsymbol{x}_j^k - \boldsymbol{x}^*\| \le \|\boldsymbol{x}_j^k - \boldsymbol{x}^k\| + \|\boldsymbol{x}^k - \boldsymbol{x}^*\| < \epsilon/2 + \epsilon/2 = \epsilon$. Since each iteration of N_j requires $\mathcal{O}(j)$ flops and function evaluations, we deduce the following result.

Theorem 37. *We can compute an ϵ-approximation of the positive solution $\boldsymbol{x}^*$ of (2) with $\mathcal{O}\big((1/\epsilon)^{1/2} \log_2 \log_2(1/\epsilon)\big)$ flops and function evaluations.*

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