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Dynamics of magneto electro elastic curved beams: Quantification of parametric uncertainties

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ABSTRACT

The objective of this paper is the evaluation of uncertainty propagation associated to several parameters in the dynamics of magneto-electro-elastic (MEE) curved beams. These MEE structures can be employed as imbedded parts in high performance technological systems to control motions and/or attenuate vibrations, for energy harvesting, etc. Although a lot of research connected with these structures was done for dynamics and statics, it is remarkable the scarcity of articles analyzing random dynamics of MEE structure, provided that many models have uncertainties associated to their parameters: loads and/or material properties, among others. A theory for MEE curved beams is derived and assumed as the deterministic model. The response is calculated by means of a finite element formulation. The probabilistic model is constructed appealing to the finite element formulation of the deterministic approach, by adopting random variables for the uncertain parameters selected. The probability density functions of the random variables are derived with the Maximum Entropy Principle. The Monte Carlo method is used to perform simulations with independent realizations. Studies are carried out in order to evaluate the influence of Magneto-elastic and/or piezoelectric coupling in the dynamics of MEE curved beams in both contexts: the deterministic and the stochastic.

1. Introduction

The MEE material are a kind of smart composites exhibiting various coupling effects that can be of useful in many high-tech structural applications. That is why many investigation on the mechanics of MEE structures have received considerable attention of the research community since the last 10 years.

Especial composite materials consisting of piezoelectric and magnetostrictive components are used in smart structures such as sensors, actuators, hydrophones, etc. The smart structures provide remarkable capabilities of sensing and reacting to external actions and/or disturbances, also satisfying reliability, light weight and the appropriate performances demanded in high-tech structural applications [1,2].

In the last fifteen years many researchers developed new models and technical theories for studying the mechanical response (statics, dynamics, instability, etc) of the so-called magneto electro elastic (MEE) structures. An interesting variety of models of MEE structures has been introduced principally for piezoelectric and piezomagnetic plates and shells [2,3]. In these articles the static behavior of multilayered MEE strips and plates was analyzed by means of 3D formulation and by subjecting the specimens to simpler loads, that is, to sinusoidally distributed magnetic, electric and mechanic loads. Pan and coworkers [4–6], Wu and Lu [7] and Tsai et al. [8] among others carried out extensive studies in the dynamics responses of shells and plates appealing to 3D formulations. There was also a research on MEE shells with simply or doubly curved profile [9,10].

According to a extensive bibliographical review, it is notorious the limited quantity of articles related to the study of MEE slender structures in the context of a beam model or technical theory. The papers of Milazzo and coworkers [11–13] are, apparently, the very first in which studies about the dynamics of MEE beams (for enhanced Bernoulli–Euler and/or Timoshenko theories) have been carried out. Besides these works are quite recent. However to the best of authors' knowledge, articles related to static/dynamic behavior of MEE curved beams are apparently absent.

On the other hand it should be recognized that many aspects related to the construction of MEE materials and/or structures

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are connected with a variable source of uncertainty that can substantially alter the response of the structure. Possible sources of uncertainty can be found in material properties, boundary conditions, loads [15], the hypotheses of model or the model itself [16], etc. In order to characterize the uncertain response in dynamics of structures there is a bunch of alternatives that can be collected in two master sets: parametric probabilistic approach (PPA) [15] and non-parametric probabilistic approach (NPPA) [16]. In the first case the source of uncertainty are the parameters of the model in the second case the model as a whole. In the PPA the uncertain parameters are associated to a random variable whose probability density function (PDF) is defined according to given information about them (mean values, standard deviation, bounds, etc.).

Thus, the scope of this research is directed toward offering some contributions in the mechanics/dynamics of curved MEE beams and especially to quantify the propagation of the uncertain in the dynamic response of curved and also straight MEE beams. In this context, the present article is arranged according to the following scheme: As a first step the hypotheses of the constitutive model are enunciated and the deterministic structural model is presented, then an equivalent MEE curved beam model is constructed on the basis of curved beam models previously developed by the first author [18,19] which are conceived in the context of first order shear theories. A finite element formulation is proposed and then employed to carry out calculations of the deterministic model. Subsequently, the probabilistic model is constructed employing the previous finite element formulation in which the random variables are incorporated. The PDF of the random variables (some elastic, electric properties, elastic foundations, etc) are deduced by employing the Maximum Entropy Principle [14,17] subjected to given known information such as expected values and/or COV. Then the Monte Carlo method is employed to simulate realizations, the statistical analysis is done and the results presented in the form of frequency response functions or other graphics of statistical interest.

2. Formulation of the deterministic model

2.1. Basic hypotheses of the structural model

The Magneto-electro-elastic structure of this article is a thin curved strip supported on elastic foundation as the one shown in Fig. 1 with the reference system located in the geometric centroid of the cross section. The curved beam has a circumferential length of $L = R\beta$, a radial thickness of h, width of b and a constant radius of curvature R.

The deterministic model for this study is based on the following assumptions: (a) The motion of the curved beam is constrained in the curvature plane (XY), (b) Shear flexibility is considered, (c) the material is supposed to be poled in the radial direction and it consists of a mixture in given proportions of $BaTiO_3$ and $CoFe_2O_4$, (d) the electric and magnetic fields are determined through their corresponding potentials which are prescribed on the cylindrical surfaces (i.e. $y = R \pm h/2$, or $y = R_i, y = R_o$); (e) The radial components of the electric and magnetic fields are substantially greater than the circumferential components ($Ex \ll Ey$ $Hx \ll Hy$), (f) A generic elastic foundation (characterized with spring coefficients) is assumed, (g) the structural damping is considered as "a posteriori" incorporation in the finite element formulation.

Employing the hypotheses, the displacement field can be derived [19] as:

$$u_x(x,y,t) = u_{xc} - y\left(\theta_z - \frac{u_{xc}}{R}\right)$$
$$u_y(x,y,t) = u_{yc} \tag{1}$$

where u_{xc} and u_{yc} are the circumferential and radial displacements of the reference point **C** whereas θ_z rotation parameter. The representative strain components can be written in the following form [19]:

$$\varepsilon_{xx} = (\varepsilon_{D1} - y\varepsilon_{D2})\mathcal{F}$$
$$\gamma_{xy} = \varepsilon_{D3}\mathcal{F} \tag{2}$$

where:

$$\varepsilon_{D1} = u'_{xc} + \frac{u_{yc}}{R}, \quad \varepsilon_{D2} = \theta'_z - \frac{u'_{xc}}{R}, \quad \varepsilon_{D3} = u'_{yc} - \theta_z, \quad \mathcal{F} = \frac{R}{R+y} \tag{3}$$

In Eq. (3), the apostrophes represent derivatives with respect to the spatial variable x. Moreover, ε_{D1} can be interpreted as the generalized circumferential strain, ε_{D2} as the generalized bending curvature and ε_{D3} as the generalized shear strain.

The mechanical equilibrium equations of the curved MEE beam supported on the elastic foundation can be written in the following form:

$$-Q'_x + \frac{M'_z}{R} + k_1 u_{xc} + \mathcal{M}_1\left(\ddot{u}_{xc}, \ddot{u}_{yc}, \ddot{\theta}_z\right) - \mathcal{P}_1(x) = 0$$
$$-Q'_y + \frac{Q_x}{R} + k_2 u_{yc} + \mathcal{M}_2\left(\ddot{u}_{xc}, \ddot{u}_{yc}, \ddot{\theta}_z\right) - \mathcal{P}_2(x) = 0 \tag{4}$$
$$-M'_z - Q_y + k_3\theta_z + \mathcal{M}_3\left(\ddot{u}_{xc}, \ddot{u}_{yc}, \ddot{\theta}_z\right) - \mathcal{P}_3(x) = 0$$

with the corresponding boundary conditions:

$$-\widetilde{Q}_x + \frac{\widetilde{M}_z}{R} + Q_x - \frac{M_z}{R} = 0, \quad or \quad u_{xc} = 0$$
$$-\widetilde{Q}_y + Q_y = 0, \quad or \quad u_{yc} = 0 \tag{5}$$
$$-\widetilde{M}_z + M_z = 0, \quad or \quad \theta_z = 0$$

In the previous expressions, Q_x is the axial force, Q_y is the shear force M_z is the bending moment, $\mathcal{P}_i, i = 1,2,3$ represent distributed forces and moments, $\mathcal{M}_i, i = 1,2,3$ are inertial forces, whereas $\widetilde{Q}_x, \widetilde{Q}_y$ and $\widetilde{M}_z$ are prescribed forces at the beam ends. These entities are defined as:

$$\{Q_x, Q_y, M_z\} = \int_A \{\sigma_{xx}, \sigma_{xy}, -y\sigma_{xx}\} dA \tag{6}$$

$$\begin{Bmatrix} \mathcal{M}_1 \\ \mathcal{M}_2 \\ \mathcal{M}_3 \end{Bmatrix} = \begin{bmatrix} J^{\rho}_{11} & 0 & J^{\rho}_{13} \\ 0 & J^{\rho}_{22} & 0 \\ J^{\rho}_{13} & 0 & J^{\rho}_{33} \end{bmatrix} \begin{Bmatrix} \ddot{u}_{xc} \\ \ddot{u}_{yc} \\ \ddot{\theta}_z \end{Bmatrix} \tag{7}$$

In Eq. (7) $J^{\rho}_{ik}, i,k \to 1,2,3$ are inertia constants that are described extensively in Appendix A.

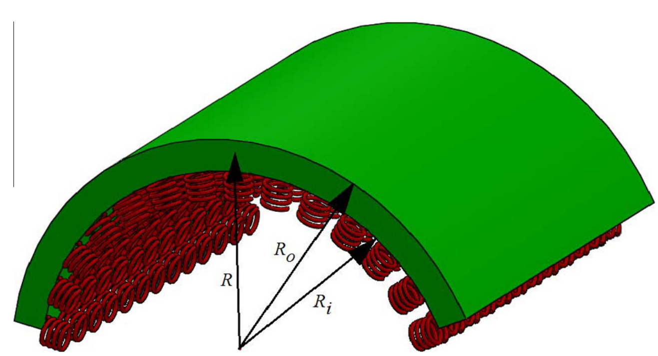

Fig. 1. Diagram of the Curved MEE beam.

2.2. Deduction of potentials and constitutive equations

The constitutive equations of a magneto-electro-elastic solid under the hypothesis of plane stress – assuming $\sigma_{yy} \ll \sigma_{xx}$ and employing the hypothesis (d) – can be written in the following matrix form [23,11,12]:

$$\begin{Bmatrix} \sigma_{xx} \\ \sigma_{xy} \\ D_x \\ D_y \\ B_x \\ B_y \end{Bmatrix} = \begin{bmatrix} c^*_{11} & 0 & -e^*_{21} & -q^*_{21} \\ 0 & c^*_{66} & 0 & 0 \\ 0 & e^*_{16} & 0 & 0 \\ e^*_{21} & 0 & \eta_{22} & d_{22} \\ 0 & q^*_{16} & 0 & 0 \\ q^*_{21} & 0 & d_{22} & \mu_{22} \end{bmatrix} \begin{Bmatrix} \varepsilon_{xx} \\ \gamma_{xy} \\ E_y \\ H_y \end{Bmatrix} \tag{8}$$

and σ_{ij} are the stresses, ϵ_{ij} and γ_{xy} are the axial and shear strain components; D_i and B_i are the components of the electric displacement and the magnetic flux, respectively; E_i and H_i are the components of the electric field and the magnetic field, respectively; c^*_{ik} are modified elastic coefficients; η_{ij} are the dielectric coefficients; μ_{ij} are the magnetic permeability coefficients; e^*_{ij} are the modified piezoelectric coefficients; q^*_{ij} are the modified piezomagnetic coefficients and d_{ij} are the magnetoelectric coefficients.

It is interesting to note that from Eq. (8), the electric displacement and magnetic flux in x-direction (D_x and B_x, respectively) depend only on the shear strain γ_{xy}. These constitutive expressions allow the calculation of the electric and the magnetic potentials in explicit form (see reference [11] for detailed issues).

The electric and magnetic fields are defined in terms of the electric and magnetic potentials (i.e. $\overline{E} = -\bar{\nabla}\Phi$ and $\overline{H} = -\bar{\nabla}\Psi$, with $\bar{\nabla}$ as the gradient operator). The equilibrium equations of electrostatics and magnetostatics are [23,13]:

$$\begin{aligned} \bar{\nabla} \cdot \overline{D} &= 0 \\ \bar{\nabla} \cdot \overline{B} &= 0 \end{aligned} \tag{9}$$

where $\overline{D} = \{D_x, D_y, D_z\}, \overline{B} = \{B_x, B_y, B_z\}$, and $\bar{\nabla}\cdot$ is the divergence operator (in curvilinear coordinates).

Now, employing Eq. (8) together with Eq. (9) and after some algebraic handling it is possible to arrive to:

$$\begin{aligned} \Phi &= \mathcal{D}_1\Phi_0 + \mathcal{D}_2\Phi_1 + \mathcal{D}_3\varepsilon'_{D3} + \mathcal{D}_4\varepsilon_{D2} \\ \Psi &= \mathcal{B}_1\Psi_0 + \mathcal{B}_2\Psi_1 + \mathcal{B}_3\varepsilon'_{D3} + \mathcal{B}_4\varepsilon_{D2} \end{aligned} \tag{10}$$

where the coefficients $\mathcal{D}_i, \mathcal{B}_i$, $i = 1, \ldots, 4$ defined extensively in Appendix B.

According to the definition of the internal forces given in Eq. (6) and employing Eq. (8) it is possible to derive the following expression of the internal forces in terms of strain components and electro/magnetic potentials:

$$\bar{\mathbf{Q}} = \mathbf{M}_{CA}\bar{\varepsilon}_D + \mathbf{M}_{CB}\bar{\varepsilon}'_D + \mathbf{M}_{CE}\bar{\mathbf{P}}_{\Phi\Psi} \tag{11}$$

where:

$$\begin{aligned} &\bar{\mathbf{Q}} = \{Q_x, M_z, Q_y\}^T, \bar{\varepsilon}_D = \{\varepsilon_{D1}, \varepsilon_{D2}, \varepsilon_{D3}\}^T \\ &\bar{\mathbf{P}}_{\Phi\Psi} = \{\Phi_0, \Phi_1, \Psi_0, \Psi_1\}^T \\ &\mathbf{M}_{CA} = \begin{bmatrix} J_{X1} & J_{X2} & 0 \\ J_{M1} & J_{M2} & 0 \\ 0 & 0 & J_{Y3} \end{bmatrix}, \quad \mathbf{M}_{CB} = \begin{bmatrix} 0 & 0 & J_{X4} \\ 0 & 0 & J_{M4} \\ 0 & 0 & 0 \end{bmatrix} \\ &\mathbf{M}_{CE} = \begin{bmatrix} J_{X5} & J_{X6} & J_{X7} & J_{X8} \\ J_{M5} & J_{M6} & J_{M7} & J_{M8} \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{aligned} \tag{12}$$

besides $J_{Xi}, J_{Mi}, J_{Yi}, i = 1, \ldots, 8$ are constants that depend on the elastic, electric and magnetic properties (extensive expressions can be followed in Ref. [20]).

2.3. Weak formulation and finite element approach of MEE curved model

Substituting Eq. (11) in Eq. (4) and incorporating the boundary conditions given in Eq. (4) it is possible to write the following weak form of the equilibrium equations of the MEE curved beam model:

$$\begin{aligned} &\int_L [Q_x\delta\varepsilon_{D1} + Q_y\delta\varepsilon_{D2} + M_z\delta\varepsilon_{D3} + \mathcal{M}_1\delta u_{xc} + \mathcal{M}_2\delta u_{yc} + \mathcal{M}_3\delta\theta_z]dx \\ &+ \int_L [(k_1u_{xc} - \mathcal{P}_1)\delta u_{xc} + (k_2u_{yc} - \mathcal{P}_2)\delta u_{yc} - (k_3\theta_z + \mathcal{P}_3)\delta\theta_z] \\ &- \left[\left(Q_x - \frac{M_z}{R}\right)\delta u_{xc} + Q_y\delta u_{yc} - M_z\delta\theta_z\right]\Bigg|_{x=0}^{L=0} = 0 \end{aligned} \tag{13}$$

The weak variational formulation (13) represents the equilibrium mechanics of a MEE curved beam. The solution of this formulation for given problems (dynamics and/or statics) can be tackled by means of many approaches: Generalized direct variational approaches, finite elements approaches among many other alternatives.

A Finite Element formulation can be derived through discretization of Eq. (13). The discretization is carried out using isoparametric elements with five nodes and shape functions of quartic order.

Each kinematic variable $u_{xc} \equiv U_1, u_{yc} \equiv U_2$ and $\theta_z \equiv U_3$, can be interpolated within the element by means of following compact form [21]:

$$U_i = \mathbf{F}_i\bar{\mathbf{U}}_e, \quad i = 1, \ldots, 3 \tag{14}$$

where the matrix $\mathbf{F}$ $[3 \times 15]$ collects the shape functions $f_j(\bar{x})$, $j = 1, \ldots, 5$, for the isoparametric elements of quartic order. Each variable $U_i, i = 1, \ldots, 3$ is interpolated with the same shape functions in $\mathbf{F}$. $\bar{\mathbf{U}}_e$ is the vector of kinematic nodal variables.

Now substituting Eq. (14) in Eqs. (3) and (11) and then in Eq. (13) it is possible to derive the following equation in the elementary domain:

$$\begin{aligned} &\delta\mathbf{U}_e^T\left[\int_{l_e}\mathbf{F}^T\mathbf{D}_{D1}^T\mathbf{M}_{CA}\mathbf{D}_{D1}\mathbf{F}l_e d\xi\right]\mathbf{U}_e + \delta\mathbf{U}_e^T\left[\int_{l_e}\mathbf{F}^T\mathbf{D}_{D1}^T\mathbf{M}_{CB}\mathbf{D}'_{D1}\mathbf{F}l_e d\xi\right]\mathbf{U}_e \\ &\quad + \delta\mathbf{U}_e^T\left[\int_{l_e}\mathbf{F}^T\mathbf{D}_{D1}^T\mathbf{M}_{CE}\bar{\mathbf{P}}_{\Phi\Psi}l_e d\xi\right] + \delta\mathbf{U}_e^T\left[\int_{l_e}\mathbf{F}^T\mathbf{M}_{CI}\mathbf{F}l_e d\xi\right]\ddot{\mathbf{U}}_e \\ &+ \delta\mathbf{U}_e^T\left[\int_{l_e}\mathbf{F}^T\mathbf{M}_{CF}\mathbf{F}l_e d\xi\right]\mathbf{U}_e - \delta\mathbf{U}_e^T\left[\int_{l_e}\mathbf{F}^T\bar{\mathcal{P}}l_e d\xi\right] - \delta\mathbf{U}_e^T\left(\mathbf{F}^T\bar{\mathbf{P}}_{BC}\right)\Big|_0^{l_e} = 0 \end{aligned} \tag{15}$$

where l_e is the length of the element, $\xi = x/l_e$ is the internal coordinate, $\mathbf{M}_{CA}, \mathbf{M}_{CB}, \mathbf{M}_{CE}$ and $\bar{\mathbf{P}}_{\Phi\Psi}$ are defined in Eq. (12), $\mathbf{M}_{CI}$is defined in Eq. (7), $\bar{\mathcal{P}}$ is the vector of distributed forces, $\bar{\mathbf{F}}_{BC}$ is a vector of prescribed forces at the ends of the element, the matrix $\mathbf{M}_{CF} = diag(k_1, k_2, k_3)$ is the matrix of elastic foundation constants and $\mathbf{D}_{D1}$ is a differential operator.

Employing the usual assembly procedures of the finite element method with the matrix structure of Eq. (15) and incorporating the "a posteriori" proportional damping ($\mathbf{C}_{SD} = \eta_1\mathbf{M} + \eta_2\mathbf{K}$ [19]) one gets:

$$\mathbf{K}\bar{\mathbf{W}} + \mathbf{C}_{SD}\dot{\bar{\mathbf{W}}} + \mathbf{M}\ddot{\bar{\mathbf{W}}} = \bar{\mathbf{F}}_{\mathbf{M}} - \bar{\mathbf{F}}_{\mathbf{em}}. \tag{16}$$

where $\mathbf{K}$ and $\mathbf{M}$ are the global matrices of elastic stiffness and mass, respectively; whereas $\bar{\mathbf{W}}, \dot{\bar{\mathbf{W}}}$ and $\ddot{\bar{\mathbf{W}}}$ are the global vectors of nodal displacements, velocities and accelerations, finally $\bar{\mathbf{F}}_{\mathbf{M}}$ and $\bar{\mathbf{F}}_{\mathbf{em}}$ are the global vectors of forces due to mechanical components strictly and forces due to electromagnetic loads applied in boundaries and/or the curved beam span, respectively.

Table 1
Natural Frequencies: comparison of the present model reduced to the case of straight beam.

Bounds	Clamped–free				Clamped–clamped			
Mode	Type	Present	Annigeri	Milazzo	Type	Present	Annigeri	Milazzo
1	*Fle*	169	170	169	*Fle*	1052	1055	1053
2	*Fle*	1041	1044	1043	*Fle*	2805	2806	2800
3	*Fle*	2833	2835	2831	*Fle*	5277	5267	5250
4	*Ext*	3867	3902	–	*Ext*	7734	7813	–
5	*Fle*	5341	5337	5323	*Ext*	8321	8291	–
6	*Fle*	8441	8423	8385	*Fle*	11815	11759	11675
7	*Ext*	11602	11705	–	*Fle*	15469	15573	15423
8	*Fle*	12005	11967	11887	*Ext*	15657	15624	–

From the previous Eq. (16) two possible issues are analyzed in the present investigation: transient analysis in the frequency domain and eigenvalues. Thus, the response in the frequency domain, after a Fourier transformation of the discrete model given in Eq. (16) is written as [25]:

$$\widehat{\mathbf{W}}(\omega) = \left[-\omega^2\mathbf{M} + i\omega\mathbf{C}_{SD} + \mathbf{K}\right]^{-1}\widehat{\mathbf{F}}(\omega), \tag{17}$$

where $\widehat{\mathbf{W}}$ and $\widehat{\mathbf{F}}$ are the Fourier transform of the displacement vector and force vector, respectively; whereas ω is the circular frequency measured in [rad/s]. Moreover, Eq. (16) can be reduced to calculate the dynamic eigenvalues (or natural frequencies) by neglecting damping and all applied forces and by assuming harmonic motion:

$$\mathbf{K}\bar{\mathbf{W}} - \omega^2\mathbf{M}\bar{\mathbf{W}} = \bar{\mathbf{0}}. \tag{18}$$

where ω is now the eigenvalue for free vibrations. Although Eqs. (17) and (18) are deduced without the presence of electromagnetic forces, they contain the coupled MEE nature by means of the constitutive relations given in Eq. (12).

3. Construction of the probabilistic model

The construction of the probabilistic model is connected with the employment of the finite element formulation of the previously developed deterministic model, as a mean expected response. The Maximum Entropy Principle (MEP) is used to derive the probability density functions (PDF) of the random variables associated with the uncertain parameters [14]. This particular is quite sensitive in stochastic analysis and PDF's should be deduced according to the given information (normally and sensitively scarce) about the uncertain parameters. The deterministic model developed has many parameters that can be uncertain, however the most relevant could be the material properties and the spring constants of the elastic foundation.

In the present problem the random variables $V_i, i = 1, 2, 3 \dots 13$ are introduced such that they represent the equivalent material properties $(i = 1, \dots, 10)$ and the elastic foundation parameters $(i = 11, 12, 13)$. The expected value of the random variables is known and it has the nominal value of the parameters in the deterministic model, i.e.: $\mathcal{E}\{V_i\} = \underline{V}_i, i = 1, 2, 3 \dots 13$. Moreover the random variables have bounded supports whose upper and lower limits can be represented in terms of given information (standard deviation or coefficient of variation). Provided that there is no information about the correlation or dependency of material properties, random variables $V_i, i = 1, \dots, 13$, according to MEP, are assumed independent and non correlated [15,17,24]. Consequently, taking into account the previous conditions, the PDF's of the random variables can be written as:

$$p_{V_i}(v_i) = \mathfrak{S}_{[\mathcal{L}_{V_i}\mathcal{U}_{V_i}]}(v_i)\frac{1}{2\sqrt{3}\underline{V}_i\delta_{V_i}}, \quad i = 1, \dots, 13 \tag{19}$$

where $\mathfrak{S}_{[\mathcal{L}_{V_i}\mathcal{U}_{V_i}]}(v_i)$ is the generic support function, whereas $\mathcal{L}_{V_i}$ and $\mathcal{U}_{V_i}$ are the lower and upper bounds of the random variable V_i. $\underline{V}_i$ is the expect value (or deterministic value) of the *i*th random variable, whereas δ_{V_i} is its coefficient of variation. The Matlab function unifrnd$\left(\underline{V}_i\left(1-\delta_{V_i}\sqrt{3}\right), \underline{V}_i\left(1+\delta_{V_i}\sqrt{3}\right)\right)$ can be used to generate realizations of random variables $V_i, i = 1, \dots, 13$. Then, using Eq. (19) in the construction of the matrices of finite element formulation given in Eq. (17), the stochastic finite element model can be written as:

$$\widehat{\mathbb{W}}(\omega) = \left[-\omega^2\mathbf{M} + i\omega\mathbb{C}_{RD} + \mathbb{K}\right]^{-1}\widehat{\mathbf{F}}(\omega). \tag{20}$$

Notice that in Eq. (20) the math-blackboard typeface is employed to indicate stochastic entities.

The Monte Carlo method is used the simulate the stochastic dynamics, which implies the calculation of a deterministic system for each independent realization of random variables $V_i, i = 1, 2, .., 13$. The convergence of the stochastic response $\widehat{\mathbb{W}}$ can be calculated with the following expression:

$$conv(N_{MS}) = \sqrt{\frac{1}{N_{MS}}\sum_{j=1}^{N_{MS}}\int_{\Omega}\left\|\widehat{\mathbb{W}}_j(\omega) - \widehat{\mathbf{W}}(\omega)\right\|^2 d\omega}, \tag{21}$$

where N_{MS} is the number of Monte Carlo samplings and Ω is the frequency band of analysis. Clearly, $\widehat{\mathbb{W}}$ is the response of the stochastic model and $\widehat{\mathbf{W}}$ the response of the mean model or deterministic model.

4. Studies on mee curved beams

4.1. Validation and comparative studies

In this section a comparison and validation of the deterministic MEE curved beam model with respect to other approaches is performed. The first example corresponds to a comparison of the present MEE curved beam model reduced to the case of straight beam (i.e. $R \to \infty$), with respect to the beam model of Milazzo et al. [11] and the 2D finite element approach of Biju et al. [22], and Annigeri et al. [23]. The material properties of the MEE composite with 40% $BaTiO_3$ and 60% $CoFe_2O_4$ are taken from the work of Annigeri et al. [23] and the equivalent constants for the beam formulation are: $c^*_{11} = 120.67$ GPa, $c^*_{66} = 45$ GPa, $e^*_{21} = 6.52$ C/m^2, $e^*_{16} = 0$, $q^*_{21} = 32.66$ N/(Am), $q^*_{16} = 180$ N/(Am), $\eta_{22} = -8.97 \times 10^{-9}$ Ns/(CV), $d_{22} = 8.85 \times 10^{-9}$ F/m, $\mu_{22} = 7.54 \times 10^{-3}$ Ns^2/C^2 and $\rho = 5600$ Kg/m^3. These values are nearly the same of those calculated by Milazzo et al. [11] with a difference in percentage no higher than 0.7%. The dimensions of the beam are $L = 0.3$ m, $h = 0.02$ m and $b = 1.0m$ (in order to compare with the 2D FEM approach [23]). The boundary conditions are clamped–free or clamped–clamped.

Table 2
Natural Frequencies of a clamped–free curved MEE beam (considering and neglecting MEE coupling): comparison of the present model with 2D FEM approach (without piezoelectric effect).

Relations	$L/R = 1.0$			$L/R = 0.3$		
Mode	2D FEM	w/o coup	With coup	2D FEM	w/o coup	with coup
1	170	169	173	167	166	170
2	921	921	938	1010	1011	1030
3	2629	2636	2682	2754	2764	2814
4	4358	4349	4363	3928	3923	3924
5	5292	5318	5398	5216	5253	5343
6	8183	8266	8398	8215	8302	8437
7	11488	11585	11664	11574	11602	11611
8	11983	12016	12118	11714	11840	12012

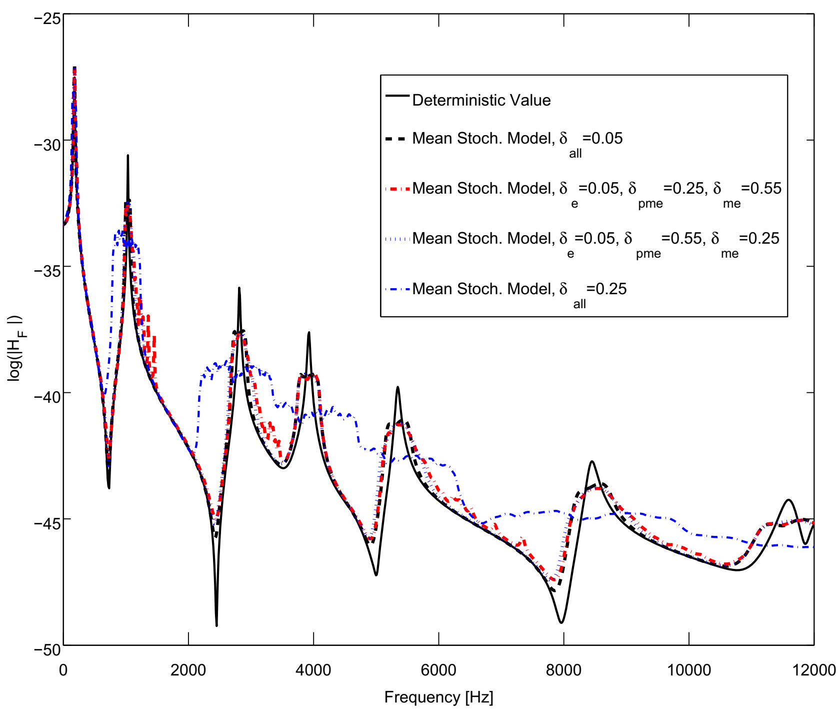

Fig. 2. Deterministic FRF and Stochastic mean response with different dispersion parameters in all properties.

In Table 1 it is shown the comparison of the first 8 natural frequencies calculated with the present model and the models derived by Milazzo et al. [11] and Annigeri et al. [23]. It is observed that the present model can recover both extensional and bending modes with difference in percentage no higher than 1.0 % with respect to Annigeri's results [23] associated to a 2D FEM approach.

In Table 2 the first 8 frequencies of a curved MEE beam are shown and a comparison between various approaches and effects is also presented. The curved beam is clamped–free and has the same length, height and width of the previous example. In fact the present 1D model is compared with a 2D approach (performed under a plane-stress state in a general purpose finite element platform). Two alternatives in the 1D model are evaluated: considering full electromagnetic coupling in beam coefficients (i.e. in J_{Xi} and $J_{Mi}, i = 1, \ldots, 4$) or neglecting that effect by imposing $e^*_{21} = 0$, $e^*_{16} = 0, q^*_{21} = 0, q^*_{16} = 0$. It is possible to note that 2D results are nearly compatible with the case where the full coupling is neglected with a difference in percentage lower than 1.0 %. On the other hand, in the case of full MEE coupling in the calculation of beam sectional properties it is seen an increase of 3.0 to 4.0 % in the frequencies when compared with the previous two cases.

4.2. Uncertainties in the dynamic response

In this section a study about the propagation of uncertainty in the frequency response function (FRF) associated to several parameters is carried out. The example fr the analysis is a clamped–free curved beam with the same dimensions of the precedent paragraph and $L/R = 0.3$. The FRF measure is characterized by $H_F = u(L)/F(L)$ where $F(L)$ is a unitary shock force [15].

Fig. 2 shows the deterministic FRF compared with four stochastic responses. The stochastic responses correspond to cases where the variability is associated with the material properties (the parameter of the elastic foundation os assumed constant in this calculation). There is also another distinction topic: the parameters related to elastic properties (δ_e), the parameters related to magneto-electric coupling properties (δ_{me}) and the parameters related to magneto-piezo-electric properties (δ_{pme}). Then recalling Fig. 2, one can see that the three cases, in which the elastic parameters ($\delta_e = 0.05$) are lower, manifest nearly the same response. Nevertheless little pikes can be found in some locations (e.g. after the second frequency pike as one can see in Fig. 3) in the case when

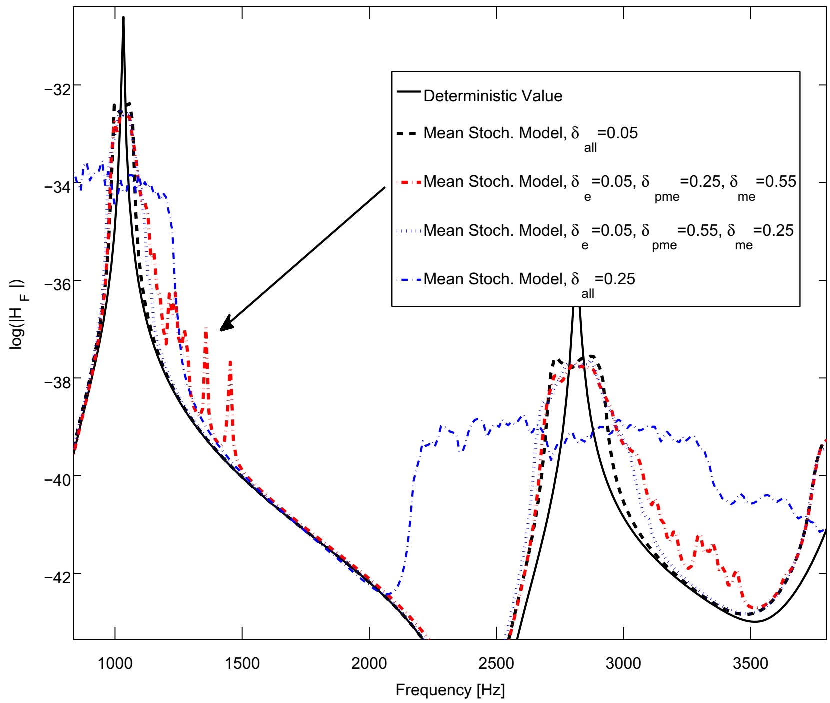

Fig. 3. Zoom of the FRF nearly the second frequency.

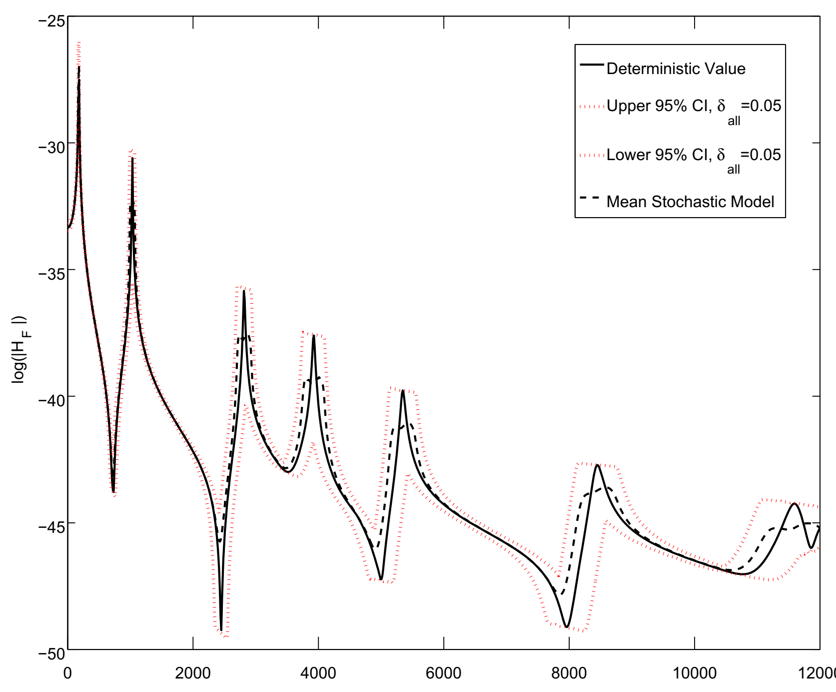

Fig. 4. Uncertainty propagation due to a low dispersion parameter in all properties.

the uncertainty in the magneto-electric parameters is extreme, i.e. $\delta_{me} = 0.55$. If the uncertainty, in the parameters associated with elastic property, is higher the mean stochastic response diverges substantially from the other three that have the lower variation in the elastic properties.

In Figs. 4–6 one can see the propagation of uncertainty associated to several parameters of the structure, i.e. the deterministic response is shown together with the mean stochastic response and the 95% confidence interval bounds. Effectively, in Figs. 5 and 6 one can see how sensitive is the response to the variation of magneto-electric properties identified with a coefficient of variation $\delta_{em} = 0.55$ which is a little bit lower than the possible limit of the selected random variable (i.e. $\delta_{em} < \sqrt{3}$, that identifies a 2nd order random variable). On the other hand, a nearly extreme

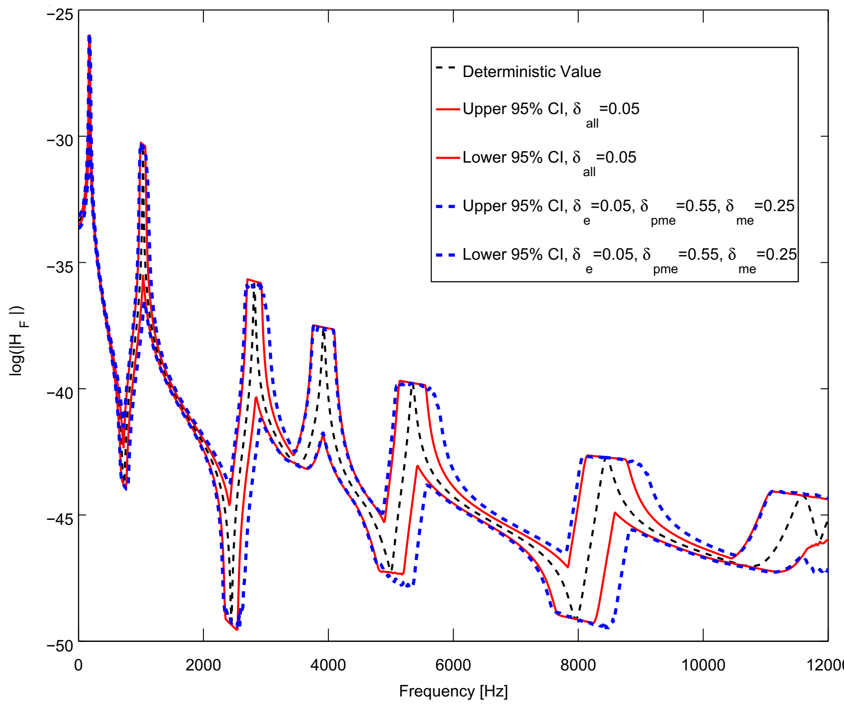

Fig. 5. Comparison of uncertainty propagation due high dispersion parameter in magneto-electric properties.

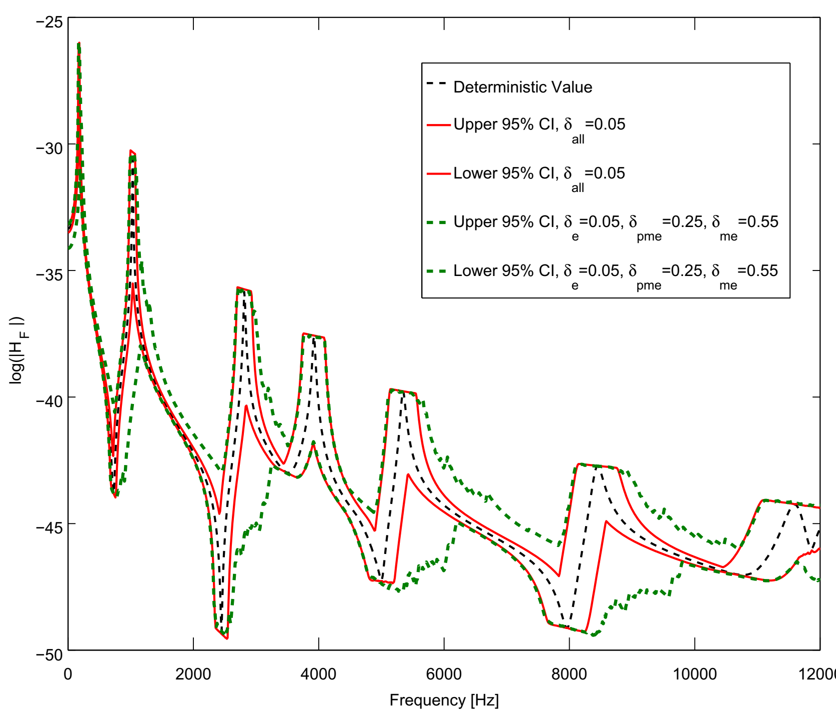

Fig. 6. Comparison of uncertainty propagation due high dispersion parameter in piezo-electric/piezo magnetic properties.

variation in the coupled piezo-electric/piezo magnetic parameters, identified by $\delta_{pme} = 0.55$, is not so sensitive in comparison to the case of standard variability, that is related to a variation of the order of $\delta_{all} = \delta_{V_i} = 0.05$ in all material parameters.

In Fig. 7 it is shown the effect of uncertainty propagation in the frequency response due to variation in the elastic foundation coefficients, and maintaining the material properties in their nominal deterministic values. As one can see an extreme variation of the coefficients of variations ($\delta_{ef} \equiv \delta_{11} \equiv \delta_{12} \equiv \delta_{13}$) does not seem to be sensitive in the dynamic response of the structure.

5. Conclusions

In this article a new model for studying vibrations in MEE curved beams has been introduced. The model has been implemented in the context of finite element methodologies and

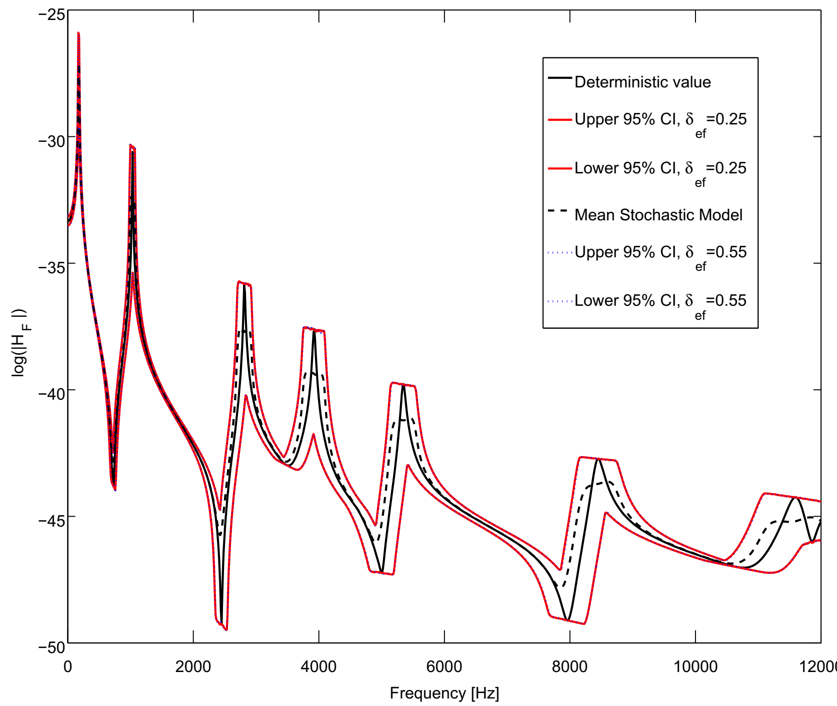

Fig. 7. Uncertainty propagation due to the elastic foundation.

employed to calculate vibration patterns of curved MEE beams. The model can reproduce results of straight beams with or without electromagnetic coupling. Also the curved beam model compares well with 2D approaches (plane stress state) when the piezoelectric effect is neglected. Some preliminary studies for quantification of uncertainty due to given parameters of the model have been carried out. The propagation of uncertainty due to material properties proved to be substantially more sensitive than the one related to the elastic foundation. The uncertainty in the elastic properties proved to be more sensitive than the other properties. However this is an extreme case. Nevertheless if the elastic properties are appropriately known the uncertainty propagation due to the magnetic-electric properties is more sensitive than the corresponding to the coupled piezo-electric/piezo magnetic properties. Although in the case of elastic foundations the analysis of uncertainty could be performed through stochastic fields, it should be the matter of future research based on the present one.

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Appendix A

The inertia constants $J^{\rho}_{ik}, i,k \to 1,2,3$ are calculated in the following compact form:

$$J^{\rho}_{ik} = \int_A \rho \bar{g}^{(a)}_i \bar{g}^{(a)}_k \mathcal{F}^{-1} dA + \int_A \rho \bar{g}^{(b)}_i \bar{g}^{(b)}_k \mathcal{F}^{-1} dA \tag{22}$$

in which $\bar{g}^a$ and $\bar{g}^b$ are defined as:

$$\begin{aligned}\bar{g}^{(a)} &= \{\mathcal{F}^{-1}, 0, -y\} \\ \bar{g}^{(b)} &= \{0, 1, 0\}\end{aligned} \tag{23}$$

The MEE beam equivalent constants are defined calculated as:

$$\begin{aligned}&J_{X1} = \int_A c^*_{11} \mathcal{F} dA, \quad J_{X2} = \int_A \left[-c^*_{11} y \mathcal{F} + e^*_{21} \frac{\partial \mathcal{D}_4}{\partial y} + q^*_{21} \frac{\partial \mathcal{B}_4}{\partial y}\right] dA, \\ &J_{X4} = \int_A \left[e^*_{21} \frac{\partial \mathcal{D}_3}{\partial y} + q^*_{21} \frac{\partial \mathcal{B}_3}{\partial y}\right] dA, \quad J_{Y3} = \int_A c^*_{66} \mathcal{F} dA\end{aligned} \tag{24}$$

$$\begin{aligned}&J_{M1} = -\int_A c^*_{11} y \mathcal{F} dA, J_{M2} = \int_A \left[c^*_{11} y^2 \mathcal{F} - e^*_{21} y \frac{\partial \mathcal{D}_4}{\partial y} - q^*_{21} y \frac{\partial \mathcal{B}_4}{\partial y}\right] dA, \\ &J_{M4} = -\int_A y \left[e^*_{21} \frac{\partial \mathcal{D}_3}{\partial y} + q^*_{21} \frac{\partial \mathcal{B}_3}{\partial y}\right] dA,\end{aligned} \tag{25}$$

$$\begin{aligned}&\{J_{X5}, J_{X6}, J_{X7}, J_{X8}\} = \int_A \{e^*_{21}, e^*_{21}, q^*_{21}, q^*_{21}\} \cdot \frac{\partial}{\partial y} \{\mathcal{D}_1, \mathcal{D}_2, \mathcal{B}_1, \mathcal{B}_2\} dA, \\ &\{J_{M5}, J_{M6}, J_{M7}, J_{M8}\} = -\int_A y \{e^*_{21}, e^*_{21}, q^*_{21}, q^*_{21}\} \cdot \frac{\partial}{\partial y} \{\mathcal{D}_1, \mathcal{D}_2, \mathcal{B}_1, \mathcal{B}_2\} dA\end{aligned} \tag{26}$$

Appendix B

Coefficients $\mathcal{D}_i$, and $\mathcal{B}_i, i = 1, \ldots, 4$ are calculated in the subsequent form (by means of a Taylor expansion in R):

$$\begin{aligned}&\mathcal{D}_1 = \mathcal{B}_1 = \left(\frac{1}{2} - \frac{y}{h}\right) + \frac{-\frac{h}{8} + \frac{y^2}{2h}}{R} + \frac{\frac{hy}{12} - \frac{y^3}{3h}}{R^2} \\ &\mathcal{D}_2 = \mathcal{B}_2 = \left(\frac{1}{2} + \frac{y}{h}\right) + \frac{\frac{h}{8} - \frac{y^2}{2h}}{R} + \frac{-\frac{hy}{12} + \frac{y^3}{3h}}{R^2}\end{aligned} \tag{27}$$

$$\begin{aligned}&\mathcal{D}_3 = A_{11} \left[\left(-\frac{1}{8} h^2 + \frac{y^2}{2}\right) + \frac{h^2 y - 4y^3}{8R} + \frac{-5h^4 - 24h^2 y^2 + 176y^4}{384R^2}\right] \\ &\mathcal{D}_4 = A_{12} \left[\frac{1}{8}\left(h^2 - 4y^2\right) + \frac{-\frac{1}{12} h^2 y + \frac{y^3}{3}}{R} + \frac{h^4 + 8h^2 y^2 - 48y^4}{192R^2}\right]\end{aligned} \tag{28}$$

$$\mathcal{B}_3 = A_{21}\left[\left(-\frac{1}{8}h^2 + \frac{y^2}{2}\right) + \frac{h^2 y - 4y^3}{8R} + \frac{-5h^4 - 24h^2y^2 + 176y^4}{384R^2}\right]$$
$$\mathcal{B}_4 = A_{22}\left[\frac{1}{8}\left(h^2 - 4y^2\right) + \frac{-\frac{1}{12}h^2 y + \frac{y^3}{3}}{R} + \frac{h^4 + 8h^2y^2 - 48y^4}{192R^2}\right] \tag{29}$$

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