Packing and Hausdorff measures of Cantor sets associated with series

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Abstract

We study a generalization of Morán's sum sets, obtaining information about the h-Hausdorff and h-packing measures of these sets and certain of their subsets.

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1 Introduction

In [Mo 89] Morán introduced the notion of a sum set,

$$C_a = \left\{ \sum_{i=1}^{\infty} \varepsilon_i a_i : \varepsilon_i = 0, 1 \right\},$$

the set of all possible subsums of the series $\sum a_n$ where $a = (a_n)$ is a sequence of vectors in $\mathbb{R}^p$ with summable norms. The classical Cantor middle-third set is one example with $a_i = 3^{-i}2$. Assuming a suitable separation condition, in [Mo 94] Morán related the h-Hausdorff measure of C_a to the quantities $R_n = \sum_{i>n} \|a_i\|$.

In this paper, we generalize Morán's sum set notion to permit a greater diversity in the geometry. (See (1) for the definition of the generalization.) For example, our generalization includes Cantor-like sets in $\mathbb{R}$ which have the property that the Cantor intervals of a given level (but not necessarily the gaps)

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are all of the same length. Moreover, unlike Morán's sets, our generalized sum sets can have Hausdorff dimension greater than one.

We obtain the analogue of Morán's results on h-Hausdorff measures for these generalized sum sets and prove dual results for h-packing measures. We show that for any of these sum sets there is a doubling dimension function h for which the sum set has both finite and positive h-Hausdorff and h-packing measure. We give formulas for the Hausdorff and packing dimensions, and show that given any α less than the Hausdorff dimension (or β less than the packing dimension) there is a sum subset that has Hausdorff dimension α (or packing dimension β). In fact, there is even a sum subset with both Hausdorff dimension α and packing dimension β provided α/β is dominated by the ratio of the Hausdorff dimension to the packing dimension of the original set. Furthermore, if the Hausdorff and/or packing measure is finite and positive (in the corresponding dimension), then we can choose this sum subset to have finite and positive Hausdorff and/or packing measure.

2 Preliminaries

Let $s_n > 0$ with $\sum_n s_n < \infty$. Fix $N \in \mathbb{N}$ and for each $n \in \mathbb{N}$ let the *nth digit set* $\mathcal{D}^n = \{0 = d_1^n, d_2^n, \ldots, d_N^n\} \subset \mathbb{R}^p$ be given. We define $C_{s,\mathcal{D}}$ by

$$C_{s,\mathcal{D}} = \{\sum_{i=1}^{\infty} s_i b_i : b_i \in \mathcal{D}^i\}, \tag{1}$$

the set of all possible sums with choices drawn from $\mathcal{D}^n$ and scaled by s_n. Morán's sum set is the special case when $s_i = ||a_i||$, $N = 2$ and $\mathcal{D}^i = \{0, a_i/||a_i||\}$. This generalized sum set is the main object of study in this paper.

For each n define

$$\kappa_n = \max\{\|d_i^n - d_j^n\| : 0 \le i, j \le N, i \ne j\}$$

and

$$\tau_n = \min\{\|d_i^n - d_j^n\| : 0 \le i, j \le N, i \ne j\}.$$

In Morán's case, $\kappa_n = \tau_n = 1$. We assume that $\kappa := \sup_n \kappa_n < \infty$, as well as $\tau := \inf \tau_n > 0$; the intent is that the sequence s_n controls the decay rate, not the (possibly varying) geometry of the digit sets $\mathcal{D}^n$. In addition, we assume the rapid decay condition

$$\sup_n \frac{\kappa R_n}{\tau s_n} = M < 1, \tag{2}$$

where $R_n = \sum_{i>n} s_i$. This is the analogue of Morán's separation condition. The quantity R_n is very important for describing the geometry of $C_{s,\mathcal{D}}$.

In certain situations where we have precise information about the geometry of $\mathcal{D}^n$, it is possible to assume something weaker than (2) and still have a suitable separation property to allow for dimensions to be calculated; see Example 8.

Example 1. *1. A very simple example is the classical Cantor set with $s_n = 2 \cdot 3^{-n}$ and $\mathcal{D} = \{0, 1\}$.*

2. Consider a finite set $\mathcal{D} \subset \mathbb{R}^p$, a real number $r < d/(2D)$ (where $d = \min \tilde{\mathcal{D}}$, $D = \max \tilde{\mathcal{D}}$ and $\tilde{\mathcal{D}} = \{\|d - d'\| : d, d' \in \mathcal{D}, d \neq d'\}$), a matrix $O \in \mathbb{R}^{p \times p}$ orthogonal and the contractions $S_d(x) = rO(x + d)$. The attractor of this IFS is $C_{s,D}$ with $s_n = r^n$ and $\mathcal{D}^n = O^n \mathcal{D}$.

We now examine some basic properties of $C_{s,\mathcal{D}}$. First we argue that $C_{s,\mathcal{D}}$ is a compact and perfect set. To do this, let $\Xi = \{1, 2, \ldots, N\}^{\mathbb{N}}$ with the product topology induced by the discrete topology on each factor. Further, for $n \in \mathbb{N}$ let $\Xi^n = \{1, \ldots, N\}^n$. We note that Ξ is a totally disconnected, perfect metric space. Define the function $\Phi : \Xi \to \mathbb{R}^p$ by

$$\Phi(\sigma) = \sum_i s_i d^i_{\sigma_i}.$$

Then the range of Φ is $C_{s,\mathcal{D}}$. Since Ξ is compact and perfect, we need only show that Φ is continuous and injective to show that $C_{s,\mathcal{D}}$ is compact and perfect. Let $\Phi_n : \Xi \to \mathbb{R}^p$ be defined by $\Phi_n(\sigma) = \sum_{i \le n} s_i d^i_{\sigma_i}$. Then Φ_n is constant on each of the sets $\Xi_\alpha = \{\sigma \in \Xi : \sigma_i = \alpha_i, 1 \le i \le n\}$ for any fixed $\alpha \in \Xi^n$. This means that each Φ_n is continuous. Furthermore, $\|\Phi_n(\sigma) - \Phi(\sigma)\| \le \kappa R_n$ and thus $\Phi_n \to \Phi$ uniformly on Ξ and so Φ is also continuous. Thus $C_{s,\mathcal{D}}$ is compact.

If n is the first place where σ and σ' disagree,

$$\begin{aligned} \|\Phi(\sigma) - \Phi(\sigma')\| &= \|\sum_i s_i(d^i_{\sigma(i)} - d^i_{\sigma'(i)})\| \\ &\ge \|s_n(d^n_{\sigma(n)} - d^n_{\sigma'(n)})\| - \|\sum_{i>n} s_i(d^i_{\sigma(i)} - d^i_{\sigma'(i)})\| \\ &\ge s_n\tau - \kappa R_n > 0. \end{aligned} \tag{3}$$

This means that Φ is injective and is thus a homeomorphism, so that $C_{s,\mathcal{D}}$ is also totally disconnected and perfect.

For a given $n \in \mathbb{N}$ and $\sigma \in \Xi^n$, we define

$$x_\sigma = \sum_{i \le n} s_i d^i_{\sigma_i}$$

and

$$C_{\sigma,n} = x_\sigma + \{\sum_{i>n} s_i b_i : b_i \in \mathcal{D}^i\}.$$

Our condition (2) ensures the non-overlapping of the sets $C_{\sigma,n}$.

Using this notation, we see two very important facts. First, $C_{\sigma,n} = x_\alpha - x_\sigma + C_{\alpha,n}$ for any $\sigma, \alpha \in \Xi^n$. That is, for a fixed n the collection of $C_{\sigma,n}$ are all translates of each other. Secondly, we can decompose $C_{s,\mathcal{D}}$ into N^n copies of $C_{\sigma,n}$ as

$$C_{s,\mathcal{D}} = \bigcup_{\sigma \in \Xi^n} C_{\sigma,n} = \{x_\sigma : \sigma \in \Xi^n\} + C_{1,n},$$

where by $1 \in \Xi^n$ we mean the element all of whose terms equal to 1.

An elementary estimate gives that

$$|C_{\sigma,n}| \leq \| \sum_{i>n} s_i b_i - \sum_{i>n} s_i b'_i \| \leq \kappa \sum_{i>n} s_i = \kappa R_n \tag{4}$$

where $|C|$ means the diameter of the set C.

3 Hausdorff and packing measures

We first recall some facts about Hausdorff and packing measures (see [Ro 98, Ma 95]). For us, a *dimension function* is a continuous non-decreasing function $h : [0, \infty) \to [0, \infty)$ with $h(0) = 0$. It is said to be *doubling* if there is some constant $c > 0$ so that $h(2x) \leq c\, h(x)$ for all $x > 0$.

For two dimension function f, g we say that $f \prec g$ if

$$\lim_{t \to 0+} g(t)/f(t) = 0.$$

For each $\delta > 0$, a δ-covering of a set E is a countable collection $\{B_i\}$ of subsets of $\mathbb{R}^p$ with diameters dominated by δ, that is $|B_i| \leq \delta$, and for which $E \subseteq \cup_i B_i$. We define

$$\mathcal{H}^h_\delta(E) = \inf\{\sum_i h(|B_i|) : \{B_i\} \text{ is a } \delta\text{-covering of } E\}$$

and the *Hausdorff h-measure* as

$$\mathcal{H}^h(E) = \lim_{\delta \to 0} \mathcal{H}^h_\delta(E).$$

Notice that in the definition of $\mathcal{H}^h_\delta$ it is sufficient to consider coverings by balls.

Now we turn to the h-packing measure $\mathcal{P}^h$. A δ-packing of a set E is a disjoint family of open balls $\{B(x_i, r_i)\}$ with $x_i \in E$ and $r_i \leq \delta$. The h-packing pre-measure is given by

$$\mathcal{P}^h_0(E) = \lim_{\delta \to 0} \mathcal{P}^h_\delta(E)$$

where

$$\mathcal{P}^h_\delta = \sup\{\sum_i h(|B_i|) : \{B_i\} \text{ is a } \delta\text{-packing of } E\}.$$

Unfortunately $\mathcal{P}^h_0$ is not a measure as it is in general not countably additive. Thus we need one more step to construct the packing measure $\mathcal{P}^h$,

$$\mathcal{P}^h(E) = \inf\{\sum_i \mathcal{P}^h_0(E_i) : E \subset \bigcup_i E_i\}.$$

The next Theorem gives estimates for the Hausdorff and packing measures of $C_{s,\mathcal{D}}$. The first two claims about the Hausdorff measure of $C_{s,\mathcal{D}}$ are given in [Mo 94] for the special case of $\mathcal{D}^n$ containing two digits.

Theorem 2. *Suppose that h is a doubling dimension function.*

1. *If $\liminf N^n h(\kappa R_n) = \alpha$ then $\mathcal{H}^h(C_{s,\mathcal{D}}) \leq \alpha$.*
2. *If $\liminf N^n h(\kappa R_n) = \alpha > 0$ then $\mathcal{H}^h(C_{s,\mathcal{D}}) > 0$.*
3. *If $\limsup N^n h(\kappa R_n) = \alpha < \infty$ then $\mathcal{P}^h(C_{s,\mathcal{D}}) \leq N\alpha$.*
4. *If $\limsup N^n h(\kappa R_n) = \alpha > 0$ then $\mathcal{P}^h(C_{s,\mathcal{D}}) > 0$.*

Remark 3. *If h is doubling and $0 < \liminf N^n h(R_n) < \infty$, then $0 < \mathcal{H}^h(C_{s,\mathcal{D}}) < \infty$ and so $C_{s,\mathcal{D}}$ is an h-Hausdorff set. Similarly, if $\limsup N^n h(R_n)$ is positive and finite, then $C_{s,\mathcal{D}}$ is an h-packing set. Finally, if $\liminf N^n h(R_n)$ is positive and $\limsup N^n h(R_n)$ is finite, then $C_{s,\mathcal{D}}$ is both an h-Hausdorff and an h-packing set.*

Proof. Item 1) is trivial by considering the covering $C_{I,n}$ for all $I \in \{0, 1, \ldots, N\}^n$, which consists of N^n sets all of diameter at most κR_n.

To prove the rest of the statements, we will use the fact that there is a Borel measure μ supported on $C_{s,\mathcal{D}}$ for which $\mu(C_{\sigma,n}) = N^{-n}$ for each σ and n. This measure is often called the *natural probability measure*.

2): Let $\beta < \alpha$ so that we have $N^n h(\kappa R_n) > \beta$ for all large n. Now choose $x \in C_{s,\mathcal{D}}$ and $\delta > 0$ and let n be such that $\kappa R_n < \delta \leq \kappa R_{n-1}$. By a simple modification of Lemma 2 in [Mo 89] there is a $q \in \mathbb{N}$ so that the number of $C_{\sigma,n}$ which intersect $B(x, \delta)$ is less than q (independent of B and δ). (This is where the condition (2) is used.) But then we have

$$\mu(B(x,\delta)) \leq q\mu(C_{\sigma,n}) = q\, N^{-n} < \frac{q}{\beta} h(\kappa R_n) < \frac{q h(\delta)}{\beta}.$$

By the mass distribution principle (see [Fal 86]), we have $\mathcal{H}^h(C_{s,\mathcal{D}}) \geq \alpha/q$.

3) Let $\beta > \alpha$ so that we have $N^n h(\kappa R_n) < \beta$ for all large n. Now choose $x \in C_{s,\mathcal{D}}$ and $\delta > 0$ and let n be such that $\kappa R_n < \delta \leq \kappa R_{n-1}$. We know that $x \in C_{\sigma,n}$ for some σ and, since $|C_{\sigma,n}| \leq \kappa R_n < \delta$, we have that $C_{\sigma,n} \subseteq B(x, \kappa R_n) \subseteq B(x, \delta)$. But then

$$\mu(B(x,\delta)) \geq \mu(C_{\sigma,n}) = N^{-n} = \frac{N^{-(n-1)}}{N} > \frac{h(\kappa R_{n-1})}{N\beta} \geq \frac{h(\delta)}{N\beta},$$

since h is a nondecreasing function. But then we have that

$$\liminf \mu(B(x,\delta))/h(\delta) \geq (N\alpha)^{-1}$$

and so $\mathcal{P}^h(C_{s,\mathcal{D}}) \leq N\alpha$ by Theorem 3.16 in [C 95].

4) Let $0 < \beta < \alpha$. Then there are n_j so that $N^{n_j} h(\kappa R_{n_j}) > \beta$ for all j. Let $x \in C_{s,\mathcal{D}}$ be given. For any j we have $x \in C_{\sigma_j, n_j}$ for some σ_j. By the same simple modification of Lemma 2 in [Mo 89], there is a $q \in \mathbb{N}$ so that for any

$\delta > 0$ and any ball B of radius δ, if $m \in \mathbb{N}$ is the smallest value with $\kappa R_m < \delta$ then the number of $C_{I,m}$ which intersect B is less than q (independent of B and δ). Let $\delta = \kappa R_{n_j - 1}$, so $\kappa R_{n_j} < \delta = \kappa R_{n_j - 1}$. Then

$$\mu(B(x, \kappa R_{n_j})) \leq \mu(B(x, \delta)) \leq q\, \mu(C_{\sigma_j, n_j}) = q\, N^{-n_j} < q\, h(\kappa R_{n_j})/\beta$$

and thus $\liminf \mu(B(x,\delta))/h(\delta) \leq q/\alpha$. By Theorem 3.16 in [C 95], it follows that $\mathcal{P}^h(C_{s,\mathcal{D}}) \geq c\alpha/q$, where c is the doubling constant for h. □

Remark 4. *Since $\kappa s_{n+1} < \kappa R_n \leq M\tau s_n < \kappa s_n$, for any doubling dimension function h, we could instead relate the two quantities, $\liminf N^n h(s_n)$ and $\limsup N^n h(s_n)$, to the h-Hausdorff and h-packing measure of $C_{s,\mathcal{D}}$.*

Theorem 5. *For any sequence s_n and collections of digits $\mathcal{D}^n$ which satisfy (2), there is a doubling dimension function h for which $C_{s,\mathcal{D}}$ is simultaneously both an h-Hausdorff set and an h-packing set.*

Proof. Following the pattern in [CMMS 04, Section 5], we define the function $h : [0, \kappa R_0] \to \mathbb{R}$ by $h(0) = 0$ and $h(x) = 1/f^{-1}(x)$ where $f(x)$ is given by

$$f(x) = \kappa R_n + \frac{\kappa R_{n+1} - \kappa R_n}{N^{n+1} - N^n}(x - N^n), \quad x \in [N^n, N^{n+1}).$$

Clearly h is non-decreasing and continuous, so we only need to show that h is doubling. For $x > 0$, let $n, m \in \mathbb{N}$ be such that $\kappa R_{m+1} < x \leq \kappa R_m < \kappa R_n \leq 2x < \kappa R_{n-1}$. Then $\kappa R_i \leq \tau M s_i \leq \kappa \frac{\tau M}{\kappa} R_{i-1}$ for all i. Letting $\theta = \tau M/\kappa < 1$,

$$\theta^{n-m} \leq \frac{\kappa R_n}{\kappa R_m} < \frac{2x}{x} = 2$$

and so we have $m - n \leq -\ln(2)/\ln(\theta)$. As $f(N^j) = \kappa R_j$,

$$\frac{h(2x)}{h(x)} = \frac{f^{-1}(x)}{f^{-1}(2x)} \leq \frac{N^{m+1}}{N^{n-1}} \leq N^{2 - \ln(2)/\ln(\theta)},$$

and so h is doubling.

Since $N^n h(\kappa R_n) = 1$ for all n, we have $C_{s,\mathcal{D}}$ is an h-Hausdorff set and an h-packing set for this dimension function h, as desired. □

The next theorem is a simple consequence of some known results. However, it shows that the set of dimensional subsets of $C_{s,\mathcal{D}}$ is an initial segment in the partially ordered set of all doubling dimension functions.

Theorem 6. *Let f, h be doubling dimension functions and assume $f \prec h$.*

1. *If $0 < \mathcal{H}^h(C_{s,\mathcal{D}}) < \infty$, then for any $t > 0$ there is a compact and perfect subset $E \subset C_{s,\mathcal{D}}$ so that $\mathcal{H}^f(E) = t$.*

2. *If $0 < \mathcal{P}^h(C_{s,\mathcal{D}}) < \infty$, then for any $t > 0$ there is a compact and perfect subset $E \subset C_{s,\mathcal{D}}$ so that $\mathcal{P}^f(E) = t$.*

Proof. 1. From Theorem 40 in [Ro 98], we have that $\mathcal{H}^f(C_{s,\mathcal{D}}) = \infty$. Then by Theorem 2 in [La 67] there is some closed subset $E' \subset C_{s,\mathcal{D}}$ for which $\mathcal{H}^f(E') = t$. As E' is a closed subset of a perfect set, it is the union of a perfect set E and a countable set, so $\mathcal{H}^f(E) = \mathcal{H}^f(E') = t$ and E is a perfect subset of $C_{s,\mathcal{D}}$.

2. By the same argument as Theorem 40 in [Ro 98], but adapted to packing measures, we have that $\mathcal{P}^f(C_{s,\mathcal{D}}) = \infty$. Now, if we obtain a closed subset $E' \subset C_{s,\mathcal{D}}$ for which $\mathcal{P}^f(E') = t$, then we find a perfect subset in a similar way to the case 1 before. In [JP 95], Joyce and Preiss proved that if a set has infinite h-packing measure (for any given $h \in \mathcal{D}$), then the set contains a compact subset with finite h-packing measure. With a simple modification of their proof, (in particular their Lemma 6), we obtain a set of finite packing measure greater than t. By Lyapunov's convexity theorem, there is a subset whose h-packing measure is exactly t [Ru 91, Theorem 5.5]. □

We now specialize to the "usual" dimension functions $h_s(x) = x^s$ and let $\dim_H$ and $\dim_P$ denote the "usual" Hausdorff and packing dimension. In analogy with the case of a "cut-out" Cantor subset of $\mathbb{R}$ (see [BT 54, CMMS 04, GMS 07], we have the following Proposition.

Proposition 7. *We have that*

$$\dim_H(C_{s,\mathcal{D}}) = \liminf \frac{-n\ln(N)}{\ln(s_n)} \quad \textit{and} \quad \dim_P(C_{s,\mathcal{D}}) = \limsup \frac{-n\ln(N)}{\ln(s_n)}.$$

Proof. First, we note that

$$\liminf \frac{-n\ln(N)}{\ln(\kappa R_n)} = \liminf \frac{-n\ln(N)}{\ln(R_n)} = \liminf \frac{-n\ln(N)}{\ln(s_n)},$$

with a similar equality for the limit superior.

If $\beta > \alpha := \liminf \frac{-n\ln(N)}{\ln(\kappa R_n)}$, then there is a subsequence (n_j) so that $N^{n_j}(\kappa R_{n_j})^\beta < 1$. Thus $\liminf N^n(\kappa R_n)^\beta < 1$ and so $\dim_H(C_{s,\mathcal{D}}) \le \alpha$ by Theorem 2.

Conversely, if $\gamma < \alpha$, then for large n we have $N^n(R_n)^\gamma > 1$ and thus $\liminf N^n(R_n)^\gamma > 1$ and so $\dim_H(C_{s,\mathcal{D}}) \ge \alpha$ by Theorem 2.

The proof for packing dimension is similar. □

Example 8. *For any $\alpha \in [0,p)$, it is possible to construct a sum set, $C_{s,\mathcal{D}} \subset \mathbb{R}^p$, with $\dim_H(C_{s,\mathcal{D}}) = \alpha$. The simplest way of doing this is to choose $\mathcal{D}^n = \{(\epsilon_1, \epsilon_2, \ldots, \epsilon_p) : \epsilon_i \in \{0,1\}\}$, the set of all corners of a p-dimensional unit cube, and set $s_n = \lambda^n$ where $\lambda = 2^{-p/\alpha}$. This will generate a self-similar set, $C_{s,\mathcal{D}}$, that is a product of classical Cantor sets. The problem is that condition (2) requires that $\lambda < 1/(1+\sqrt{p})$, which does not allow the full range of dimensions (and, in fact, gets worse as p increases). However, from the simple geometry of this example, we can see that the sets $C_{\sigma,n}$ are non-overlapping provided $s_n > R_n$. Under this (weaker) assumption, $C_{s,\mathcal{D}}$ is a self-similar set satisfying the open set condition and hence its dimensions are as stated in the previous proposition. This separation condition allows for any $\lambda \in [0,1/2)$.*

In the case of the "usual" dimension functions h_s, Theorem 6 has a stronger form in that not only is there a Cantor subset with the correct dimension but this subset corresponds to all the subsums of a subsequence of (s_n).

Theorem 9. *Suppose that* $\dim_H(C_{s,\mathcal{D}}) = A$ *and* $dim_P(C_{s,\mathcal{D}}) = B$*. Then for any* $0 \leq \alpha \leq A$ *and* $0 \leq \beta \leq B$*, with* $\alpha/A \leq \beta/B$*, there is a subsequence* (t_n) *of* (s_n) *such that* $\dim_H(C_{t,\mathcal{D}}) = \alpha$ *and* $\dim_P(C_{t,\mathcal{D}}) = \beta$*.*

Proof. We will assume $0 < \alpha < A$, $0 < \beta < B$ and leave the details of the endpoint cases for the reader. Choose n_i and m_i to be disjoint sequences of indices such that

$$\lim_i \frac{-n_i \ln(N)}{\ln(s_{n_i})} = A \quad \text{and} \quad \lim_i \frac{-m_i \ln(N)}{\ln(s_{m_i})} = B.$$

If necessary, we take subsequences in order to assure that $n_1 \geq 100$, $m_i \geq 2^{n_i}$, and $n_{i+1} \geq 2^{m_i}$. To obtain the new sequence t_k, we remove terms from s_n in segments, each in a "uniform" manner with some density $\xi \in (0,1)$. To explain, suppose the segment is the set of indices $\{q, q+1, \ldots, \ell\} \subset \mathbb{N}$. Then to uniformly remove terms with density ξ from this segment, we remove all the terms of the form $q + \lfloor i/\xi \rfloor$ for $i = 0, \ldots, \lfloor \xi(\ell - q) - 1 \rfloor$ (to make sure we do not remove ℓ). Note that removing with density ξ is the same as retaining with density $1 - \xi$.

From the set of indices $\{1, 2, \ldots, n_1\}$, we remove terms in a "uniform" way with density $1 - \frac{\alpha}{A}$. Then from the set of indices $\{n_1 + 1, \ldots, m_1\}$ we remove terms in a "uniform" way with density $1-\frac{\beta}{B}$. We continue alternating, removing terms with density $1 - \frac{\alpha}{A}$ from $\{m_i + 1, \ldots, n_{i+1}\}$ and with density $1 - \frac{\beta}{B}$ from $\{n_i + 1, \ldots, m_i\}$. Call the resulting sequence t_ℓ where we have $t_\ell = s_n$, with $\ell = n\Theta(n)$ where $\Theta : \mathbb{N} \to [\alpha/A, \beta/B]$ is a measure of the "local scaling" of the index. From the construction we have $\Theta(n_j) \approx \alpha/A$, $\Theta(m_j) \approx \beta/B$, Θ is increasing on $\{n_i + 1, \ldots, m_i\}$ and decreasing on $\{m_i + 1, \ldots, n_{i+1}\}$. Further,

$$\frac{-\ell \ln(N)}{\ln(t_\ell)} = \theta(n) \frac{-n \ln(N)}{\ln(s_n)}.$$

From here it is straightforward to show that $\liminf \frac{-\ell \ln(N)}{\ln(t_\ell)} = \alpha$ and also that $\limsup \frac{-\ell \ln(N)}{\ln(t_\ell)} = \beta$, as desired. The condition $\alpha/A \leq \beta/B$ is used to check the new liminf and limsup. Since the original sequence satisfies condition (2), it is easy to see that any subsequence will as well. □

Of course, this construction does not guarantee that $C_{t,\mathcal{D}}$ will satisfy $0 < \mathcal{H}^\alpha(C_{t,\mathcal{D}}) < \infty$ even if it has the proper dimension. Comparing Theorem 6 with Theorem 9, we trade the ability to specify the $\mathcal{H}^t$-measure of the subset with the ability to ensure that the subset is of a particularly nice form, in Theorem 9 being the full set of subsums of some subsequence. However, if we assume a bit more on $C_{t,\mathcal{D}}$ we can obtain a substantially stronger result.

Theorem 10.

1. *Suppose that* $0 < \mathcal{H}^A(C_{s,\mathcal{D}}) < \infty$. *Then for any* $0 \le a \le A$ *there is a subsequence* (t_n) *of* (s_n) *such that* $0 < \mathcal{H}^a(C_{t,\mathcal{D}}) < \infty$.

2. *Suppose that* $0 < \mathcal{P}^B(C_{s,\mathcal{D}}) < \infty$. *Then for any* $0 \le b \le B$ *there is a subsequence* (t_n) *of* (s_n) *such that* $0 < \mathcal{P}^b(C_{t,\mathcal{D}}) < \infty$.

3. *Suppose that* $0 < \mathcal{H}^A(C_{s,\mathcal{D}}) < \infty$ *and* $0 < \mathcal{P}^B(C_{s,\mathcal{D}}) < \infty$. *Then for any* $0 \le a \le A$ *and* $0 \le b \le B$ *with* $a/A \le b/B$, *there is a subsequence* (t_n) *of* (s_n) *such that* $0 < \mathcal{H}^a(C_{t,\mathcal{D}}) < \infty$ *and* $0 < \mathcal{P}^b(C_{t,\mathcal{D}}) < \infty$.

Proof. We prove the third statement as it is the most involved. The other two are similar. As in Theorem 9 we work with s_n rather than R_n.

Let $m_i, n_i \in \mathbb{N}$ be such that $\lim N^{m_j} s_{m_j}^B = \limsup N^n s_n^B = S$ and $\lim N^{n_j} s_{n_j}^A = \liminf N^n s_n^A = I$. In addition, we assume that $n_j < m_j < n_{j+1} < m_{j+1}$, $m_j/n_j \to \infty$, and $n_{j+1}/m_j \to \infty$. The two cases $a/A = b/B$ and $a/A < b/B$ require different techniques and so we do them separately.

Case 1: $a/A = b/B$

If $a/A = 1$, then there is nothing to prove. We define our subsequence (t_n) by defining the indexing function $\pi : \mathbb{N} \to \mathbb{N}$ such that $t_n = s_{\pi(n)}$. Define $\hat{\pi} : \mathbb{N} \to \mathbb{N}$ by $\hat{\pi}(i) = \lfloor (A/a)i \rfloor$. If $m_j, n_j \in \hat{\pi}(\mathbb{N})$ for all j, then we let $\pi = \hat{\pi}$. Otherwise, suppose that $m_j \notin \hat{\pi}(\mathbb{N})$. Then $i := \lfloor m_j a/A \rfloor < m_j a/A$, so we define $\pi(i) = m_j$. We do the same procedure for any $n_j \notin \hat{\pi}(\mathbb{N})$. Since $A/a > 1$ we know that $\hat{\pi}$ is injective. If we assume that $|n_j - m_k| > 2A/a$ for all j and k then π is also guaranteed to be injective. Since $[k, k + A/a + 1] \cap \hat{\pi}(\mathbb{N})$ is nonempty for any k, we know that $-1 \le \pi(i) - (A/a)i \le A/a + 1$ or, more useful for us,

$$-1 - \frac{a}{A} + \pi(i)\left(\frac{a}{A}\right) \le i \le \frac{a}{A} + \pi(i)\left(\frac{a}{A}\right).$$

This means that

$$\begin{aligned} N^i t_i^a &= N^i s_{\pi(i)}^a \ge N^{-1-a/A} N^{\pi(i)(a/A)} s_{\pi(i)}^{(a/A)A} = N^{-1-a/A}\left(N^{\pi(i)} s_{\pi(i)}^A\right)^{a/A} \\ &\ge N^{-1-a/A}(I-\epsilon)^{a/A} > 0, \end{aligned}$$

for large enough i. Thus $\liminf N^i t_i^a > 0$ and so $\mathcal{H}^a(C_{t,\mathcal{D}}) > 0$. By construction, there is a sequence $q_j \in \mathbb{N}$ so that $\pi(q_j) = n_j$ and so

$$N^{q_j} t_{q_j}^a \le N^{a/A} N^{\pi(q_j)(a/A)} s_{n_j}^{(a/A)A} = N^{a/A}\left(N^{n_j} s_{n_j}^A\right)^{a/A} \le N^{a/A}(I+\epsilon)^{a/A} < \infty.$$

Thus $\mathcal{H}^a(C_{t,\mathcal{D}}) < \infty$ as well. The proof that $0 < \mathcal{P}^b(C_{t,\mathcal{D}}) < \infty$ is similar.

Case 2: $a/A < b/B$

Let

$$\gamma_0 = \frac{\frac{B}{b} - 1}{\frac{A}{a} - 1}$$

and then choose $\delta > 0$ so that $\gamma := \gamma_0 + \delta < 1$. Define

$$n_j' = \left\lfloor \frac{n_j}{\gamma} \right\rfloor \quad \text{and} \quad m_j' = \lfloor \gamma m_j \rfloor$$

and notice that $n'_j > n_j$ and $m'_j < m_j$. For notational ease, let

$$P_j = m_j - \left\lceil (1 - \frac{b}{B}) m_j \right\rceil, \qquad P'_j = m'_j - \left\lceil (1 - \frac{b}{B}) m'_j \right\rceil,$$

$$Q_j = n_j - \left\lfloor (1 - \frac{a}{A}) n_j \right\rfloor, \quad \text{and} \quad Q'_j = n'_j - \left\lfloor (1 - \frac{a}{A}) n'_j \right\rfloor.$$

Further, let

$$d_j = \frac{P'_j + \lceil (1 - \frac{b}{B}) m_j \rceil - (Q'_j + \lfloor (1 - \frac{a}{A}) n_j \rfloor)}{P'_j - Q'_j}$$

and

$$e_j = \frac{Q_{j+1} + \lfloor (1 - \frac{a}{A}) n_{j+1} \rfloor - (P_j + \lceil (1 - \frac{b}{B}) m_j \rceil)}{Q_{j+1} - P_j}.$$

Define $\pi : \mathbb{N} \to \mathbb{N}$ by

$$\pi(i) = \begin{cases} i + \lfloor (1 - \frac{a}{A}) n_j \rfloor & \text{if } Q_j \leq i < Q'_j \\ i + \lceil (1 - \frac{b}{B}) m_j \rceil & \text{if } P'_j < i \leq P_j \\ Q'_j + \lfloor (1 - \frac{a}{A}) n_j \rfloor + \lfloor k d_j \rfloor & \text{if } i = Q'_j + k, k = 0, \ldots, P'_j - Q'_j \\ P_j + \lceil (1 - \frac{b}{B}) m_j \rceil + \lceil k e_j \rceil & \text{if } i = P_j + k, k = 1, \ldots, Q_{j+1} - P_j - 1. \end{cases}$$

We define $t_i = s_{\pi(i)}$. The choice of n_j, m_j, and γ ensure that $Q_j < Q'_j < P'_j < P_j < Q_{j+1}$. We also have $\pi(Q_j) = n_j$ and $\pi(P_j) = m_j$. It is straightforward but quite tedious to check that π is injective and also that

$$\frac{B}{b} \leq \frac{\pi(i)}{i} \leq \frac{A}{a} \tag{5}$$

for all large i. We remark that the strict inequality $a/A < b/B$ is necessary in order to show (5) for the last two cases in the definition of π.

Thus for $\epsilon > 0$ small and all large enough i, we have

$$N^i t_i^a = N^i s_{\pi(i)}^a \geq N^{\pi(i)(a/A)} s_{\pi(i)}^{(a/A)A} = \left(N^{\pi(i)} s_{\pi(i)}^A \right)^{a/A} \geq (I - \epsilon)^{a/A} > 0,$$

and thus $\mathcal{H}^a(C_{t,\mathcal{D}}) > 0$. For $i = Q_j$, we have $\pi(i) = n_j$ and $Q_j \leq (a/A) n_j + 1$ and so

$$N^i t_i^a = N^{Q_j} s_{n_j}^a \leq N^{(a/A) n_j} s_{n_j}^a N = N \left(N^{n_j} s_{n_j}^A \right)^{a/A} \leq N (I + \epsilon)^{a/A} < \infty$$

and so $\mathcal{H}^a(C_{t,\mathcal{D}}) < \infty$. The argument that $0 < \mathcal{P}^b(C_{t,\mathcal{D}}) < \infty$ is similar. □

Example 11. *The simple Example 1 will show that in general we cannot find a subsequence which will give a subset of arbitrary measure. Recall that it was* $s_n = 2/3^n$ *and* $\mathcal{D}^n = \{0, 1\}$ *for all* n*. It is known that* $\dim_H C_{s,\mathcal{D}} = d = \ln(2)/\ln(3)$ *and* $\mathcal{H}^d(C_{s,\mathcal{D}}) = 1$*. The key observation is that if* t_n *is a subsequence of* s_n *constructed by removing only* K *terms from* s_n*, then* $\mathcal{H}^d(C_{t,\mathcal{D}}) = 2^{-K}$*, since* $C_{s,\mathcal{D}}$ *is the union of* 2^K *disjoint copies of* $C_{t,\mathcal{D}}$ *(these copies correspond to the possible subsums of the removed terms). But this means that it is impossible to find a subsequence* t_n *with* $\mathcal{H}^d(C_{t,\mathcal{D}}) = 1/3$.

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