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Collapse, Plurals and Sets

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Abstract. This paper raises the question under what circumstances a plurality forms a set. My main point is that not always all things form sets. A provocative way of presenting my position is that, as a result of my approach, there are more pluralities than sets. Another way of presenting the same thesis claims that there are ways of talking about objects that do not always collapse into sets. My argument is related to expressive powers of formal languages. Assuming classical logic, I show that if all plurality form a set and the quantifiers are absolutely general, then one gets a trivial theory. So, by reductio, one has to abandon one of the premiss. Then, I argue against the collapse of the pluralities into sets. What I am advocating is that the *thesis of collapse* limits important applications of the plural logic in model theory, when it is assumed that the quantifiers are absolutely general.

Keywords: Pluralities; absolute generality; sets; hierarchies.

We often say that some things form a set. For instance, every house in Beacon Hill may form a set. Also, all antimatter particles in the universe, all even numbers, all odd numbers, and in general all natural numbers do so. Naturally, following this line of thought, one might think that the plurality of all things constitutes a set. And although natural language allows us, by means of its plural constructions, to talk about objects without grouping them in one entity, there are also nominalization devices to turn constructions involving high order expressive resources into others that only make use of first order ones. For example, the predicate "to be a natural number" can be transformed into "the property of being a natural number", and the plural expression "the antimatter particles" into "the collection of antimatter particles". Such transformations motivate some version of the Collapse Thesis. If one follows this line —as with the Beacon Hill houses and the numbers— every time we have some things, there will be a set constituted by them. Despite the initial plausibility of the thesis, in this article I will defend a completely different stance. According to my point of view, not all things always form sets. One provocative way of presenting my position would be to say that, as a result of this, there are more pluralities than sets. Another perhaps less attractive way to put it is to say that there are devices to talk about objects that do not always collapse into sets. My argument is related to the expressive needs of formal languages. I defend that the collapse thesis imposes limitations on important applications of logic to model theory, when one assumes that quantifiers are absolutely general.

1. Pluralities and formal theories

Natural languages contain singular terms, quantifiers and predicates (e.g. 'it', 'something' and "... is in Boston'). But it also contains plural terms, quantifiers and predicates (e.g. 'they', 'some things' and '... are scattered on the floor'). Philosophers, logicians and linguists (Yi 1999, 2005; Oliver and Smiley 2001; Rayo 2003; and McKay 2006) have become increasingly interested in plurals over the past couple of decades. A plural predicate is a predicate taking a plural variable in one or more of its argument places. According to Pluralism, ' (xx)' can be *jointly* satisfied or unsatisfied by several individuals at once. Plural expressions such as 'some things' are a primary concern in plural logic. In classical singular first order logic, we only have singular variables like x and y, which are bounded by singular universal (' x') and existential (' x') quantifiers. Besides these expressive devices, in plural logic we have plural variables, like xx and yy, which can be bounded by plural quantifiers. In plural languages, the universal plural quantifier xx is interpreted (intuitively) as 'any objects such that' and the existential plural quantifier xx as 'there are some things such that'. (Boolos 1984 and 1985.) Then according to Pluralism, all that is required for the truth of ' xx (xx)' is that there be individuals in the range of one's singular quantifiers that jointly satisfy ' (xx)'. As a result, pluralists believe that the use of plural quantifiers carries no commitment to entities outside the range of one's singular quantifiers. In this way, the *plural* quantification is understood as a new form of quantification over the same old thing (those in the first-order domain).

There are several reasons for motivating the adoption of plural resources. For example, the Geach-Kaplan sentence (see Boolos 1984) 'Some critics admire only one another' does not allow for a first-order rendering on the intended interpretation. The second-order formalization is:

$$X[\quad z(Xz \quad Cz) \quad xXx \quad x,y((Xx \quad Axy) \quad (x = y \quad Xy))]$$

Of course, it can be formalized in first-order set theory as well, but the original sentence does not seem to commit to sets in any way. In formal presentations of plural logic, a plural logical predicate is also included. The expression "$u \quad xx$" is interpreted (intuitively) as "u is one of the xx". Then, adopting plural quantification, this sentence can be formalized as:

$$xx[\quad u(u \quad xx \quad Cu) \,\&\, \quad u \quad v(u \quad xx \,\&\, Auv \quad v \quad xx \,\&\, u = v)]$$

without any commitment to sets.

One important question is about the formal principles that involve pluralities. Plural quantifiers obey rules analogous to the classical introduction and elimination rules. For example, in the case of the existential plural quantifier, the introduction and elimination rules are as follows:

Existential Introduction

$$(xx) \quad xx(\ (xx))$$

Existential Elimination

$$\frac{\quad}{(xx)} = \frac{\quad}{xx(\ (xx))}$$

(where 'xx' does not occur free in or). These rules are completely analogous to the rules governing the singular quantifiers. So. it can hardly be denied that they too qualify as logical.

Nevertheless, as well as set theory for sets, one has to add some particular axioms about pluralities. In this way, plural formal theories include the plural naïve comprehension schema:

COMP-P: $u \ u \quad xx \ u(u \quad xx \quad (u))$

where is a formula in the that contains 'u' and possibly other variables free but contains no occurrence of 'xx'. Which intuitively this axiom says that, for any condition (u), assuming there is at least one , some things are such that something is one of them just in case it satisfies that condition. Further, in order fully to capture the idea that all pluralities are non-empty, one also adopts the axiom:

$$xx \ u(u \quad xx)$$

Finally, the following axiom schema of extensionality:

$$xx \ yy[\ u(u \quad xx \quad u \quad yy) \quad (\ (xx) \quad (yy))]$$

ensures that all coextensive pluralities are indiscernible.

Proponents of plural quantification claim that these theories allow plural locutions to be formalized in a way that is fundamentally different from the old set-theoretic paraphrases. In particular, plural theory is not *ontologically committed* to any entities beyond those already accepted in the ordinary first order domain. It has also the property of being universally applicable: the theory of plural quantification can be applied to any realm of discourse, no matter what objects this discourse is concerned with. This distinguishes the plural theories both from set theory and from second-order logic with the usual set theory semantics. Moreover, the theory of plural quantification presupposes no extra-logical ideas to be understood. Plurals can be understood directly. Our understanding of it does not consist in an understanding of extra-logical ideas, such as ideas from set theory or other mathematical theories.

In sum, singular and plural quantifiers have introduction and elimination rules that are completely analogous. Plural systems, as well as the first-order logic, are ontologically innocent, universally applicable and cognitively neutral. However, some people challenge the status of COMP-P as a truth of plural logic for different reasons, claiming it should not be included in the system, because this principle implies a contradiction. The most promising proposal adopts some form of collapse: the thesis that for any plurality of objects, there is a set constituted by those objects. The idea is that there is a collapse from pluralities into sets. "Pluralities and sets fit naturally together". (Linnebo 2010, p.151.) That is, the advocate of Collapse maintains

COLLAPSE: $xx \quad y\mathrm{FORM}(xx, y)$

where the plural predicate "Form(xx, y)" is an abbreviated way to say that the things xx form a set if and only if $u(u \quad xx \quad u \quad y)$. Recall that Collapse is compatible with the possibility of pluralities and sets not being identical. The thesis says that for any objects, there is a set constituted by them.

In the next part, I will present the *knock-down argument*. This argument can be used to reject COMP-P using COLLAPSE. This argument comes from a semantic version of Russell's paradox. I am going to show that COMP-P and COLLAPSE jointly entail triviality. Then, I will offer some reasons to reject COLLAPSE in order to allow to support COMP-P.

2. The knock-down argument

Lets suppose that a plural theory (a formal classical logical system) includes the plural naive comprehension schema, and that we adopt an absolutely unrestricted reading of the existential plural quantifier in it.[1] Then, for each condition (u), there are some things that are all and only the things satisfying the condition. An instance of this schema allows us to consider the objects rr which are all and only the sets that aren't members of themselves:

1. $u(u \quad rr \quad u \quad u)$

But, by collapse, the rr form the Russell set:

2. $r \ u(u \quad r \quad u \quad rr)$

From 1 and 2, the following characterization of the Russell set r is obtained:

3. $u(u \quad r \quad u \quad u)$

And by universal quantifier elimination in 3, with respect to r, we get:

4. *r* *r* *r* *r*

If classical logic is accepted, 4 entails a contradiction.

Nonetheless, it is not clear what should we conclude from that argument. Everybody agrees that it is a *reductio*. But there is no consensus regarding the object of such reductio. There are several options.

In the first place, as I said above, the argument implicitly assumes an unrestricted interpretation of the quantifiers. If we didn't, it would be possible to say that their scope has changed in the course of the argument. In particular, one could say that the quantifier *r* in 2 has a bigger domain than the quantifier in 1.[2] If that were the case, the step from 3 to 4 would not be valid, since it involves an instantiation of the quantifier regarding an object which cannot be assumed to be on its domain. Of course, this means that one way to block the argument is to reject that there is such thing as genuine unrestricted quantification. This line of thought has got important defendants. For example, it could be argued that there are grounds for questioning the very intelligibility of unrestricted quantification. This line of attack could be based on traditional arguments for semantic indeterminacy arising from the Löwenheim–Skolem theorem. In particular, it is well known that first-order logic cannot distinguish unrestricted quantification from restricted quantification. This means that no theory can discriminate between interpretations with an all-inclusive domain and interpretations with a restricted domain. Anything unrestricted quantification supported may say, if true by his lights, can be so interpreted as to hold true of a restricted domain. Nonetheless, it is clear that this indeterminacy only affects to first order languages. As it is well known, the Löwenheim–Skolem theorem canĄft be generalized to higher-order languages. So, this objection is very limited. Other influential strategy for casting doubt on the unrestricted quantification derives from the work of Michael Dummett. He claims that "indefinitely extensible concept is one such that, if we can form a definite conception of a totality all of whose members fall under the concept, we can, by reference to that totality, characterize a larger totality all of whose members fall under it" (Dummett 1993, p.441). Picking up on this idea, Parsons (1974) and Glanzberg (2004) (2006) state that the possibility of indefinitely extending quantification domains implies the unfeasibility of unrestricted quantification over their instances. The picture arising in its place is that of an open series of domains, each of them extending beyond its predecessor, but none of which is capable of comprising everything. An especially pressing problem for this approach concerns the proper formulation of their view (Williamson 2006; Fine 2006). A proper expression of generality of this approach cannot presuppose that unrestricted quantification is possible. So the warm-up formulation, i.e.

It is not possible to quantify over absolutely everything, will be self-defeating if ‘everything’ ranges, in effect, over absolutely everything. As Lewis affirms: “Maybe [who rejects the genuine unrestricted quantification] replies that some mystical censor stops us from quantifying over absolutely everything without restriction. (...) He violates his own stricture in the very act of proclaiming it.” (Lewis 1990, p.68.) The same strategy is following by McGee. He says: “The reason [the skepticism about the genuine unrestricted quantification] is not a serious worry is that the thesis that, for any discussion, there are things that lie outside the universe of discourse of that discussion is a position that cannot be coherently maintained. Consider the discussion we are having right now. We cannot coherently claim that there are things that lie outside the universe of our discussion, for any witness to the truth of that claim would have to lie outside the claim’s universe of discourse.” (McGee 2000, p.55.) So, unrestricted quantification seems not only possible but necessary. Other examples as:

Everything is self-identical.

seems to show that unrestricted quantification is also necessary for expressing thesis about *absolutely* everything.

In the second place, the argument use COMP-P and COLLAPSE. But, of course, it also uses classical logic to get triviality. Then, an option could involve the rejection of some classical principles. At least three projects stand out as deserving of recognition: the paraconsistent approach of Priest and others, (Priest 1979) the paracomplete approach of Field and others (Field 2008) and views for which the transitivity of implication fails (Weir 1998; 2005). I am not to going to consider these options for reasons of brevity. But it is clear that there may be good reasons for avoid this route because a *substantial* weakening of classical logic will be required: typically paraconsistent logics lack modus ponens, paracomplete logics lack identity and systems of logic without transitivity have problems to analyze validity as truth-preservation. So, there are two opposite ways to answer the paradox: one of them consists in abandoning COMP-P and defend COLLAPSE; the other one follows the contrary route.

TRIVIALITY

COLLAPSE: $xx\ \ yFORM(xx, y)$ COMP-P: $u\ \ (u)\ \ \ \ xx\ \ u(u\ \ \ xx\ \ \ \ \ (u))$

Unrestricted quantifiers Classical logic

My own answer to the argument follows this last path. My position states that, because of reasons connected with the expressibility of languages, we have enough motives to adopt COMP-P. Henceforth, if we want to restore consistency and accept unrestricted quantification, we must abandon COLLAPSE.

3. Implausibility of real collapse between plural forms and sets

According to my position, the strategy of adopting COLLAPSE and restricting COMP-P suffers many difficulties. On the first place, the adoption of Collapse is implausible because it does not allow set theoretical models with absolute generality of quantification. Basically this thesis turns unrestricted quantification over sets unstable. Let's recall a renowned argument one must answer in defending COLLAPSE. Assume one of the philosophical interests of model theory is to give formal languages a generalized semantics. That is, a theory about all the ways to interpret linguistic devices. One motivation related to this interest could be the following: given any language, the definition of logical consequence presupposes the possibility to quantify over every way of interpreting the non logical expressions. If model theory weren't capable of representing absolutely every one of these interpretations, the aforementioned definition would not be extensionally adequate. We can add to this the thesis that one of those interpretations is maximally expressive, the so called "absolute generality" thesis (Rayo & Uzquiano 2006). According to it, there are uses of quantification by means of which we can talk about absolutely every object. There are two aspects of this. On one hand, the commitment with the existence of languages furnished with enough expressive resources to be able to quantify in a completely unrestricted way. On the other hand, the endorsement of an absolutely general domain over which the variables are linked to the quantifiers to get their values. In sum, we support the approach that accepts languages with maximally expressive resources.

Consider a first order language intended to deal with all sets. Formulate that language in the theory ZF. Clearly, if one adopts the thesis according to which every possible interpretation is captured by some set theoretic model, then doing generalized semantics for the language of ZF is just doing applied set theory. But the absolutely general interpretation of a first order language requires a domain that collects everything there is. Nevertheless, if interpretations are sets —and because according to the theorems of ZF there is no universal set— there is no model that captures the absolutely general interpretation of that language (Barrio 2007, 2009, and Rayo 2003).

Another way of making the same point is to appeal to the iterative conception of the universe of sets (Linnebo 2013). The iterative conception gives us a consistent characterization of the set theoretic universe as a cumulative hierarchy. According to this approach, sets build up a hierarchy. In the inferior level are the urelements, or, in the absence of them, the empty set. New sets appear as we go up the hierarchy, each of them belonging to a determinate level. There are two kinds of levels: successors and limit. In each successor stage we get every set formed by objects in the previous stages. In the limit stage we get the union of all sets below that stage. The hierarchy extends as long as possible. That is, at least prima facie, there is no upper bound, or a maximum stage. Every time we fix a determinate limit, it seems possible to go beyond it. In contrast with the ongoing hierarchy of the iterative approach, an absolutely general interpretation of ZF axioms and theorems requires the ability to talk about every set. Put in another way, the scope of the quantifiers in ZF formulae such as "the empty sets has no members whatsoever" or "there is no set of all sets" seems to ask for a definite totality that ZF claims it doesn't exist.

The COLLAPSE defendant finds herself in serious trouble when she aims to express unrestricted quantification among sets. Such thesis seemingly leads to the idea of capturing interpretations in the universe of sets. Hence, domains of quantification must be sets. But since the universe of sets is iterative, when we talk about a universal domain we cannot be referring to a set that can be found in the hierarchy. Because of that, if it is to be assumed that interpretations are captured by sets, the idea of a universal set would lead to contradiction. Therefore, every quantification presupposing domains to be set-like entities cannot be absolutely general. And so it would not be possible for a COLLAPSE defendant to do generalized semantics.

Notice that an appeal to the Kreisel squeezing argument (see Kreisel 1967) would not be of much help for the COLLAPSE defendant. According to the theorem, in the case of first order logical theories, it is enough (in the sense of an extensionally adequate definition of logical consequence) to consider set theoretic interpretations. Nonetheless, the proof of the theorem relies on the completeness of the theory. That is why the theorem cannot be applied to a theory which complies with the conditions of Gödel's Theorems. In particular, it does not apply to higher order languages. For these cases, there is the possibility of an extensional divergence arising between set theoretic and non set theoretic ways of interpreting the expressions.

Summing up, if we want to do generalized semantics and allow absolutely unrestricted interpretation of the quantifier, we must reject the thesis that interpretations are captured by set theoretic models. By requiring that the domains of quantification be set-sized, COLLAPSE rules out any interpretation whose domain is too big to form a set. As a result, COLLAPSE is unable to capture all the intuitive interpretations of the language, such as those in which the domain of quantification contains absolutely everything. This is especially problematic if, as in the case of plural logic with standard semantics, the completeness theorem fails, and thus there is no assurance via Kreisel's squeezing argument that the mere appeal to interpretations with a set-sized domain yields an extensionally adequate relation of logical consequence. That's why COLLAPSE turns unrestricted quantification unstable.

On the contrary, if COLLAPSE is not adopted, there is a promising answer to this argument. Ever since the work of Boolos,[3] appealing to pluralities is a usual way to resolve the tension between the apparent continuous expansion of the hierarchy underlying the iterative conception and the semantic requirements needed to make unrestricted interpretations of the quantifiers possible: we can talk about all the sets without presupposing that such totality forms a set.[4] In this way, a model is not seen as a set, but as a plurality of objects. There are indeed many different manners in which this idea can be implemented. Without giving much technical details, we can say that the language in which the interpretations of ZF formulae are given must include expressive resources enough to quantify over pluralities and express plural predicates. It is important to point out that relying on these expressive resources does not suppose any kind of ontological rise. We can refer to the domain of ZF by talking about "the sets" without committing to any new entities. A plurality of objects is formed just by the objects that constitute it.

Of course, if we were to collapse pluralities into sets, that kind of option would not be available. We wouldn't be able to talk about a domain of certain objects, simply by talking about the objects themselves. Denying COLLAPSE, we can talk about the universal domain simply by talking about some things such that everything is one of them. That is:

$$xx \;\; u(u \quad xx)$$

where the plural existential quantifier and the singular universal quantifier are interpreted in an absolutely unrestricted way. It is evident that this move requires rejecting the All-in-One Principle.[5] According to this principle, in order to quantify over certain objects, one must presuppose those objects constitute a complete collection: a thing that has said objects as members.

All-in-One: To quantify over some objects presupposes that those objects constitute a set or set-like collection that has those objects as its elements.

The adoption of the COLLAPSE thesis, along with this principle, limits the possibility of quantifying over all sets without forming a new set. The plausibility of the All-in-One principle is linked with the need to fix the quantifier scope in a precise manner. To quantify over absolutely all objects whatsoever requires — per impossible — that there be a set- or set-like domain that has absolutely everything as an element. The needs — at least, the ontological needs — of absolutely general quantification are met simply by there being some things that severally comprise absolutely everything. But, as well known, according to set theory, this is not possible: this entity cannot exist. Nonetheless, the development of Boolos semantics shows we can fix the scope of the quantifiers, even in an absolutely general interpretation, without collecting all those objects into a new entity. Rejection of COLLAPSE opens the possibility of talking about some things that satisfy a condition without them forming a thing. One way of making this point is to call attention to the fact that giving up COLLAPSE allows us to replace the all-in-one principle for some sort of plural quantification principle, known as All-in-Many (Uzquiano2009):

All-in-Many: In order to quantify over objects which satisfy some condition, one must presuppose there are some things which are all and only the things satisfying the condition.

That is, the existence of some predicative way of fixing the quantification domain that is not an object. If it were, it would fall under the unrestricted quantification, restoring the aforementioned problems.[6]

In sum, adopting All-in-Many principle and rejecting COLLAPSE one may characterize a plural unrestricted reading of the quantifiers as the claim that quantifiers sometimes range over *some things* that comprise absolutely everything. If one accepts COLLAPSE, then this strategy is not available.

On the second place, when facing the knock-down argument, the COLLAPSE defendant needs to restrict the scope of COMP-P to restore consistency, if she is willing to accept an unrestricted reading of the quantifiers. Nevertheless, I think there isn't any candidate that is at the same time non arbitrary, plausible, and does not restrict the scope of absolute generality. That is, given the reductio pushed by the knock-down argument, the COLLAPSE defendant must abandon COMP-P if she wants to avoid contradiction. But, for the answer to be complete, she must offer an explanation of why COMP-P fails.

A path available for the COLLAPSE defendant would be to dig deeper into the extensional nature of sets and pluralities. In this direction, Linnebo (2010) argues that, given the common extensional nature, both sets and pluralities fit together. COLLAPSE would be a natural consequence, streaming from the fact that plural logic guarantees every plurality is always absolutely determined by its extension. Because of that, there seems to be no reason to deny that:

$$u(u \quad y \qquad u \quad xx)$$

That is, at least part of the nature of sets and pluralities is constituted by the elements belonging to those entities. In Uzquiano's words:

> What motivates the move is the intuitive thought that all it takes for some objects to form a set is for them to exist. Once the members have been characterized, their set has been specified. A set is, after all, completely characterized by its elements. Once we have them, there is nothing to bar them to form a set. No further fact is required. So COLLAPSE has great initial appeal, and one may prefer to sacrifice plural comprehension in order to restore consistency. In particular, if we allow some sets to be all and only the non-self-membered sets, then nothing else stands in the way of their set, which, we know, will yield a contradiction (Uzquiano 2009, p.307).

And so, given that the nature of sets and pluralities is intimately related to their members, if COLLAPSE were true, the restriction of COMP-P could be motivated on the set theoretic hierarchy. That is, a condition successfully generates some objects only when applied to members of some set in the cumulative hierarchy. In this sense, and in the same fashion as sets, COMP-P could be replaced by the following axiom of plural separation (Uzquiano 2009):

$$\text{SEP-P:}\quad t\ \ xx\ \ u(u\quad xx\qquad (u\quad t\qquad (u)))$$

If t is a set and a condition, then there are some objects which are all and only the members of t satisfying . This principle allows us to separate objects from a set by means of a certain condition they satisfy. However, as Uzquiano notices, the adoption of SEP-P gives a direct route from the Russellian argument to the non existence of a comprehensive domain. By SEP-P one can get:

$$xx\ \ u(u\quad xx\qquad (u = u))$$

And by COLLAPSE, those objects form a set. But by SEP-P, some of those objects are the non self-membered sets. And since:

$$xx\ \ u(u\quad xx\qquad (u\quad u))$$

a contradiction follows. Besides, a serious problem with this strategy is that an explanation of which instances of COMP-P are safe and which aren't is conceptually prior to an explanation of what sets there are.

Another alternative, also considered by Uzquiano (2009), consists in accepting only predicative instances of COMP-P. This way out, inspired by Dummett's diagnosis of the Russell paradox, (Dummett 1991) has nonetheless serious problems.

The main one is, as Burguess notes, that the impredicative restriction of the plural comprehension schema restrains us from obtaining enough mathematics. By adopting Frege's Law V,

$$F\ \ G(ext(F) = ext(G)\qquad x(Fx\qquad Gx))$$

Along with non predicative second order logic, we cannot go beyond Robinson arithmetic. In particular, any system which adopts any form of induction needs to liberalize the predicative conditions.

In short, it is not easy to see how the COLLAPSE defendant could justify imposing restrictions to COMP-P. The natural appeal to the set theoretic hierarchy doesn't seem to help, since neither of the apparent open paths leads to a non arbitrary and sufficiently expressive answer.

Moreover, even if pluralities and sets did satisfy their respective extensionality axiom, I don't favor the idea that such fact authorizes the collapse. According to my position, matters of abstract entities existence are internal to their suitable axiomatizations. What sets are there? It is a question that is answered by analyzing ZF axioms and the model theoretic structures that satisfy them. What pluralities are there? It is a question that is answered by analyzing axioms of plural logic and their models. Clearly, there are some entities that make those axioms true. Nevertheless, there is no empty plurality, or single-element plurality, even though there is an empty set and its singleton. On top of that, unlike sets, pluralities satisfy the axiom of naïve plural comprehension:

$$xx \quad u(u \quad xx \qquad (u))$$

without incurring in Russellian inconsistencies. It is enough to avoid contradiction that we don't collapse pluralities into sets, and that we keep two alternative quantifying domains. Nonetheless, sets cannot satisfy the naïve comprehension axiom:

$$x \quad u(u \quad x \qquad (u))$$

And, according to ZF, they satisfy other axioms that despite guaranteeing the existence of certain sets, are not as liberal as to allow that every condition determines a set.

Notice that my position against COLLAPSE does not hang on any supposition about the size of pluralities and sets. That is, I am not committed with rejecting the collapse because pluralities are too big to form a set. This line of thought could be inspired by certain analysis of the Russell paradox: in order to avoid a contradiction, we should limit the size of sets. We do not have to do such thing with pluralities. Because of that, there is certain direction someone who rejects COLLAPSE might take: some pluralities are too big compared to sets. Nevertheless, I concur with Linnebo in that there is no acceptable concept of limitation of size. Any appeal to a particular cardinality to limit sizes will always be arbitrary. The iterative universe of sets grows without limits and, in principle, I do not see any reason, besides ZF axioms, to restrict the existence of sets, no matter what their size is.[7] My point is that pluralities do not collapse into sets because the axioms that regulate them are different from those regulating sets. In particular, pluralities satisfy COMP-P without generating contradiction. Such a thing cannot happen with sets. If every condition were to determine a set, we would have a direct path to Russell's paradox.

One way of reinforcing these reasons against COLLAPSE appeals to the standard semantics for high order languages. Usually, Peano arithmetic is formulated in a first order language PA_1. Because of that, the principle of mathematical induction is formalized through the following schema:

$$(\quad (0) \qquad x(\quad (x) \qquad (Sx))) \qquad x(\quad (x))$$

Where (x) can be replaced by any open formula in the language of arithmetic, with x free. We would like (x) to be instantiated by genuine arithmetical conditions. Nonetheless, different arithmetical axiomatizations can be obtained, depending on how liberal we are with respect to which instances are genuine. Of course, formally we must replace the induction schema with a formula involving high order quantification. But what the range of these quantifiers may be depends on the underlying expressive resources. Usually, some kind of COMP axiom is used in order to determine legitimate instances. Being more or less liberal about the instances yields different subsystems of

second order arithmetic PA_2. However, if the COMP axiom is restricted, it is not possible to avoid non standard models, as a direct result of the compactness theorem. It is well known that Dedekind showed that all models of PA_2 are isomorphic (to the standard model).[8] Of course, it is crucial to obtain this result that assignments to second order variables take values from the full collection of subsets of the domain. In this case, non standard models are avoided.[9] Because of that, it is only by means of standard semantics for high order languages that we can get categorical PA. That is, only if unrestricted COMP-P is adopted, we have categoricity of models of arithmetic.[10]

Finally it is important to note that I am not affirming that never some things form a set. Some things form a set and some other things do not. For example, the empty set and its singleton form a set. The sets do not. But we need to talk about sets in order to get enough expressive power to get the intended model of the set theory. Of course, one important question is under what conditions a plurality forms a set. It is clear to me that this problem depends on our conception about sets. For example, if one adopt the iterative conception about sets, one can support the claim:

ITER-SET: Some xx form a set iff there is some á such that the xx all occur at some point in the hierarchy of sets.

Nevertheless, ITER-SET could be considered too restricted. Non-well-founded sets are described consistently by Aczel. In this description, he modifies ZF by removing Foundation, and adding some (Aczel 1988) anti-foundation axiom securing the existence of non-well-founded sets. Then, maybe some xx could form an Aczel's set. But my point is that not always a plurality form a set. And my response to the question about when a plurality do not form a set depends on the universe of sets is. When does a given plurality form a set? When our best set theory confirms that it does. When does not? When for reasons expressive our plurals constructions allow to talk consistently about objects without forming a set.

In conclusion, in this section I have defended the thesis that there are expressive reasons that make collapse from plural resources to singular ones undesirable. Firstly the adoption of Collapse is implausible because it does not allow set theoretical models with absolute generality of quantification. Secondly when facing the knock-down argument, the COLLAPSE defendant needs to restrict the scope of COMP-P to restore consistency, if she is willing to accept an unrestricted reading of the quantifiers. Nevertheless, I think there isn't any candidate that is at the same time non arbitrary, plausible, and does not restrict the scope of absolute generality. Thirdly, as I noted, I am not supporting that never some things form a set. Some things form a set and some other things do not. But, the problem is to adopt the thesis according to which always some things form a set. In this case, plurals constructions lost the most important: their capability to talk beyond objects that can form a set.

4. Implausibility of the possibilist version of Collapse

Recently, Linnebo (2010) has defended the thesis that sets and pluralities hierarchies are potential (not really complete, but it is always possible to obtain new sets and expand it).[11] In this direction, he proposes to adopt a modal version of collapse and COMP-P, which captures the potential character of the hierarchies. The idea is simple. Linnebo starts with the set theoretic hierarchy. Possible stages in the process of set formation are in this perspective thought of as possible worlds. The potential character of the hierarchy is made explicit by introducing two modal operators: and . The idea is to treat both operators as primitive, their behavior being regulated by some modal logic. Just for heuristic purposes, it is useful to interpret as "it is possible to go on to form sets so as to make it the case that ", and as "no matter what sets we go on to form it will remain the case that ."

The key point of his proposal can be found in the models of the logic. Linnebo suggests adopting a modal logic of the kind of $S4.2$, which is satisfied by structures with a reflexive, anti-symmetric, transitive, well-founded, directed and maximal accessibility relation between worlds (or stages of formation).[12]

Linnebo defends his point of view by saying his solutions provide an explanation of why the axiom of plural comprehension has to be restricted when (in the context of set theory) quantifiers are understood as having an implicit modal character. And at the same time, it allows us to maintain a modal version of COLLAPSE. According to Linnebo "plural collapse has an elegant modal implementation, which can be used to provide a novel motivation for ZF set theory." In this modal implementation, COLLAPSE states that:

$$xx \quad y\text{Form}(xx, y)$$

That is, given any objects xx, it is possible to form a set which has exactly those objects as elements. The modal translation of the unrestricted comprehension schema is:

P-COMP*: $\quad xx \quad u(u \quad xx \qquad (u))$

That is, given any condition (u), it is possible that there are some objects such that (no matter whether they form a set), they are all and only the objects satisfying the condition. According to Linnebo, in the knock-down context, if a non modal reading of the quantifiers is assumed —as I do in this article— COLLAPSE is false. However, it is possible to give another reading of the quantification, one that restores consistency and prevents collapse from being knocked down. In this respect, he says:

> For given some objects xx, the existence of a set with precisely xx as elements is only potential. This means that we may go on to form such as set. But it does not mean that such a set has actually been formed, which is what collapse asserts.

And then he goes on to say:

> When the quantifiers are understood as having an implicit modal character, Collapse is true and (COMPP) is false. And when the quantifiers are taken at face value, Collapse is false and (COMP-P) is true (Linnebo 2010, p.13).

If Linnebo is right, and every plurality potentially defines a set, even if the arguments I have given in the previous section are correct, they are not enough to prevent every form of collapse. In some sense, although not every plurality collapses into a set, there is no room for pluralities to lie outside the set theoretic hierarchy, for every plurality may form a set.

Even though Linnebo's picture seems attractive, I think it is not problem-free. A general reaction about this strategy is that it is not clear how to understand the modalities involved. Naturally — as Linnebo himself admits— modalities should not be understood ontologically, for sets are abstract entities that, if existent, they exist in every possible world. However, in Linnebo's picture they seem to have a contingent existence. In particular, given the inherent potentiality of the proposed hierarchy, the system allows the existence of the elements of a set without the existence of the set itself. Otherwise, it would not be possible for pluralities to not actually collapse. This feature is characteristic of the system $S4.2$, adopted by Linnebo. The model contrasts with the one presented by Fine (1981), who adopts the system S5 for a similar purpose. In this case, necessarily if the elements of a set exist, the set exists as well. Regarding this, I don't see how, being purely extensional entities, sets could fail to exist when all of their members are there. To emphasize this, it seems that to model a potential hierarchy not only makes existence a contingent matter, but it also requires an explanation of how can a set move from potentiality to actuality.

Moreover, it seems plausible to say that if we don't have an absolute modality — absolutely general quantification over worlds — we cannot quantify over absolutely every set. But if some sort of indefinite extensibility over worlds is adopted, it does not seem possible to quantify over absolutely every set. What I mean is that the potential picture in Linnebo's modal system fits better with abandoning absolute generality than with adopting it.

It is true that in a world we can make general statements about sets. But, applied to a certain world, COLLAPSE is false. It is not true that every plurality in a world forms a set in that world. And since there is modal collapse, the universe of sets expands and goes on to form enough sets to collapse any plurality. But then, if at some point of the process were it possible to quantify unrestrictedly over every set, the process would be over.

A different way of making the same point is to note that the proposal prevents applications of plural logic meant to express set theory models with an absolutely general interpretation of the quantifiers. For the potential collapse of pluralities into sets precludes them from expressing the whole universe of sets, if we want to avoid contradiction.

One final thing I want to mention is that COMP-P

$$u \quad (u) \qquad xx \;\; u(u \quad xx \qquad (u))$$

is a truth of plural logic. However, it turns out not to be a necessary truth according to the potentialist perspective, as is shown by this principle:

P-COMP*: $\quad xx \quad u(u \quad xx \qquad (u))$

This seems very strange. Usually logical truths are necessary. Then, the defender of P-COMP* should explain this puzzling result.

5. How to restore consistency by abandoning COLLAPSE

In this article I have argued against COLLAPSE. I have tried to show that adoption of such a thesis leads to profound expressive limitations in languages: adopting unrestricted quantification, it doesn't seem possible to give semantics for set theory. At the same time, restrictions to COMP-P — necessary to restore consistency — crucially limit the expressive resources of high order languages, if COLLAPSE is not given up. The possibility of expressing the categoricity result of PA_2 mandates not to limit COMP-P. On the other hand, the modal version of COLLAPSE, although attractive, is not safe from problems. Basically, and beyond any questions regarding the interpretation of the modalities involved, it turns unrestricted quantification over sets unstable.

It remains then the question of what to do with the knock-down argument if COMP-P is adopted and COLLAPSE is rejected. The idea is that the Russell plurality rr fails to form a set. For that reason, in any considerations involving pluralities and sets, first order quantifiers x and x must not have the same interpretation domain as the plural quantifiers xx and xx. The paradox arises when, assuming COLLAPSE, the Russell plurality rr forms a set. This way we can talk about the Russell plurality rr without there being an object — falling under the first order quantifier — which is the set form by the things that constitute that plurality. Consistency is restored, allowing us to preserve the expressive resources needed to have legitimate high order quantification.

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Resumo. Este artigo trata da questão sobre quais as circunstâncias em que uma pluralidade forma um conjunto. Meu ponto principal é que nem sempre todas as coisas formam conjuntos. Um a maneira provocativa de apresentar minha posição é que, como um resultado de minha abordagem, existem mais pluralidades do que conjuntos. Outra maneira de apresentar a mesma tese afirma que existem maneiras de falar de objetos que nem sempre colapsam em conjuntos. Meu argumento está relacionado com o poder expressivo de linguagens formais. Assumindo a lógica clássica, mostro que se toda pluralidade forma um conjunto e os quantificadores são absolutamente gerais, então obtemos uma teoria trivial. Portanto, por reduction, devemos abandonar uma das premissas. Então, argumento contra o colapso de pluralidades em conjuntos. O que estou advogando é que a *tese do colapso* limita importantes aplicações da lógica plural na teoria de modelos, quando é assumido que os quantificadores são absolutamente gerais.

Palavras-chave: Pluralidades; generalidade absoluta; conjuntos; hierarquias.

Notes

[1] Absolute generality about quantifiers is the view that quantifiers in natural or artificial languages sometimes range over a domain comprising absolutely everything.

[2] Relativism about quantifiers is the opposing view that quantifiers in natural or artificial languages never range over a domain comprising absolutely everything.
[3] Boolos has developed an ingenious formal device, alternative to Tarski's, in which interpretations are not objects themselves, but values or second order variables assignments. Boolos' idea was to use a high order language which allows to talk about certain individuals (whether or not they form a set) that play the role of a model for the language. These, and not sets, codify a specification of the individuals over which the quantifiers range. An application of this idea is to talk about a domain of "all sets", without committing with such an entity. And like that, more expressive power can be achieved, and contradictions can be avoided. G. Boolos (1985).
[4] See Rayo 2006. For the classic idea, see Boolos 1984, 1985, and Lewis 1990.
[5] Cartwright (1994) claims that "The general principle appears to be that to quantify over certain objects is to presuppose that these objects constitute a "collection" or a "completed collection" — some one thing of which those objects are members. I call this the All-in-One Principle".
[6] Of course, the use of more expressive resources on the metalanguage (predicates and plural quantification) opens up the discussion about expressive conditions of that language, and about the resources necessary to provide semantics to it. Rayo (2006) has shown that "in the presence of absolutely general quantification, proper semantic theorizing is essentially unstable: it is impossible to provide a suitably general semantics for a given language in a language of the same logical type." p.220. Because of that, we are forced to ascend on the hierarchy of pluralities, in order to accomplish our goal. All that is needed is an unlimited hierarchy of plural expressions to provide semantics to any of the plural languages we can face. For an explanation of the hierarchies in the context of theory of types, sets and plurals, see Linnebo, O. & Rayo, A (2012).
[7] Recently McGee (1997) has shown that ZF axioms formulated in second order, plus the axiom of choice with urelements (ZFCU2) allow to characterize the structure of the universe of pure sets up to isomorphism. This result could be interpreted as establishing the "size" of the universe of sets. Nonetheless, the result is obtained by assuming unrestricted quantification, and does not provide a way to decide whether there are seven inaccessible cardinals, less or more. In any case, I do not need to refer to McGee's controversial result in order to defend my position.
[8] For an informal proof of the result, see Shapiro 1991, p.82–3.
[9] When standard models are used, n-ary predicative variables range over the entire class of relations on the domain. A standard model of a second order language is a structure similar to that of first order languages: a domain and an interpretation function assigning appropriate semantic values to the non-logical expressions. When some domain is specified, it fixes the range of both kind of variables: first and second order. The set of all subsets of the domain determines the set of assignments to second order variables.
[10] As I showed at the first part of this paper, beyond arithmetical examples, there are examples in natural language which help to illustrate the same point. In this direction, adopting COLLAPSE, by limiting P-COMP, leaves us without a sound interpretation of Kaplan's sentence: "Some critics admire only one another". But, adopting my view, this sentence can be formalized using plural expressions.
[11] This approach is also followed by James Studd (2015).

[12] The first three properties of the relation guarantee that the process is partially ordered. Linnebo, O. (2013).