

Frobenius property for fusion categories of small integral dimension

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Let k be an algebraically closed field of characteristic zero. In this paper, we prove that fusion categories of Frobenius–Perron dimensions 84 and 90 are of Frobenius type. Combining this with previous results in the literature, we obtain that every weakly integral fusion category of Frobenius–Perron dimension less than 120 is of Frobenius type.

Keywords: Fusion category; Frobenius property; Frobenius–Perron dimension.

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1. Introduction and Main Results

A fusion category \mathcal{C} is called of *Frobenius type* if for every simple object X of \mathcal{C} , the Frobenius–Perron dimension of X divides the Frobenius–Perron dimension of \mathcal{C} , that is, the ratio $\text{FPdim } \mathcal{C} / \text{FPdim } X$ is an algebraic integer. A classical result

of Frobenius asserts that if \mathcal{C} is the category of finite-dimensional representations of a finite group, then \mathcal{C} is of Frobenius type. Kaplansky conjectured that the representation category of every finite-dimensional semisimple Hopf algebra is of Frobenius type; this is the sixth one of ten questions posed by Kaplansky in 1975 [11, Appendix 2]. In general, the conjecture is still open.

A more general related question was raised in [6, Question 1]: it was asked in that paper if there exists a fusion category \mathcal{C} which does not have the *strong Frobenius property*, that is, such that there is a simple object in an indecomposable module category over \mathcal{C} whose Frobenius–Perron dimension does not divide the Frobenius–Perron dimension of \mathcal{C} . Recall from [6] that a fusion category \mathcal{C} is called *weakly group-theoretical* if it is tensor Morita equivalent to a nilpotent fusion category. By [6, Theorem 1.5] every weakly group-theoretical fusion category does have the strong Frobenius property. Then every such fusion category is of Frobenius type.

In Theorems 4.1 and 4.2, we study types for integral fusion categories of Frobenius–Perron dimensions 84 and 90. Combining this with the results of the paper [6] for fusion categories of Frobenius–Perron dimensions $p^a q^b$, pqr and 60, we obtain the following theorem.

Theorem 1.1. *Let \mathcal{C} be a fusion category of integer Frobenius–Perron dimension less than 120. Then \mathcal{C} is of Frobenius type. Furthermore, if $\text{FPdim } \mathcal{C} > 1$ and $\mathcal{C} \not\cong \text{rep } \mathbb{A}_5$, then \mathcal{C} has nontrivial invertible objects.*

Theorem 1.1 will be proved in Sec. 4. Recall that, according to the definition given in [6], a fusion category is called *simple* if it has no nontrivial proper fusion subcategories. As a consequence of Theorem 1.1, if $\text{FPdim } \mathcal{C} \leq 119$ and $\text{FPdim } \mathcal{C} \neq 60$ or p , where p is a prime number, then \mathcal{C} is not simple as a fusion category. Combined with the results of the paper [6], the theorem implies that the only weakly integral simple fusion categories of Frobenius–Perron dimension ≤ 119 are the categories $\text{rep } \mathbb{A}_5$ of finite-dimensional representations of the alternating group \mathbb{A}_5 and the pointed fusion categories $\mathcal{C}(\mathbb{Z}_p, \omega)$ of finite-dimensional \mathbb{Z}_p -graded vector spaces, where p is a prime number, with associativity constraint determined by a 3-cocycle $\omega \in H^3(\mathbb{Z}_p, k^*)$.

The paper is organized as follows. In Sec. 2, we recall some basic definitions and results on fusion categories. Some useful lemmas are also contained in this section. In Sec. 3, we consider integral fusion categories and state several results on the possible simple Frobenius–Perron dimensions that will be used later. Most of them have appeared in the literature in the context of representations of semisimple Hopf algebras. In Sec. 4, we prove our main results on fusion categories of small dimension; the computations in this section are partly handled by a computer.

2. Preliminaries

Throughout this paper, we shall work over an algebraically closed base field k of characteristic zero. Recall that a fusion category over k is a k -linear semisimple

rigid tensor category \mathcal{C} with finitely many isomorphism classes of simple objects, finite-dimensional hom spaces, and such that the unit object $\mathbf{1}$ of \mathcal{C} is simple. We refer the reader to [5] for the main notions about fusion categories used throughout.

Let \mathcal{C} be a fusion category over k . Let also $\text{Irr}(\mathcal{C})$ and $\text{G}(\mathcal{C})$ denote the set of isomorphism classes of simple and invertible objects of \mathcal{C} , respectively. Then $\text{Irr}(\mathcal{C})$ is a basis of the Grothendieck ring $K_0(\mathcal{C})$ of \mathcal{C} and $\text{G}(\mathcal{C})$ is a subgroup of the group of units of $K_0(\mathcal{C})$.

The *Frobenius–Perron dimension* of $x \in \text{Irr}(\mathcal{C})$ is the Frobenius–Perron eigenvalue of the matrix of left multiplication by x in the Grothendieck ring of \mathcal{C} . Thus FPdim extends to a ring homomorphism $\text{FPdim} : K_0(\mathcal{C}) \rightarrow \mathbb{R}$. This is the unique ring homomorphism that takes positive values in all elements of $\text{Irr}(\mathcal{C})$. The Frobenius–Perron dimension of \mathcal{C} is the number $\text{FPdim } \mathcal{C} = \sum_{x \in \text{Irr}(\mathcal{C})} (\text{FPdim } x)^2$.

If X is an object of \mathcal{C} , $\text{FPdim } X$ is the Frobenius–Perron dimension of the class of X in $K_0(\mathcal{C})$. We have $\text{FPdim } X \geq 1$, for all objects X of \mathcal{C} . Moreover $\text{FPdim } X = 1$ if and only if X is an invertible object.

Let $y \in K_0(\mathcal{C})$ and write $y = \sum_{x \in \text{Irr}(\mathcal{C})} m(x, y)x$, where $m(x, y) \in \mathbb{Z}$. The integer $m(x, y)$ is called the *multiplicity* of x in y . This extends to a bilinear form $m : K_0(\mathcal{C}) \times K_0(\mathcal{C}) \rightarrow \mathbb{Z}$. If x and y represent the class of the objects X and Y of \mathcal{C} , respectively, then we have $m(x, y) = \dim \text{Hom}_{\mathcal{C}}(X, Y)$.

Let $x, y, z \in K_0(\mathcal{C})$. Then we have $m(x, y) = m(x^*, y^*)$, and

$$m(x, yz) = m(y^*, zx^*) = m(y, xz^*).$$

Let $x, y \in \text{Irr}(\mathcal{C})$. Then for each $g \in \text{G}(\mathcal{C})$ we have $m(g, xy) = 1$ if and only if $y = x^*g$ and 0 otherwise. In particular, $m(g, xy) = 0$ if $\text{FPdim } x \neq \text{FPdim } y$. Let $x \in \text{Irr}(\mathcal{C})$. Then for all $g \in \text{G}(\mathcal{C})$, $m(g, xx^*) > 0$ if and only if $m(g, xx^*) = 1$ if and only if $gx = x$. The set of isomorphism classes of such invertible objects will be denoted $G[x]$. Thus $G[x]$ is a subgroup of $\text{G}(\mathcal{C})$ of order at most $(\text{FPdim } x)^2$. In particular, for all $x \in \text{Irr}(\mathcal{C})$, we have a relation

$$xx^* = \sum_{g \in G[x]} g + \sum_{y \in \text{Irr}(\mathcal{C}), \text{FPdim } y > 1} m(y, xx^*)y.$$

In fact, the group $\text{G}(\mathcal{C})$ acts on the set $\text{Irr}(\mathcal{C})$ by left multiplication. This action preserves Frobenius–Perron dimensions and, for $x \in \text{Irr}(\mathcal{C})$, $G[x]$ is the stabilizer of x in $\text{G}(\mathcal{C})$.

For every $\alpha \in \mathbb{R}_+$, we shall use the notation $\text{Irr}_\alpha(\mathcal{C})$ to indicate the set of isomorphism classes of simple objects of \mathcal{C} of Frobenius–Perron dimension α . So that $\text{G}(\mathcal{C}) = \text{Irr}_1(\mathcal{C})$. Notice that, if $|\text{Irr}_\alpha(\mathcal{C})| = 1$ for some $\alpha \in \mathbb{R}_+$, and $x \in \text{Irr}_\alpha(\mathcal{C})$, then $G[x] = \text{G}(\mathcal{C})$.

A fusion subcategory of \mathcal{C} is a full tensor subcategory \mathcal{D} such that if X is an object of \mathcal{C} isomorphic to a direct summand of an object Y of \mathcal{D} , then X is in \mathcal{D} . If \mathcal{D} is a fusion subcategory of \mathcal{C} , then \mathcal{D} is in fact a fusion category and $\text{FPdim } \mathcal{D}$ divides $\text{FPdim } \mathcal{C}$, that is, the quotient $\text{FPdim } \mathcal{C} / \text{FPdim } \mathcal{D}$ is an algebraic integer.

Fusion subcategories of \mathcal{C} correspond to *fusion subrings* of the Grothendieck ring of \mathcal{C} , that is, subrings R with the property that for all $a \in R$ and for all $x \in \text{Irr}(\mathcal{C})$ such that $m(x, a) \neq 0$, we have $x \in R$. In other words, R is a subring which is spanned by a subset of $\text{Irr}(\mathcal{C})$. A subset X of $\text{Irr}(\mathcal{C})$ spans a fusion subring of $K_0(\mathcal{C})$ if and only if the product of elements of X decomposes as a sum of elements of X .

The group $G(\mathcal{C})$ of invertible objects of \mathcal{C} generates a fusion subcategory \mathcal{C}_{pt} of \mathcal{C} , which is the unique largest pointed fusion subcategory of \mathcal{C} . In particular, the order of $G(\mathcal{C})$ coincides with $\text{FPdim } \mathcal{C}_{\text{pt}}$ and therefore it divides $\text{FPdim } \mathcal{C}$.

Let G be a finite group. A fusion category \mathcal{C} is called a G -extension of a fusion category \mathcal{D} if it admits a faithful grading $\mathcal{C} = \bigoplus_{g \in G} \mathcal{C}_g$ by the group G , satisfying that the tensor product of \mathcal{C} maps $\mathcal{C}_g \times \mathcal{C}_h \rightarrow \mathcal{C}_{gh}$ and $(\mathcal{C}_g)^* = \mathcal{C}_{g^{-1}}$, and such that the trivial homogeneous component \mathcal{C}_e is equivalent to \mathcal{D} .

Recall from [9] that a fusion category \mathcal{C} admits a canonical faithful grading $\mathcal{C} = \bigoplus_{g \in U(\mathcal{C})} \mathcal{C}_g$, whose trivial component \mathcal{C}_e coincides with the *adjoint fusion subcategory* \mathcal{C}_{ad} . The group $U(\mathcal{C})$ is called the *universal grading group* of \mathcal{C} . Any faithful grading $\mathcal{C} = \bigoplus_{g \in G} \mathcal{C}_g$ by a group G comes from a group epimorphism $U(\mathcal{C}) \rightarrow G$.

Lemma 2.1. *Let \mathcal{C} be a fusion category and let $\mathcal{Z}(\mathcal{C})$ be its Drinfeld center. Consider the group homomorphism $F_0 : G(\mathcal{Z}(\mathcal{C})) \rightarrow G(\mathcal{C})$ induced by the forgetful functor $F : \mathcal{Z}(\mathcal{C}) \rightarrow \mathcal{C}$. Then the following hold:*

- (i) \mathcal{C} is faithfully graded by the group \widehat{N} , where N is the kernel of F_0 .
- (ii) Suppose $U(\mathcal{C}) = 1$. Then the group homomorphism F_0 is injective.

Proof. (i) Observe that F induces by restricting a tensor functor $F : \langle N \rangle \rightarrow \text{Vec}$, where $\langle N \rangle$ is the (pointed) fusion subcategory of $\mathcal{Z}(\mathcal{C})$ generated by N . Therefore, $\langle N \rangle$ is equivalent as a fusion category to the category of representations of the commutative Hopf algebra k^N . Thus part (i) follows from [5, Proposition 5.10].

(ii) The assumption implies that \mathcal{C} admits no faithful group grading. Therefore (ii) follows from (i) (observe that the group of invertible objects of $\mathcal{Z}(\mathcal{C})$ and thus also its subgroup N are abelian, whence $\widehat{N} \cong N$). □

The fusion category \mathcal{C} is called *weakly integral* if $\text{FPdim } \mathcal{C}$ is a natural number. If $\text{FPdim } X \in \mathbb{Z}$, for all object X of \mathcal{C} , \mathcal{C} is called *integral*. Suppose \mathcal{C} is a weakly integral fusion category. It follows from [9, Theorem 3.10] that either \mathcal{C} is integral, or \mathcal{C} is a \mathbb{Z}_2 -extension of a fusion subcategory \mathcal{D} . In particular, if $\text{FPdim } \mathcal{C}$ is odd or if $\mathcal{C} = \mathcal{C}_{\text{ad}}$, then \mathcal{C} is necessarily integral.

Lemma 2.2. *Let $x \in \text{Irr}(\mathcal{C})$. Then the following hold:*

- (i) The order of $G[x]$ divides $(\text{FPdim } x)^2$.
- (ii) The order of $G(\mathcal{C})$ divides $n(\text{FPdim } x)^2$, where n is the number of non-isomorphic simple objects of Frobenius–Perron dimension $\text{FPdim } x$.

Proof. It is enough to show (i). Consider the fusion subcategory $\mathcal{D} \subseteq \mathcal{C}_{\text{pt}}$ generated by $G[x]$. Regard \mathcal{C} as module category over \mathcal{D} with respect to the action given by the tensor product of \mathcal{C} . If X is a simple object of \mathcal{C} representing x , then the definition of $G[x]$ implies that $g \otimes X \cong X$, for all simple object g of \mathcal{D} . Therefore, the full abelian subcategory \mathcal{M} whose objects are isomorphic to direct sums of copies of X is an indecomposable \mathcal{D} -module subcategory of \mathcal{C} . Since X is the unique simple object of \mathcal{M} up to isomorphism, then $\text{FPdim } \mathcal{M} = (\text{FPdim } x)^2$. Part (i) now follows from [5, Proposition 8.15 and Remark 8.17]. \square

Corollary 2.3. *Suppose that $G(\mathcal{C})$ is of prime order p . Assume in addition that $p > |\text{Irr}_\alpha(\mathcal{C})| > 0$, for some $\alpha \neq 1$. Then $G[x] = G(\mathcal{C})$, for all $x \in \text{Irr}_\alpha(\mathcal{C})$. In particular, p divides α^2 .*

Proof. Let $x \in \text{Irr}_\alpha(\mathcal{C})$. Decomposing the set $\text{Irr}_\alpha(\mathcal{C})$ into disjoint orbits under the action of $G(\mathcal{C})$, we get that $G[x] \neq \mathbf{1}$. Then necessarily $G[x] = G(\mathcal{C})$. It follows from Lemma 2.2 that p divides α^2 . \square

Remark 2.4. In the notation of the proof of Lemma 2.2, the rank-one module category \mathcal{M} corresponds to a fiber functor over \mathcal{D} . Being a pointed fusion category, this implies that \mathcal{D} is equivalent as a fusion category to the category $\mathcal{C}(G[x]) = \mathcal{C}(G[x], \mathbf{1})$ of finite-dimensional $G[x]$ -graded vector spaces.

The proof of Corollary 2.3 shows moreover that, under the assumptions of the lemma, we have an equivalence of fusion categories $\mathcal{C}_{\text{pt}} \cong \mathcal{C}(\mathbb{Z}_p)$.

Let $1 = d_0, d_1, \dots, d_s$, $s \geq 0$, be positive real numbers such that $1 = d_0 < d_1 < \dots < d_s$, and let n_1, n_2, \dots, n_s be positive integers. We shall say that \mathcal{C} is of type $(d_0, n_0; d_1, n_1; \dots; d_s, n_s)$ if, for all $i = 0, \dots, s$, n_i is the number of the non-isomorphic simple objects of Frobenius–Perron dimension d_i .

Hence, if \mathcal{C} is of type $(d_0, n_0; d_1, n_1; \dots; d_s, n_s)$, then n_0 equals the order of $G(\mathcal{C})$ and we have a relation

$$\text{FPdim } \mathcal{C} = n_0 + d_1^2 n_1 + \dots + d_s^2 n_s. \tag{1}$$

For each $i = 1, \dots, s$, let $x_i^l \in \text{Irr}(\mathcal{C})$, $1 \leq l \leq n_i$, such that $\text{FPdim } x_i^l = d_i$. Decomposing the tensor product $x_i^l \otimes x_i^{l*}$ into a direct sum of simple objects, and comparing dimensions, we get a relation

$$d_i^2 = |G[x_i^l]| + \sum_{j=1}^s m_j d_j,$$

where $m_j = \sum_t m(x_j^t, x_i^l \otimes x_i^{l*})$, for all j . Hence $m_j \geq 0$, and in addition $m_j = \sum_t m(x_j^t, x_i^l \otimes x_i^{l*}) = \sum_t m(x_i^l, x_j^t \otimes x_i^l) \leq \sum_t \text{FPdim } x_j^t = n_j d_j$.

Lemma 2.5. *Let $x, x' \in \text{Irr}(\mathcal{C})$. Then the following are equivalent:*

- (i) $x^* x' \in \text{Irr}(\mathcal{C})$.
- (ii) For all $\mathbf{1} \neq y \in \text{Irr}(\mathcal{C})$, either $m(y, x x^*) = 0$ or $m(y, x' x'^*) = 0$.

Proof. See [1, Lemma 6.1]. □

We state the following lemma for future use. It is a consequence of the solvability of fusion categories of Frobenius–Perron dimension $p^a q^b$ [6, Theorem 1.6].

Lemma 2.6. *Suppose $\text{FPdim } \mathcal{C} = p^a q^b$, where p and q are prime numbers, $a, b \geq 0$, such that $a + b \neq 0$. Then the group $G(\mathcal{C})$ is not trivial.*

Proof. The proof is by induction on $\text{FPdim } \mathcal{C}$. The assumption implies that \mathcal{C} is solvable [6, Theorem 1.6]. Then \mathcal{C} is either a G -equivariantization or a G -extension of a fusion category \mathcal{D} , where G is a cyclic group of prime order. If \mathcal{C} is a G -equivariantization, then \mathcal{C} contains a fusion subcategory equivalent to $\text{rep } G$, hence the lemma follows in this case. Otherwise, \mathcal{C} has a faithful grading $\mathcal{C} = \bigoplus_{g \in G} \mathcal{C}_g$, where $\mathcal{C}_e = \mathcal{D}$. Since $\text{FPdim } \mathcal{D}$ divides $\text{FPdim } \mathcal{C}$, then either $\text{FPdim } \mathcal{D} = 1$ and \mathcal{C} is pointed, or $1 < \text{FPdim } \mathcal{D} = p^{a'} q^{b'}$, with $|G| \text{FPdim } \mathcal{D} = \text{FPdim } \mathcal{C}$. In the first case we are done. In the last case, \mathcal{D} has nontrivial invertible objects, by induction. Hence so does \mathcal{C} . This finishes the proof of the lemma. □

3. Integral Fusion Categories

Along this section \mathcal{C} will denote an integral fusion category over k . We collect here some useful facts about the structure of \mathcal{C} that will be useful in the proof of our main results. Some of these results have been established in the literature in the case where \mathcal{C} is the category of finite-dimensional representations of a semisimple Hopf algebra, but their proofs only make use of the properties of the Grothendieck ring explained in Sec. 2. Therefore, these proofs also work *mutatis mutandis* in the fusion category setting, and thus they are omitted in our exposition.

In what follows we assume that \mathcal{C} is of type $(1, n_0; d_1, n_1; \dots; d_s, n_s)$.

Lemma 3.1. *Let $d = \text{gcd}(d_1, \dots, d_s)$. Then d divides n_0 .*

In particular, if n_0 is a prime number and $d \neq 1$, then $d = p$. On the other hand, if \mathcal{C} is of type $(1, n_0; d, n)$, then d divides n_0 (see [2, Lemma 5.2], [12, Theorem 5.1(b)]), hence in this case \mathcal{C} is of Frobenius type.

Proof. Let $x \in \text{Irr}(\mathcal{C})$ such that $\text{FPdim } x > 1$. Decomposing the product xx^* and taking Frobenius–Perron dimensions, we get that d divides the order of $G[x]$. This implies the lemma. □

Lemma 3.2. *Let \mathcal{C} be an integral fusion category. Suppose that one of the following conditions hold:*

- (a) $\text{Irr}_4(\mathcal{C}) = \emptyset$ and $G[x] \neq \mathbf{1}$, for all $x \in \text{Irr}_2(\mathcal{C})$, or
- (b) $G[x] \cap G[x'] \neq \mathbf{1}$, for all $x, x' \in \text{Irr}_2(\mathcal{C})$.

Then \mathcal{C} has a fusion subcategory of type $(1, n_0; 2, n)$, where $n = |\text{Irr}_2(\mathcal{C})|$.

Proof. Assume first that (a) holds. Let $x, y \in \text{Irr}_2(\mathcal{C})$. Note that the invertible objects appearing in xy with positive multiplicity form a coset of the stabilizer $G[x]$ in $G(\mathcal{C})$, and therefore there is an even number of them. The remaining simple constituents (if any) of xy should therefore belong to $\text{Irr}_2(\mathcal{C})$.

In view of Lemma 2.5, assumption (b) also implies that the product of any two elements $x, y \in \text{Irr}(\mathcal{C})$ such that $\text{FPdim } x, \text{FPdim } y \leq 2$ decomposes a sum of elements of $\text{Irr}(\mathcal{C})$ of Frobenius–Perron dimension ≤ 2 . See [14, Theorem 2.4.2].

Then we have shown that both assumptions (a) and (b) imply that the set $\{x \in \text{Irr}(\mathcal{C}) \mid \text{FPdim } x \leq 2\}$ spans a fusion subring of $K_0(\mathcal{C})$, corresponding to a fusion subcategory of the prescribed type. \square

Lemma 3.3. *Suppose that $\text{Irr}_2(\mathcal{C})$ is odd and the order of $G(\mathcal{C})$ is divisible by 4. Then \mathcal{C} has a non-pointed fusion subcategory of Frobenius–Perron dimension 8.*

Proof. It follows from [14, Proposition 2.1.3]. \square

The following theorem is a restatement of [16, Theorem 11] in the context of fusion categories. The theorem has found many applications in the classification of low-dimensional semisimple Hopf algebras, since such semisimple Hopf algebras often have irreducible characters of degree 2.

Theorem 3.4. *Suppose $x \in \text{Irr}(\mathcal{C})$ is such that $\text{FPdim } x = 2$. Then at least one of the following holds:*

- (i) $G[x] \neq \mathbf{1}$.
- (ii) \mathcal{C} has a fusion subcategory \mathcal{D} of type $(1, 2; 2, 1; 3, 2)$, such that $x \notin \text{Irr}(\mathcal{D})$ which has an invertible object g of order 2 such that $gx \neq x$.
- (iii) \mathcal{C} has a fusion subcategory of type $(1, 3; 3, 1)$ or $(1, 1; 3, 2; 4, 1; 5, 1)$.

It follows that if $G[x] = \mathbf{1}$, then $\text{FPdim } \mathcal{C}$ is divisible by 12, 24, or 60. In particular, $\text{FPdim } \mathcal{C}$ is always even.

In the next lemmas, we list some consequences of Theorem 3.4.

Lemma 3.5. (i) *Suppose that the order of $G(\mathcal{C})$ is odd. Assume that $\text{Irr}_2(\mathcal{C}) \neq \emptyset$. Then $\text{Irr}_3(\mathcal{C}) \neq \emptyset$. Further, if $\text{Irr}_4(\mathcal{C}) = \emptyset$ or $\text{Irr}_5(\mathcal{C}) = \emptyset$, then \mathcal{C} has a fusion subcategory of type $(1, 3; 3, 1)$ and hence 12 divides $\text{FPdim } \mathcal{C}$.*

- (ii) *Suppose that $G(\mathcal{C}) = \mathbf{1}$ and $\text{Irr}_2(\mathcal{C}) \neq \emptyset$. Then $\text{Irr}_3(\mathcal{C})$, $\text{Irr}_4(\mathcal{C})$ and $\text{Irr}_5(\mathcal{C})$ are non-empty and 60 divides $\text{FPdim } \mathcal{C}$.*
- (iii) *Suppose that $G(\mathcal{C})$ is of prime order $p \neq 3$ and $\text{Irr}_2(\mathcal{C}) \neq \emptyset$. If $\text{Irr}_4(\mathcal{C}) = \emptyset$, then $G(\mathcal{C})$ is of order 2 and \mathcal{C} has a fusion subcategory of Frobenius–Perron dimension $2 + 4|\text{Irr}_2(\mathcal{C})|$.*
- (iv) *Suppose that $G(\mathcal{C})$ is of prime order p . Assume that $\text{Irr}_2(\mathcal{C}) \neq \emptyset$ and $\text{FPdim } \mathcal{C}$ is not divisible by 12. Then $p = 2$ and \mathcal{C} has a fusion subcategory of Frobenius–Perron dimension $2 + 4|\text{Irr}_2(\mathcal{C})|$.*

- Proof.** (i) The assumption implies that $G[x] = \mathbf{1}$, for all $x \in \text{Irr}_2(\mathcal{C})$. Moreover, \mathcal{C} cannot have fusion subcategories of type $(1, 2; 2, 1; 3, 2)$ or $(1, 3; 3, 1)$. Part (i) then follows from Theorem 3.4.
- (ii) Follows directly from Theorem 3.4.
- (iii) Let $x \in \text{Irr}_2(\mathcal{C})$. Assume first that $G[x] = \mathbf{1}$. By [14, Remark 2.2.2(i)], there exist $y \in \text{Irr}_3(\mathcal{C})$ such that $|G[y]| = 3$. Then 3 divides $|G(\mathcal{C})|$, which is a contradiction. Therefore $G[x] \neq \mathbf{1}$ and hence $G[x] = G(\mathcal{C})$ is of order 2. Since this holds for all $x \in \text{Irr}_2(\mathcal{C})$, part (iii) follows from Lemma 3.2.
- (iv) Let $x \in \text{Irr}_2(\mathcal{C})$. Since 12 does not divide $\text{FPdim } \mathcal{C}$, Theorem 3.4 implies that $G[x] \neq \mathbf{1}$. Hence $G[x] = G(\mathcal{C})$, for all $x \in \text{Irr}_2(\mathcal{C})$, and therefore $p = 2$. Part (iv) follows from Lemma 3.2. \square

Lemma 3.6. *Suppose that $\text{FPdim } x = 2$ for some $x \in \text{Irr}(\mathcal{C})$. Let n_0 be the order of $G(\mathcal{C})$ and let $s = |\text{Irr}_2(\mathcal{C})|$. Assume one of the following conditions hold:*

- (i) *12 does not divide $\text{FPdim } \mathcal{C}$ and $\text{Irr}_4(\mathcal{C}) = \emptyset$.*
- (ii) *$n_0 = 2$ and $\text{FPdim } \mathcal{C}$ is not divisible by 24 or 60.*
- (iii) *$n_0 = 2$ and $\text{Irr}_3(\mathcal{C}) = \emptyset$.*
- (iv) *$\text{Irr}_3(\mathcal{C}) = \text{Irr}_4(\mathcal{C}) = \emptyset$.*

Then \mathcal{C} has a fusion subcategory of type $(1, n_0; 2, s)$. In particular, $\text{FPdim } \mathcal{C}$ is divisible by $n_0 + 4s$.

- Proof.** (i) Suppose that 12 does not divide $\text{FPdim } \mathcal{C}$. Then Theorem 3.4 shows that $G[x] \neq \mathbf{1}$ for all $x \in \text{Irr}_2(\mathcal{C})$. Hence (i) follows from Lemma 3.2.
- (ii) In this case \mathcal{C} cannot have fusion subcategories of type $(1, 3; 3, 1)$. It follows from Theorem 3.4 that $G[x] = G(\mathcal{C})$, for all $x \in \text{Irr}_2(\mathcal{C})$. Then (ii) follows from Lemma 3.2.
- (iii) By Theorem 3.4, $G[x] \neq \mathbf{1}$, for all $x \in \text{Irr}_2(\mathcal{C})$. Hence (iii) follows from Lemma 3.2.
- (iv) Since $\text{Irr}_3(\mathcal{C}) = \emptyset$, Theorem 3.4 implies that $G[x] \neq \mathbf{1}$, for all $x \in \text{Irr}_2(\mathcal{C})$. Then $G[x] = G(\mathcal{C})$, for all $x \in \text{Irr}_2(\mathcal{C})$, and (iv) follows also from Lemma 3.2. \square

The next proposition is a restatement of some results of this section as well as other results in the literature, in terms of fusion category types. It provides us with conditions that can be easily handled by a computer.

Proposition 3.7. *Let \mathcal{C} be an integral fusion category over k .*

- (i) *If 60 does not divide $\text{FPdim } \mathcal{C}$, then \mathcal{C} cannot be of type $(1, 1; 2, m; \dots)$.*
- (ii) *If 8 does not divide $\text{FPdim } \mathcal{C}$, then \mathcal{C} cannot be of type $(1, n_0; 2, m; \dots)$, where n_0 is divisible by 4 and m is odd.*
- (iii) *If n_0 does not divide $\text{FPdim } \mathcal{C}$ or $n_i d_i^2$, for some $2 \leq i \leq s$, then \mathcal{C} cannot be of type $(1, n_0; d_2, n_2; \dots; d_s, n_s)$.*
- (iv) *If t does not divide n_0 , then \mathcal{C} cannot be of type $(1, n_0; t, n)$.*

- (v) If $s \leq 2$, then \mathcal{C} cannot be of type $(1, 1; d_1, n_1; \dots, d_s, n_s)$.
- (vi) If neither 24 nor 60 nor $2 + 4m$ divide $\text{FPdim } \mathcal{C}$, then \mathcal{C} cannot be of type $(1, 2; 2, m; \dots)$.
- (vii) If neither 12 nor $n_0 + 4m$ divide $\text{FPdim } \mathcal{C}$, and \mathcal{C} does not have simple objects of Frobenius–Perron dimension 4, then \mathcal{C} cannot be of type $(1, n_0; 2, m; \dots)$.
- (viii) If $n_0 + 4m$ does not divide $\text{FPdim } \mathcal{C}$ and $d \geq 5$, then \mathcal{C} cannot be of type $(1, n_0; 2, m; d, n; \dots)$.

Proof. Part (i) follows from Theorem 3.4. Part (ii) follows from Lemma 3.3. Part (iii) follows from Lemma 2.2. Part (iv) follows from Lemma 3.1. Part (v) follows from [17, Lemma 11]. Parts (vi)–(viii) follow from Lemma 3.6. □

We end this section with some further applications of Theorem 3.4 that will be used later on (c.f. the proof of Theorem 4.2).

Proposition 3.8. *Let $n \geq 1$ be an odd natural number and let \mathcal{C} be an integral fusion category such that $\text{FPdim } \mathcal{C} = 2n$. Assume in addition that $\text{Irr}_2(\mathcal{Z}(\mathcal{C})) \neq \emptyset$. Then \mathcal{C} is a \mathbb{Z}_p -extension or a \mathbb{Z}_p -equivariantization of a fusion category \mathcal{D} , for some prime number p .*

Proof. It will be enough to show that $\mathcal{Z}(\mathcal{C})$ has a nontrivial Tannakian subcategory \mathcal{E} . In this case, $\mathcal{E} \cong \text{rep } \Gamma$ where the order of Γ divides $\text{FPdim } \mathcal{C} = 2n$: indeed, $\mathcal{E} \subseteq \mathcal{E}'$, where \mathcal{E}' denotes the Müger’s centralizer of \mathcal{E} in $\mathcal{Z}(\mathcal{C})$ and, by [13, Theorem 3.2], $\text{FPdim } \mathcal{E} \text{FPdim } \mathcal{E}' = (\text{FPdim } \mathcal{C})^2$. In particular, Γ is solvable (see e.g. [10, Theorem 1.35]), and it follows that \mathcal{E} , and hence also $\mathcal{Z}(\mathcal{C})$, contains a Tannakian subcategory of prime dimension. By [6, Propositions 2.9 and 2.10], \mathcal{C} is an equivariantization or an extension of a fusion category \mathcal{D} .

Consider the forgetful functor $F : \mathcal{Z}(\mathcal{C}) \rightarrow \mathcal{C}$. Suppose first that the universal grading group $U(\mathcal{C})$ is not trivial. Since \mathcal{C} is a $U(\mathcal{C})$ -extension of its adjoint fusion subcategory \mathcal{C}_{ad} , it follows from [6, Proposition 2.9(ii)] that $\mathcal{Z}(\mathcal{C})$ contains the category $\text{rep } U(\mathcal{C})$ as a Tannakian subcategory. The above discussion shows that the proposition holds in this case. Therefore, we may assume that $U(\mathcal{C})$ is trivial. It follows from Lemma 2.1(ii) that F induces an injective group homomorphism $G(\mathcal{Z}(\mathcal{C})) \rightarrow G(\mathcal{C})$. Then the order of $G(\mathcal{Z}(\mathcal{C}))$ cannot be divisible by 4 (since 4 does not divide $\text{FPdim } \mathcal{C}$). By [6, Theorem 2.11], the Frobenius–Perron dimensions of simple objects of $\mathcal{Z}(\mathcal{C})$ divide $\text{FPdim } \mathcal{C}$. Hence $\mathcal{Z}(\mathcal{C})$ has no simple objects of Frobenius–Perron dimension 4.

Suppose that $\mathcal{Z}(\mathcal{C})$ has a fusion subcategory \mathcal{D} of type $(1, 3; 3, 1)$. Let $\mathcal{Z}_2(\mathcal{D}) \subseteq \mathcal{D}$ denote the Müger center of \mathcal{D} . If $\mathcal{Z}_2(\mathcal{D})$ is trivial, then \mathcal{D} is non-degenerate and in view of [13, Theorem 4.2], [3, Theorem 3.13], there is an equivalence of braided fusion categories $\mathcal{Z}(\mathcal{C}) \cong \mathcal{D} \boxtimes \mathcal{D}'$, where \mathcal{D}' is the Müger centralizer of \mathcal{D} . In particular, \mathcal{D}' must have simple objects of Frobenius–Perron dimension 2. But this is impossible since $\text{FPdim } \mathcal{D}'$ is odd [6, Theorem 2.11]. Hence $\mathcal{Z}_2(\mathcal{D})$ is not trivial, and therefore $\text{rep } \mathbb{Z}_3 \subseteq \mathcal{Z}_2(\mathcal{D})$. But $\text{rep } \mathbb{Z}_3$, being an odd-dimensional

symmetric fusion category, is Tannakian [3, Corollary 2.50(i)]. Hence we are done in this case. We may thus assume that $\mathcal{Z}(\mathcal{C})$ has no fusion subcategory \mathcal{D} of type $(1, 3; 3, 1)$.

Let $x \in \text{Irr}_2(\mathcal{Z}(\mathcal{C}))$. Since $\text{FPdim } \mathcal{Z}(\mathcal{C})$ is not divisible by 24 and $\text{Irr}_4(\mathcal{Z}(\mathcal{C})) = \emptyset$, it follows from Theorem 3.4 and the discussion in the previous paragraph, that $G[x] \neq \mathbf{1}$. Since the order of $G(\mathcal{Z}(\mathcal{C}))$ is not divisible by 4, then $G[x]$ is of order 2, for all $x \in \text{Irr}_2(\mathcal{Z}(\mathcal{C}))$. Furthermore, the abelian group $G(\mathcal{Z}(\mathcal{C}))$ has a unique subgroup of order 2, and therefore $G[x] = G[x'] \simeq \mathbb{Z}_2$, for all $x, x' \in \text{Irr}_2(\mathcal{Z}(\mathcal{C}))$. Lemma 3.2 implies that $\mathcal{Z}(\mathcal{C})$ has a fusion subcategory \mathcal{D} of type $(1, n_0; 2, m)$, where $n_0 = |G(\mathcal{C})|$ and $m \geq 1$. Since \mathcal{D} is braided, then \mathcal{D} contains a Tannakian subcategory, namely, the Müger center of the fusion subcategory generated by a self-dual object of $\text{Irr}_2(\mathcal{D})$ (see [15, Lemma 4.7]). This finishes the proof of the proposition. \square

Corollary 3.9. *Let \mathcal{C} be an integral fusion category such that $\text{FPdim } \mathcal{C} = 90$. Suppose that \mathcal{C} has a fusion subcategory \mathcal{D} of Frobenius–Perron dimension 6. Then \mathcal{C} is weakly group-theoretical.*

Proof. By Proposition 3.8, we may assume that $\text{Irr}_2(\mathcal{Z}(\mathcal{C})) = \emptyset$. Indeed, if $\mathcal{Z}(\mathcal{C})$ had a simple object of Frobenius–Perron dimension 2, then \mathcal{C} would be an equivariantization or an extension of a fusion category \mathcal{D} , whose Frobenius–Perron dimension divides 90 and is less than 90. In view of the results of [6], the fusion category \mathcal{D} (and then also \mathcal{C}) must be weakly group-theoretical. As in the proof of Proposition 3.8, it will be enough to show that $\mathcal{Z}(\mathcal{C})$ has a nontrivial Tannakian subcategory. To do this, we shall follow the lines of the proof of [6, Theorem 9.16].

Let $I : \mathcal{C} \rightarrow \mathcal{Z}(\mathcal{C})$ be the left adjoint of the forgetful functor $F : \mathcal{Z}(\mathcal{C}) \rightarrow \mathcal{C}$. Then $A = I(\mathbf{1})$ is a commutative algebra in $\mathcal{Z}(\mathcal{C})$ such that $\mathcal{C} \cong \mathcal{Z}(\mathcal{C})_A$ as fusion categories [6]. We may assume that A contains no nontrivial invertible object of $\mathcal{Z}(\mathcal{C})$: indeed, if $m(g, A) \neq 0$, where $\mathbf{1} \neq g \in G(\mathcal{Z}(\mathcal{C}))$, then $m(\mathbf{1}, F(g)) \neq 0$ and therefore $F(g) \cong \mathbf{1}$. By Lemma 2.1, \mathcal{C} is faithfully graded by \widehat{N} , where $N = \langle g \rangle$ is the subgroup generated by g (in particular, $\mathcal{Z}(\mathcal{C})$ contains a Tannakian subcategory) and we are done.

The fusion subcategory $\mathcal{D} \subseteq \mathcal{C}$ corresponds to a subalgebra B of A such that $\text{FPdim } B = \text{FPdim } \mathcal{C} / \text{FPdim } \mathcal{D} = 15$. In view of [6, Theorem 2.11], the Frobenius–Perron dimensions of simple objects of $\mathcal{Z}(\mathcal{C})$ divide 90. Hence the possible decompositions of B as an object of $\mathcal{Z}(\mathcal{C})$ are the following:

$$\mathbf{1} \oplus X_3 \oplus X'_3 \oplus X''_3 \oplus X_5, \quad \mathbf{1} \oplus X_3 \oplus X_5 \oplus X_6, \quad \mathbf{1} \oplus X_5 \oplus X_9, \quad (2)$$

where X_3, X'_3, X''_3 are simple objects of Frobenius–Perron dimension 3, and X_5, X_6, X_9 are simple objects of Frobenius–Perron dimensions 5, 6 and 9, respectively.

It follows that $\mathcal{Z}(\mathcal{C})$ has simple objects of prime power dimension, and therefore it contains a nontrivial symmetric subcategory [6, Corollary 7.2]. We may assume $\mathcal{Z}(\mathcal{C})$ has a unique nontrivial symmetric subcategory \mathcal{E} , which is equivalent to the

category of super vector spaces. Note that $\mathcal{E} \subseteq \mathcal{Z}(\mathcal{C})_{\text{pt}}$ and $\text{FPdim}(\mathcal{E}) = 2$. Let $\mathcal{Z} = \mathcal{E}' \subseteq \mathcal{Z}(\mathcal{C})$ be the Müger centralizer of \mathcal{E} . Then $\mathcal{E} \subseteq \mathcal{Z}$, since \mathcal{E} is symmetric, and \mathcal{Z} is a slightly degenerate (as $\mathcal{Z}_2(\mathcal{E}') = \mathcal{E}$) fusion subcategory of Frobenius–Perron dimension $(\text{FPdim} \mathcal{C})^2/2$.

In particular the group $G = G(\mathcal{Z}(\mathcal{C})) \neq \mathbf{1}$ and, since $G(\mathcal{Z}(\mathcal{C}))$ is isomorphic to the universal grading group of $\mathcal{Z}(\mathcal{C})$ [9, Theorem 6.2], there is a faithful G -grading on $\mathcal{Z}(\mathcal{C})$, with trivial component $\mathcal{Z}(\mathcal{C})_{\text{ad}}$. By [9, Corollary 6.8], we have $\mathcal{Z}(\mathcal{C})'_{\text{ad}} = \mathcal{Z}(\mathcal{C})_{\text{pt}}$. Since $\mathcal{E} \subseteq \mathcal{Z}(\mathcal{C})_{\text{pt}}$, then $\mathcal{Z}(\mathcal{C})'_{\text{pt}} = \mathcal{Z}(\mathcal{C})_{\text{ad}} \subseteq \mathcal{Z}$.

Note that if the slightly degenerate integral braided category \mathcal{Z} has a simple object of odd prime power dimension, then it follows from [6, Proposition 7.4] that \mathcal{Z} , and thus also $\mathcal{Z}(\mathcal{C})$, contains a nontrivial Tannakian subcategory. Then we are done in this case.

We may therefore assume that \mathcal{Z} has no simple object of odd prime power dimension. Consider first the case where $\text{Irr}_3(\mathcal{Z}(\mathcal{C})) \neq \emptyset$ and let X be a simple object with $\text{FPdim} X = 3$. We have $X \otimes X^* \in \mathcal{Z}(\mathcal{C})_{\text{ad}}$. Since $\mathcal{Z}(\mathcal{C})_{\text{ad}} \subseteq \mathcal{Z}$, then $X \otimes X^*$ has no simple constituents of odd prime power dimension. This implies that $X \otimes X^* = \bigoplus_{a \in G[X]} a \oplus Y$, where $|G[X]| = 3$ and Y is a simple object of Frobenius–Perron dimension 6, or $X \otimes X^* = \bigoplus_{a \in G[X]} a$ with $|G[X]| = 9$.

Since $G[X] \subseteq G(\mathcal{Z}(\mathcal{C})) \cap \mathcal{Z}(\mathcal{C})_{\text{ad}} \subseteq \mathcal{Z}(\mathcal{C})'_{\text{ad}} \cap \mathcal{Z}(\mathcal{C})_{\text{ad}}$, then the fusion subcategory $\langle G[X] \rangle$ generated by $G[X]$ is symmetric and therefore Tannakian, because it is odd dimensional [3, Corollary 2.50(i)].

We may therefore assume that $\text{Irr}_3(\mathcal{Z}(\mathcal{C})) = \emptyset$. Then the Frobenius–Perron dimensions of simple objects of $\mathcal{Z}(\mathcal{C})$ are among the numbers 1, 5, 6, 9, 10, 15, 18, 30 and 45. Moreover, in view of the possible decompositions in (2), $\text{Irr}_5(\mathcal{Z}(\mathcal{C}))$ and $\text{Irr}_9(\mathcal{Z}(\mathcal{C}))$ are both non-empty.

Let $X \in \mathcal{Z}(\mathcal{C})$ be a simple object of Frobenius–Perron dimension 5 and let \mathcal{B} denote the fusion subcategory generated by X . We claim that \mathcal{B}' is a nontrivial proper fusion subcategory of $\mathcal{Z}(\mathcal{C})$. Indeed, there must exist a simple object Y of Frobenius–Perron dimension not divisible by 5 such that X and Y projectively centralize each other. Otherwise $S_{X,Y} = 0$ for all such simple objects [6, Lemma 7.1]. As in the proof of [6, Corollary 7.2], the orthogonality of columns of the S -matrix imply a relation

$$0 = \sum_{Y \in \text{Irr}(\mathcal{Z}(\mathcal{C}))} \frac{S_{X,Y}}{5} \text{FPdim} Y = 1 + \sum_{\substack{\mathbf{1} \neq Y \\ (\text{FPdim} Y, 5) \neq 1}} \frac{S_{X,Y}}{5} \text{FPdim} Y,$$

which is impossible. Thus there exists a simple object Y with $\text{FPdim} Y = 6, 9$ or 18 and such that X and Y projectively centralize each other. Then $Y \otimes Y^* \in \mathcal{B}'$ and the claim follows (note that $\mathcal{B}' \neq \mathcal{Z}(\mathcal{C})$ since otherwise $\text{Vec} = \mathcal{Z}(\mathcal{C})' = \mathcal{B}'' = \mathcal{B}$).

Let $\mathcal{D} = \mathcal{B}'$. If \mathcal{D} is degenerate, then its Müger center $\mathcal{Z}_2(\mathcal{D})$ is a symmetric fusion subcategory of $\mathcal{Z}(\mathcal{C})$ and therefore $\mathcal{Z}_2(\mathcal{D}) = \mathcal{E} \subseteq \mathcal{D}$. But this implies that $\mathcal{B} = \mathcal{D}' \subseteq \mathcal{E}' = \mathcal{Z}$, thus X is a simple object of odd prime power dimension of \mathcal{Z} and we are done.

We may therefore assume that \mathcal{D} is non-degenerate. Hence $\mathcal{Z}(\mathcal{C}) = \mathcal{D} \boxtimes \mathcal{D}' = \mathcal{D} \boxtimes \mathcal{B}$, and \mathcal{B} is non-degenerate. Since \mathcal{B} has a simple object of prime dimension 5, it follows from [6, Corollary 7.2] that \mathcal{B} contains a symmetric fusion subcategory. Hence $\mathcal{E} \subseteq \mathcal{B}$. In particular, $\mathcal{E} \not\subseteq \mathcal{D}$ and we obtain that \mathcal{D} has no simple objects of prime power dimension. This implies that all simple objects of Frobenius–Perron dimension 5 and 9 belong to \mathcal{B} .

Let Z be a non-invertible simple constituent of $Y \otimes Y^*$, where Y is a simple object with $\text{FPdim } Y = 6, 9$ or 18 that projectively centralizes X . Then $Z \in \mathcal{D}$ and the Frobenius–Perron dimension of Z is either $6, 10, 15, 18, 30$ or 45 . On the other hand, if $X_5, X_9 \in \mathcal{B}$ are simple objects of \mathcal{B} of Frobenius–Perron dimensions 5 and 9, respectively, then $Z \boxtimes X_5$ and $Z \boxtimes X_9$ are simple objects of $\mathcal{Z}(\mathcal{C}) = \mathcal{D} \boxtimes \mathcal{B}$ of Frobenius–Perron dimension $5 \text{ FPdim } Z$ and $9 \text{ FPdim } Z$. This contradicts [6, Theorem 2.11(i)] because both these numbers cannot divide 90. This finishes the proof of the corollary. \square

4. Fusion Categories of Small Dimension

The computations in this section were partly handled by a computer. Some of them were done using [7]. For example, it is easy to write a computer program by which one finds out all possible positive integers $1 = d_1, d_2, \dots, d_s$ and n_1, n_2, \dots, n_s such that $84 = \sum_{i=1}^s n_i d_i^2$, and then one can exclude those which are not possible types by using Proposition 3.7.

Theorem 4.1. *Let \mathcal{C} be an integral fusion category of Frobenius–Perron dimension 84. Then \mathcal{C} is of Frobenius type. Further, the group $G(\mathcal{C})$ is of order 2, 3, 4, 6, 12, 21 or 28 and \mathcal{C} is of one of the following types:*

- $(1, 2; 2, 3; 3, 2; 4, 1; 6, 1), (1, 2; 2, 3; 3, 6; 4, 1), (1, 2; 3, 2; 4, 4),$
- $(1, 3; 2, 18; 3, 1), (1, 3; 2, 9; 3, 5), (1, 3; 2, 9; 3, 1; 6, 1),$
- $(1, 3; 2, 6; 3, 1; 4, 3), (1, 3; 3, 9), (1, 3; 3, 5; 6, 1), (1, 3; 3, 1; 6, 2),$
- $(1, 4; 2, 20), (1, 4; 2, 16; 4, 1), (1, 4; 2, 12; 4, 2), (1, 4; 2, 8; 4, 3), (1, 4; 2, 4; 4, 4),$
- $(1, 4; 2, 2; 3, 8), (1, 4; 2, 2; 6, 2), (1, 4; 2, 2; 3, 4; 6, 1), (1, 4; 4, 5),$
- $(1, 6; 2, 3; 3, 2; 4, 3), (1, 6; 2, 6; 3, 2; 6, 1), (1, 6; 2, 6; 3, 6), (1, 6; 2, 15; 3, 2),$
- $(1, 12; 2, 18), (1, 12; 3, 8), (1, 12; 6, 2), (1, 12; 3, 4; 6, 1), (1, 12; 2, 6; 4, 3),$
- $(1, 21; 3, 7), (1, 28; 2, 14).$

Proof. Assume first that \mathcal{C} is of Frobenius type. In view of the results in Sec. 3, a computer program gives the prescribed ones as the only possible types in dimension 84. In fact, using Proposition 3.7, the list of all possible types can be reduced to the type $(1, 1; 3, 2; 4, 1; 7, 1)$ and the types appearing in the claim. But the type $(1, 1; 3, 2; 4, 1; 7, 1)$ can be discarded as follows: Let $x_3 \in \text{Irr}_3(\mathcal{C})$. Then

$x_3x_3^* = \mathbf{1} + 2x_4$, where $x_4 \in \text{Irr}_4(\mathcal{C})$. From $m(x_4, x_3x_3^*) = m(x_3, x_4x_3) = 2$, we have $x_4x_3 = 2x_3 + 2x_3'$, where $x_3 \neq x_3' \in \text{Irr}_3(\mathcal{C})$. Then $m(x_3', x_4x_3) = m(x_4, x_3'x_3^*) = 2$, which means that $x_3'x_3^* = 2x_4 + \mathbf{1}$. So $x_3 = x_3'$, a contradiction.

Now assume that \mathcal{C} is not of Frobenius type. It follows from Proposition 3.7 that \mathcal{C} has one of the following types: $(1, 1; 3, 1; 5, 1; 7, 1)$, $(1, 2; 3, 2; 8, 1)$, $(1, 4; 4, 1; 8, 1)$, $(1, 2; 4, 2; 5, 2)$. We shall exclude these types below.

Type $(1, 1; 3, 1; 5, 1; 7, 1)$. Let $x_3, x_5, x_7 \in \text{Irr}(\mathcal{C})$ of Frobenius–Perron dimensions 3, 5 and 7, respectively. From $x_3x_3^* = \mathbf{1} + x_3 + x_5$, we have $m(x_5, x_3x_3^*) = m(x_3, x_5x_3) = 1$, which means that $x_5x_3 = x_3 + x_5 + x_7$. Then $m(x_5, x_5x_3) = m(x_5, x_3x_5) = m(x_3, x_5^2) = 1$, which means that $x_5^2 = \mathbf{1} + x_3 + 3x_7$. Then $m(x_7, x_5^2) = m(x_5, x_7x_5) = 3$, which means that $x_7x_5 = 3x_5 + 2x_3 + 2x_7$. Then $m(x_3, x_7x_5) = m(x_7, x_3x_5) = 2$, which means that $x_3x_5 = 2x_7 + \mathbf{1}$. This contradicts Schur’s Lemma.

Type $(1, 2; 3, 2; 8, 1)$. Let $x_3 \in \text{Irr}_3(\mathcal{C})$. Then $x_3x_3^* = \mathbf{1} + x_8$, where x_8 is the unique simple object of Frobenius–Perron dimension 8. From $m(x_8, x_3x_3^*) = m(x_3, x_8x_3) = 1$, we have $x_8x_3 = x_3 + mx_3' + nx_8$, where $3m + 8n = 21$ and $x_3 \neq x_3' \in \text{Irr}_3(\mathcal{C})$. From $m(x_3', x_8x_3) = m(x_8, x_3'x_3^*) = m$, we know that $m \leq 1$. Hence, the equation $3m + 8n = 21$ cannot hold true.

Type $(1, 4; 4, 1; 8, 1)$. Let $x_4, x_8 \in \text{Irr}(\mathcal{C})$ of Frobenius–Perron dimensions 4 and 8, respectively, and $G(\mathcal{C}) = \{\mathbf{1}, g_1, g_2, g_3\}$. Then

$$x_4^2 = x_4x_4^* = \mathbf{1} + g_1 + g_2 + g_3 + 3x_4 \quad \text{or} \quad x_4x_4^* = \mathbf{1} + g_1 + g_2 + g_3 + x_4 + x_8.$$

If the first possibility holds, then the set $G(\mathcal{C}) \cup \text{Irr}_4(\mathcal{C})$ spans a fusion subring of $K_0(\mathcal{C})$. Hence, \mathcal{C} has a fusion subcategory of Frobenius–Perron dimension 20. This is impossible since 20 does not divide $\text{FPdim } \mathcal{C}$. If the second possibility holds, then $m(x_8, x_4x_4^*) = m(x_4, x_8x_4) = 1$, which means that $x_8x_4 = x_4 + mx_8$, where m is a non-negative integer. This is impossible.

Type $(1, 2; 4, 2; 5, 2)$. Let $x_4 \in \text{Irr}_4(\mathcal{C})$ and let $G(\mathcal{C}) = \{\mathbf{1}, g\}$. Then there must exist $z \in \text{Irr}_5(\mathcal{C})$ such that $m(z, x_4x_4^*) = 1, 2$ or 3 .

If $m(z, x_4x_4^*) = 3$ then $m(x_4, zx_4) = 3$. This means that $zx_4 = 3x_4 + 2y$, where $x_4 \neq y \in \text{Irr}_4(\mathcal{C})$. Then $m(y, zx_4) = m(z, yx_4^*) = 2$. This means that $yx_4^* = 2z + u$, where u is an object of dimension 6 such that $m(z, u) = 0$. Notice that, in our case, $x_4x_4^* = \mathbf{1} + 3z$ and hence $y = gx_4$. Hence, $yx_4^* = gx_4x_4^* = g(\mathbf{1} + 3z)$, which contradicts with $m(z, yx_4^*) = 2$.

If $m(z, x_4x_4^*) = 2$ then $m(x_4, zx_4) = 2$. This means that $zx_4 = 2x_4 + 3y$, where $x_4 \neq y \in \text{Irr}_4(\mathcal{C})$. Then $m(y, zx_4) = m(z, yx_4^*) = 3$. This implies that $yx_4^* = 3z + g$, and hence $y = gx_4$. Hence, $yx_4^* = gx_4x_4^* = 3z + g$, which means that $x_4x_4^* = 3gz + \mathbf{1}$. This contradicts the assumption that $m(z, x_4x_4^*) = 2$.

If $m(z, x_4x_4^*) = 1$ then $m(x_4, zx_4) = 1$. This means that $zx_4 = x_4 + 4y$, where $x_4 \neq y \in \text{Irr}_4(\mathcal{C})$. Then $m(y, zx_4) = m(z, yx_4^*) = 4$. This is a contradiction. This completes the proof of the theorem. \square

Theorem 4.2. *Let \mathcal{C} be an integral fusion category of Frobenius–Perron dimension 90. Then \mathcal{C} is of Frobenius type. Moreover, the group $G(\mathcal{C})$ is of order 2, 6, 9, 10, 15, 18, 30 or 45 and \mathcal{C} is of one of the following types:*

- (1, 2; 2, 4; 3, 8), (1, 2; 2, 4; 6, 2), (1, 2; 2, 4; 3, 4; 6, 1), (1, 2; 2, 22),
- (1, 6; 2, 3; 3, 8), (1, 6; 2, 3; 6, 2), (1, 6; 2, 3; 3, 4; 6, 1), (1, 6; 2, 21),
- (1, 9; 3, 9), (1, 9; 3, 1; 6, 2), (1, 9; 3, 5; 6, 1), (1, 9; 9, 1), (1, 10; 2, 20), (1, 15; 5, 3),
- (1, 18; 2, 18), (1, 18; 3, 8), (1, 18; 3, 4; 6, 1), (1, 18; 6, 2), (1, 30; 2, 15), (1, 45; 3, 5).

Proof. Assume first that \mathcal{C} is of Frobenius type. Besides of the types listed in the claim, a computer calculation combined with Proposition 3.7 yields the additional types (1, 5; 2, 10; 3, 5), (1, 9; 2, 9; 3, 5), (1, 3; 2, 3; 5, 3), (1, 9; 2, 9; 3, 1; 6, 1). These four types can be excluded by Lemma 3.5.

Now assume that \mathcal{C} is not of Frobenius type. By Proposition 3.7, \mathcal{C} has one of the following types:

- (1, 1; 3, 1; 4, 1; 8, 1), (1, 2; 2, 2; 4, 1; 8, 1), (1, 2; 4, 1; 6, 2),
- (1, 2; 3, 8; 4, 1), (1, 6; 3, 4; 4, 3), (1, 2; 3, 4; 4, 1; 6, 1),
- (1, 2; 2, 2; 4, 5), (1, 2; 2, 1; 4, 3; 6, 1), (1, 2; 2, 1; 3, 2; 4, 1; 5, 2),
- (1, 2; 2, 1; 3, 4; 4, 3), (1, 3; 2, 3; 3, 3; 4, 3), (1, 6; 4, 3; 6, 1), (1, 6; 2, 9; 4, 3).

Note that the type (1, 3; 2, 3; 3, 3; 4, 3) is discarded by Lemma 3.5(i). In addition, since every weakly group-theoretical fusion category is of Frobenius type, then the types (1, 6; 3, 4; 4, 3), (1, 6; 4, 3; 6, 1) and (1, 6; 2, 9; 4, 3) are discarded by Corollary 3.9. Similarly, the types (1, 2; 2, 1; 4, 3; 6, 1), (1, 2; 2, 1; 3, 2; 4, 1; 5, 2) and (1, 2; 2, 1; 3, 4; 4, 3) are discarded by Lemma 3.5(iv) and Corollary 3.9. We shall exclude the remaining types below.

Type (1, 1; 3, 1; 4, 1; 8, 1). Let $x_3, x_4, x_8 \in \text{Irr}(\mathcal{C})$ of Frobenius–Perron dimensions 3, 4 and 8, respectively. Then $x_3x_3^* = \mathbf{1} + 2x_4$ or $x_3x_3^* = \mathbf{1} + x_8$. In the first case, $m(x_4, x_3x_3^*) = m(x_3, x_4x_3) = 2$. This means that $x_4x_3 = 2x_3 + z$, where $\text{FPdim } z = 6$ and z is a sum of simple objects of Frobenius–Perron dimension 4 or 8, which is impossible. In the second case, $m(x_8, x_3x_3^*) = m(x_3, x_8x_3) = 1$. This means that $x_8x_3 = x_3 + u$, where $\text{FPdim } u = 21$ and u is a sum of simple objects of Frobenius–Perron dimension 4 or 8. It is also impossible.

Type (1, 2; 2, 2; 4, 1; 8, 1). Let $x_4 \in \text{Irr}_4(\mathcal{C})$. Then there exists $x_2 \in \text{Irr}_2(\mathcal{C})$ such that $m(x_2, x_4x_4^*) = 1$ or 2. If $m(x_2, x_4x_4^*) = 1$ then $x_2x_4 = x_4 + x'_2 + x''_2$, where $x'_2, x''_2 \in \text{Irr}_2(\mathcal{C})$. Then $m(x'_2, x_2x_4) = m(x'_2, x_4x_2^*) = m(x_4, x'^*_2x_2) = 1$. This means that $x'_2 \neq x''_2$ and hence $m(x_2, x_2x_4) = 1$. Then $m(x_4, x'^*_2x_2) = 1$, which is impossible. Therefore $m(x_2, x_4x_4^*) = 2$, for every $x_2 \in \text{Irr}_2(\mathcal{C})$. Then $x_4x_4^* = \mathbf{1} + g + 2x_2 + y$, where $\text{FPdim } y = 10$ and $m(x_2, y) = 0$. Taking Frobenius–Perron dimensions, we see that this is impossible.

Type (1, 2; 4, 1; 6, 2). Let x_4 be the unique element of $\text{Irr}_4(\mathcal{C})$. Then $x_4x_4^* = \mathbf{1} + g + 2x_4 + x_6$, where $\{\mathbf{1}, g\} = G(\mathcal{C})$ and $x_6 \in \text{Irr}_6(\mathcal{C})$. It follows that $m(x_6, x_4x_4^*) = m(x_4, x_6x_4) = 1$ and $x_6x_4 = x_4 + z$, where $m(x_4, z) = 0$ and $\text{FPdim } z = 20$. It is impossible.

Type (1, 2; 3, 8; 4, 1). Let x_4 be the unique element of $\text{Irr}_4(\mathcal{C})$. Then there must exist an element x_3 of $\text{Irr}_3(\mathcal{C})$ such that $m(x_3, x_4x_4^*) = m(x_4, x_3x_4) = 1, 2$ or 3 . If $m(x_4, x_3x_4) = 1$ or 2 , then the decomposition of x_3x_4 gives rise to a contradiction. Therefore, $m(x, x_4x_4^*) = 3$ or 0 , for every $x \in \text{Irr}_3(\mathcal{C})$. Then the decomposition of $x_4x_4^*$ gives rise to a contradiction.

Type (1, 2; 3, 4; 4, 1; 6, 1). Let $x_4 \in \text{Irr}_4(\mathcal{C})$. If $m(x_3, x_4x_4^*) > 0$ for some $x_3 \in \text{Irr}_3(\mathcal{C})$, then $m(x_4, x_3x_4) = 1, 2$ or 3 . If $m(x_4, x_3x_4) = 1$ or 2 , then the decomposition of x_3x_4 gives rise to a contradiction. If $m(x_4, x_3x_4) = 3$ then $x_4x_4^* = \mathbf{1} + g + 3x_3 + z$, where $\{\mathbf{1}, g\} = G(\mathcal{C})$ and $\text{FPdim } z = 5$. It is impossible. Therefore, $x_4x_4^* = \mathbf{1} + g + 2x_4 + x_6$, where $x_6 \in \text{Irr}_6(\mathcal{C})$. It follows that $m(x_6, x_4x_4^*) = m(x_4, x_6x_4) = 1$ and $x_6x_4 = x_4 + z$, where $\text{FPdim } z = 20$ and $m(x_4, z) = 0$. It is also impossible since z is the sum of elements of $\text{Irr}_3(\mathcal{C})$ and $\text{Irr}_6(\mathcal{C})$.

Type (1, 2; 2, 2; 4, 5). Write $G(\mathcal{C}) = \{\mathbf{1}, g\}$. Because $|\text{Irr}_4(\mathcal{C})| = 5$ is odd, there exists a self-dual $x_4 \in \text{Irr}_4(\mathcal{C})$. Moreover, $G[x_4] = G(\mathcal{C})$. Counting degrees, we find that there exists only one element of $\text{Irr}_2(\mathcal{C})$ appearing in the decomposition of $x_4x_4^*$ with multiplicity 1, say x_2 . Clearly, x_2 is self-dual. Let $x_2 \neq x_2' \in \text{Irr}_2(\mathcal{C})$. Since $G[x_2] = G[x_2'] = G(\mathcal{C})$, Lemma 2.5 shows that x_2x_2' and $x_2'x_2$ are not irreducible. The fusion rules of elements of $\text{Irr}(\mathcal{C})$ show that $x_2x_2' = x_2'x_2 = x_2 + x_2'$.

From $m(x_2, x_4^2) = m(x_4, x_2x_4) = 1$, we have $x_2x_4 \stackrel{(1)}{=} x_4 + x_4'$, where $x_4 \neq x_4' \in \text{Irr}_4(\mathcal{C})$. Then $m(x_4', x_2x_4) = m(x_2, x_4'x_4) \stackrel{(2)}{=} 1$. Counting degrees, $x_4'x_4 = x_2 + x_2' + \omega$ is the only possible decomposition of $x_4'x_4$, where $\text{FPdim } \omega = 12$ and $m(x_2', \omega) = 0$. Then $m(x_2', x_4'x_4) = m(x_4', x_2'x_4) = 1$, which means that $x_2'x_4 \stackrel{(3)}{=} x_4' + x_4''$, where $x_4' \neq x_4'' \in \text{Irr}_4(\mathcal{C})$. From $m(x_2', x_4'x_4) = m(x_4, x_2'x_4') = 1$, we have $x_2'x_4' = x_4 + x_4'''$, where $x_4 \neq x_4''' \in \text{Irr}_4(\mathcal{C})$. Multiplying equality (1) on the left by x_2 , we have $x_2^2x_4 = x_2x_4 + x_2x_4'$.

If $x_2^2 = \mathbf{1} + g + x_2$, then $(\mathbf{1} + g + x_2)x_4 = 2x_4 + x_2x_4 = x_2x_4 + x_2x_4'$. This shows that $x_2x_4' = 2x_4$. It follows that $m(x_4, x_2x_4') = m(x_2, x_4x_4') = m(x_2, x_4'x_4) = 2$. This contradicts equality (2).

If $x_2^2 = \mathbf{1} + g + x_2'$, then $(\mathbf{1} + g + x_2')x_4 = 2x_4 + x_2'x_4 = 2x_4 + x_4' + x_4'' = x_2x_4 + x_2x_4' = x_4 + x_4' + x_2x_4'$. This shows that $x_2x_4' = x_4 + x_4''$. Multiplying on the left by x_2' , we have

$$\begin{aligned} x_2'x_2x_4' &= (x_2 + x_2')x_4' = x_2x_4' + x_2'x_4' \\ &= x_4 + x_4'' + x_4 + x_4''' \\ &= x_2'x_4 + x_2'x_4'' = x_4' + x_4'' + x_2'x_4''. \end{aligned}$$

This shows that $x'_4 + x'_2x''_4 = 2x_4 + x'''_4$. Hence, $x'_4 = x'''_4$ and $x'_2x''_4 = 2x_4$. Then $m(x_4, x'_2x''_4) = m(x_4, (x'_4)^*x'_2) = m((x'_4)^*, x_4x'_2) = m(x''_4, x'_2x_4) = 2$. This contradicts equality (3). \square

Proof of Theorem 1.1. Let \mathcal{C} be a fusion category and suppose $\text{FPdim } \mathcal{C} = N$ is a natural number and $N < 120$. In view of the results of [6], every fusion category of Frobenius–Perron dimension 60, $p^a q^b$ or pqr , where p, q and r are prime numbers, $a, b \geq 0$, is weakly group-theoretical. Indeed, if $\text{FPdim } \mathcal{C} = 60$, this is shown in [6, Theorem 9.16]. If $\text{FPdim } \mathcal{C} = p^a q^b$, then \mathcal{C} is solvable and thus weakly group-theoretical [6, Theorem 1.6]. If $\text{FPdim } \mathcal{C} = pqr$, then either \mathcal{C} is integral and thus group-theoretical [6, Theorem 9.2], or \mathcal{C} is a \mathbb{Z}_2 -extension of a fusion subcategory \mathcal{D} and we may assume that $p = 2$ and $\text{FPdim } \mathcal{D} = qr$. Then $\text{FPdim } \mathcal{D}$ is odd and \mathcal{D} is group-theoretical by [4]. Hence \mathcal{C} is weakly group-theoretical also in this case. In particular, all such fusion categories have the strong Frobenius property [6, Theorem 1.5].

We may therefore assume that $N = 84 = 2^2 \cdot 3 \cdot 7$ or $90 = 2 \cdot 3^2 \cdot 5$. If \mathcal{C} is not integral, then \mathcal{C} is a \mathbb{Z}_2 -extension of a fusion subcategory \mathcal{D} [9, Theorem 3.10] with $\text{FPdim } \mathcal{D} = 42$ or 45 , respectively. By the previous discussion, \mathcal{D} is weakly group-theoretical, and therefore so is \mathcal{C} . Hence in this case \mathcal{C} has the strong Frobenius property and in particular, it is of Frobenius type. Finally, if \mathcal{C} is integral, then \mathcal{C} is of Frobenius type, by Theorems 4.1 and 4.2.

Suppose next that $\text{FPdim } \mathcal{C} > 1$ and $\mathcal{C} \not\cong \text{rep } \mathbb{A}_5$. It follows from [6, Theorem 1.6], that if $\text{FPdim } \mathcal{C} = p^a q^b$, where p and q are prime numbers, then \mathcal{C} has nontrivial invertible objects (see Lemma 2.6).

If $\text{FPdim } \mathcal{C} = pqr$, where p, q and r are distinct prime numbers, then either \mathcal{C} is a \mathbb{Z}_2 -extension of a fusion subcategory \mathcal{D} or \mathcal{C} is group-theoretical. In the first case, we know that \mathcal{D} has nontrivial invertible objects, whence so does \mathcal{C} . Thus, we may assume that \mathcal{C} is group-theoretical, that is, \mathcal{C} is equivalent as a fusion category to the category $\mathcal{C}(G, \omega, H, \psi)$ of $k_\psi H$ -bimodules in $\mathcal{C}(G, \omega)$, where G is a group of order pqr , $H \subseteq G$ is a subgroup, $\omega \in Z^3(G, k^*)$ and $\psi \in C^2(H, k^*)$ are such that $\omega|_H = d\psi$. We may assume that $H \neq 1$ (otherwise \mathcal{C} is pointed and we are done). Then the group \widehat{H} of linear characters on H is also nontrivial, and it follows from [8, Theorem 5.2] that $G(\mathcal{C}) \neq \mathbf{1}$ also in this case.

Suppose that $\text{FPdim } \mathcal{C} = 60$. Since, by assumption, $\mathcal{C} \not\cong \text{rep } \mathbb{A}_5$, [6, Theorem 9.12] implies that \mathcal{C} has a proper fusion subcategory \mathcal{D} . The previous discussion shows that $G(\mathcal{D}) \neq \mathbf{1}$ and therefore also $G(\mathcal{C}) \neq \mathbf{1}$.

It remains to consider the cases where $\text{FPdim } \mathcal{C} = 84$ or 90 . As before, we may assume that \mathcal{C} is integral. The result follows in this case from Theorems 4.1 and 4.2, respectively. \square

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