Random Sampling Applied to the Measurement of a DC Signal Immersed in Noise

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***Abstract*—This paper introduces the use of random sampling for the recovery of dc signals immersed in noise. This technique avoids the use of antialiasing filters even if the disturbance frequencies are higher than the maximum sampling frequency available. The use of random sampling and a moving average (MA) filter for the measurement of dc signals is mathematically and experimentally demonstrated.**

***Index Terms*—A/D acquisition, dc measurement, random sampling, signal processing.**

I. Introduction

MEASUREMENTS of a dc component immersed in an ac disturbance require either analog or digital processing in order to recover the actual value of the dc component. When digital processing is used, the signal to be measured has to be sampled with a frequency higher than the double of the highest input frequency (Shannon Theorem). There are certain cases in which the acquisition system has restrictions to meet this requirement. For example, the acquisition time of the system could not be short enough. This is even more likely if the same A/D converter acquires numerous multiplexed channels. In these cases, a bulky and expensive low-frequency antialiasing filter per input is needed. Furthermore, the use of a single filter at the multiplexer output results in unacceptable settling times even for low-resolution acquisition system.

Elimination of antialiasing filters leads to distortion in the measurement because the information resulting from the uniform sampling can be wrongly interpreted as a dc. Moreover, a low-frequency distortion is more noticeable when the sampling frequency is close to any disturbance harmonics. These errors are bigger when the correlation between the interfering signal and the sampling function is higher. For this reason, it is worth considering the use of random sampling instead of uniform. The purpose of the random sampling is to reduce or even eliminate the correlation between the disturbance and the sampling signal.

The issue of nonuniform sampling was already addressed by Steiglitz [1], and later on by Oppenheim and Johnson [2]. They analyzed its possible application for estimating spectral functions. Filicori *et al.* [3] applied a random sampling strategy and an associated filtering algorithm for the efficient implementation of broad-band measurement instruments based on nonlinear signal conversion. Recently, random sampling has been used for measuring spectral functions [4], [5]. Other authors, such as Kan *et al.* [6] and Frey *et al.* [7], used it to analyze the distortions of the random jitter that occur in those systems where uniform sampling is required.

In the following sections, the elimination of antialiasing filters by using random sampling is analyzed. In Section II, the mathematical formulation of the proposal is developed. The formulation considers a moving average (MA) filter as digital processing. Numerical simulations and experimental results are presented in Sections III and IV, respectively.

II. Mathematical Formulation

The mathematical formulation of the method is presented in this section. The analysis considers that the signal to be eliminated is a pure sinusoidal. This is compatible with an impulsive disturbance, which means signals of the type

$$f(t) = \sum^{n} \cos(n\omega t). \tag{1}$$

The interference sinusoidal signal has the following characteristics:

- frequency f unknown;
- random phase ϕ, with a uniform probability density function (uniform pdf), ranging from $-\pi$ to π;
- amplitude B.

Being A, the amplitude of the dc component to recover, the input signal is

$$x(t) = A + B\cos(2\pi f t + \phi). \tag{2}$$

The random sampling is carried out using an ideal sampling function $\delta(t)$, illustrated in Fig. 1. The sampling interval of $\delta(t)$ is T_i

$$T_i = T_{A/D} + \tau_i \tag{3}$$

where $T_{A/D}$ is the minimum sampling interval that the acquisition system can produce. (This is limited by technological factors, usually the acquisition time) and τ_i is a random time with a uniform pdf

$$\begin{aligned} f(\tau_i) = &\quad \frac{1}{T_S}, \quad 0 < \tau_i < T_s \\ f(\tau_i) = &\quad 0 \text{ for any other } \tau_i \end{aligned} \tag{4}$$

where T_S is the maximum value of τ_i.

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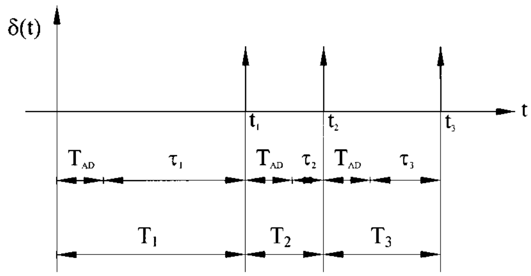

Fig. 1. Proposed sampling function.

The value of T_S sets the correlation between the sampling and input signals. If T_S is zero, the sampling period is constant and randomness disappears. In such cases, coherence occurs between the sampling function and those input signals whose frequencies are multiples of $1/T_{A/D}$. Therefore, T_S must be long enough to ensure a low degree correlation between the sampling function and any input signal.

The output of the MA filter μ_n is an estimation of the dc value and it is given by

$$\mu_n = \frac{1}{n}\sum_{k=1}^{n} x_k \tag{5}$$

where x_k is the instantaneous value of the input at the sampling time t_k and n is the number of samples in the MA filter. The values x_k are given by

$$x_k = A + B\cos\left(2\pi f \sum_{i=1}^{k} T_i + \phi\right). \tag{6}$$

It can be seen that (6) includes $k+1$ random values: ϕ and the sampling intervals T_i, for $i = 1$ to k.

Using (6), (5) becomes

$$\mu_n = \frac{1}{n}\sum_{k=1}^{n}\left\{A + B\cos\left(2\pi f\sum_{i=1}^{k} T_i + \phi\right)\right\}. \tag{7}$$

The expected value of μ_n is calculated in order to check the quality of the proposed estimate. The expected value of a function of several random variables is [8]

$$E[\mu_n] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} \mu_n f_{\phi,T_1,T_2,\ldots,T_n}\, d\phi\, dT_1\, dT_2\cdots dT_n \tag{8}$$

where

$f_{\phi,T_1,T_2,\ldots,T_n}$ joint pdf of μ_n;

ϕ phase of the perturbation;

$T_1, T_2, \ldots T_n$ successive sampling periods of the n samples.

Random variables ϕ, T_1, $T_2, \ldots, T_n$ are statistically independent. Consequently, the joint pdf yields [8]

$$f_{\phi,T_1,T_2,\ldots,T_n} = f_\phi f_{T_1} f_{T_2}\cdots f_{T_n}. \tag{9}$$

Substituting the pdf of ϕ, $T_1, T_2, \ldots, T_n$ in (9) results in

$$\begin{aligned} f_{\phi,T_1,T_2,\ldots,T_n} &= \frac{1}{2\pi\, T_s^n} \quad \pi \le \phi \le \pi,\\ T_{A/D} &\le T_1, T_2, \ldots, T_n \le T_{A/D} + T_s\\ f_{\phi,T_1,T_2,\ldots,T_n} &= 0 \text{ otherwise.} \end{aligned} \tag{10}$$

The introduction of (7) and (10) in (8), leads to

$$E[\mu_n] = A + \frac{B}{n2\pi T_S^n}\int_{T_{A/D}}^{T_{A/D}+T_S}\int_{T_{A/D}}^{T_{A/D}+T_S}\cdots \cdots\int_{-\pi}^{\pi}\sum_{k=1}^{n}\cos\left(2\pi f\sum_{i=1}^{k} T_i + \phi\right) d\phi\, dT_1\, dT_2\cdots dT_n. \tag{11}$$

According to (3), (11) can be expressed as

$$E[\mu_n] = A + \frac{B}{n2\pi T_S^n}\int_0^{T_S}\int_0^{T_S}\cdots\int_{-\pi}^{\pi} \times \sum_{k=1}^{n}\cos\left(2\pi f\sum_{i=1}^{k}\left(T_{A/D} + \tau_i\right) + \phi\right) d\phi\, d\tau_1\, d\tau_2\cdots d\tau_n. \tag{12}$$

Finally, the solution of (12) is

$$E[\mu_n] = A. \tag{13}$$

According to (13), the result of the expected value is the dc value to be measured, whatever the values of n, $T_{A/D}$, and T_S. Therefore, uniform sampling ($T_S = 0$) and random sampling give satisfactory values the estimation.

In order to prove that random sampling produces better results than uniform sampling, the error of the former one should be smaller. The variance of the estimate is used to compare both methods and is calculated using the following expression [8]:

$$\Gamma_\mu^2 = E\left[\mu_n^2\right] - \left(E[\mu_n]\right)^2 \tag{14}$$

where $E\left[\mu_n^2\right]$ is the second-order moment of μ_n and its definition is

$$E[\mu_n^2] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\mu_n^2 f_{\phi,T_1,T_2,\ldots,T_n}\, d\phi\, dT_1\, dT_2\cdots dT_n. \tag{15}$$

By introducing (10) in (15), we obtain

$$E[\mu_n^2] = \frac{1}{2\pi T_S^n}\int_{T_{A/D}}^{T_{A/D}+T_S}\int_{T_{A/D}}^{T_{A/D}+T_S}\cdots \cdots\int_{-\pi}^{\pi}\mu_n^2\, d\phi\, dT_1\, dT_2\ldots dT_n \tag{16}$$

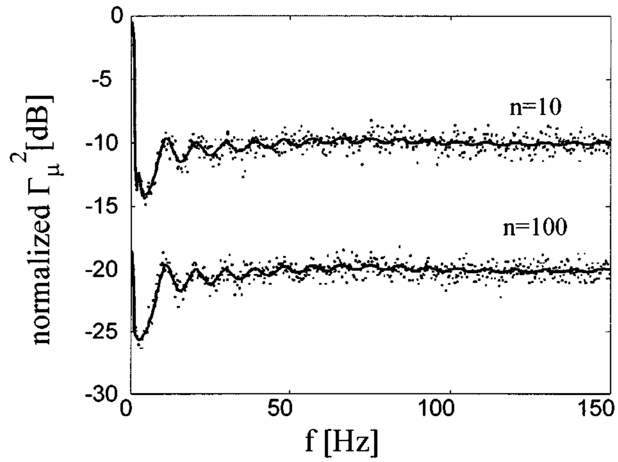

Fig. 2. Normalized variance versus frequency. The dots represent experiments, the solid lines represent (20).

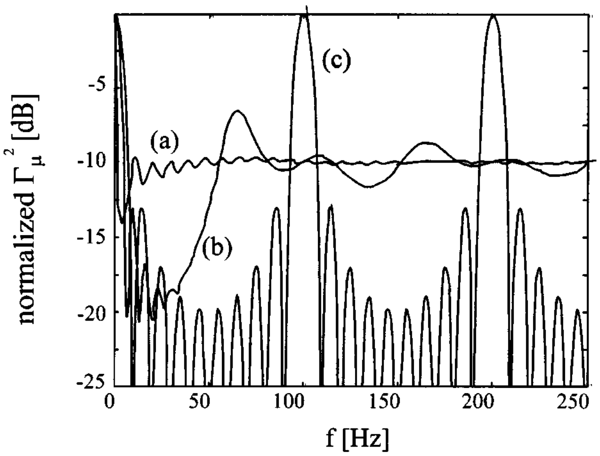

Fig. 3. Normalized variance, (20), versus frequency, for $n = 10$: (a) $T_S/T_{A/D} = 10$; (b) $T_S/T_{A/D} = 1$; and (c) $T_S/T_{A/D} = 0$.

which can be expressed as (17), shown at the bottom of the page. The solution of (17) leads to the following equation:

$$E\left[\mu_n^2\right] = A^2 + \frac{B^2}{2n} + \frac{B^2}{n^2}\sum_{k=1}^{n-1}(n-k) \times \cos\left(2\pi k f\left(T_{A/D} + \frac{T_S}{2}\right)\right) \cdot \sin c^k(fT_S). \tag{18}$$

Therefore, the variance of the estimation can be expressed as

$$\Gamma_\mu^2 = \frac{B^2}{2n} + \frac{B^2}{n^2}\sum_{k=1}^{n-1}(n-k) \times \cos\left(2\pi k f\left(T_{A/D} + \frac{T_S}{2}\right)\right) \cdot \sin c^k(fT_S). \tag{19}$$

Equation (19) indicates that the variance depends on the amplitude B, the number of samples in the MA filter, the frequency of the sinusoidal disturbance, and the value of the time periods $T_{A/D}$ and T_S.

The variance in (19) is normalized in relation to the disturbance variance $B^2/2$

$$\Gamma_\mu^2 = \frac{1}{n} + \frac{2}{n^2}\sum_{k=1}^{n-1}(n-k) \times \cos\left(2\pi k f\left(T_{A/D} + \frac{T_S}{2}\right)\right) \cdot \sin c^k(fT_S). \tag{20}$$

Fig. 2 plots the normalized variance Γ_μ^2 given by (20) as a function of the frequency f (represented by solid lines). The normalized variance of (20) presents a minimum at approximately $1/[2(T_S + 2T_{A/D})]$.

It is interesting to point out that the variance of the estimation approaches a constant value as the frequency increases. This value is equal to the disturbance variance divided by the number of samples, as it is also the case in the uniform sampled Gaussian white noise [9]. Moreover, the function represented by (20) does not show the typical peaks of the spectral overlapping at any frequency. Consequently, it is possible to estimate the value of the dc because the disturbance is properly reduced for all the frequencies.

An uncertain interval can be set from the value of the variance. The pdf of the estimation can be considered as a Gaussian distribution. Therefore, it is possible to establish, with a probability of 95%, that the error margins will be $\pm 2\Gamma(\mu)$ [10]. This uncertain interval can be made small as it is required increasing the number of samples n.

A correlation between the disturbance and the sampling function still exists in this random sampling proposal and it depends on the ratio $T_S/T_{A/D}$. The smaller this ratio is, the closer to the uniform sampling case. This is illustrated by Fig. 3. It shows normalized variance [dB] as a function of frequency for $n = 10$ and $T_{A/D} = 10$ ms, for three cases: (a) $T_S/T_{A/D} = 10$, (b) $T_S/T_{A/D} = 1$, and (c) $T_S/T_{A/D} = 0$. For $T_S/T_{A/D} = 0$, (uniform sampling) the variance peaks at frequencies that are multiple of the sampling frequency. Moreover, it can be observed that the normalized variance is significantly smaller in the uniform sampling than in the random one except for the regions around frequencies which are multiple of the sampling frequency. In these regions, the effect of the disturbing signal is very important and its elimination is the aim of the method that is proposed in this paper.

III. Numerical Verifications

Numerical verifications were done to validate the theoretical results. Numerical experiments were carried out using MATLAB. Each experiment consisted of a set of μ_{ni}, where

$$E[\mu_n^2] = \frac{1}{n^2 2\pi T_S^n}\int_0^{T_S} \cdot \int_0^{T_S} \cdots \int_{-\pi}^{\pi}\left(nA + B\sum_{k=1}^{n}\cos\left(2\pi f\sum_{i=1}^{k}\left(T_{A/D} + \tau_i\right) + \phi\right)\right)^2 d\phi\, d\tau_1\, d\tau_2 \cdots d\tau_n \tag{17}$$

$i = 1, 2, \ldots 100$, being μ_{ni} the output of the MA filter. The MA filter input was a sinusoidal signal of constant frequency f and random phase ϕ_i. This signal was sampled ten times $(n = 10)$ in a random way, with sampling intervals given by (3), $T_{A/D} = 10$ ms, and $T_S = 100$ ms. Each μ_{ni} was obtained by (5), where x_k is the sampled value of the input at t_k.

The variance of μ_n along the set of 100 μ_{ni} is given by

$$\hat{\Gamma}_{\mu}^2 = \frac{1}{100} \sum_{i=1}^{100} (\mu_{ni} - E[\mu_n])^2 \tag{21}$$

where $E[\mu_n]$ is the mean value of the μ_{ni}.

Three hundred experiments were done, for $n = 10$, sweeping the frequency from 0 to 150 Hz in steps of 1/2 Hz, and another 300 were done for $n = 100$. Fig. 2 shows the results of these experiments. The normalized variance versus frequency is represented by dots while the theoretical result from (20) is represented by the solid line. It can be seen that the numerical and theoretical results are very close. The worst case is the experiment carried out around 35 Hz with a deviation of -4 dB from the analytical expression. According to the analytical expression, the simulations show a valley between 1 and 3 Hz, which is approximately the value $1/[2(T_S + 2T_{AD})]$ deduced from (20).

An important characteristic of the variance can be seen in Fig. 2. The variance decreases and reaches a saturation value, which is sensitive to the number of samples n. The higher the number of samples, the lower the saturation value.

Results of Fig. 2 indicate that a dc signal, immersed in a sinusoidal noise of a frequency higher than the maximum acquisition frequency, can be easily measured by using random sampling and MA filters. The attenuation of the ac disturbing signal is proportional to the inverse of the number of samples in the MA filter. For $n = 10$, the variance of the remaining disturbing signal is 10% of the input ac signal variance. In terms of power, the attenuation is 10 dB for $n = 10$ and 20 dB for $n = 100$. If uniform sampling is used, the attenuation is 0 dB at frequencies multiple of sampling frequency (see Fig. 3).

IV. Experimental Results

A DSP-based acquisition board without antialiasing filter was used to verify the theoretical and numerical results. The system was configured to $T_{A/D} = 10$ ms and $T_S = 100$ ms. An input signal which is composed by a dc signal of 2 V plus an ac signal of 101 Hz and 2 V peak was acquired. The input signal was processed by an DSP-implemented MA filter. The output values were converted to analog in order to view them using an oscilloscope. Fig. 4 shows the output of the MA filter when the input signal is acquired using uniform sampling. A sinusoidal signal of 1 Hz and 2 V peak appears added to the correct value of dc. Fig. 5 shows the data acquired with random sampling plus an MA filter of $n = 10$. The figure presents a 2 V dc plus a random noise whose power is 0.2441 V^2, which is much lower than in uniform sampling. Fig. 5 also shows that only 5% of the samples are out of the range $\pm 2\Gamma(\mu)$, ($(\Gamma(\mu))^2 = 0.2441$ and then $\pm 2\Gamma(\mu) = \pm 2(0.2441)^{1/2} = \pm 0.9881$). Fig. 6 illustrates random sampling but with an MA of 100 samples. The noise power is 0.0228 V^2, that is, it is still better than that showed in Fig. 5. Values out of the range $\pm 2\Gamma(\mu)$ are 5% of the total, $(\pm 2\Gamma(\mu) = \pm 2(0.0228)^{1/2} = \pm 0.302)$. The experimental results showed a better performance as n is incremented and they are according to the theory.

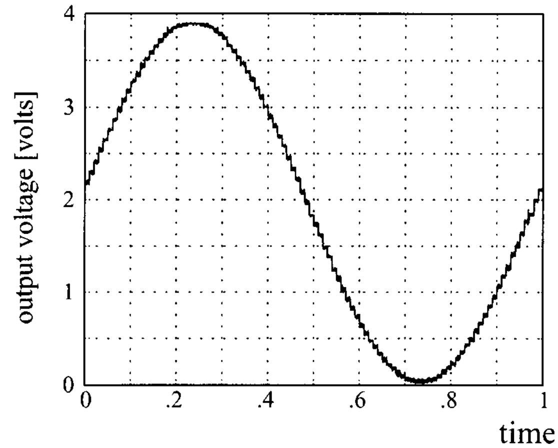

Fig. 4. Oscilloscope view of the data acquired by uniform sampling, $T_{A/D} =$ 10 ms. Signal acquired is a dc of 2 V plus a sinusoidal of 101 Hz and 2 V peak.

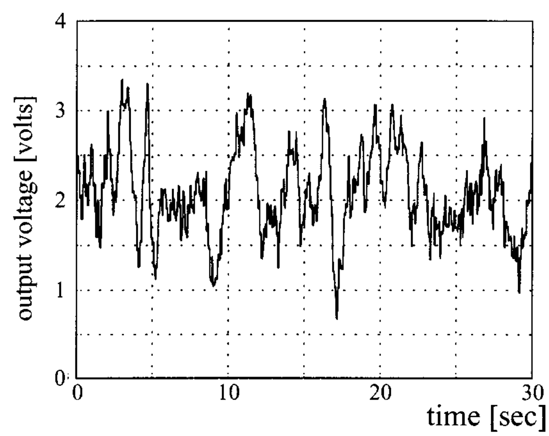

Fig. 5. Oscilloscope view of the data acquired using random sampling and MA of ten samples, $T_{A/D} = 10$ ms, $T_S/T_{A/D} = 10$. Signal acquired is a dc of 2 V plus a sinusoidal of 101 Hz and 2 V peak.

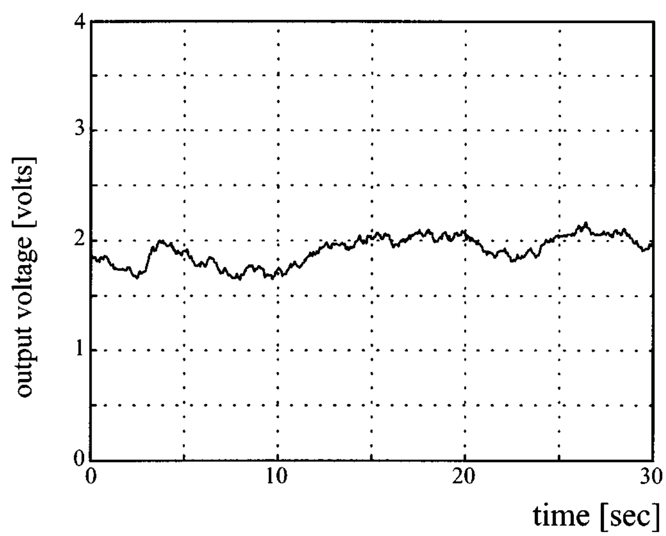

Fig. 6. Oscilloscope view of the data acquired using random sampling and MA of 100 samples, $T_{A/D} = 10$ ms, $T_S/T_{A/D} = 10$. Signal acquired is a dc of 2 V plus a sinusoidal of 101 Hz and 2 V peak.

V. CONCLUSIONS

The paper introduces the use of random sampling as a technique to avoid aliasing problems in acquisition systems. The use of an MA filter and random sampling is effective in acquiring a dc signal immersed in an impulsive noise. Antialiasing filters, which are strongly recommended if uniform sampling is used, can be avoided when the proposed method is used, even if the disturbance frequencies are higher than the maximum available sampling frequency.

The proposal is mathematically and experimentally demonstrated. The results show a reduction of the disturbing effect of the impulsive noise. The reduction depends on the number of samples included in the MA filter. The results also show that the performance of the method depends on the degree of correlation between the sampling function and the perturbation. The greater the correlation, the greater the risk of aliasing problems.

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