Frontal operators in gi-lattices

Sergio A. Celani and Hernán J. San Martín

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Abstract

We introduce a family of extensions of bounded distributive lattices. These extensions are obtained by adding two operations: an internal unary operation, and a function (called generalized implication) that maps pair of elements to ideals of the lattice. A bounded distributive lattice with a generalized implication is called gi-lattice in [4].

The main goal of this paper is to introduce and study the category of frontal gi-lattices (and some subcategories of it). This category can be seen as a generalization of the category of frontal weak Heyting algebras ([9]). In particular, we study the case of frontal gi-lattices where the generalized implication is defined as the annihilator ([11], [15]). We give a Priestley's style duality for each one of the new classes of structures considered.

1 Introduction

The class of distributive lattices with a generalized implication, or gi-lattices, was introduced in [5] as a common abstraction of the notions of annihilator ([15]), quasi-modal lattices ([4]), and weak Heyting algebras, or WH-algebras for short ([7]).

A *generalized implication* on a bounded distributive lattice A is a binary function $\Rightarrow$ that maps every pair of elements (a, b) of A to an ideal $a \Rightarrow b$ and satisfies the following conditions for every $a, b, c \in A$:

1. $(a \Rightarrow b) \cap (a \Rightarrow c) = a \Rightarrow (b \wedge c)$,

2. $(a \Rightarrow c) \cap (b \Rightarrow c) = (a \vee b) \Rightarrow c$,

3. $(a \Rightarrow b) \cap (b \Rightarrow c) \subseteq a \Rightarrow c$,

4. $a \Rightarrow a = A$.

A *distributive lattice with a generalized implication*, or *gi-lattice* for short, is a pair $\langle A, \Rightarrow \rangle$, where A is a bounded distributive lattice and $\Rightarrow$ is a generalized implication on A. For a gi-lattice $\langle A, \Rightarrow \rangle$, in general we write A in place of $\langle A, \Rightarrow \rangle$. The theory of gi-lattices is strongly connected with the theory of quasi-modal lattices ([4]). If A is a gi-lattice, then the unary function $\Box$ between A and the set of all ideals of A defined by $\Box a = 1 \Rightarrow a$, for each $a \in A$, satisfies the conditions $\Box 1 = A$ and $\Box(a \wedge b) = \Box a \cap \Box b$, for all $a, b \in A$ (where 1 is the last element of A). Thus, the pair $\langle A, \Box \rangle$ is a quasi-modal lattice ([4]).

We define a *frontal operator* in a gi-lattice A as an unary expansive operator τ preserving finite meets which also satisfies the condition

$$\tau(a) \in (b] \vee (b \Rightarrow a)$$

for every $a, b \in A$, where $(b]$ is the ideal generated by b and $\vee$ is the supremum in the lattice of ideals of A.

One of the purpose of this paper is to introduce and study frontal gi-lattices as a generalization of the frontal weak Heyting algebras studied in [9]. A weak Heyting algebra ([2],[7]) is a pair $\langle A, \rightarrow \rangle$, where A is a bounded distributive lattice and $\rightarrow$ is a binary map on A such that for all $a, b, c \in A$ the following conditions are satisfied: $(a \rightarrow b) \wedge (a \rightarrow c) = a \rightarrow (b \wedge c)$, $(a \rightarrow c) \wedge (b \rightarrow c) = (a \vee b) \rightarrow c$, $(a \rightarrow b) \wedge (b \rightarrow c) \leq a \rightarrow c$ and $a \rightarrow a = 1$. A distributive lattice with a generalized implication can be seen as an extension of the notion of weak Heyting algebra: if $\langle A, \rightarrow \rangle$ is a weak Heyting algebra, then the binary map from $A \times A$ to the set of ideals of A defined by setting

$$a \Rightarrow b := (a \rightarrow b]$$

is a generalized implication on A. In this case, if τ is an unary map on A we have that $\tau(a) \in (b] \vee (b \rightarrow a]$ iff $\tau(a) \leq b \vee (b \rightarrow a)$. Hence, a frontal operator in a gi-lattice can be seen as an extension of the notion a frontal operator in a weak Heyting algebra ([9]).

In a lattice A, for $a, b \in A$ the *annihilator of a relative to b* ([11], [15]) is defined by $\langle a, b \rangle := \{c \in A : c \wedge a \leq b\}$, which is an ideal if A is a distributive lattice. The other purpose of this paper is to introduce and study frontal operators in gi-lattices where the generalized implication is defined as

$$a \Rightarrow b := \langle a, b \rangle \, .$$

The concept of annihilator is a natural generalization of the relative pseudocomplement of an element $a \in A$ relative to an element $b \in A$: if $\langle A, \rightarrow \rangle$ is a Heyting algebra and τ is an unary map on A, we have that $\langle a, b \rangle = (a \rightarrow b]$. The main motivation to study frontal operators in Heyting algebras stemmed from topological semantics in which τ is interpreted as the co-derivative operator ([12]).

The study of frontal operators in gi-lattices provides a common framework to obtain the case of weak Heyting algebras with $a \Rightarrow b = (a \rightarrow b]$, and the case of bounded distributive lattices with $a \Rightarrow b = \langle a, b \rangle$. The paper is organized as follows. In Section 2 we recall the concepts and basic results of the Priestley duality for gi-lattices and for the category of bounded distributive lattices with a modal operator. In Section 3 we define the category of frontal gi-lattices. We give and study two classes of gi-lattices, which will be called gi-lattices with successor, and gi-lattices with gamma. In Section 4 we give a dual categorical equivalence for the category of frontal gi-lattices based on the duality for gi-lattices ([5]) and the duality for modal lattices ([8],[10] or [13]). We define frontal gi-spaces as structures $\langle X, \leq, T, R \rangle$, where $\langle X, \leq, T \rangle$ is a gi-space ([4]), $\langle X, \leq, R \rangle$ is a modal Priestley space and certain conditions are satisfied that connect the relations T, R and $\leq$. From this duality, we obtain a dual categorical equivalence similar to that given in [9] for the category of frontal weak

Heyting algebras. Then we study two equivalent categories to the gi-lattices with successor (these algebras can be seen a generalization of the weak Heyting algebras with successor studied in [9]). The first category is based on the frontal gi-spaces previously studied, i.e., the operator is interpreted by means of the relation R in the standard way. The other category is based on a particular class of gi-spaces, and in this case the modal operator is interpreted by means of the relations $\leq$ and T. We prove that these two categories are isomorphic. The final purpose of this section is to study two categorical equivalence for the category of γgi-lattices. Finally, in Section 5 we define and study the particular case of the category of frontal operators in bounded distributive lattices where the generalized implication is defined as the annihilator ([11], [15]). We also establish an equivalence for the above mentioned category, and we buid up equivalences for certain subcategories of it.

In the following we give a table with the categories we shall define and use in this paper:

Category	**Objects**	**Morphisms**
DLatGi	gi-lattices	gi-homorphisms
GiS	gi-spaces	gi-morphisms
ML	Modal lattices	Morphisms of modal lattices
MS	Modal Priestley spaces	p-morphisms
FDLatGi	Frontal gi-lattices	gi- morphisms which preserve the modal map
$\mathbf{DLatGi}_S$	gi-lattices with successor	Sgi-homomorphisms
$\mathbf{DLatGi}_\gamma$	gi-lattices with γ	γgi-homomorphisms
FGiS	Frontal gi-spaces	gi-morphisms which are p-morphisms
$\mathbf{FGi}_S$	Frontal S-spaces	Morphisms of **FGiS**
SGiS	gi-spaces with successor	Sgi-morphisms
$\mathbf{FGi}_\gamma$	Frontal γ-spaces	Morphisms of **FGiS**
γ**GiS**	gi-spaces with γ	γgi-morphisms
FBDL	Frontal lattices	Morphisms of **ML**
$\mathbf{BDL}_S$	Lattices with successor	Morphisms of **ML**
$\mathbf{BDL}_\gamma$	Lattices with γ	Morphisms of **ML**
FPS	Frontal Priestley spaces	Morphisms of **MS**
SPS	S-Priestley spaces	Certain morphisms of Priestley spaces
γ**PS**	γ-Priestley spaces	Certain morphisms of Priestley spaces

2 Preliminaries and basic results

If X is a set, then the power set of X is denoted by $\mathcal{P}(X)$. If A is a distributive lattice, then $\mathrm{Fi}(A)$ and $\mathrm{Id}(A)$ respectively denote the family of filters of A and the family of ideals of A. The filter (ideal) generated by a subset $X \subseteq A$ is denoted by $\mathrm{F}(X)$ $(\mathrm{I}(X))$. The family of the prime filters of A is denoted by $\mathbf{X}(A)$. Given a bounded distributive lattice A, let $\varphi\colon A \to \mathcal{P}(\mathbf{X}(A))$ be the Stone map defined by $\varphi(a) = \{P \in \mathbf{X}(A) : a \in P\}$, for each $a \in A$. The family $\varphi[A] = \{\varphi(a) : a \in A\}$ is closed under unions, intersections, and it contains $\emptyset$ and A; it is therefore a bounded distributive lattice.

Given a poset $\langle X, \leq \rangle$, a set $Y \subseteq X$ is said to be *upset* if it is closed under $\leq$, that is, if for every $x \in Y$ and every $y \in X$, if $x \leq y$ then $y \in Y$. The set complement of a subset $Y \subseteq X$ is denoted by Y^c or $X - Y$. For each $Y \subseteq X$, the upset (downset) generated by Y is $[Y) = \{x \in X \mid \exists y \in Y (y \leq x)\}$

$((Y] = \{x \in X \mid \exists y \in Y(x \leq y)\})$. If $Y = \{y\}$, we write $[y)$ and $(y]$ instead of $[\{y\})$ and $(\{y\}]$, respectively. A *totally order-disconnected topological space* is a triple $\langle X, \leq\rangle = \langle X, \leq, \mathcal{T}\rangle$, where $\langle X, \leq\rangle$ is a poset, $\langle X, \mathcal{T}\rangle$ is a topological space and given $x, y \in X$ such that $x \not\leq y$ there is a clopen upset U such that $x \in U$ and $y \notin U$. A *Priestley space* is a compact totally order-disconnected topological space. A morphism between Priestley spaces is a continuous and monotone function between them. If $\langle X, \leq\rangle$ is a Priestley space, the family of all clopen upsets of $\langle X, \leq\rangle$ is denoted by $\mathbf{D}(X)$, and it is a bounded distributive lattice.

The Priestley space of a bounded distributive lattice A is $\langle \mathbf{X}(A), \subseteq, \mathcal{T}_A\rangle$, where $\mathcal{T}_A$ is the topology generated by taking as a subbase the family

$$\{\varphi(a) : a \in A\} \cup \{\varphi(a)^c : a \in A\}\,.$$

Besides $A \cong \mathbf{D}(\mathbf{X}(A))$. For more details on Priestley spaces see [16].

Let A_1 and A_2 be gi-lattices. A bounded lattice homomorphism $h : A_1 \to A_2$ is a *gi-homomorphism* if $I(h[a \Rightarrow_1 b]) = h(a) \Rightarrow_2 h(b)$, for every $a, b \in A_1$. Thus, the class of gi-lattices, taken as objects, and their gi-homomorphism, taken as arrows, form a category that we denote by **DLatGi**.

Remark 1. *Let A be a gi-lattice. We define the relation $T_\Rightarrow$ on* $\mathrm{Fi}(A)$ *by*

$$(F, G) \in T_\Rightarrow \text{ iff } (\forall a, b \in A)(((a \Rightarrow b) \cap F \neq \emptyset \;\&\; a \in G) \Longrightarrow b \in G).$$

We write for future reference the following result from [5].

Lemma 1. *Let A be a gi-lattice, $a, b \in A$ and $P \in \mathbf{X}(A)$. Then $(a \Rightarrow b) \cap P = \emptyset$ iff there exists $Q \in \mathbf{X}(A)$ such that $(P, Q) \in T_\Rightarrow$, $a \in Q$ and $b \notin Q$.*

Let $\langle X, \leq\rangle$ be a Priestley space and $T \subseteq X \times X$. For every $U, V \in \mathbf{D}(X)$ we define the following sets:

$$U \to_T V = \{x \in X : T(x) \cap U \subseteq V\},$$

$$U \Rightarrow_T V = \{W \in \mathbf{D}(X) : W \subseteq U \to_T V\}.$$

Given a relation $T \subseteq X \times X$, for each $x \in X$, $T(x)$ will denote the image of $\{x\}$ by T, i.e., $T(x) = \{y \in X : (x, y) \in T\}$.

A *gi-space* is a structure $\langle X, \leq, T\rangle$, where $\langle X, \leq\rangle$ is a Priestley space and T is a binary relation on X such that:

1. $T(x)$ is a closed set, for every $x \in X$.
2. For all $U, V \in \mathbf{D}(X)$, $U \to_T V$ is an open upset of X.

Let $\langle X_1, \leq_1, T_1\rangle$ and $\langle X_2, \leq_2, T_2\rangle$ be gi-spaces. A function $f\colon X_1 \to X_2$ is a *gi-morphism* if it is a morphism of Priestley spaces (i.e., it is continuous and monotone), and

1. If $(x, y) \in T_1$, then $(f(x), f(y)) \in T_2$.
2. If $(f(x), z) \in T_2$, then there is $y \in X_1$ such that $(x, y) \in T_1$ and $f(y) = z$.

We denote by **GiS** to the category that has gi-spaces as objects and gi-morphisms as arrows.

For the next theorem we only give a sketch of the proof (see [5] for details).

Theorem 2. *The categories* **DLatGi** *and* **GiS** *are dually equivalent.*

Proof. Define a contravariant functor $(-)_* : \mathbf{DLatGi} \to \mathbf{GiS}$ as follows. If A is a gi-lattice, then $A_* = \langle \mathbf{X}(A), \subseteq, T_\Rightarrow \rangle$, where $T_\Rightarrow \subseteq \mathbf{X}(A) \times \mathbf{X}(A)$ is the relation defined in Remark (1). If $h\colon A_1 \to A_2$ is a gi-homomorphism, then the mapping $h_*\colon \mathbf{X}(A_2) \to \mathbf{X}(A_1)$ given by $h_*(P) = h^{-1}(P)$ is a gi-morphism. Next define the contravariant functor $(-)^* : \mathbf{GiS} \to \mathbf{DLatGi}$ as follows. For a gi-space $\langle X, \leq, T \rangle$, the structure $\langle X, \leq, T \rangle^* = \langle \mathbf{D}(X), \Rightarrow_T \rangle$ is a gi-lattice. If $f\colon X_1 \to X_2$ is a gi-morphism, then the map $f^*\colon \mathbf{D}(X_2) \to \mathbf{D}(X_1)$ given by $f^*(U) = f^{-1}(U)$ is a gi-homomorphism. Consequently, $(-)_*$ and $(-)^*$ are well-defined contravariant functors. Moreover, the function $\varphi\colon A \to \mathbf{D}(\mathbf{X}(A))$ is an isomorphism between the gi-lattice $\langle A, \Rightarrow \rangle$ and $(A_*)^* = \langle \mathbf{D}(\mathbf{X}(A)), \Rightarrow_{T_\Rightarrow} \rangle$. Moreover, the function $\varepsilon\colon X \to \mathbf{X}(\mathbf{D}(X))$ given by $\varepsilon(x) = \{U \in \mathbf{D}(X) : x \in U\}$ is an isomorphism between the gi-spaces $\langle X, \leq, T \rangle$ and $(\langle X, \leq, T \rangle^*)_* = \langle \mathbf{X}(\mathbf{D}(X)), \subseteq, T_{\Rightarrow_T} \rangle$. This yields the desired dual equivalence between **DLatGi** and **GiS**. □

An algebra $\langle A, \tau \rangle$ is a *modal lattice*, or a *τ-lattice*, if A is a bounded distributive lattice and τ is an unary operator defined on A such that satisfies the following equations:

1. $\tau(1) = 1$,
2. $\tau(a \wedge b) = \tau(a) \wedge \tau(b)$.

A morphism of bounded lattices which preserve the modal operator is called a *morphism of modal lattices.*

If X is a set and $R \subseteq X \times X$, for every $U \subseteq X$ we define the set

$$\tau_R(U) = \{x \in X : R(x) \subseteq U\}.$$

A *modal Priestley space* ([13], [8], or [10]) is a relational structure $\langle X, \leq, R \rangle$, where $\langle X, \leq \rangle$ is a Priestley space and R is a binary relation on X such that

1. $R(x)$ is a closed upset, for each $x \in X$.
2. $\tau_R(U) \in \mathbf{D}(X)$, for each $U \in \mathbf{D}(X)$.

Let $\langle X_1, \leq_1, R_1 \rangle$ and $\langle X_2, \leq_2, R_2 \rangle$ be two modal Priestley spaces. A *p-morphism* is a monotone and continuous mapping $f\colon X_1 \to X_2$ satisfying the following conditions:

1. If $(x, y) \in R_1$, then $(f(x), f(y)) \in R_2$.
2. If $(f(x), z) \in R_2$, then there is $y \in X_1$ such that $(x, y) \in R_1$ and $f(y) \leq_2 z$.

Remark 2. *Let $\langle A, \tau \rangle$ be a τ-lattice. We define a binary relation R_τ on $\mathbf{X}(A)$ by $(P, Q) \in R_\tau$ iff $\tau^{-1}(P) \subseteq Q$.*

We denote **ML** the category which has modal lattices as objects, and morphisms of modal lattices as arrows. We denote by **MS** to the category which has modal Priestley spaces as objects, and p-morphisms as arrows. For the next theorem we only give a sketch of the proof. The missing details can be found in [8], [10] or [13].

Theorem 3. *The categories* **ML** *and* **MS** *are dually equivalent.*

Proof. Define a contravariant functor $\mathcal{F}\colon \mathbf{ML} \to \mathbf{MS}$ as follows. If $\langle A, \tau\rangle$ is a modal lattice, then $\mathcal{F}(\langle A, \tau\rangle) = \langle \mathbf{X}(A), \subseteq, R_\tau\rangle$ is a modal Priestley space. If $h\colon \langle A_1, \tau_1\rangle \to \langle A_2, \tau_2\rangle$ is a morphism of modal lattices, then the mapping $\mathcal{F}(h)\colon \langle \mathbf{X}(A_2), \subseteq, R_{\tau_2}\rangle \to \langle \mathbf{X}(A_1), \subseteq, R_{\tau_1}\rangle$ given by $\mathcal{F}(h)(P) = h^{-1}(P)$ is a p-morphism. Next define the contravariant functor $\mathcal{G}\colon \mathbf{MS} \to \mathbf{ML}$ as follows. For a modal Priestley space $\langle X, \leq, R\rangle$, the structure $\mathcal{G}(\langle X, \leq, R\rangle) = \langle \mathbf{D}(X), \tau_R\rangle$ is a modal lattice. If $f\colon \langle X_1, \leq_1, R_1\rangle \to \langle X_2, \leq_2, R_2\rangle$ is a p-morphism, then the map $\mathcal{G}(f) : \langle \mathbf{D}(X_2), \tau_{R_2}\rangle \to \langle \mathbf{D}(X_1), \tau_{R_1}\rangle$ given by $\mathcal{G}(f)(U) = f^{-1}(U)$ is a morphism of modal lattices. Consequently, $\mathcal{F}$ and $\mathcal{G}$ are well-defined contravariant functors. If $\langle A, \tau\rangle$ is a modal lattice, then the the mapping $\varphi\colon \langle A, \tau\rangle \to \langle \mathbf{D}\left(\mathbf{X}(A)\right), \tau_{R_\tau}\rangle$ is an isomorphism of modal lattices, i.e., $\varphi\left(\tau(a)\right) = \tau_{R_\tau}\left(\varphi\left(a\right)\right)$, for all $a \in A$. Moreover, the function $\varepsilon\colon \langle X, \leq, R\rangle \to \langle \mathbf{X}(\mathbf{D}(X)), \subseteq, R_{\tau_R}\rangle$ is an isomorphism in the category of modal Priestley spaces. This yields the desired dual equivalence between **ML** and **MS**. □

3 Frontal gi-lattices

In this section we define frontal gi-lattices as a generalization of the frontal weak Heyting algebras introduced in [9]. We give two examples of them: the gi-lattices with successor and the gi-lattices with gamma.

Taking into account that the poset of ideals of a distributive lattice forms a distributive lattice, we can give the following

Definition 1. *A* frontal gi-lattice *is a pair $\langle A, \tau\rangle$ such that A is a gi-lattice and τ is an unary operator satisfying the following conditions for every $a, b \in A$:*

(W1) $\tau(a \wedge b) = \tau(a) \wedge \tau(b)$,

(W2) $a \leq \tau(a)$,

(W3) $\tau(a) \in (b] \vee (b \Rightarrow a)$.

If $\langle A, \tau\rangle$ is a frontal gi-lattice we say that τ is a *frontal operator*. We write **FDLatGi** for the category whose objects are frontal gi-lattices, and whose morphisms are gi-homomorphisms which preserve the frontal operator.

In [14] Kuznetsov introduced an operation on Heyting algebras as an attempt to build an intuitionistic version of the provability logic of Gödel-Löb, which formalizes the concept of provability in Peano Arithmetic. This unary operation, which we call *successor* ([3]), was also studied by Caicedo and Cignoli in [3] and by Esakia in [12]. In particular, Caicedo and Cignoli considered it as an example of an implicit compatible operation on Heyting algebras. In [9] the successor function was defined and studied in WH-algebras as a generalization of the successor in Heyting algebras. In the following we define the gi-lattices with successor as a generalization of the WH-algebras with successor.

Definition 2. *A* gi-lattice with successor*, or* Sgi-lattice*, is a pair* $\langle A, S\rangle$ *such that* A *is a gi-lattice and* $S\colon A \to A$ *is a function which satisfies the conditions* (**W2**), (**W3**), *and the following condition for every* $a \in A$*:*

$$S(a) \Rightarrow a \subseteq (a]. \tag{1}$$

If $\langle A, S\rangle$ is a Sgi-lattice, then the function S is called the *successor function.* Let $\mathbf{DLatGi}_S$ be the category whose objects are Sgi-lattices, and whose morphisms are gi-homomorphisms which preserve the successor. These morphisms are called *Sgi-homomorphisms.*

Lemma 4. *If* $\langle A, S\rangle$ *is a* S*gi-lattice, then* $\langle A, S\rangle$ *is a frontal gi-lattice and* S *is given by* $S(a) = min\ \{b \in A : b \Rightarrow a \subseteq (b]\}$*.*

Proof. First we prove that $S(a \wedge b) = S(a) \wedge S(b)$, for all $a, b \in A$. In order to prove that S is monotone, let $c \leq d$. Then by **(W3)**, (1) in Definition 2 and **(W2)** we have that

$$S(c \wedge d) \in (S(d)] \vee (S(d) \Rightarrow (c \wedge d)),$$

$$(S(d)] \vee (S(d) \Rightarrow (c \wedge d)) = (S(d)] \vee ((S(d) \Rightarrow c) \cap (S(d) \Rightarrow d)),$$

$$(S(d)] \vee ((S(d) \Rightarrow c) \cap (S(d) \Rightarrow d)) \subseteq (S(d)] \vee ((S(d) \Rightarrow c) \cap (S(d)]) = (S(d)],$$

so $S(c) \leq S(d)$. Thus, $S(a \wedge b) \leq S(a) \wedge S(b)$. On the other hand, $S(a) \in (S(a \wedge b)] \vee (S(a \wedge b) \Rightarrow a)$ and $S(b) \in (S(a \wedge b)] \vee (S(a \wedge b) \Rightarrow b)$. Hence, taking into account (1) in Definition 2 and **(W2)** we obtain that

$$S(a) \wedge S(b) \in (S(a \wedge b)] \vee ((S(a \wedge b) \Rightarrow a) \cap (S(a \wedge b) \Rightarrow b)),$$

$$S(a \wedge b)] \vee ((S(a \wedge b) \Rightarrow a) \cap (S(a \wedge b) \Rightarrow b)) = (S(a \wedge b)] \vee (S(a \wedge b) \Rightarrow (a \wedge b)),$$

$$S(a \wedge b)] \vee (S(a \wedge b) \Rightarrow (a \wedge b)) \subseteq (S(a \wedge b)] \vee (a \wedge b],$$

$$(S(a \wedge b)] \vee (a \wedge b] = (S(a \wedge b)],$$

so $S(a) \wedge S(b) \leq S(a \wedge b)$. Therefore we obtain the equality $S(a \wedge b) = S(a) \wedge S(b)$, for every $a, b \in A$.

Let us prove that S is given by $S(a) = \min\ \{b \in A : b \Rightarrow a \subseteq (b]\}$, for each $a \in A$. Taking into account (**W2**) and (1) in Definition 2, we conclude that $S(a) \in \{b \in A : b \Rightarrow a \subseteq (b]\}$. Let $b \in A$ such that $b \Rightarrow a \subseteq (b]$. By (**W3**) we have that $S(a) \leq b$. Thus, $S(a) = \min\ \{b \in A : b \Rightarrow a \subseteq (b]\}$. ☐

Remark 3. *Let* $\langle A, S\rangle$ *be a* S*gi-lattice and* $a \in A$*. Then* $S(a) = a$ *iff* $a = 1$*.*

Caicedo and Cignoli introduced in [3] an unary connective, which we call gamma. This operation is a variant of the Smetanich constant [17]. In [9] the gamma function was defined and studied in WH-algebras as a generalization of the gamma function in Heyting algebras. In the following we define the gi-lattices with gamma as a generalization of the WH-algebras with gamma.

Definition 3. *A* gi-lattice with γ*, or* γgi-lattice*, is a pair* $\langle A, \gamma\rangle$ *such that* A *is a gi-lattice and* $\gamma\colon A \to A$ *is a function which satisfies the condition* (**W3**) *and the following conditions for every* $a \in A$*:*

(g1) $\gamma(0) \Rightarrow 0 = \{0\}$,

(g2) $\gamma(a) = a \vee \gamma(0)$.

This function can be characterized by the conditions that define a frontal operator, the condition (**g1**) and the equation $\gamma(a) \leq a \vee \gamma(0)$. Similarly to Lemma 4, it is possible to prove that if there is γ then it takes the form $\gamma(a) = \min\{b \in A : (b \Rightarrow 0) \vee (a] \subseteq (b]\}$. Let **DLatGi**$\gamma$ be the category whose objects are γgi-lattices, and whose arrows are gi-homomorphisms which preserve the gamma operator. These morphisms are called γ*gi-homomorphisms*.

Example 1. *Let A be a bounded distributive lattice and consider the operation $\Rightarrow: A \times A \to Id(A)$ given by*

$$a \Rightarrow b = \begin{cases} A & \text{if } a \leq b \\ \{0\} & \text{if } a \not\leq b \end{cases}$$

Then $\langle A, \Rightarrow\rangle$ is a gi-lattice. Moreover, there exists successor function in $\langle A, \Rightarrow\rangle$ iff A has only one element.

4 Representation and duality

In this section we build up a dual categorical equivalence for the category of frontal gi-lattices based on the duality for gi-lattices and the duality for modal lattices. We define the frontal gi-spaces as structures $\langle X, \leq, T, R\rangle$, where $\langle X, \leq, T\rangle$ is a gi-space, $\langle X, \leq, R\rangle$ is a modal Priestley space and certain conditions are satisfied that connect the relations T, R and $\leq$. From this duality, we obtain a dual categorical equivalence. Then we study two equivalent categories to the gi-lattices with successor. The first category is based on the frontal gi-spaces previously studied, i.e., the operator is interpreted by means of the relation R in the standard way. The other category is based on a particular class of gi-spaces, and in this case the modal operator is interpreted by means of the relations $\leq$ and T. We prove that these two categories are isomorphic. Finally, we build up two categorical equivalences for the category of γgi-lattices.

Let $\langle X, \leq\rangle$ be a poset and T a binary relation. We define an auxiliary relation $\bar{T} \subseteq X \times X$ in the following way:

$$(x, y) \in \bar{T} \text{ iff } (x, y) \in T \text{ and } y \not\leq x.$$

Definition 4. *A* frontal gi-space *is a structure $\langle X, \leq, T, R\rangle$ such that:*

1. *$\langle X, \leq, T\rangle$ is a gi-space and $\langle X, \leq, R\rangle$ is a modal Priestley space.*

2. $\bar{T} \subseteq R \subseteq \leq$.

Let $\langle A, \tau\rangle$ be a frontal gi-lattice. Since τ is a modal operator, we can consider the relation $R_\tau \subseteq \mathbf{X}(A) \times X(A)$ defined in Remark 2.

Remark 4. *If A is a gi-lattice and $\tau: A \to A$ is a function satisfying* (**W1**) *and* (**W2**)*, then $\langle A, \tau\rangle$ is a τ-lattice and the structure $\langle \mathbf{X}(A), \subseteq, R_\tau\rangle$ is a modal Priestley space. Moreover, $a \leq \tau(a)$ for every $a \in A$ iff $R_\tau \subseteq \leq$.*

In the next proposition we give a characterization for the condition **(W3)**.

Proposition 1. *Let A be a gi-lattice and $\tau: A \to A$ a function satisfying* (**W1**) *and* (**W2**)*. Then $\bar{T}_\Rightarrow \subseteq R_\tau$ iff $\tau(a) \in (b] \vee (b \Rightarrow a)$, for every $a, b \in A$.*

Proof. $\Longrightarrow$) Suppose that there are $a, b \in A$ such that $\tau(a) \notin (b] \vee (b \Rightarrow a)$. Taking into account that $[\tau(a)) \cap ((b] \vee (b \Rightarrow a)) = \emptyset$, we have that there is $P \in \mathbf{X}(A)$ such that $\tau(a) \in P$, $b \notin P$ and $P \cap (b \Rightarrow a) = \emptyset$. Hence, by Lemma 1 there is $Q \in \mathbf{X}(A)$ such that $(P, Q) \in T_{\Rightarrow}$, $b \in Q$ and $a \notin Q$. As $b \in Q$ and $b \notin P$, we obtain $Q \not\subseteq P$, so $(P, Q) \in \bar{T}_{\Rightarrow}$. Thus, by hypothesis we have that $(P, Q) \in R_{\tau}$. Using that $a \in \tau^{-1}(P)$ we conclude that $a \in Q$, which is a contradiction.

$\Longleftarrow$) Let $(P, Q) \in \bar{T}_{\Rightarrow}$. Then $(P, Q) \in T_{\Rightarrow}$ and $Q \not\subseteq P$. Hence, there exists $b \in Q$ such that $b \notin P$. Let $a \in A$ such that $\tau(a) \in P$. By (**W3**) we have that $\tau(a) \in (b] \vee (b \Rightarrow a)$. Thus, there is $c \in b \Rightarrow a$ such that $\tau(a) \leq b \vee c$. As $\tau(a) \in P$ and $b \notin P$ we obtain $c \in P$, so $c \in (b \Rightarrow a) \cap P$. Thus, $(P, Q) \in T_{\Rightarrow}$, $(b \Rightarrow a) \cap P \neq \emptyset$ and $b \in Q$. In consequence, $a \in Q$ and then $(P, Q) \in R_{\tau}$. Therefore, $\bar{T}_{\Rightarrow} \subseteq R_{\tau}$. □

Let **FGiS** be the category whose objects are frontal gi-spaces, and whose morphisms are maps $f : X_1 \to X_2$ such that f is a gi-morphism and f is a p-morphism. Then by the results given in [5] for gi-lattices and the results given in [8], [10] or [13] for bounded distributive lattices with a modal operator, we obtain the following

Theorem 5. *The category* **FGiS** *is dually equivalent to the category* **FDLatGi**.

4.1 Categorical equivalences for $\mathbf{DLatGi}_S$

The following aim is to build up two categorical equivalences for the category of gi-lattices with successor. We start with the following

Definition 5. *A* frontal S-space *is a frontal gi-space $\langle X, \leq, T, R\rangle$ such that*

(**S**) *For every $U \in \mathbf{D}(X)$ and $x \in X$, if $x \in U^c$ then there exists $y \in U^c$ such that $(x, y) \in T$ and $R(y) \subseteq U$.*

The category $\mathbf{FGi}_S$ is the full subcategory of **FGiS** whose objects are frontal S-spaces.

Proposition 2. *Let A be a gi-lattice and let $S : A \to A$ be a function satisfying* (**W1**), (**W2**) *and* (**W3**). *Then the pair $\langle A, S\rangle$ is a Sgi-lattice iff $\langle X(A), \subseteq, T_{\Rightarrow}, R_S\rangle$ is a frontal S-space.*

Proof. $\Longrightarrow$) Let us prove that if $P \in \mathbf{X}(A)$ and $a \notin P$, then there exists $Q \in \mathbf{X}(A)$ such that $a \notin Q$, $(P, Q) \in T_{\Rightarrow}$ and $R_S(Q) \subseteq \varphi(a)$. Taking into account that $a \notin P$, we have that $(S(a) \Rightarrow a) \cap P = \emptyset$. Thus, by Lemma 1 there is $Q \in \mathbf{X}(A)$ such that $a \notin Q$, $S(a) \in Q$ and $(P, Q) \in T_{\Rightarrow}$. As $S(a) \in Q$, we obtain $R_S(Q) \subseteq \varphi(a)$. Hence, $\langle \mathbf{X}(A), \subseteq, T_{\Rightarrow}, R_S\rangle$ is a frontal S-space.

$\Longleftarrow$) Conversely, let us prove that $S(a) \Rightarrow a \subseteq (a]$, for any $a \in A$. Suppose that there exists $a \in A$ such that $S(a) \Rightarrow a \not\subseteq (a]$. Then there is $b \in S(a) \Rightarrow a$ such that $b \nleq a$, so there is $P \in \mathbf{X}(A)$ such that $a \notin P$ and $b \in P$. By hypothesis, there exists $Q \in \mathbf{X}(A)$ such that $a \notin Q$, $(P, Q) \in T_{\Rightarrow}$ and $R_S(Q) \subseteq \varphi(a)$ (i.e. $S(a) \in Q$). In consequence, we have that $(P, Q) \in T_{\Rightarrow}$, $(S(a) \Rightarrow a) \cap P \neq \emptyset$ and $S(a) \in Q$. Thus $a \in Q$, which is a contradiction. Hence, $S(a) \Rightarrow a \subseteq (a]$ for any $a \in A$. □

It follows from Proposition 2 and Theorem 5 the following

Theorem 6. *The category* $\mathbf{FGi}_S$ *is dually equivalent to the category* $\mathbf{DLatGi}_S$*.*

If $\langle X, \leq\rangle$ is a poset and $T \subseteq X \times X$, for each $U \subseteq X$ we define the set

$$U_T = \{x \in U^c : T(x) \cap U^c \subseteq (x]\}.$$

In the following we introduce a new type of gi-spaces that are dual to the Sgi-lattices.

Definition 6. *A* gi-space with successor*, or* Sgi-space*, is a gi-space* $\langle X, \leq, T\rangle$ *satisfying the following conditions for every* $U, V \in \mathbf{D}(X)$*:*

(a) $U \cup U_T \in \mathbf{D}(X)$

(b) *If* $x \in U^c$, *then* $T(x) \cap U_T \neq \emptyset$.

(c) *If* $(x, y) \in \bar{T}$ *and* $x \in U \cup U_T$, *then* $y \in U$.

(d) *If* $U \cup U_T \subseteq V \cup (V \to_T U)$*, then there exists* $W \in \mathbf{D}(X)$ *such that* $U \cup U_T \subseteq V \cup W$ *and* $W \subseteq V \to_T U$.

Let **SGiS** be the category whose objects are Sgi -spaces and whose morphisms are gi-morphisms $f\colon \langle X_1, \leq_1 T_1\rangle \to \langle X_2, \leq_2, T_2\rangle$ such that

$$f^{-1}(U \cup U_{T_2}) = f^{-1}(U) \cup f^{-1}(U)_{T_1},$$

for each $U \in \mathbf{D}(X_2)$. These morphisms will be called *Sgi-morphisms.*

Remark 5. *Let* $\langle X, \leq\rangle$ *be a Priestley space and* $T \subseteq X \times X$ *such that satisfies the condition* (**c**) *of Definition 6. A straightforward computation shows that the condition* (**d**) *of Definition 6 is equivalent to the following one: for every* $U, V \in \mathbf{D}(X)$, $U \cup U_T \subseteq V \cup (V \Rightarrow_T U)$.

Proposition 3. *If* $\langle X, T\rangle$ *is a Sgi-space, then* $\langle \mathbf{D}(X), \cup, \cap, \Rightarrow_T, S, \emptyset, X\rangle$ *is a Sgi-lattice, where* S *is given by* $S(U) = U \cup U_T$.

Proof. Function S is well defined and it satisfies conditions **(W1)** and **(W2)** (see proof of Proposition 5.5 of [9]). By Remark 5 we obtain that if $U, V \in \mathbf{D}(X)$, then $S(U) \in (V] \vee (V \Rightarrow U)$. Finally let us prove that $S(U) \Rightarrow U \subseteq (U]$, for every $U \in \mathbf{D}(X)$. Let $U \in \mathbf{D}(X)$, and suppose that there exists $V \in \mathbf{D}(X)$ such that $V \in S(U) \Rightarrow U$ and $V \nsubseteq U$. In particular, there is $x \in X$ such that $x \in V$ and $x \notin U$. As $x \in V$, then $T(x) \cap S(U) \subseteq U$. On the other hand, we have that $x \notin U$. So from condition **(b)** of Definition 6 there exists $y \in X$ such that $(x, y) \in T$ and $y \in U_T \subseteq S(U)$. Then $y \in T(x) \cap S(U)$, and consequently $y \in U$, which is impossible because $y \in U_T$. □

Lemma 7. *Let* $\langle A, S\rangle$ *be a Sgi-lattice. For every* $a \in A$ *we have that*

$$\varphi(a) \cup \varphi(a)_{T_\Rightarrow} = \varphi(S(a)).$$

Proof. Let $a \in A$. Let us see that $\varphi(a) \cup \varphi(a)_{T_\Rightarrow} = \varphi(S(a))$, for each $a \in A$. Let $P \in \mathbf{X}(A)$ such that $S(a) \in P$ and $a \notin P$. Let us prove that $P \in \varphi(a)_{T_\Rightarrow}$, i.e., $T_\Rightarrow(P) \cap \varphi(a)^c \subseteq (P]$. Suppose that there is $Q \in T_\Rightarrow(P) \cap \varphi(a)^c$ such that $Q \nsubseteq P$, so there is $b \in Q - P$. As $S(a) \in (b] \vee (b \Rightarrow a)$, we have that there is $c \in b \Rightarrow a$ such that $S(a) \leq b \vee c$. In particular, we obtain $c \in P$. Hence,

$c \in (b \Rightarrow a) \cap P$. Taking into account that $Q \in T_{\Rightarrow}(P)$ and $b \in Q$, we have that $a \in Q$, which is an contradiction. Therefore $\varphi(S(a)) \subseteq \varphi(a) \cup \varphi(a)_{T_{\Rightarrow}}$. Conversely, let $P \in \varphi(a) \cup \varphi(a)_{T_{\Rightarrow}}$. If $a \in P$, then $S(a) \in P$ because $a \leq S(a)$. Assume that $a \notin P$. Then $P \in \varphi(a)_{T_{\Rightarrow}}$, i.e., $T_{\Rightarrow}(P) \cap \varphi(a)^c \subseteq (P]$. Suppose that $S(a) \notin P$. Note that $S(a) \Rightarrow a \subseteq (a] \subseteq (S(a)]$, so $(S(a) \Rightarrow a) \cap P = \emptyset$. Then by Lemma 1 there exists $Q \in \mathbf{X}(A)$ such that $S(a) \in Q$, $a \notin Q$ and $(P, Q) \in T_{\Rightarrow}$. Hence, $Q \in T_{\Rightarrow}(P) \cap \varphi(a)^c$. Consequently $Q \subseteq P$ and $a \in P$, which is a contradiction. □

Proposition 4. *Let $\langle A, S \rangle$ be a Sgi-lattice. Then $\langle \mathbf{X}(A), \subseteq, T_{\Rightarrow} \rangle$ is a Sgi-space.*

Proof. Let us prove conditions **(a)**, **(b)**, **(c)** and **(d)** of Definition 6. **(a)** It follows from Lemma 7. **(b)** Let $P \in \mathbf{X}(A)$ and let $a \notin P$. Then $(S(a) \Rightarrow a) \cap P = \emptyset$. It follows from Lemma 1 that there is $Q \in T_{\Rightarrow}(P) \cap \varphi(a)^c$ such that $S(a) \in Q$. Let us see that $Q \in \varphi(a)_{T_{\Rightarrow}}$, i.e. $T_{\Rightarrow}(Q) \cap \varphi(a)^c \subseteq (Q]$. Let $D \in T_{\Rightarrow}(Q) \cap \varphi(a)^c$. If $D \nsubseteq Q$, then there is $b \in D - Q$. As $S(a) \in (b] \vee (b \Rightarrow a)$, we obtain that there is $c \in b \Rightarrow a$ such that $S(a) \leq b \vee c$. Taking into account that $S(a) \in Q$ and $b \notin Q$, we have that $c \in Q$. Hence, $c \in (b \Rightarrow a) \cap Q$ and $(P, D) \in T_{\Rightarrow}$, $(b \Rightarrow a) \cap Q \neq \emptyset$ and $b \in D$. Therefore $a \in D$, which is impossible. **(c)** Let $a \in A$ and let $P, Q \in \mathbf{X}(A)$ such that $(P, Q) \in T_{\Rightarrow}$, $Q \nsubseteq P$ and $P \in \varphi(a) \cup \varphi(a)_{T_{\Rightarrow}}$. From **(a)** we have that $\varphi(a) \cup \varphi(a)_{T_{\Rightarrow}} = \varphi(S(a))$. Then $S(a) \in P$. It follows from $Q \nsubseteq P$ that there is $b \in Q - P$. So from $S(a) \in (b] \vee (b \Rightarrow a) \in P$, we obtain that $(b \Rightarrow a) \cap P \neq \emptyset$. Thus, $(P, Q) \in T_{\Rightarrow}$ and $a \in Q$. **(d)** In this item we use Remark 5. Let $a, b \in A$. Let us prove that there is $c \in A$ such that $\varphi(S(a)) \subseteq \varphi(b) \cup \varphi(c)$ and $\varphi(c) \subseteq \varphi(b) \rightarrow_{T_{\Rightarrow}} \varphi(a)$. As $S(a) \in (b] \vee (b \Rightarrow a)$, we have that there is $c \in A$ such that $S(a) \leq b \vee c$. Hence, $\varphi(S(a)) \subseteq \varphi(b) \cup \varphi(c)$. Let $Q \in \varphi(c)$, so $c \in Q$. Let us see that $T_{\Rightarrow}(Q) \cap \varphi(b) \subseteq \varphi(a)$. Let $D \in T_{\Rightarrow}(Q) \cap \varphi(b)$. As $c \in (b \Rightarrow a) \cap Q$, we have that $(Q, D) \in T_{\Rightarrow}$, $b \in D$ and $(b \Rightarrow a) \cap Q \neq \emptyset$. Therefore $a \in D$, i.e., $D \in \varphi(a)$. □

Note that if $f\colon \langle X_1, \leq_1 T_1 \rangle \to \langle X_2, \leq_2, T_2 \rangle$ is a Sgi-morphism, then the function $f^*\colon \mathbf{D}(\mathbf{X}_2) \to \mathbf{D}(\mathbf{X}_1)$ is a Sgi-homomorphism because $f^{-1}(S_2(U)) = f^{-1}(U \cup U_{T_2}) = f^{-1}(U) \cup f^{-1}(U)_{T_1} = S_1(f^{-1}(U))$, for each $U \in \mathbf{D}(\mathbf{X}_2)$.

Proposition 5. *Let $\langle A, S_A \rangle$ and $\langle B, S_B \rangle$ be two Sgi-lattices. Let $h\colon A \to B$ be a Sgi-homomorphism. Then $h_*\colon \mathbf{X}(B) \to \mathbf{X}(A)$ is a Sgi-morphism.*

Proof. See proof of Proposition 5.8 of [9]. □

Proposition 6. *Let $\langle A, S_A \rangle$ be a Sgi-lattice. Then we have that $\varphi : \langle A, S_A \rangle \to \left\langle \mathbf{D}(\mathbf{X}(A)), S_{\mathbf{D}(\mathbf{X}(A))} \right\rangle$ is an isomorphism in $\mathbf{DLatGi}_S$.*

Proof. It follows from Proposition 3, Lemma 7 and Proposition 4. □

Proposition 7. *Let $\langle X, T \rangle$ be a Sgi-space. Then we have that $\varepsilon\colon \langle X, \leq, T \rangle \to \langle \mathbf{X}(\mathbf{D}(X)), \subseteq, T_{\Rightarrow_T} \rangle$ is an isomorphism in* **SGiS**.

Proof. Analogous to the proof of the Proposition 5.10 of [9]. □

By previous results, and by the results given in [5] for gi-lattices, we obtain the following.

Theorem 8. *The category* **SGiS** *is dually equivalent to the category* $\mathbf{DLatGi}_S$.

The next aim is to study the connection between frontal S-spaces and Sgi-spaces.

Lemma 9. *Let $\langle X, \leq, T\rangle$ be a gi-space. Let R be a binary relation on X that satisfies the following conditions for every $U \in \mathbf{D}(\mathbf{X})$ and $x \in X$:*

(i) $\bar{T} \subseteq R \subseteq \leq$.

(ii) *If $x \in U^c$, then there exists $y \in U^c$ such that $(x, y) \in T$ and $R(y) \subseteq U$.*

(iii) $\leq \circ R \subseteq R$.

Then for every $U \in \mathbf{D}(\mathbf{X})$ it holds that $\tau_R(U) = U \cup U_T$.

Proof. Analogous to the proof of Lemma 5.12 of [9]. □

The proof of the following remark is similar to the proof of Remark 5.13 of [9].

Remark 6. *If $\langle X, \leq, T\rangle$ is a Sgi-space such that $\bar{T} \subseteq \leq$, then*

$$U \cup U_T = \{x \in X : \bar{T}(x) \subseteq U\},$$

for each $U \in \mathbf{D}(X)$.

Proposition 8. *Let $\langle X, \leq, T\rangle$ be a Sgi-space. Then there exists a binary relation R_T on X such that $\langle X, \leq, T, R_T\rangle$ is a frontal S-space.*

Proof. We define a binary relation R_T on X in the following way:

$$(x, y) \in R_T \quad \text{iff} \quad \forall U \in \mathbf{D}(X) \text{ (if } x \in U \cup U_T \text{, then } y \in U).$$

A direct computation proves that $R_T(x)$ is a closed upset of X, for each $x \in X$. Besides the relation R_T satisfies conditions (**i**), (**ii**) and (**iii**) of Lemma 9. Hence, from Lemma 9 we have that $\tau_{R_T}(U) \in \mathbf{D}(X)$, for every $U \in \mathbf{D}(X)$. Therefore $\langle X, \leq, T, R_T\rangle$ is a frontal Sgi-space. For more details see the proof of Propostion 5.14 of [9]. □

The frontal S-space $\langle X, \leq, T, R_T\rangle$ built in the previous is called the *associated frontal S-space of the Sgi-space $\langle X, \leq, T\rangle$*. Note that $\tau_{R_T}(U) = U \cup U_T$, for each $U \in \mathbf{D}(X)$.

Proposition 9. *Let $\langle X, \leq, T, R\rangle$ be a frontal S-space. Then $\langle X, \leq, T\rangle$ is a Sgi-space such that $R = R_T$.*

Proof. Let us prove the conditions of Definition 6. (**a**) It follows from that for the gi-space $\langle X, \leq, T\rangle$ the relation R satisfies the conditions of Lemma 9. (**b**) Let $x \in U^c$. By Definition 5, there exists $y \in U^c$ such that $(x, y) \in T$ and $R(y) \subseteq U$. Let us see that $y \in U_T$. Let $z \in T(y) \cap U^c$. In particular, $z \notin R(y)$, and then $(y, z) \notin \overline{T}$ because $\bar{T} \subseteq R$. Thus, $z \leq y$. Consequently, $y \in U_T$. Therefore $y \in T(x) \cap U_T$, i.e., $T(x) \cap U_T \neq \emptyset$. (**c**) Let $(x, y) \in \bar{T}$ and $x \in U \cup U_T$. Then $(x, y) \in T$ and $y \nleq x$. Suppose that $x \in U$- Hence, $y \in U$. Now let $x \in U_T$. Thus, $x \in U^c$ and $U^c \cap T(x) \subseteq (x]$. If $y \in U^c$, then $y \in U^c \cap T(x)$. So $y \leq x$, which is a contradiction. (**d**) By Theorem 6, we have that in the gi-lattice $\langle \mathbf{D}(X), \Rightarrow_T\rangle$ there exists successor function. Moreover, by Lemma 9 the successor function S in $\mathbf{D}(X)$ is given by $S(U) = U \cup U_T$. Taking into account that $S(U) \in (V] \vee (V \Rightarrow_T U)$, we have that there exists $W \in \mathbf{D}(X)$ such that $U \cup U_T \subseteq V \cup W$ and $W \subseteq V \rightarrow_T V$. Thus, by Remark 5 it holds condition (**d**) of Definition 6. Therefore $\langle X, \leq, T\rangle$ is a Sgi-space. Let us see that $R \subseteq R_T$. Let $(x, y) \in R$ and $x \in U \cup U_T$. By Lemma 9 we obtain $\tau_R(U) = U \cup U_T$. Using that $x \in \tau_R(U)$ we conclude that $R(x) \subseteq U$. Therefore $y \in U$. Finally let us prove that $R_T \subseteq R$. From Lemma 9 we have that

$$\tau_R(U) = U \cup U_T = \tau_{R_T}(U)$$

for each $U \in \mathbf{D}(X)$. Let $(x, y) \in R_T$ and suppose that $(x, y) \notin R$. As $R(x)$ is a closed upset, there exists $U \in \mathbf{D}(X)$ such that $R(x) \subseteq U$ and $y \notin U$. So, $x \in \tau_R(U) = \tau_{R_T}(U)$, i.e., $R_T(x) \subseteq U$, which is a contradiction. □

In the next proposition we show that a morphism between two Sgi-spaces can be characterized as a gi-morphism that is a p-morphism with respect to the associated frontal S-spaces.

Proposition 10. *Let $\langle X_1, \leq_1, T_1\rangle$ and $\langle X_2, \leq_2, T_2\rangle$ be two Sgi-spaces. Let $f\colon X_1 \to X_2$ be a gi-morphism. Then f is a Sgi-morphism iff f is a p-morphism between the associated frontal S-spaces $\langle X_1, \leq_1, T_1, R_{T_1}\rangle$ and $\langle X_2, \leq_2, T_2, R_{T_2}\rangle$.*

Proof. Similar to the proof of the Proposition 5.16 of [9]. □

Similar ideas used in the proof of Proposition 10 show the following

Proposition 11. *Let $\langle X_1, \leq_1, R_1, T_1\rangle$ and $\langle X_2, \leq_2, R_2, T_2\rangle$ be two frontal S-spaces. Let $f\colon X_1 \to X_2$ be a gi-morphism. Then f is a p-morphism iff f is a Sgi-morphism.*

Then we obtain the following

Theorem 10. *The categories* **SGiS** *and* $\mathbf{FGi}_S$ *are isomorphic.*

4.2 Categorical equivalences for $\mathbf{DLatGi}_\gamma$

The next goal is to establish two categorical equivalences for the category of gi-lattices with gamma. In order to make it possible, we start with the following

Definition 7. *A frontal gi-space $\langle X, \leq, T, R\rangle$ is a* frontal γ-space *if the following conditions are satisfied:*

(γ_1) *For every $x \in X$ there exists $y \in X$ such that $(x, y) \in T$ and $R(y) = \emptyset$.*

(γ_2) *For every $x \in X$, $R(x) = \emptyset$ or $x \in R(x)$.*

We write $\mathbf{FGi}_\gamma$ for the full subcategory of **FGiS** whose objects are frontal γ-spaces.

Proposition 12. *Let A be a gi-lattice and $\gamma\colon A \to A$ a function satisfying* (**W1**), (**W2**) *and* (**W3**). *Then the pair $\langle A, \gamma\rangle$ is a γgi-lattice iff $\langle \mathbf{X}(A), \subseteq, T_{\Rightarrow}, R_\gamma\rangle$ is a frontal γgi-space.*

Proof. $\Longrightarrow$) The proof of condition (γ_1) of Definition 7 is similar to the proof of Lemma 2 (taking $a = 0$). In order to prove condition (γ_2) of Definition 7, let $P \in \mathbf{X}(A)$ such that $R_\gamma(P) \neq \emptyset$. It implies that $\gamma^{-1}(P) \neq A$. As $\gamma^{-1}(P)$ is a proper filter, $0 \notin \gamma^{-1}(P)$. Let us prove that $\gamma^{-1}(P) \subseteq P$. Let $\gamma(a) \in P$. Then $a \vee \gamma(0) \in P$. But $0 \notin \gamma^{-1}(P)$, so $a \in P$. Thus, $(P, P) \in R_\gamma$.

$\Longleftarrow$) By the proof of Lemma 2 and taking $a = 0$, we have that $\gamma(0) \Rightarrow 0 = \{0\}$. Let us prove that $\gamma(a) \leq a \vee \gamma(0)$, for any $a \in A$. Suppose that $\gamma(a) \not\leq a \vee \gamma(0)$, then there exists $P \in \mathbf{X}(A)$ such that $\gamma(a) \in P$, $a \notin P$ and $\gamma(0) \notin P$. Hence $\gamma^{-1}(P)$ is a proper filter, i.e., $R_\gamma(P) \neq \emptyset$. Then $(P, P) \in R_\gamma$. However $\gamma(a) \in P$ and then $a \in P$, which is a contradiction. Therefore, $\langle A, \gamma \rangle$ is a γgi-lattice. □

Thus, by Proposition 12 and Theorem 5 we obtain the following

Theorem 11. *The category* $\mathbf{DLatGi}_\gamma$ *is dually equivalent to the category* $\mathbf{FGi}_\gamma$*.*

Recall that if X is a set and $T \subseteq X \times X$, then $\emptyset_T = \{x \in X : T(x) \subseteq (x]\}$. In what follows we will provide other duality for the category of γgi-lattices.

Definition 8. *A* gi-space with gamma*, or* γgi-space*, is a gi-space* $\langle X, \leq, T \rangle$ *satisfying the following conditions for every* $U, V \in \mathbf{D}(X)$*:*

(a) $U \cup \emptyset_T \in \mathbf{D}(X)$.

(b) $T(x) \cap \emptyset_T \neq \emptyset$*, for every* $x \in X$.

(c) *If* $(x, y) \in \bar{T}$ *and* $x \in U \cup \emptyset_T$*, then* $y \in U$.

(d) *There exists* $W \in \mathbf{D}(X)$ *such that* $U \cup \emptyset_T \subseteq V \cup W$ *and* $W \subseteq V \to_T \emptyset$.

Let $\gamma\mathbf{GiS}$ be the category whose objects are γgi-spaces $\langle X, \leq, T \rangle$, and whose morphisms are gi-morphisms $f\colon \langle X_1, \leq_1 T_1 \rangle \to \langle X_2, \leq_2, T_2 \rangle$ such that $f^{-1}(U \cup \emptyset_{T_2}) = f^{-1}(U) \cup f^{-1}(\emptyset)_{T_1}$, for each $U \in D(X_2)$. These morphisms will be called *γgi-morphisms.*

Proposition 13. *If* $\langle X, T \rangle$ *is a* γ*gi-space, then* $\langle \mathbf{D}(X), \cup, \cap, \Rightarrow_T, \gamma, \emptyset, X \rangle$ *is a* γ*gi-lattice, where* γ *is defined by* $\gamma(U) = U \cup \emptyset_T$*, for each* $U \in \mathbf{D}(X)$.

Proof. The proof is similar to the proof of Proposition 3. □

The proof of the following lemma is analogous to the proof of Lemma 7.

Lemma 12. *Let* $\langle A, \gamma \rangle$ *be a* γ*gi-lattice. For every* $a \in A$ *we have that* $\varphi(a) \cup \varphi(0)_{T_\Rightarrow} = \varphi(\gamma(a))$.

Proposition 14. *If* $\langle A, \gamma \rangle$ *is a* γ*gi-lattice, then* $\langle \mathbf{X}(A), \subseteq, T_\Rightarrow \rangle$ *is a* γ*gi-space.*

Proof. The proof is similar to the proof of Proposition 4. Condition **(a)** follows from Lemma 12. To prove condition **(b)** of Definition 8, take a prime filter P in A. Since $(\gamma(0) \Rightarrow 0) \cap P = \{0\} \cap P = \emptyset$, there exists $Q \in \mathbf{X}(A)$ such that $(P, Q) \in T_\Rightarrow$ and $\gamma(0) \in Q$. Let us see that $T_\Rightarrow(Q) \subseteq (Q]$. Suppose that there is $D \in T_\Rightarrow(Q)$ but $D \nsubseteq Q$. Then there is $b \in D - Q$. As $\gamma(0) \in (b] \vee (b \Rightarrow 0)$, we have that $\gamma(0) \leq b \vee c$ for some $c \in b \Rightarrow 0$. Hence, $D \in T_\Rightarrow(Q)$, $Q \cap (b \Rightarrow 0) \neq \emptyset$ and $b \in D$, so we get $0 \in D$, which is impossible. Thus, $T_\Rightarrow(P) \cap \emptyset_{T_\Rightarrow} \neq \emptyset$. It is condition **(b)** of Definition 8. In the next let us prove condition **(c)** of

Definition 8. Let $P, Q \in \mathbf{X}(A)$, and $a \in A$ such that $(P, Q) \in T_{\Rightarrow}$, $Q \nsubseteq P$, and $P \in \varphi(a) \cup \varphi(0)_{T_{\Rightarrow}} = \varphi(\gamma(a))$. Then $\gamma(a) \in P$, and there is $b \in Q - P$. Taking into account that $\gamma(a) \in (b] \vee (b \Rightarrow a)$, we have that $(P, Q) \in T_{\Rightarrow}$, $(b \Rightarrow a) \cap P \neq \emptyset$ and $b \in Q$. Therefore $a \in Q$, i.e., $Q \in \varphi(a)$. It is condition **(c)** of Definition 8. Finally, the proof of condition **(d)** of Definition 8 is similar to the proof of condition **(d)** of Definition 6 which was made in Proposition 4. □

Note that if $f\colon \langle X_1, \leq_1 T_1\rangle \to \langle X_2, \leq_2, T_2\rangle$ is a γgi-morphism, then the function $f^*\colon \mathbf{D}(X_2) \to \mathbf{D}(X_1)$ is a γgi-homomorphism because $f^{-1}(\gamma(U)) = f^{-1}(U \cup \emptyset_{T_2}) = f^{-1}(U) \cup f^{-1}(\emptyset)_{T_1} = \gamma(f^{-1}(U))$, for each $U \in \mathbf{D}(X_2)$.

Proposition 15. *Let $\langle A, \gamma_A\rangle$ and $\langle B, \gamma_B\rangle$ be γgi-lattices. Let $h\colon A \to B$ be a γgi-homomorphism. Then $h_*\colon \mathbf{X}(B) \to \mathbf{X}(A)$ is a γgi-morphism.*

Proof. The proof is similar to the proof of Proposition 5. □

Proposition 16. *Let $\langle A, \gamma_A\rangle$ be a γgi-lattice. Then we have that $\varphi : \langle A, \gamma_A\rangle \to \left\langle \mathbf{D}(\mathbf{X}(A)), \gamma_{\mathbf{D}(\mathbf{X}(A))}\right\rangle$ is a an isomorphism in* $\mathbf{DLatGi}_{\gamma}$.

Proof. It follows from Proposition 13, Lemma 12 and Proposition 14. □

Proposition 17. *Let $\langle X, T\rangle$ be a γgi-space. Then we have that $\varepsilon\colon \langle X, \leq, T\rangle \to \langle \mathbf{X}(\mathbf{D}(X)), \subseteq, T_{\Rightarrow_T}\rangle$ is an isomorphism in* $\gamma\mathbf{GiS}$.

Proof. The proof is similar to the proof of Proposition 7. □

Then we obtain the following

Theorem 13. *The category $\gamma\mathbf{GiS}$ is dually equivalent to the category* $\mathbf{DLatGi}_{\gamma}$.

In the following we study the connection between frontal γ-spaces and γgi-spaces.

Lemma 14. *Let $\langle X, \leq, T\rangle$ be a gi-space and R a binary relation on X such that the following conditions are satisfied:*

(i) $\bar{T} \subseteq R \subseteq \leq$.

(ii) *For every $x \in X$ there exists $y \in X$ such that $(x, y) \in T$ and $R(y) = \emptyset$.*

(iii) $\leq \circ R \subseteq R$.

(iv) *For every $x \in X$, $R(x) = \emptyset$ or $x \in R(x)$.*

Then for every $U \in \mathbf{D}(X)$ it holds that $\tau_R(U) = U \cup \emptyset_T$.

Proof. Let $U \in \mathbf{D}(X)$. The inclusion $U \cup \emptyset_T \subseteq \tau_R(U)$ can be proved using the same idea as in the proof of Lemma 9. Conversely, let $x \in \tau_R(U)$ and suppose that $x \notin \emptyset_T$. Thus, there is $y \in X$ such that $y \in T(x)$ and $y \nleq x$. As $\bar{T} \subseteq R$, we have that $y \in R(x)$. By condition (**iv**) we obtain $x \in R(x)$. Besides $x \in \tau_R(U)$ and in consequence $x \in U$. □

Proposition 18. *Let $\langle X, \leq, T\rangle$ be a γgi-space. Then there exists a binary relation R_T in X such that $\langle X, \leq, T, R_T\rangle$ is a frontal γ-space.*

Proof. We define a binary relation on X in the following way:

$$(x,y)\in R_T \quad \text{iff} \quad \forall U\in \mathbf{D}(X)(\text{if } x\in U\cup\emptyset_T, \text{ then } y\in U).$$

Note that $R_T(x)$ is a closed upset of X, for every $x \in X$. Let us prove that R_T satisfies conditions (**i**)-(**iv**) of Lemma 14. (**i**) The fact that $\bar{T} \subseteq R_T$ is consequence of (**c**) of Definition 8, and $R_T \subseteq \leq$ is proved like in Lemma 9. The proof of items (**ii**) and (**iii**) is similar to the proof of Lemma 9. (**iv**) Let $x \in X$. Suppose that $x \notin R_T(x)$. Hence, there exists $V \in \mathbf{D}(X)$ such that $x \in V \cup \emptyset_T$ and $x \notin V$. So $x \in \emptyset_T$, i.e., $T(x) \subseteq (x]$. By condition (**ii**) of Lemma 14, we have that there is $y \in T(x)$ with $R(y) = \emptyset$. Thus, $y \leq x$, so $R_T(x) \subseteq R_T(y) = \emptyset$. Hence, $R_T(x) = \emptyset$. It follows from Lemma 14 that $\tau_{R_T}(U) \in \mathbf{D}(X)$ for every $U \in \mathbf{D}(X)$. Therefore we have that $\langle X, \leq, T, R_T\rangle$ is a frontal γ-space. □

The frontal γ-space $\langle X, \leq, T, R_T\rangle$ built in the previous proof is called the *associated frontal γ-space of the γgi-space $\langle X, \leq, T\rangle$*. Note that $\tau_{R_T}(U) = U\cup\emptyset_T$, for each $U \in \mathbf{D}(X)$.

Proposition 19. *If $\langle X, \leq, T, R\rangle$ is a frontal γ-space, then $\langle X, \leq, T\rangle$ is a γgi-space such that $R = R_T$.*

Proof. (**a**) It follows from Lemma 14. (**b**) Let $x \in X$. Then by condition (γ_1) there exists $y \in X$ such that $(x, y) \in T$ and $R(y) = \emptyset$. Let us prove that $y \in T(x) \cap \emptyset_T$. Let $z \in T(y)$, so $(y, z) \in T$. In particular $z \notin R(y)$, so $z \leq y$ (because if $z \not\leq y$ then we would have that $(y, z) \in R$, which is a contradiction). Thus, $y \in \emptyset_T$. (**c**) Let $U \in \mathbf{D}(X)$, $(x, y) \in \bar{T}$ and $x \in U \cup \emptyset_T$. If $x \in \emptyset_T$ then $y \leq x$, wich is a contradiction. Thus, $x \in U$. Taking into account that $x \leq y$, we have that $y \in U$. The proof that $R = R_T$ is similar to the proof of Proposition 9. □

The proofs of the following two propositions are similar to the proofs of propositions 10 and 11, respectively.

Proposition 20. *Let $\langle X_1, \leq, T_1\rangle$ and $\langle X_2, \leq, T_2\rangle$ be two γgi-spaces. Let $f\colon X_1 \to X_2$ be a gi-morphism. Then f is a γgi-morphism iff f is a p-morphism between the associated frontal γ-spaces $\langle X_1, T_1, R_{T_1}\rangle$ and $\langle X_2, T_2, R_{T_2}\rangle$.*

Proposition 21. *Let $\langle X_1, \leq_1, R_1, T_1\rangle$ and $\langle X_2, \leq_2, R_2, T_2\rangle$ be two frontal γ-spaces. Let $f\colon X_1 \to X_2$ be a gi-morphism. Then f is a p-morphism iff f is a γgi-morphism.*

Therefore we conclude the following

Theorem 15. *The categories $\gamma\mathbf{GIS}$ and $\mathbf{FGI}_\gamma$ are isomorphic.*

5 Frontal bounded distributive lattices

In a lattice A the *annihilator of a relative to b* is defined by

$$\langle a, b\rangle := \{c \in A : c \wedge a \leq b\}.$$

Several authors have studied annihilators in lattices ([11], [15]). In particular, Mandelker ([15]) proved that a lattice A is distributive iff $\langle a, b\rangle$ is an ideal for all

$a, b \in A$. The concept of annihilator is a natural generalization of the relative pseudocomplement $a \to b$ of an element $a \in A$ relative to an element $b \in A$. If A is a bounded distributive lattice, then the function $\Rightarrow: A \times A \to Id(A)$ defined by

$$a \Rightarrow b := \langle a, b \rangle$$

satisfies the conditions of a generalized implication. In consequence, the notion of generalized implication is an extension of the concept of annihilator in distributive lattices.

In this section we define and study the case of frontal operators in bounded distributive lattices where the generalized implication is defined as the annihilator, and we consider two particular cases of them: the successor function and the gamma function. Finally we establish a categorical equivalence for the category of bounded distributive lattices with frontal operators, and for the subcategory of bounded distributive lattices with successor and the subcategory of bounded distributive lattices with gamma.

Definition 9. *A* frontal lattice *is an algebra (A, τ) such that A is a bounded distributive lattice and $\tau : A \to A$ is a function satisfying the conditions* (**W1**), (**W2**), *and the following condition for every $a, b \in A$:*

(LW3) $\tau(a) \in (b] \vee \langle b, a \rangle$.

If $\langle A, \tau \rangle$ is a frontal lattice we say that τ is a frontal operator*.*

We write **FBDL** for the category whose objects are frontal lattices, and whose morphisms are homomorphisms of bounded lattices which preserve the frontal operator. This category is a subcategory of the category of modal algebras. In the next paragraph we define some particular classes of frontal gi-lattices.

Definition 10. *A* lattice with successor*, or* S-lattice*, is a pair $\langle A, S \rangle$ such that A is a bounded distributive lattice and $S \colon A \to A$ is a function which satisfies the conditions* (**W2**), (**LW3**), *and the following one:*

$$\langle S(a), a \rangle \subseteq (a].$$

If $\langle A, S \rangle$ is a S-lattice, then the function S is called *successor*. By Lemma 4 we have the successor function in bounded distributive lattices is a frontal operator. Let $\mathbf{BDL}_S$ be the category whose objects are S-lattices, and whose morphisms are homomorphisms of bounded lattices which preserve the modal operator. These morphisms are called *S-homomorphisms.* In every S-lattice we have that $\langle S(a), a \rangle = (a]$, for each $a \in A$.

Let A be a bounded distributive lattice. We say that a function $T : A \to A$ satisfies the following condition:

$$(\mathbf{P})\ \langle T(a), a \rangle \subseteq (T(a)] \text{ for every } a \in A.$$

For every $a \in A$ we define $E_a = \{b \in A : \langle b, a \rangle \subseteq (b]\}$.

Proposition 22. *Let A be a bounded distributive lattice and $S : A \to A$ a function. The following three conditions are equivalent:*

(1) *$\langle A, S \rangle$ is a S-lattice.*

(2) *Function S satisfies condition* (**P**), *and for every $a, b \in A$ there is $c \in A$ with $S(a) \leq b \vee c$ and $c \wedge b \leq a$.*

(3) *Function S is given by $S(a) = min\ E_a$, and for every $a, b \in A$ there is $c \in A$ with $S(a) \leq b \vee c$ and $c \wedge b \leq a$.*

Proof. (1) $\Longrightarrow$ (2). Let $a \in A$. By (**W2**) we obtain $\langle S(a), a\rangle \subseteq (a] \subseteq (S(a)]$. Let $a, b \in A$. Then by (**LW3**) we have that there is $c \in A$ with $S(a) \leq b \vee c$ and $c \wedge b \leq a$.

(2) $\Longrightarrow$ (1). Let $a \in A$. Taking into account condition (**P**), we have that $\langle S(a), a\rangle \subseteq (S(a)]$. Let $b \in \langle S(a), a\rangle$, i.e. $b \wedge S(a) \leq a$. By hypothesis we have that $b \in (S(a)]$, so $b \leq a$. Hence, $b \in (a]$ and $\langle S(a), a\rangle \subseteq (a]$. The equation (**W2**) follows from that for every $a \in A$ we have that $a \in \langle S(a), a\rangle \subseteq (S(a)]$. The condition (**LW3**) follows from that if $a, b \in A$, then there is $c \in A$ with $S(a) \leq b \vee c$ and $c \wedge b \leq a$.

(2) $\Longrightarrow$ (3). It follows from Lemma 4.

(3) $\Longrightarrow$ (2). It is immediate. □

Definition 11. *A* lattice with γ, *or* γ-lattice, *is a pair $\langle A, \gamma\rangle$ such that A is a bounded distributive lattice and $\gamma\colon A \to A$ is a function which satisfies the condition* (**LW3**) *and the following conditions:*

(Lg1) $\langle \gamma(0), 0\rangle = \{0\}$,

(Lg2) $\gamma(a) = a \vee \gamma(0)$.

Let $\mathbf{BDL}_\gamma$ be the category whose objects are γ-lattices, and whose morphisms are homomorphisms of bounded lattices which preserve the modal operator. These morphisms are called *γ-homomorphisms*.

In the following we consider two examples of bounded distributive lattices which are not Heyting algebras, and we study their relation with the existence of successor function.

Example 2. *Let $A = (\mathbb{N} \times \mathbb{N}) \oplus \{\omega\}$, where $\mathbb{N}$ is the set of natural numbers and $\oplus$ is the ordinal sum of posets ([1]). Let us see that there is successor function (we use the notation of Prop. 22).*

Direct computations show that for every $n, m \in \mathbb{N}$ we have that $E_{(n,m)} = \{(p, q) \in \mathbb{N} \times \mathbb{N} : n < p\ \&\ m < q\} \cup \{\omega\}$ and $E_\omega = \{\omega\}$. Thus, these sets have minimum. On the other hand let $a, b \in A$. If $a = \omega$, define $c = \omega$. If $b = \omega$, consider $c = (0, 0)$. If $a, b \in A - \{\omega\}$, then $a = (n, m)$ and $b = (p, q)$ for some $n, m, p, q \in \mathbb{N}$. Hence $S(n, m) = (n + 1, m + 1)$. If $n + 1 \leq p$ and $m + 1 \leq q$, consider $c = (0, 0)$. If $n + 1 \leq p$ and $m + 1 > q$, take $c = (0, m + 2)$. The other two cases are similar.

Example 3. *Let $A = ([0, 1) \times [0, 1)) \oplus \{(1, 1)\}$. For every $(a, b) \in [0, 1) \times [0, 1)$ we have that $E_{(a,b)} = \{(c, d) \in [0, 1) \times [0, 1)) : a < c\ \&\ b < d\} \cup \{(1, 1)\}$. We have that there is not successor function in A because the set $E_{(a,b)}$ does not have minimum.*

5.1 Representation and duality

In the following we give some defintions and results in order to give an equivalence for the category **FBDL**.

Definition 12. *A* frontal Priestley space *is a structure* $\langle X, \leq, R\rangle$ *such that:*

1. $\langle X, \leq, R\rangle$ *is a modal Priestley space.*

2. $< \subseteq R \subseteq \leq$.

Here $<$ is the strict order associated to the order $\leq$. Let **FPS** be the category whose objects are frontal Priestley spaces, and whose morphisms are functions $f\colon X_1 \to X_2$ such that f is a morphism of Priestley spaces and f is a p-morphism.

Lemma 16. *Let A be a bounded distributive lattice and $\Rightarrow\colon A \times A \to Id(A)$ the function given by $a \Rightarrow b = \langle a, b\rangle$. Then $T_{\Rightarrow} = \leq$ and $\bar{T}_{\Rightarrow} = <$.*

Proof. Let $(P, Q) \in T_{\Rightarrow}$ and suppose that $P \nsubseteq Q$. Then there is $a \in P$ such that $a \in P$ and $a \notin Q$. Hence, $\langle 1, a\rangle \cap P \neq \emptyset$, $(P, Q) \in T_{\Rightarrow}$ and $1 \in Q$, so $a \in Q$, which is a contradiction. Then $T_{\Rightarrow} \subseteq \leq$. Conversely, let $P, Q \in \mathbf{X}(A)$ and $a, b \in A$ such that $P \subseteq Q$, $\langle a, b\rangle \cap P \neq \emptyset$ and $a \in Q$. Thus, there exists $c \in P$ such that $c \wedge a \leq b$. However $c \in P \subseteq Q$, so $b \in Q$. In consequence, $(P, Q) \in T_{\Rightarrow}$ and $\leq \subseteq T_{\Rightarrow}$. Therefore $T_{\Rightarrow} = \leq$. □

Lemma 17. *Let $\langle X, \leq\rangle$ be a Priestley space and $T = \leq$. Then $\langle X, \leq, T\rangle$ is a gi-space. Moreover, for every $U, V \in \mathbf{D}(X)$ we have that $U \Rightarrow_T V = \langle U, V\rangle$.*

Proof. Let $\langle X, \leq\rangle$ be a Priestley space and $T = \leq$. Straightforward computations proves that $\langle X, \leq, T\rangle$ is a gi-space. In the following let us prove that for every $U, V \in \mathbf{D}(X)$ we have that $U \Rightarrow_T V = \langle U, V\rangle$. Let $W \in U \Rightarrow_T V$, so $W \subseteq U \to_T V$. Let $x \in W \cap U$, so $x \in T(x) \cap U \subseteq V$. That is, $W \cap U \subseteq V$. Hence, $U \Rightarrow_T V \subseteq \langle U, V\rangle$. Conversely, let $W \in \langle U, V\rangle$. Then $W \cap U \subseteq V$. Let us see that $W \subseteq U \to_T V$. Let $x \in W$ and $y \in T(x) \cap U$. Taking into account that $x \leq y$ and $y \in U$, we have that $y \in W \cap U \subseteq V$. Then $T(x) \cap U \subseteq V$. Thus, $\langle U, V\rangle \subseteq U \Rightarrow_T V$. Therefore $U \Rightarrow_T V = \langle U, V\rangle$. □

By the previous lemmas and Theorem 5 we have the following

Theorem 18. *The category* **FBDL** *is dually equivalent to the category* **FPS**.

Finally we give some defintions and results in order to give an equivalences for the categories $\mathbf{BDL}_S$ and $\mathbf{BDL}_\gamma$.

Lemma 19. *Let A be a bounded distributive lattice, $\Rightarrow\colon A \times A \to Id(A)$ the function given by $a \Rightarrow b = \langle a, b\rangle$ and $S : A \to A$ a function. Then $\langle A, S\rangle$ is a Sgi-lattice iff $\langle A, S\rangle$ is a S-lattice.*

Proof. It follows from a direct computation. □

Remark 7. *If $\langle X, \leq\rangle$ is a poset and $U \subseteq X$, we write U_M for the set of maximal elements of U. If U is an upset, then $U \cup (U^c)_M$ is an upset. If $T \subseteq X \times X$ is the relation $\leq$, then $U_T = (U^c)_M$.*

Definition 13. *A* S-Priestley space *is a structure* $\langle X, \leq\rangle$ *such that:*

1. $\langle X, \leq\rangle$ *is a Priestley space.*

2. *The set* $U \cup (U^c)_M$ *is clopen, for every* $U \in \mathbf{D}(X)$.

Note that if $\langle X, \leq\rangle$ is a Priestley space and $U \in \mathbf{D}(X)$, then $U \cup (U^c)_M$ is clopen iff $(U^c)_M$ is clopen.

Lemma 20. *Let* $\langle X, \leq\rangle$ *be a Priestley space and* $T = \leq$. *Then* $\langle X, \leq, T\rangle$ *is a* S*gi-space iff* $\langle X, \leq\rangle$ *is a* S*-Priestley space.*

Proof. $\Longrightarrow$) It is immediate.

$\Longleftarrow$) Let $\langle X, \leq\rangle$ be a S-Priestley space and $T \subseteq X \times X$ given by $T = \leq$. By Lemma 17, we have that $\langle X, \leq, T\rangle$ is a gi-space. Let us prove that $\langle X, \leq, T\rangle$ satisfies Definition 6. **(a)** It follows from Remark 7. **(b)** Let $U \in \mathbf{D}(X)$ and $x \in U^c$. Taking into account that $\langle X, \leq\rangle$ is a Priestley space, we have that there is $y \in (U^c)_M$ such that $x \leq y$. Then $T(x) \cap U_T \neq \emptyset$. **(c)** Let $U \in \mathbf{D}(X)$, $(x, y) \in \bar{T}$ and $x \in U \cup U_T$. Hence, $x \leq y$ and $y \not\leq x$. If $x \in U$, then $y \in U$. If $x \in U_T = (U^c)_M$, then $y \in U$. Therefore $y \in U$. **(d)** Let $U, V \in \mathbf{D}(X)$. We define the set

$$W = U \cup (U_T \cap V^c) = U \cup [(U^c)_M \cap V^c].$$

In the following let us prove that $W \in \mathbf{D}(X)$, $U \cup U_T \subseteq V \cup W$ and $W \subseteq (V \to_T U)$. The set $(U^c)_M$ is clopen. Hence, W is clopen. Let $x, y \in X$ such that $x \in W$ and $x \leq y$. If $y \in U$, then $y \in W$. If $y \notin U$, then $x \in (U^c)_M \cap V^c$. Hence, $x = y \in W$. Thus, $W \in \mathbf{D}(X)$. Let $x \in U \cup (U^c)_M$. If $x \in U$, then $x \in W \subseteq V \cup W$. If $x \in (U^c)_M$ and $x \in V$, then $x \in V \subseteq V \cup W$. If $x \in (U^c)_M$ and $x \notin V$, then $x \in (U^c)_M \cap V^c \subseteq W$. Thus, $U \cup (U^c)_M \subseteq V \cup W$. Finally, note that $V \to_T U = \{x \in X : [x) \cap V \subseteq U\}$ and let $x \in W$. If $x \in U$, then $x \in V \to_T U$. Let $x \in (U^c)_M$. Let us show that $[x) \cap V \subseteq U$. In order to prove it, let $y \in [x) \cap V$. Thus, $x \leq y$ and $y \in V$. Suppose that $y \notin U$, so $x \in (U^c)_M \cap V^c$. Hence, $y = x \notin V$, which is a contradiction. Therefore $W \subseteq (V \to_T U)$. □

If $\langle X, \leq\rangle$ is a S-Priestley space, then the successor function in $\mathbf{D}(X)$ takes the form $S(U) = U \cup (U^c)_M$. We write **SPS** for the category whose objects are S-Priestley spaces, and whose morphisms are morphisms of Priestley spaces $f\colon \langle X_1, \leq_1\rangle \to \langle X_2, \leq_2\rangle$ such that $f^{-1}(U \cup (U^c)_M) = f^{-1}(U) \cup (f^{-1}(U^c))_M$, for each $U \in \mathbf{D}(X_2)$.

Then using Lemma 19, Lemma 20 and Theorem 18 we obtain the following

Theorem 21. *The category* **SPS** *is dually equivalent to the category* $\mathbf{BDL}_S$.

Straightforward computations show the following

Lemma 22. *Let* A *be a bounded distributive lattice,* $\Rightarrow\colon A \times A \to Id(A)$ *the function given by* $a \Rightarrow b = \langle a, b\rangle$ *and* $\gamma : A \to A$ *a function. Then* $\langle A, \gamma\rangle$ *is a* γ*gi-lattice iff* $\langle A, \gamma\rangle$ *is a* γ*-lattice.*

Now we give the following

Definition 14. *A* γ-Priestley space *is a structure* $\langle X, \leq\rangle$ *such that:*

1. *$\langle X, \leq \rangle$ is a Priestley space.*

2. *The set $U \cup X_M$ is clopen, for every $U \in \mathbf{D}(X)$.*

Lemma 23. *Let $\langle X, \leq \rangle$ be a Priestley space and $T = \leq$. Then $\langle X, \leq, T \rangle$ is a γgi-space iff $\langle X, \leq \rangle$ is a γ-Priestley space.*

Proof. Similar to the proof of Lemma 20. □

If $\langle X, \leq \rangle$ is a γ-Priestley space, then the gamma function in $\mathbf{D}(X)$ takes the form $\gamma(U) = U \cup X_M$. We write $\gamma\mathbf{PS}$ for the category whose objects are γ-Priestley spaces, and whose morphisms are morphisms of Priestley spaces $f \colon \langle X_1, \leq_1 \rangle \to \langle X_2, \leq_2 \rangle$ such that $f^{-1}(U \cup Y_M) = f^{-1}(U) \cup X_M$, for each $U \in \mathbf{D}(X_2)$.

Theorem 24. *The category $\gamma\mathbf{PS}$ is dually equivalent to the category $\mathbf{BDL}_\gamma$.*

Proof. It follows from Lemma 22, Lemma 23 and Theorem 18. □

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Sergio Arturo Celani.
CONICET and Departamento de Matemática,
Facultad de Ciencias Exactas, UNCPBA.
Pinto 399,
Tandil (7000),
Argentina.
scelani@exa.unicen.edu.ar

Hernán Javier San Martín.
CONICET and Departamento de Matemática,
Facultad de Ciencias Exactas, UNLP.
Casilla de correos 172,
La Plata (1900),
Argentina.
hsanmartin@mate.unlp.edu.ar