STUDIA MATHEMATICA 230 (2) (2015)

Biduals of tensor products in operator spaces

by

Verónica Dimant (Buenos Aires) and
Maite Fernández-Unzueta (Guanajuato)

Abstract. We study whether the operator space $V^{**} \overset{\alpha}{\otimes} W^{**}$ can be identified with a subspace of the bidual space $(V \overset{\alpha}{\otimes} W)^{**}$, for a given operator space tensor norm. We prove that this can be done if α is finitely generated and V and W are locally reflexive. If in addition the dual spaces are locally reflexive and the bidual spaces have the completely bounded approximation property, then the identification is through a complete isomorphism. When α is the projective, Haagerup or injective norm, the hypotheses can be weakened.

1. Introduction and preliminaries. Whenever V and W are reflexive operator spaces and α is an operator space tensor norm then clearly $V^{**} \overset{\alpha}{\otimes} W^{**}$ can be completely isometrically embedded in $(V \overset{\alpha}{\otimes} W)^{**}$. We are interested in this contention for non-reflexive operator spaces V and W. Thus, the objective of this paper is to provide conditions on the operator spaces and the operator tensor norms to guarantee that, indeed, the tensor product of two bidual operator spaces can be regarded as a subspace of the bidual of the operator space tensor product.

One main step in this direction is to prove the existence of a natural completely bounded mapping Θ_α which realizes the identification. This is done in Proposition 4.6 for a finitely generated tensor norm α, when the spaces are assumed to be locally reflexive. In the case of the Haagerup tensor norm and the projective operator space tensor norm, the identification mapping exists and is a complete contraction, without any assumption on the operator spaces V, W (see diagram (4.2) and Proposition 3.2).

When $\otimes_h$ is the Haagerup tensor product of operator spaces, Θ_h defines a canonical completely isometric embedding of $V^{**} \overset{h}{\otimes} W^{**}$ in $(V \overset{h}{\otimes} W)^{**}$. This

2010 *Mathematics Subject Classification*: Primary 47L25; Secondary 46M05, 47L22.
Key words and phrases: operator spaces, tensor products, bilinear mappings.
Received 13 May 2015; revised 29 December 2015.
Published online 4 February 2016.

DOI: 10.4064/sm8292-1-2016

was proved for C^*-algebras in [16, Theorem 4.1] and for general operator spaces in [13, Theorem 2.2].

The case of the operator space projective tensor product is studied in Section 3. The approach we follow in this case is like that in [5] for Banach spaces (see also [9, Section 3]).

In [13], the authors also study the projective operator space tensor product case. Their approach is based on the decomposition of jointly completely bounded maps defined on exact operator spaces, stated in [19]. They prove in [13, Theorem 2.7] that for exact operator spaces, there is a Banach space embedding from $V^{**} \widehat{\otimes} W^{**}$ into $(V \widehat{\otimes} W)^{**}$.

Here, we establish that Θ is a complete isomorphism onto its image when the dual spaces are assumed to be locally reflexive and one of the bidual spaces has the completely bounded approximation property, CBAP (Corollary 3.5). Furthermore, when the dual spaces are assumed to be locally reflexive and both bidual spaces have CBAP, we prove that Θ is a complete isomorphism which induces the *complete compact extension property* (CCEP) of $V^{**} \widehat{\otimes} W^{**}$ as a subspace of $(V \widehat{\otimes} W)^{**}$ (Theorem 3.6). We also obtain an analogous statement in Theorem 4.9 for the mapping Θ_α when α is a finitely generated operator space tensor norm.

We finish the paper with an application concerning the so called "unique norm-preserving extension problem": when Θ_α is a complete isometry, we give a characterization of the uniqueness of the norm preserving extensions of a bilinear scalar valued mapping (Theorem 5.2).

Throughout the article V and W will be operator spaces. The space of $n \times n$ matrices of elements of V will be denoted by $M_n(V)$. When V is the scalar field, we just write M_n (and $M_{n\times m}$ for the space of $n \times m$ scalar matrices); T_n will denote its Banach dual. The space of completely bounded linear mappings from V to W will be denoted by $\mathcal{CB}(V, W)$. These spaces will be assumed to be endowed with their usual operator space structure.

We recall the duality and some definitions and notation related to tensor products and bilinear mappings in operator spaces. We refer to [11, 20] for all the necessary background and notation from operator space theory, to [4] for tensor products of operator spaces and to [8] for some specific results on bilinear ideals on operator spaces.

The operator space structure of the dual space V^* is given by the identification $V^* \cong \mathcal{CB}(V, \mathbb{C})$. It has the property that $M_n(V^*) \cong T_n(V)^*$ and $T_n(V^*) \cong M_n(V)^*$ are complete isometric isomorphisms (see [11, Proposition 7.1.6]), where $T_n(V) \cong T_n \widehat{\otimes} V$ and $\widehat{\otimes}$ denotes the operator space projective tensor product, which we now introduce:

The *operator space projective tensor norm* of $u \in M_n(V \otimes W)$ is defined as

$$\|u\|_\wedge = \inf\{\|\alpha\| \cdot \|v\| \cdot \|w\| \cdot \|\beta\|\},$$

where the infimum is taken over all the representations of u as $u = \alpha(v \otimes w)\beta$ with $v \in M_p(V)$, $w \in M_q(W)$, $\alpha \in M_{n\times p\cdot q}$, $\beta \in M_{p\cdot q\times n}$ for some $p, q \in \mathbb{N}$.

The *operator space injective tensor norm* of $u \in M_n(V \otimes W)$ is defined as

$$\|u\|_\vee = \sup\{\|(f \otimes g)_n(u)\| : f \in M_p(V^*),\ g \in M_q(W^*),\ \|f\| \le 1,\ \|g\| \le 1\}.$$

The *operator space projective tensor product* $V \widehat{\otimes} W$ and the *operator space injective tensor product* $V \check{\otimes} W$ are the completion of $(V \otimes W, \|\cdot\|_\wedge)$ and the completion of $(V \otimes W, \|\cdot\|_\vee)$, respectively.

The *Haagerup tensor product* $V \overset{h}{\otimes} W$ is the completion of $V \otimes W$ under the *Haagerup tensor norm* which is defined for $u \in M_n(V \otimes W)$ as

$$\|u\|_h = \inf\{\|v\| \cdot \|w\| : u = v \odot w,\ v \in M_{n\times r}(V),\ w \in M_{r\times n}(W),\ r \in \mathbb{N}\}.$$

Denoting by $\mathcal{JCB}(V \times W)$ and $\mathcal{MB}(V \times W)$ the spaces of jointly completely bounded bilinear forms and multiplicatively bounded bilinear forms, the following completely isometric identifications are known:

$$\mathcal{JCB}(V \times W) \cong (V \widehat{\otimes} W)^* \cong \mathcal{CB}(V, W^*), \qquad \mathcal{MB}(V \times W) \cong (V \overset{h}{\otimes} W)^*.$$

Recall that an operator space V is λ-*locally reflexive* if for every finite-dimensional operator space L and every complete contraction $\varphi : L \to V^{**}$ there exists a net $\{\varphi_\gamma\} \subset \mathcal{CB}(L, V)$ such that $\|\varphi_\gamma\|_{cb} \le \lambda$ and

$$\langle \varphi_\gamma(u), v^*\rangle \to \langle \varphi(u), v^*\rangle \quad \text{ for all } u \in L,\ v^* \in V^*.$$

When $\lambda = 1$, we will use the term *locally reflexive* instead of *1-locally reflexive*. Most of the results of the article stated for locally reflexive operator spaces remain valid for arbitrary local reflexivity constants, with straightforward changes.

2. Approximations and complementation. In order to address the problem of when the natural mapping $\Theta : V^{**}\widehat{\otimes}W^{**} \to (V\widehat{\otimes}W)^{**}$ (described properly in Proposition 3.1) is completely isometric, we begin by describing the completely bounded approximation property in dual and bidual spaces, in interplay with the local reflexivity property.

Recall (see [10, 11]):

DEFINITION 2.1. V has the *completely bounded approximation property* (CBAP) if there exist a constant $K > 0$ and a net (φ_γ) of finite rank completely bounded mappings $\varphi_\gamma : V \to V$ with $\|\varphi_\gamma\|_{cb} \le K$ and $\varphi_\gamma(v) \to v$ for all $v \in V$.

If this holds with $K = 1$ we say that V has the *completely metric approximation property* (CMAP).

Note that when V has CBAP then $(\varphi_\gamma)_n(v) \to v$ for all $v \in M_n(V)$.

CBAP on an operator space implies BAP on the underlying Banach space. In general, the converse is not true (see, for instance, [1, 18]).

The space $\mathcal{L}(H)$ has neither CBAP, nor BAP, but there are many interesting examples of operator spaces, even non-reflexive, with CBAP or CMAP. One way to construct such examples is to consider a (non-reflexive dual or bidual) Banach space X with the metric approximation property and to endow it with a homogeneous operator space structure (see [20, 9.2] for the definition), like $\min(X)$ or $\max(X)$.

CBAP admits a somewhat better formulation in the case of bidual spaces (see [5, Corollary 1] for the Banach space statement). To prove it, we need to state the following (probably) known result:

Lemma 2.2 (CBAP in duals). *Let V be a locally reflexive operator space such that V^* has CBAP (with constant K). Then there exists a net (ξ_β) of w^*-continuous finite rank mappings $\xi_\beta : V^* \to V^*$ with $\|\xi_\beta\|_{cb} \le K$ and $\xi_\beta(v^*) \to v^*$ for all $v^* \in V^*$.*

Proof. Since V^* has CBAP, there exists a net (φ_γ) of finite rank mappings on V^* such that $\|\varphi_\gamma\|_{cb} \le K$ and $\varphi_\gamma(v^*) \to v^*$ for all $v^* \in V^*$. Let us consider $\varphi_\gamma : V^* \to \operatorname{ran}(\varphi_\gamma) = E_\gamma \subset V^*$ and $\varphi_\gamma^* : E_\gamma^* \to V^{**}$. Due to the local reflexivity of V and the finite dimension of E_γ, the following operator space analogue of Dean's identity [6] can be derived by taking duals from [11, Theorem 14.3.1] (or just considering the definition of local reflexivity [20, Definition 18.1]):

$$\mathcal{CB}(E_\gamma^*, V^{**}) \simeq \mathcal{CB}(E_\gamma^*, V)^{**}.$$

Through this identification, the mappings φ_γ^* can be seen in $\mathcal{CB}(E_\gamma^*, V)^{**}$ and thus we can use the Goldstine theorem and a standard net argument to obtain the desired result.

Indeed, for any γ, a finite set $F \subset V^*$ and $\varepsilon > 0$, we write $\beta = (\gamma, F, \varepsilon)$ and consider the set of all β ordered in the usual way. For each $\beta = (\gamma, F, \varepsilon)$, there exists $\psi_\beta \in \mathcal{CB}(E_\gamma^*, V)$ with $\|\psi_\beta\|_{cb} \le K$ and $\|v^* \circ \psi_\beta - v^* \circ \varphi_\gamma\| \le \varepsilon$ for all $v^* \in F$ (this is a consequence of the w^*-denseness of the ball of $\mathcal{CB}(E_\gamma^*, V)$ in the ball of $\mathcal{CB}(E_\gamma^*, V)^{**}$ and the fact that E_γ is finite-dimensional).

Now, if $\xi_\beta = \psi_\beta^* \in \mathcal{CB}(V^*, E_\gamma)$, then ξ_β is w^*-continuous, $\|\xi_\beta\|_{cb} \le K$ and $\xi_\beta(v^*) \to v^*$ for all $v^* \in V^*$. ∎

From the lemma, we get the following:

Proposition 2.3 (CBAP in biduals). *Let V be an operator space such that V^* is locally reflexive and V^{**} has CBAP (with constant K). Then there exists a net $(\psi_\gamma) \subset \mathcal{CB}(V, V^{**})$ of finite rank mappings with $\|\psi_\gamma\|_{cb} \le K$ and $\psi_\gamma^{**}(v^{**}) \to v^{**}$ for all $v^{**} \in V^{**}$.*

Proof. Since V^* is locally reflexive and V^{**} has CBAP, the previous lemma provides a bounded net (φ_γ) in $\mathcal{CB}(V^{**},V^{**})$ of w^*-continuous finite rank mappings. Since each φ_γ can be expressed as

$$\varphi_\gamma = \sum_{i=1}^{N} v_i^* \otimes v_i^{**} \quad \text{ with } v_i^* \in V^* \text{ and } v_i^{**} \in V^{**},$$

we can define $\psi_\gamma \in \mathcal{CB}(V,V^{**})$ to be the restriction of φ_γ to V. Then ψ_γ is a finite rank mapping that satisfies $\psi_\gamma^{**} = \varphi_\gamma$. ∎

For the sake of completeness, we include the proof of the following lemma. The analogous statement in the more general case of finitely generated operator space tensor norms is given in Lemma 4.8.

LEMMA 2.4 (CBAP in projective tensor products). *Let V and W be operator spaces having CBAP with constants K_V and K_W, respectively. Then $V \mathbin{\widehat{\otimes}} W$ has CBAP with constant $K_V \cdot K_W$.*

Proof. Let $(\varphi_\gamma) \subset \mathcal{CB}(V,V)$ and $(\psi_\delta) \subset \mathcal{CB}(W,W)$ be nets of finite rank mappings approximating the identities of V and W, respectively, with $\|\varphi_\gamma\| \le K_V$ and $\|\psi_\delta\| \le K_W$. Let us consider the net $(\Phi_{(\gamma,\delta)})$ where $\Phi_{(\gamma,\delta)} = \varphi_\gamma \otimes \psi_\delta : V \mathbin{\widehat{\otimes}} W \to V \mathbin{\widehat{\otimes}} W$ and the index set (γ,δ) is ordered canonically. It is clear that each $\Phi_{(\gamma,\delta)}$ is a finite rank mapping and $\|\Phi_{(\gamma,\delta)}\| \le K_V \cdot K_W$. To see that they approximate the identity, it is enough to check their values on elementary tensors:

$$\begin{aligned}
\|\Phi_{(\gamma,\delta)}(v\otimes w) - v\otimes w\| &= \sup_{\phi \in B_{(V\widehat{\otimes}W)^*}} |\phi(\Phi_{(\gamma,\delta)}(v\otimes w)) - \phi(v\otimes w)| \\
&= \sup_{\phi \in B_{(V\widehat{\otimes}W)^*}} |\phi(\varphi_\gamma(v)\otimes\psi_\delta(w)) - \phi(v\otimes w)| \\
&= \sup_{\phi \in B_{(V\widehat{\otimes}W)^*}} |\phi((\varphi_\gamma(v)-v)\otimes\psi_\delta(w)) + \phi(v\otimes(\psi_\delta(w)-w))| \\
&\le \sup_{\phi \in B_{(V\widehat{\otimes}W)^*}} (\|\phi\|\,\|\varphi_\gamma(v)-v\|\,\|\psi_\delta\|\,\|w\| + \|\phi\|\,\|v\|\,\|w-\psi_\delta(w)\|).
\end{aligned}$$

Hence, $\|\Phi_{(\gamma,\delta)}(v\otimes w) - v\otimes w\| \to 0$. ∎

In Theorem 3.6 it will be proved that, under appropriate hypotheses, the space $V^{**} \mathbin{\widehat{\otimes}} W^{**}$ has a stronger property than being just an operator subspace of $(V \mathbin{\widehat{\otimes}} W)^{**}$. To be precise, we introduce the following definition, which is an operator space analogue of the so called *compact extension property* of Banach spaces (see [14]).

DEFINITION 2.5. Let V, W be operator spaces such that V is completely isomorphic to a subspace of W through a mapping $i : V \hookrightarrow W$. We say that V *has the complete compact extension property in* W (CCEP) *through the mapping* i if there exists $\lambda > 0$ such that for every operator space Z

and every compact mapping $K \in \mathcal{CB}(V,Z)$ there exists a compact mapping $\widetilde{K} \in \mathcal{CB}(W,Z)$ with $\|\widetilde{K}\|_{cb} \le \lambda \|K\|_{cb}$ that makes commutative the diagram

$$\begin{tikzcd} V \arrow[rr, hook, "i"] \arrow[dr, "K"'] & & W \arrow[dl, "\widetilde{K}"] \\ & Z & \end{tikzcd}$$

Note that it is not necessary to require the existence of a constant λ in the previous definition. Indeed, if each compact operator $K \in \mathcal{CB}(V,Z)$ admits a compact extension $\widetilde{K} \in \mathcal{CB}(W,Z)$, then such a λ necessarily exists. The proof follows a classical argument by Lindenstrauss [17, Theorem 2.2].

The next lemma gives a condition under which we can derive CCEP for operator spaces with CBAP. The analogous result for Banach spaces is in [5, Lemma 4].

Lemma 2.6. *Let V and W be operator spaces and $\psi : V \hookrightarrow W$ a linear mapping in $\mathcal{CB}(V,W)$. Suppose that we have a net $(\varphi_\gamma) \subset \mathcal{CB}(V,V)$ satisfying $\varphi_\gamma(v) \to v$ for all $v \in V$, and there exist $\lambda > 0$ and a net $(\widetilde{\varphi}_\gamma) \subset \mathcal{CB}(W,V)$ with $\|\widetilde{\varphi}_\gamma\|_{cb} \le \lambda$ such that the following diagram commutes:*

$$\begin{tikzcd} V \arrow[rr, "\psi"] \arrow[dr, "\varphi_\gamma"'] & & W \arrow[dl, "\widetilde{\varphi}_\gamma"] \\ & V & \end{tikzcd}$$

Then:

(i) *V is completely isomorphic to the subspace $\psi(V) \subset W$.*
(ii) *V has CCEP in W through the mapping ψ with constant λ.*

Proof. (i) For every $v \in M_n(V)$,

$$\|v\| \le \|v - (\varphi_\gamma)_n(v)\| + \|(\varphi_\gamma)_n(v)\|.$$

Since $(\varphi_\gamma)_n(v) \to v$ for all $v \in M_n(V)$, and $\|(\varphi_\gamma)_n(v)\| = \|(\widetilde{\varphi}_\gamma)_n(\psi_n(v))\| \le \lambda \|\psi_n(v)\|$, we obtain

$$\lambda^{-1}\|v\| \le \|\psi_n(v)\| \le \|\psi_n\| \cdot \|v\|.$$

Therefore, for all n, ψ_n is an isomorphism onto its image and so ψ is a complete isomorphism.

(ii) By (i), we can see V inside W through ψ. Now, for any operator space Z, let $K \in \mathcal{CB}(V,Z)$ be a compact completely bounded mapping. For each $w \in W$, the element $(K \circ \widetilde{\varphi}_\gamma)(w)$ belongs to the image by K of a ball in V of radius $\lambda\|w\|$. This holds for every γ and so the net $(((K \circ \widetilde{\varphi}_\gamma)(w))_{w\in W})_\gamma$ is contained in a relatively compact set in the product topology. Thus, there is a convergent subnet

$$((K \circ \widetilde{\varphi}_\delta)(w))_{w\in W} \to (\widetilde{K}(w))_{w\in W}.$$

This allows us to define the linear mapping

$$\widetilde{K} : W \to Z, \qquad w \mapsto \widetilde{K}(w).$$

Since $\widetilde{K}(B_W) \subset K(\lambda B_V)$, it is clear that $\widetilde{K}$ is compact. Also,

$$\widetilde{K}(\psi(v)) = \lim_{\delta} (K \circ \widetilde{\varphi}_\delta)(\psi(v)) = \lim_{\delta} K(\varphi_\delta(v)) = K(v).$$

Hence, the diagram of the definition of CCEP commutes.

Finally, let us see that $\widetilde{K}$ is completely bounded. For each $w \in M_n(W)$,

$$\widetilde{K}_n(w) = \lim_{\delta} K_n((\widetilde{\varphi}_\delta)_n(w)), \qquad \|K_n((\widetilde{\varphi}_\delta)_n(w))\| \le \|K_n\| \cdot \|(\widetilde{\varphi}_\delta)_n\| \cdot \|w\|,$$

so $\|\widetilde{K}_n(w)\| \le \|K\|_{cb} \cdot \lambda \cdot \|w\|$ for all w. Thus, $\|\widetilde{K}\|_{cb} \le \|K\|_{cb} \cdot \lambda$. ∎

In the case of Banach spaces, it is proved in [14, Theorem 3.5] that the *compact extension property*, originally introduced in [17], is equivalent to the *local complementability* of the subspace and to the existence of an extension morphism between the dual spaces. At some point, the so called *principle of local reflexivity* is mainly used, so it is not to be expected that an analogous statement remains valid for arbitrary operator spaces. Anyway, one of the implications holds, as the following proposition shows, while for the relationship with local complementability we need an extra hypothesis (see Proposition 2.10).

PROPOSITION 2.7. *Let V and W be operator spaces with a completely isomorphic inclusion $i : V \hookrightarrow W$. If V has CCEP in W through i then there exists a completely bounded linear mapping $L : V^* \to W^*$ such that $L(v^*)(i(v)) = v^*(v)$ for all $v \in V$ and $v^* \in V^*$.*

Proof. As usual we denote $\mathrm{FIN}(V^*) = \{F \subset V^* : F$ is a finite-dimensional subspace$\}$. For each $F \in \mathrm{FIN}(V^*)$, we set $Z_F = V/F^\perp$ and let $q_F : V \to Z_F$ be the quotient mapping. Since q_F has finite-dimensional range, it is a compact mapping, with $\|q_F\|_{cb} \le 1$. Thus, by hypothesis there exists a compact completely bounded map $K_F \in \mathcal{CB}(W, Z_F)$ with $\|K_F\|_{cb} \le \lambda$ (where λ is the constant of CCEP of V in W) such that $K_F \circ i = q_F$.

Now, by means of the complete isometry $Z_F^* \simeq F$ we can think that the adjoint of the mapping K_F is defined in F, that is, $K_F^* : F \to W^*$. This mapping has the following properties: $\|K_F^*\|_{cb} \le \lambda$ and

$$(K_F^* f)(i(v)) = f((K_F \circ i)(v)) = f(q_F(v)) = f(v) \quad \text{for all } f \in F,\ v \in V.$$

For each $v^* \in V^*$, we consider the net $(K_F^*(v^*))_{F \in \mathrm{FIN}(V^*)}$, where we set $K_F^*(v^*) = 0$ when $v^* \notin F$. Note that with this definition the mapping K_F^* is no more linear. Anyway, for every $F \in \mathrm{FIN}(V^*)$, $(K_F^*(v^*))_{v^* \in V^*}$ belongs to $\prod_{v^* \in V^*} \lambda \cdot \|v^*\| \cdot B_{W^*}$, which is a relatively compact set in the w^*-product topology. Thus, the net $((K_F^*(v^*))_{v^*\in V^*})_F$ has a subnet $((K_G^*(v^*))_{v^*\in V^*})_G$ which converges to an element $(L(v^*))_{v^*\in V^*}$ in the w^*-product topology. This means

$$L(v^*)(w) = \lim_G K_G^*(v^*)(w) \quad \text{for all } w \in W,\ v^* \in V^*.$$

In this way we have defined a mapping $L : V^* \to W^*$ that is linear. Also, since

$$\begin{aligned} |\langle\langle (K_G^*)_n(v^*), w\rangle\rangle| &\le \|(K_G^*)_n\| \cdot \|v^*\| \cdot \|w\| \\ &\le \lambda \cdot \|v^*\| \cdot \|w\| \quad \text{for all } v^* \in M_n(G),\ w \in M_n(W), \end{aligned}$$

where $\langle\langle\ ,\ \rangle\rangle$ means the *matrix pairing* according to [11, (1.1.27)], it follows that $\|L\|_{cb} \le \lambda$ and so L is completely bounded. Finally,

$$L(v^*)(i(v)) = \lim_G K_G^*(v^*)(i(v)) = v^*(v) \quad \text{for all } v \in V,\ v^* \in V^*. \ \blacksquare$$

REMARK 2.8. Note that the existence of a completely bounded extension mapping $L : V^* \to W^*$ implies that V^* is completely isomorphic to a complemented subspace of W^*.

Now, when a subspace has a kind of local reflexivity property (namely, V is LCC in V^{**} according to the definition below), then the local complementability and the compact extension property are equivalent in the operator space framework.

DEFINITION 2.9. Let V be a subspace of the operator space W. We say that V *is locally completely complemented in* W (LCC) if there exists $c > 0$ such that for any finite-dimensional subspace $E \subset W$, and any $\epsilon > 0$, there exists $\varphi \in \mathcal{CB}(E, V)$ such that $\|\varphi\|_{cb} < c + \epsilon$ and $\varphi(z) = z$ whenever $z \in E \cap V$.

PROPOSITION 2.10. *Let $V \subset W$. Consider the following statements:*

(i) *V is LCC in W.*
(ii) *V has the complete compact extension property* (*CCEP*) *in W.*

Then (i) *implies* (ii), *and if V is LCC in V^{**} then* (i) *and* (ii) *are equivalent.*

Proof. For the first implication consider a compact operator $K \in \mathcal{CB}(V, Z)$. Assuming that V is LCC in W, we can choose, for each $E \in \mathrm{FIN}(W)$, an operator $\varphi_E : E \to V$ as in Definition 2.9. Proceeding as in the proof of the previous proposition, we define $\psi_E(w) := K(\varphi_E(w))$ for $w \in E$ (and $\psi_E(w) := 0$ otherwise). The net $((\psi_E(w))_{w\in W})_{E\in\mathrm{FIN}(W)}$ is contained in the relatively compact set (for the w^*-product topology) $\prod_{w\in W} K((c+\epsilon)\|w\| B_V)$, which has a limit point ψ. The mapping $\tilde{K} : W \to Z$ defined as $\tilde{K}(w) := \psi(w)$ is a compact complete extension of K.

Now, assume that V has CCEP in W. Then the adjoint mapping of L in Remark 2.8, $L^* : W^{**} \to V^{**}$, is a complete contractive projection. If in addition V is LCC in V^{**}, there exists $c > 0$ such that for each $\epsilon > 0$ and each finite-dimensional $F \subset V^{**}$, we can take a mapping $\psi_F : F \to V$ satisfying the conditions in Definition 2.9. Let $E \subset W$ be a finite-dimensional subspace and let i_W denote the natural inclusion $W \subset W^{**}$. Set $F := L^{**} \circ i_W(E) \subset V^{**}$. Then $\varphi_E := \psi_F \circ L^{**} \circ i_W : E \to V$ satisfies $\varphi_E(v) = v$ if $v \in E \cap V$ and $\|\varphi_E\|_{cb} \leq c + \epsilon$. ∎

Two comments are in order. First, the local reflexivity of V and the fact that V is LCC in V^{**} are not equivalent. For instance, $\mathcal{L}(H)$ is locally completely complemented in $\mathcal{L}(H)^{**}$ but not locally reflexive.

Second, we recall that H. Rosenthal introduced in [21, Definition 2.32] the notion of a *completely locally complemented subspace*, a property which is, in principle, stronger that the one in Definition 2.9. For Banach spaces both properties coincide (see [14, Lemma 3.2]).

3. Bidual of the projective tensor product. For V and W operator spaces, the completely isometric identification $\mathcal{JCB}(V \times W) \simeq \mathcal{CB}(V, W^*)$ allows us to extend each $\phi \in \mathcal{JCB}(V \times W)$ to $\overline{\phi} \in \mathcal{JCB}(V^{**} \times W^{**})$ in a canonical way. Indeed, if ϕ is identified with $\varphi \in \mathcal{CB}(V, W^*)$ then $\overline{\phi}$ is the bilinear mapping associated to φ^{**}. Note that $\|\phi\|_{jcb} = \|\varphi\|_{cb} = \|\varphi^{**}\|_{cb} = \|\overline{\phi}\|_{jcb}$ and so $\overline{\phi}$ is a norm-preserving extension of ϕ to the biduals.

Moreover, we have

PROPOSITION 3.1. *The map*

$$\mathcal{JCB}(V \times W) \hookrightarrow \mathcal{JCB}(V^{**} \times W^{**}), \qquad \phi \to \overline{\phi},$$

is a completely isometric injection.

Proof. We have already shown that this is an isometric injection. To see that it is a complete isometry, note that the n-amplification of this mapping coincides with the following procedure: to each $\phi \in M_n(\mathcal{JCB}(V \times W)) \simeq \mathcal{JCB}(V \times W, M_n)$ we associate $\varphi \in \mathcal{CB}(V, \mathcal{CB}(W, M_n)) \simeq \mathcal{CB}(V, M_n(W^*))$. Now, φ^{**} is in

$$\mathcal{CB}(V^{**}, M_n(W^*)^{**}) \simeq \mathcal{CB}(V^{**}, M_n(W^{***})) \simeq \mathcal{CB}(V^{**}, \mathcal{CB}(W^{**}, M_n))$$

and identifies with $\overline{\phi} \in \mathcal{JCB}(V^{**} \times W^{**}, M_n) \simeq M_n(\mathcal{JCB}(V^{**} \times W^{**}))$. ∎

From the above considerations, we have naturally defined the following mapping:

$$\begin{aligned} \Theta : V^{**} \widehat{\otimes} W^{**} &\to (V \widehat{\otimes} W)^{**} \simeq \mathcal{JCB}(V \times W)^*, \\ v^{**} \otimes w^{**} &\mapsto (\phi \mapsto \overline{\phi}(v^{**}, w^{**})). \end{aligned}$$

PROPOSITION 3.2. *Θ is a well defined completely bounded linear map with $\|\Theta\|_{cb} = 1$.*

Proof. It is clear that $\Theta(u)$ belongs to $\mathcal{JCB}(V\times W)^*$ with $\|\Theta(u)\| \le \|u\|$ for any $u \in V^{**} \widehat{\otimes} W^{**}$. Thus, Θ is a well defined continuous linear mapping.

To see the complete boundedness of Θ, consider, for each $n \in \mathbb{N}$, the n-amplification

$$\Theta_n : M_n(V^{**} \widehat{\otimes} W^{**}) \to M_n((V \widehat{\otimes} W)^{**}) \simeq M_n(\mathcal{JCB}(V \times W)^*).$$

Given $u \in M_n(V^{**} \widehat{\otimes} W^{**})$, we want to show

$$\|\Theta_n(u)\|_{M_n((V\widehat{\otimes}W)^{**})} = \|\Theta_n(u)\|_{M_n(\mathcal{JCB}(V\times W)^*)} \le \|u\|_{M_n(V^{**}\widehat{\otimes}W^{**})}.$$

We do this by means of the duality

$$M_n(\mathcal{JCB}(V \times W)^*) \simeq T_n(\mathcal{JCB}(V \times W))^*.$$

For $\phi \in T_n(\mathcal{JCB}(V \times W))$ we denote by $\overline{\phi}$ the matrix $(\overline{\phi}_{ij})$. Appealing to Proposition 3.1 and [11, Theorem 4.1.8] we know that

$$\|\overline{\phi}\|_{T_n(\mathcal{JCB}(V^{**}\times W^{**}))} \le \|\phi\|_{T_n(\mathcal{JCB}(V\times W))}.$$

Hence,

$$\begin{aligned}|\langle \Theta_n(u), \phi\rangle| = |\langle \overline{\phi}, u\rangle| &\le \|\overline{\phi}\|_{T_n(\mathcal{JCB}(V^{**}\times W^{**}))}\|u\|_{M_n(V^{**}\widehat{\otimes}W^{**})} \\ &\le \|\phi\|_{T_n(\mathcal{JCB}(V\times W))}\|u\|_{M_n(V^{**}\widehat{\otimes}W^{**})}.\end{aligned}$$

This implies $\|\Theta_n(u)\|_{M_n(\mathcal{JCB}(V\times W)^*)} \le \|u\|_{M_n(V^{**}\widehat{\otimes}W^{**})}$ and so $\|\Theta_n\| \le 1$ for all n.

Moreover, by evaluating Θ_n on a diagonal matrix in $M_n(V^{**} \widehat{\otimes} W^{**})$, with $v \otimes w$ in the diagonal for any $v \in V$ and $w \in W$, we can show that actually $\|\Theta_n\| = 1$ for every n. Therefore, $\|\Theta\|_{cb} = 1$. ∎

One way to determine when Θ is completely isometric is to use some facts about duality between the projective and the injective operator space tensor norms. Specifically, we bear in mind two natural mappings: the complete isometry $\Lambda : V^* \check{\otimes} W^* \to (V \widehat{\otimes} W)^*$ (see [11, Proposition 8.1.2]) and $\Omega : V^{**} \widehat{\otimes} W^{**} \to (V^* \check{\otimes} W^*)^*$, as well as the canonical commutative diagram

(3.1)
$$\begin{tikzcd} V^{**} \widehat{\otimes} W^{**} \arrow[rr, "\Theta"] \arrow[dr, "\Omega"'] & & (V \widehat{\otimes} W)^{**} \arrow[dl, "\Lambda^*"] \\ & (V^* \check{\otimes} W^*)^* & \end{tikzcd}$$

where Λ^* (which in the Banach space setting is called the *Borel transform*) is the adjoint of Λ. Once we know that Ω is a complete isometry, the same will hold for Θ.

A weak form of the CMAP, namely W*MAP, was introduced in [10]. There, it was proved that V has W*MAP if and only if the natural map $V \widehat{\otimes} W \to (V^* \check{\otimes} W^*)^*$ is a complete isometry for every operator space W [10, Theorem 2.2]. Note that this mapping is a restriction of Ω in diagram (3.1).

Also, by [10, Theorem 2.1], the canonical mapping $V^* \widehat{\otimes} E^* \to (V \check{\otimes} E)^*$ is a complete isometry for every finite-dimensional operator space E if and only if V is locally reflexive. Thus, in order to prove that our mapping Ω is a complete isometry too, it seems natural to assume local reflexivity of both V and W. The respective facts for Banach spaces can be found in [7, 16.3 Corollary 2].

THEOREM 3.3. *Let V and W be locally reflexive operator spaces such that V^* (or W^*) has CMAP. Then the mapping $\Omega : V^* \widehat{\otimes} W^* \to (V \check{\otimes} W)^*$ is completely isometric.*

Proof. The mapping Ω is always completely contractive. The point here is to prove that, under our hypothesis, $\|u\| \le \|\Omega_n(u)\|$ for every $u \in M_n(V^* \widehat{\otimes} W^*)$. Clearly, it is enough to consider matrices u in the uncompleted projective tensor product $(V^* \otimes W^*, \wedge)$. Let $u = (u_{ij})$, where, for each i, j,

$$u_{ij} = \sum_{k=1}^{N_{ij}} v^*_{ijk} \otimes w^*_{ijk} \in V^* \otimes W^*.$$

Since $(M_n(V^* \widehat{\otimes} W^*))^* \simeq T_n((V^* \widehat{\otimes} W^*)^*) \simeq T_n(\mathcal{CB}(V^*, W^{**}))$, there exists $T \in T_n(\mathcal{CB}(V^*, W^{**}))$ such that

$$\|T\|_{T_n(\mathcal{CB}(V^*,W^{**}))} = 1, \qquad |\langle T, u\rangle| = \Big|\sum_{i,j} \langle T_{ij}, u_{ij}\rangle\Big| = \|u\|_{M_n(V^* \widehat{\otimes} W^*)}.$$

Suppose that V^* has CMAP (the proof when W^* has CMAP is analogous) and let $\varepsilon > 0$. Since V is locally reflexive we can apply Lemma 2.2 to obtain a completely bounded w^*-continuous finite rank map $\varphi_\varepsilon : V^* \to V^*$ satisfying $\|\varphi_\varepsilon\|_{cb} \le 1$ and $\|\varphi_\varepsilon(v^*_{ijk}) - v^*_{ijk}\| < \varepsilon/M$, where $M = \sum_{i,j,k} \|w^*_{ijk}\|$.

For each i, j, the range of $T_{ij} \circ \varphi_\varepsilon$ is a finite-dimensional subspace of W^{**}. Now, by the local reflexivity of W there is a completely bounded map ψ : $\mathrm{span}\{\mathrm{ran}(T_{ij} \circ \varphi_\varepsilon) : 1 \le i, j \le n\} \to W$ with $\|\psi\|_{cb} \le 1 + \varepsilon$ such that

$$\psi(w^{**})(w^*_{ijk}) = w^{**}(w^*_{ijk})$$

for all i, j, k and $w^{**} \in \mathrm{span}\{\mathrm{ran}(T_{ij} \circ \varphi_\varepsilon) : 1 \le i, j \le n\}$. Thus, the composition $\psi \circ T_{ij} \circ \varphi_\varepsilon$ can be seen as an element of $V \otimes W$ with $\|\psi \circ T_{ij} \circ \varphi_\varepsilon\|_{V \check{\otimes} W} = \|\psi \circ T_{ij} \circ \varphi_\varepsilon\|_{\mathcal{CB}(V^*,W)}$. This gives

$$\begin{aligned}\|\psi \circ T \circ \varphi_\varepsilon\|_{T_n(V \check{\otimes} W)} &= \|\psi \circ T \circ \varphi_\varepsilon\|_{T_n(\mathcal{CB}(V^*,W))} \\ &\le \|\psi\|_{cb} \|T\|_{T_n(\mathcal{CB}(V^*,W^{**}))} \|\varphi_\varepsilon\|_{cb} \le 1 + \varepsilon.\end{aligned}$$

Hence,

$$\begin{aligned}\|\Omega_n(u)\|_{M_n(V \check{\otimes} W)^*} &\ge \frac{1}{1+\varepsilon} |\langle \psi \circ T \circ \varphi_\varepsilon, \Omega_n(u)\rangle| \\ &= \frac{1}{1+\varepsilon} \Big|\sum_{i,j} \langle \psi \circ T_{ij} \circ \varphi_\varepsilon, u_{ij}\rangle\Big|\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{1+\varepsilon}\Big|\sum_{i,j,k} \psi(T_{ij}(\varphi_\varepsilon(v^*_{ijk})))(w^*_{ijk})\Big| \\
&= \frac{1}{1+\varepsilon}\Big|\sum_{i,j,k} T_{ij}(\varphi_\varepsilon(v^*_{ijk}))(w^*_{ijk})\Big| \\
&\geq \frac{1}{1+\varepsilon}\Big[\Big|\sum_{i,j,k} T_{ij}(v^*_{ijk})(w^*_{ijk})\Big| - \Big|\sum_{i,j,k} T_{ij}(v^*_{ijk} - \varphi_\varepsilon(v^*_{ijk}))(w^*_{ijk})\Big|\Big] \\
&\geq \frac{1}{1+\varepsilon}\Big[|\langle T, u\rangle| - \sum_{i,j,k} \|T_{ij}\|\,\|v^*_{ijk} - \varphi_\varepsilon(v^*_{ijk})\|\,\|w^*_{ijk}\|\Big] \\
&\geq \frac{1}{1+\varepsilon}(\|u\|_{M_n(V^*\widehat{\otimes} W^*)} - \varepsilon).
\end{aligned}
$$

Since ε can be chosen arbitrarily small, the proof is complete. ∎

Some comments are in order. First, observe that if we assume CBAP instead of CMAP in Theorem 3.3, we conclude that Ω is a complete isomorphism onto its image.

Second, if $\mathcal{N}$ and $\mathcal{I}$ denote, respectively, the ideals of completely nuclear and completely integral bilinear mappings (see the definitions in [8]), then Theorem 3.3 reads:

Corollary 3.4. *Let V and W be operator spaces such that V^* (or W^*) has CMAP and both V and W are locally reflexive. Then the natural inclusion $\mathcal{N}(V \times W) \hookrightarrow \mathcal{I}(V \times W)$ is completely isometric.*

Proof. From [8, Proposition 3.14] we know that in this case $\mathcal{N}(V\times W) = V^* \widehat{\otimes} W^*$, and from [8, Proposition 3.11], we always have $\mathcal{I}(V \times W) = (V \check{\otimes} W)^*$. Thus, Ω corresponds to the natural inclusion between the ideals. ∎

Third, by the relation between Ω and Θ explained just below the commutative diagram (3.1), we obtain:

Corollary 3.5. *Let V and W be operator spaces such that V^{**} (or W^{**}) has CMAP (respectively CBAP) and both V^* and W^* are locally reflexive. Then the mapping $\Theta : V^{**} \widehat{\otimes} W^{**} \to (V \widehat{\otimes} W)^{**}$ is a complete isometry (resp. a complete isomorphism onto its image).*

If in the corollary it is further assumed that both dual spaces have CBAP, we get a stronger conclusion about the embedding of $V^{**}\widehat{\otimes}W^{**}$ in $(V\widehat{\otimes}W)^{**}$. It is worth noting that the proofs of Corollary 3.5 and of the following theorem are different: the latter does not use the duality of the tensor norms, but the approximation properties in dual spaces and tensor products stated in Section 2. For the analogous result in the Banach space setting see [5, Theorem 1].

THEOREM 3.6. *Let V and W be operator spaces such that both V^{**} and W^{**} have CBAP with constants K_1 and K_2 (respectively CMAP) and both V^* and W^* are locally reflexive. Then $V^{**} \widehat{\otimes} W^{**}$ has CCEP in $(V \widehat{\otimes} W)^{**}$ through the mapping Θ. Hence, $\mathcal{JCB}(V^{**} \times W^{**})$ is completely isomorphic (respectively isometric) to a complemented subspace of $\mathcal{JCB}(V \times W)^{**}$.*

Proof. By Corollary 2.3 there exist nets $(\varphi_\gamma) \subset \mathcal{CB}(V, V^{**})$ and $(\psi_\delta) \subset \mathcal{CB}(W, W^{**})$ of finite rank mappings with $\|\varphi_\gamma\|_{cb} \leq K_1$ and $\|\psi_\delta\|_{cb} \leq K_2$ such that

$$\begin{aligned} \|\varphi_\gamma^{**} v^{**} - v^{**}\| \to 0 \quad &\text{for all } v^{**} \in V^{**}, \\ \|\psi_\delta^{**} w^{**} - w^{**}\| \to 0 \quad &\text{for all } w^{**} \in W^{**}. \end{aligned}$$

Now, the proof of Lemma 2.4 shows that the net $(\Phi_{(\gamma,\delta)} = \varphi_\gamma^{**} \otimes \psi_\delta^{**})$ realizes CBAP in $V^{**} \widehat{\otimes} W^{**}$ with constant $K_1 \cdot K_2$.

Due to Lemma 2.6, the proof will be finished once we find mappings $\widetilde{\Phi}_{(\gamma,\delta)} \in \mathcal{CB}((V \widehat{\otimes} W)^{**}, V^{**} \widehat{\otimes} W^{**})$ with $(\|\widetilde{\Phi}_{(\gamma,\delta)}\|_{cb})$ bounded such that the following diagram commutes:

$$\begin{tikzcd} V^{**} \widehat{\otimes} W^{**} \arrow[rr, "\Theta"] \arrow[dr, "\Phi_{(\gamma,\delta)}"'] & & (V \widehat{\otimes} W)^{**} \arrow[dl, "\widetilde{\Phi}_{(\gamma,\delta)}"] \\ & V^{**} \widehat{\otimes} W^{**} & \end{tikzcd}$$

Let us see that we achieve this goal through $\widetilde{\Phi}_{(\gamma,\delta)} = (\varphi_\gamma \otimes \psi_\delta)^{**}$. Since the mappings $\varphi_\gamma \otimes \psi_\delta \in \mathcal{CB}(V \widehat{\otimes} W, V^{**} \widehat{\otimes} W^{**})$ have finite rank, it is clear that the $\widetilde{\Phi}_{(\gamma,\delta)}$ also have finite rank and belong to $\mathcal{CB}((V \widehat{\otimes} W)^{**}, V^{**} \widehat{\otimes} W^{**})$. Also,

$$\|\widetilde{\Phi}_{(\gamma,\delta)}\|_{cb} = \|\varphi_\gamma \otimes \psi_\delta\|_{cb} \leq \|\varphi_\gamma\|_{cb} \cdot \|\psi_\delta\|_{cb} \leq K_1 \cdot K_2.$$

It only remains to prove the commutativity of the diagram, that is, $\widetilde{\Phi}_{(\gamma,\delta)} \circ \Theta = \Phi_{(\gamma,\delta)}$; it is enough to check it on elementary tensors.

For $v^{**} \in V^{**}$, $w^{**} \in W^{**}$ and $\phi \in \mathcal{JCB}(V^{**} \times W^{**}) = (V^{**} \widehat{\otimes} W^{**})^*$ we have

$$\begin{aligned} &\phi\big(\widetilde{\Phi}_{(\gamma,\delta)}(\Theta(v^{**} \otimes w^{**}))\big) \\ &\quad = \phi\big((\varphi_\gamma \otimes \psi_\delta)^{**}(\Theta(v^{**} \otimes w^{**}))\big) = \Theta(v^{**} \otimes w^{**})((\varphi_\gamma \otimes \psi_\delta)^*(\phi)) \\ &\quad = \overline{(\varphi_\gamma \otimes \psi_\delta)^*(\phi)}(v^{**}, w^{**}) = \overline{\phi \circ (\varphi_\gamma \otimes \psi_\delta)}(v^{**}, w^{**}) \\ &\quad = \overline{\phi}(\varphi_\gamma^{**} v^{**}, \psi_\delta^{**} w^{**}) = \phi(\varphi_\gamma^{**} v^{**}, \psi_\delta^{**} w^{**}) \\ &\quad = \phi((\varphi_\gamma^{**} \otimes \psi_\delta^{**})(v^{**} \otimes w^{**})) = \phi(\Phi_{(\gamma,\delta)}(v^{**} \otimes w^{**})). \end{aligned}$$

Since this is valid for every $v^{**} \in V^{**}$, $w^{**} \in W^{**}$ and $\phi \in \mathcal{JCB}(V^{**} \times W^{**}) = (V^{**} \widehat{\otimes} W^{**})^*$, we are done.

The last part of the statement is now a consequence of Lemma 2.6(ii), Proposition 2.7 and Remark 2.8.

The statement with the CMAP hypothesis follows as well, taking into account that in this case $K_1 = K_2 = 1$. ∎

4. Bidual of a tensor product with other tensor norms. In this section we address the analogous problem of whether an element in $V^{**} \overset{\alpha}{\otimes} W^{**}$ can be identified with an element in $(V \overset{\alpha}{\otimes} W)^{**}$, for an operator space tensor norm α. In order to have a mapping $\Theta_\alpha : V^{**} \overset{\alpha}{\otimes} W^{**} \to (V \overset{\alpha}{\otimes} W)^{**}$ properly defined, we need to establish some properties of operator space tensor norms and to impose conditions on the spaces involved. We recall the notion of operator space tensor norm as defined in [8]:

Definition 4.1. We say that α is an *operator space tensor norm* if α is an operator space matrix norm on each tensor product of operator spaces $V \otimes W$ that satisfies the following two conditions:

(a) α is a cross matrix norm, that is, $\alpha(v \otimes w) = \|v\| \cdot \|w\|$ for all $v \in M_p(V)$, $w \in M_q(W)$ and $p, q \in \mathbb{N}$.
(b) α has the "completely metric mapping property": for every $r_1 \in \mathcal{CB}(U_1, V)$, $r_2 \in \mathcal{CB}(U_2, W)$, the operator $r_1 \otimes r_2 : (U_1 \otimes U_2, \alpha) \to (V \otimes W, \alpha)$ is completely bounded and $\|r_1 \otimes r_2\|_{cb} \le \|r_1\|_{cb} \cdot \|r_2\|_{cb}$.

We denote by $V \overset{\alpha}{\otimes} W$ the completion of $(V \otimes W, \alpha)$.

Definition 4.2. We say that an operator space tensor norm α is *finitely generated* if for any operator spaces V, W and every $u \in M_n(V \otimes W)$,

$$\begin{aligned}&\|u\|_{M_n(V \otimes_\alpha W)}\\ &\quad = \inf\{\|u\|_{M_n(E \otimes_\alpha F)} : E \in \mathrm{FIN}(V),\ F \in \mathrm{FIN}(W),\ u \in M_n(E \otimes F)\}.\end{aligned}$$

Now we prove an operator space version of [7, Extension Lemma 6.7] which we need in order to have an appropriate generalization of Proposition 3.1 to other tensor norms α (see Corollary 4.5).

Lemma 4.3 (Right extension lemma). *Let V and W be operator spaces with W locally reflexive and let α be a finitely generated operator space tensor norm. Then the following mapping is a complete isometry:*

$$(V \overset{\alpha}{\otimes} W)^* \to (V \overset{\alpha}{\otimes} W^{**})^*, \qquad \phi \mapsto \phi^\wedge,$$

where $\phi^\wedge(v \otimes w^{**}) = w^{**}(L_\phi(v))$ and $L_\phi : V \to W^*$ is given by $L_\phi(v)(w) = \phi(v, w)$.

Proof. Denoting by $J_W : W \to W^{**}$ the canonical inclusion, it is clear that the mapping $\mathrm{id} \otimes J_W : V \overset{\alpha}{\otimes} W \to V \overset{\alpha}{\otimes} W^{**}$ is completely bounded with

$\|\mathrm{id}\otimes J_W\|_{cb}\le 1$. Thus, from the duality $M_n((V\overset{\alpha}{\otimes}W)^*)\simeq T_n(V\overset{\alpha}{\otimes}W)^*$ and [11, Theorem 4.1.8] it is easily seen that every $\phi\in M_n((V\overset{\alpha}{\otimes}W)^*)$ satisfies

$$\|\phi\|_{M_n((V\overset{\alpha}{\otimes}W)^*)}\le\|\phi^\wedge\|_{M_n((V\overset{\alpha}{\otimes}W^{**})^*)}.$$

For the reverse inequality, let $\phi\in M_n((V\overset{\alpha}{\otimes}W)^*)$. By the complete identification $M_n((V\overset{\alpha}{\otimes}W^{**})^*)\simeq\mathcal{CB}(V\overset{\alpha}{\otimes}W^{**},M_n)$ and by [11, Proposition 2.2.2] we have

$$\begin{aligned}\|\phi^\wedge\|_{M_n((V\overset{\alpha}{\otimes}W^{**})^*)}&=\|\phi^\wedge\|_{\mathcal{CB}(V\overset{\alpha}{\otimes}W^{**},M_n)}=\|\phi_n^\wedge\|\\&=\sup_{u\in B_{M_n(V\overset{\alpha}{\otimes}W^{**})}}\|\phi_n^\wedge(u)\|.\end{aligned}$$

For this last supremum it is enough to consider matrices u in the uncompleted tensor product. Let $u\in M_n(V\otimes W^{**})$ and take finite-dimensional subspaces $F_1\subset V$ and $F_2\subset W^{**}$ such that $u\in M_n(F_1\otimes F_2)$.

Since $\phi\in M_n((V\overset{\alpha}{\otimes}W)^*)\simeq\mathcal{CB}(V\overset{\alpha}{\otimes}W,M_n)$ it has an associated linear mapping $L_\phi:V\to\mathcal{CB}(W,M_n)=M_n(W^*)$ satisfying $\phi^\wedge(v\otimes w^{**})=(w^{**})_n(L_\phi(v))$ for all $v\in V$ and $w^{**}\in W^{**}$. Now, there exists a finite-dimensional subspace $E\subset W^*$ such that $L_\phi(F_1)\subset M_n(E)$. Equivalently, $L_{\phi_{ij}}(F_1)\subset E$, for all i,j, where ϕ_{ij} denote the entries of the matrix ϕ.

By the local reflexivity of W, given $\varepsilon>0$, there is a completely bounded map $\varphi:F_2\to W$ such that

$$\begin{aligned}&\|\varphi\|_{cb}<1+\varepsilon,\\&\varphi(w^{**})(L_{\phi_{ij}}(v))=w^{**}(L_{\phi_{ij}}(v))\quad\text{for all } i,j \text{ and all } w^{**}\in F_2,\ v\in F_1.\end{aligned}$$

This can be translated into the equality $\phi_{ij}(v\otimes\varphi(w^{**}))=\phi_{ij}^\wedge(v\otimes w^{**})$, and so $\phi(v\otimes\varphi(w^{**}))=\phi^\wedge(v\otimes w^{**})$ for all $w^{**}\in F_2$ and $v\in F_1$. Hence, $\phi_n^\wedge(u)=\phi_n((\mathrm{id}\otimes\varphi)_n(u))$ and we obtain

$$\begin{aligned}\|\phi_n^\wedge(u)\|&\le\|\phi_n\|\cdot\|(\mathrm{id}\otimes\varphi)_n\|_{cb}\cdot\|u\|_{M_n(F_1\overset{\alpha}{\otimes}F_2)}\\&\le\|\phi_n\|\cdot(1+\varepsilon)\cdot\|u\|_{M_n(F_1\overset{\alpha}{\otimes}F_2)}.\end{aligned}$$

This holds for any $\varepsilon>0$ and all finite-dimensional subspaces $F_1\subset V$ and $F_2\subset W^{**}$ such that $u\in M_n(F_1\otimes F_2)$. Since α is finitely generated we clearly derive

$$\|\phi_n^\wedge\|\le\|\phi_n\|\quad\text{and so}\quad\|\phi^\wedge\|_{M_n((V\overset{\alpha}{\otimes}W^{**})^*)}\le\|\phi\|_{M_n((V\overset{\alpha}{\otimes}W)^*)}.$$

This finishes the proof. ■

An analogous argument leads to:

LEMMA 4.4 (Left extension lemma). *Let V and W be operator spaces with V locally reflexive and let α be a finitely generated operator space tensor* *norm. Then the following map is a complete isometry:*

$$(V \overset{\alpha}{\otimes} W)^* \to (V^{**} \overset{\alpha}{\otimes} W)^*, \qquad \phi \mapsto {}^\wedge\phi,$$

where ${}^\wedge\phi(v^{**} \otimes w) = v^{**}(R_\phi(w))$ *and* $R_\phi : W \to V^*$ *is given by* $R_\phi(w)(v) = \phi(v, w)$.

Corollary 4.5. *Let V and W be locally reflexive operator spaces and let α be a finitely generated operator space tensor norm. Then the following maps are complete isometries:*

$$(V \overset{\alpha}{\otimes} W)^* \to (V^{**} \overset{\alpha}{\otimes} W^{**})^*, \qquad \phi \mapsto {}^\wedge(\phi^\wedge),$$
$$(V \overset{\alpha}{\otimes} W)^* \to (V^{**} \overset{\alpha}{\otimes} W^{**})^*, \qquad \phi \mapsto ({}^\wedge\phi)^\wedge.$$

A priori, these two isometries may be different. However, in many cases they coincide, as is the case for Banach spaces (see Remark 5.3).

When α is the projective operator space tensor norm, the extension called $\overline{\phi}$ in Section 3 coincides with ${}^\wedge(\phi^\wedge)$. Note that in this case we proved this result in Proposition 3.1 without requiring the local reflexivity hypothesis.

For an operator space tensor norm α, the set $\mathfrak{A}_\alpha$ of bilinear mappings determined by the relation $\mathfrak{A}_\alpha(V \times W, X) \cong \mathcal{CB}(V \overset{\alpha}{\otimes} W, X)$ defines an operator space bilinear ideal [8, Proposition 2.4]. Thus, Corollary 4.5 also says that, under the hypothesis of local reflexivity, every bilinear form $\phi \in \mathfrak{A}_\alpha(V \times W)$ admits an extension $\overline{\phi} \in \mathfrak{A}_\alpha(V^{**} \times W^{**})$ with $\|\phi\|_{\mathfrak{A}_\alpha} = \|\overline{\phi}\|_{\mathfrak{A}_\alpha}$.

With the same proof as for the projective operator space tensor product (Proposition 3.2), we obtain

Proposition 4.6. *Let V and W be locally reflexive operator spaces and let α be a finitely generated operator space tensor norm. Then the map*

$$\Theta_\alpha : V^{**} \overset{\alpha}{\otimes} W^{**} \to (V \overset{\alpha}{\otimes} W)^{**} \simeq \mathfrak{A}_\alpha(V \times W)^*,$$
$$v^{**} \otimes w^{**} \mapsto (\phi \mapsto {}^\wedge(\phi^\wedge)(v^{**}, w^{**})),$$

is completely bounded with $\|\Theta_\alpha\|_{cb} = 1$.

In the particularly relevant cases of the injective and the Haagerup tensor norms, Θ_α is, in fact, a complete isometry:

The injective tensor norm case. Let us see that under the conditions of Proposition 4.6, when α is the injective operator space tensor norm $\|\cdot\|_\vee$, the mapping $\Theta_\vee$ is a complete isometry. Indeed, the version for the injective norm of diagram (3.1) is

(4.1)
$$\begin{tikzcd} V^{**} \check{\otimes} W^{**} \arrow[rr, "\Theta_\vee"] \arrow[dr, "\Lambda"'] & & (V \check{\otimes} W)^{**} \arrow[dl, "\Omega^*"] \\ & (V^* \widehat{\otimes} W^*)^* & \end{tikzcd}$$

Since the mapping $\Lambda : V^{**} \check{\otimes} W^{**} \to (V^* \widehat{\otimes} W^*)^*$ is completely isometric and both $\Theta_\vee$ and Ω are complete contractions, $\Theta_\vee$ is also a complete isometry. Thus, we have

PROPOSITION 4.7. *For V and W locally reflexive operator spaces the map $\Theta_\vee : V^{**} \check{\otimes} W^{**} \to (V \check{\otimes} W)^{**}$ is a complete isometry.*

It has been proved in [11, Chapter 14] that the mapping $\Theta_\vee$ is a complete isometry in another situation: If one of the spaces, say V, satisfies the so called *condition C* (that is, if it is exact and locally reflexive), then, for any W, $\Theta_\vee$ is a complete isometry.

The Haagerup tensor norm case. When α is the Haagerup tensor norm, the map $\Theta_h : V^{**} \overset{h}{\otimes} W^{**} \to (V \overset{h}{\otimes} W)^{**}$ can be defined and proved to be completely isometric without any hypothesis on V and W. This was done in [13, Theorem 2.2] and [15, Remark 2.11]. The proof in [15] can be written using a commutative diagram analogous to (3.1) and the fact that the Haagerup tensor norm is self-dual.

Indeed, defining the embedding $\Theta_h : V^{**} \overset{h}{\otimes} W^{**} \to (V \overset{h}{\otimes} W)^{**}$ by setting $\Theta_h(v^{**} \otimes w^{**})(\phi) = \widetilde{\phi}(v^{**}, w^{**})$, where $\widetilde{\phi}$ is the (unique) separately w^*-continuous extension of ϕ (see [3, 1.6.7]), it is clear that Θ_h is a complete contraction. Now, the fact that the mappings $\Omega_h : V^{**} \overset{h}{\otimes} W^{**} \to (V^* \overset{h}{\otimes} W^*)^*$ and $\Lambda_h : V^* \overset{h}{\otimes} W^* \to (V \overset{h}{\otimes} W)^*$ are complete isometries (see, for instance [11, Theorem 9.4.7]) implies that Θ_h is also a complete isometry, through the following commutative diagram:

$$\begin{tikzcd} V^{**} \overset{h}{\otimes} W^{**} \arrow[rr, "\Theta_h"] \arrow[dr, "\Omega_h"'] & & (V \overset{h}{\otimes} W)^{**} \arrow[dl, "\Lambda_h^*"] \\ & (V^* \overset{h}{\otimes} W^*)^* & \end{tikzcd} \tag{4.2}$$

We return now to the general case of any finitely generated operator space tensor norm α. To deduce that Θ_α is a completely bounded inclusion that produces the CCEP, we need again to impose an approximation hypothesis. We omit the proofs since, once we know that Θ_α is properly defined, they follow as their projective analogues, Lemma 2.4 and Theorem 3.6:

LEMMA 4.8 (CBAP in tensor products). *Let V and W be operator spaces with CBAP with constants K_V and K_W respectively, and let α be a finitely generated operator space tensor norm. Then $V \overset{\alpha}{\otimes} W$ has CBAP with constant $K_V \cdot K_W$.*

THEOREM 4.9. *Let V and W be operator spaces such that both V^{**} and W^{**} have CBAP with constants K_1 and K_2 (respectively CMAP) and all the spaces V, V^*, W and W^* are locally reflexive. Let α be a finitely generated operator space tensor norm. Then:*

(i) *The mapping $\Theta_\alpha : V^{**} \overset{\alpha}{\otimes} W^{**} \to (V \overset{\alpha}{\otimes} W)^{**}$ is a complete isomorphism (resp. complete isometry) and $V^{**} \overset{\alpha}{\otimes} W^{**}$ has CCEP in $(V \overset{\alpha}{\otimes} W)^{**}$ through Θ_α.*
(ii) *The space $\mathfrak{A}_\alpha(V^{**} \times W^{**})$ is completely isomorphic (resp. completely isometric) to a complemented subspace of $\mathfrak{A}_\alpha(V \times W)^{**}$.*

5. Unique norm preserving extensions. A long-standing problem in the Banach space setting is under which conditions a unique norm-preserving extension of a mapping exists. From one of the possible points of view on this matter, there is a classical result due to Godefroy [12] that characterizes when a linear mapping has a unique norm-preserving extension to the bidual space. This result has been extended to polynomial ideals in [2, 9].

As an application of the previous sections, we address the unique norm-preserving extension problem for extensions of bilinear mappings to bidual spaces. By Proposition 3.1, each $\phi \in M_n(\mathcal{JCB}(V \times W))$ has an extension to $V^{**} \times W^{**}$, which we called $\overline{\phi}$, satisfying $\|\phi\|_{jcb} = \|\overline{\phi}\|_{jcb}$. Also, if V and W are locally reflexive operator spaces and α is a finitely generated operator space tensor norm, then by Corollary 4.5, there are norm-preserving extensions of every $\phi \in M_n(\mathfrak{A}_\alpha(V \times W))$ to the biduals. More precisely, we know that $({}^\wedge\phi)^\wedge$ and ${}^\wedge(\phi^\wedge)$ are extensions of ϕ to $M_n(\mathfrak{A}_\alpha(V^{**} \times W^{**}))$ that satisfy $\|({}^\wedge\phi)^\wedge\|_{\mathfrak{A}_\alpha} = \|{}^\wedge(\phi^\wedge)\|_{\mathfrak{A}_\alpha} = \|\phi\|_{\mathfrak{A}_\alpha}$.

The following statement, which concerns the existence of norm-preserving extensions, is in some sense converse to Proposition 4.6.

LEMMA 5.1. *Let V and W be operator spaces and α be a finitely generated operator space tensor norm. Assume that there exists a complete contraction $\Theta_\alpha : V^{**} \overset{\alpha}{\otimes} W^{**} \to (V \overset{\alpha}{\otimes} W)^{**}$ such that $\Theta_\alpha(v \otimes w)(\phi) = \phi(v \otimes w)$. Then, for every $\phi \in M_n(\mathfrak{A}_\alpha(V \times W))$, the mapping*

$$\overline{\phi} := (\Theta_\alpha^*)_n|_{M_n(\mathfrak{A}_\alpha(V\times W))}(\phi)$$

is an extension of ϕ such that $\|\overline{\phi}\|_{\mathfrak{A}_\alpha} = \|\phi\|_{\mathfrak{A}_\alpha}$.

Proof. The restriction of Θ_α^* to $\mathfrak{A}_\alpha(V \times W)^*$ is a complete contraction. Consequently, $\|\overline{\phi}\|_{\mathfrak{A}_\alpha} \le \|\phi\|_{\mathfrak{A}_\alpha}$. The other inequality follows from the fact that α has the "completely metric mapping property" (condition (b) in Definition 4.1). ∎

In the next theorem, the notation $\overline{\phi}$ is as in Lemma 5.1.

THEOREM 5.2. *Let V, W, α and*

$$\Theta_\alpha : V^{**} \overset{\alpha}{\otimes} W^{**} \to (V \overset{\alpha}{\otimes} W)^{**} \simeq \mathfrak{A}_\alpha(V \times W)^*$$

be as in the preceding lemma. If Θ_α is a complete isometry then, for a given $\phi \in M_n(\mathfrak{A}_\alpha(V\times W))$ with $\|\phi\|_{\mathfrak{A}_\alpha} = 1$, the following conditions are equivalent:

(i) *There is a unique extension of ϕ to $V^{**} \times W^{**}$ preserving the $\mathfrak{A}_\alpha$-norm.*
(ii) *For any net $(\phi_\gamma)_\gamma \in M_n(\mathfrak{A}_\alpha(V \times W))$ with $\|\phi_\gamma\|_{\mathfrak{A}_\alpha} \le 1$ for all γ that satisfies $\phi_\gamma(v,w) \to \phi(v,w)$ for every $(v,w) \in V \times W$ we have $\overline{\phi}_\gamma(v^{**},w^{**}) \to \overline{\phi}(v^{**},w^{**})$ for every $(v^{**},w^{**}) \in V^{**} \times W^{**}$.*

Proof. To prove that (i) implies (ii), let $(\phi_\gamma)_\gamma \in M_n(\mathfrak{A}_\alpha(V \times W))$ be a net with $\|\phi_\gamma\|_{\mathfrak{A}_\alpha} \le 1$ for all γ such that $\phi_\gamma(v,w) \to \phi(v,w)$ for every $(v,w) \in V \times W$. Since the net $(\overline{\phi}_\gamma)_\gamma$ is contained in the unit ball of

$$M_n(\mathfrak{A}_\alpha(V^{**} \times W^{**})) \cong M_n((V^{**} \overset{\alpha}{\otimes} W^{**})^*) \cong (T_n(V^{**} \overset{\alpha}{\otimes} W^{**}))^*,$$

which is w^*-compact, there exists a subnet (still denoted $(\overline{\phi}_\gamma)_\gamma$) w^*-converging to an element ψ in the unit ball of $M_n(\mathfrak{A}_\alpha(V^{**} \times W^{**}))$.

Hence, for every $(v^{**},w^{**}) \in V^{**} \times W^{**}$, $\overline{\phi}_\gamma(v^{**},w^{**}) \to \psi(v^{**},w^{**})$. Since $\overline{\phi}_\gamma(v,w) = \phi_\gamma(v,w)$ for each $(v,w) \in V \times W$ and $\phi_\gamma(v,w) \to \phi(v,w)$ we derive that $\psi|_{V\times W} = \phi$. Also, the fact that $\|\psi\|_{\mathfrak{A}_\alpha} \le 1 = \|\phi\|_{\mathfrak{A}_\alpha}$ says that ψ is a $\mathfrak{A}_\alpha$-norm-preserving extension of ϕ. By (i), we have $\psi = \overline{\phi}$. A canonical subnet argument finishes the proof.

For the reverse implication, let $\psi \in M_n(\mathfrak{A}_\alpha(V^{**}\times W^{**}))$ be an extension of ϕ with $\|\psi\|_{\mathfrak{A}_\alpha} = 1$. Through the complete isometry Θ_α, the operator space $V^{**} \overset{\alpha}{\otimes} W^{**}$ can be seen as a subspace of $(V \overset{\alpha}{\otimes} W)^{**}$. Thus, the matrix $\psi \in M_n(\mathfrak{A}_\alpha(V^{**}\times W^{**})) \cong \mathcal{CB}(V^{**}\overset{\alpha}{\otimes}W^{**}, M_n)$ has a norm-preserving extension to $\widetilde{\psi} \in \mathcal{CB}((V \overset{\alpha}{\otimes} W)^{**}, M_n) \cong M_n(\mathfrak{A}_\alpha(V \times W)^{**})$. Using an operator space version of Goldstine's theorem, we know that there exists a net $(\phi_\gamma)_\gamma \subset M_n(\mathfrak{A}_\alpha(V \times W))$ with $\|\phi_\gamma\| \le 1$ for all γ, satisfying

$$u_n(\phi_\gamma) \to \widetilde{\psi}(u) \quad \text{for all } u \in \mathfrak{A}_\alpha(V \times W)^*.$$

Now, for any $(v,w) \in V \times W$, the elementary tensor $v \otimes w$ belongs to $\mathfrak{A}_\alpha(V \times W)^*$. Then

$$\phi_\gamma(v,w) = (v \otimes w)_n(\phi_\gamma) \to \widetilde{\psi}(v \otimes w) = \psi(v,w) = \phi(v,w).$$

Applying (ii) and the equality $\overline{\phi}_\gamma(v^{**},w^{**}) = (v^{**} \otimes w^{**})_n(\phi_\gamma)$ we obtain, for every $(v^{**},w^{**}) \in V^{**} \times W^{**}$,

$$\overline{\phi}_\gamma(v^{**},w^{**}) \to \overline{\phi}(v^{**},w^{**}), \quad \overline{\phi}_\gamma(v^{**},w^{**}) \to \widetilde{\phi}(v^{**} \otimes w^{**}) = \psi(v^{**},w^{**}).$$

Therefore $\psi = \overline{\phi}$, and this completes the proof. ∎

Remark 5.3. Some comments are in order.

- When V and W are locally reflexive and ϕ satisfies (i) of the previous theorem, then, in particular, $^\wedge(\phi^\wedge) = (^\wedge\phi)^\wedge$.
- *Projective operator space tensor product:* When $\alpha = \|\cdot\|_\wedge$, the hypothesis of the previous theorem (i.e. Θ being completely isometric) holds when V^* and W^* are locally reflexive and V^{**} (or W^{**}) has CMAP, by Corollary 3.5.
- *Injective operator space tensor product:* When $\alpha = \|\cdot\|_\vee$, the hypothesis of the previous theorem (i.e. $\Theta_\vee$ being completely isometric) holds when V and W are locally reflexive, by Proposition 4.7.
- *Haagerup tensor product:* When $\alpha = \|\cdot\|_h$, there is a completely isometric natural inclusion $\Theta_h : V^{**} \overset{h}{\otimes} W^{**} \to (V \overset{h}{\otimes} W)^{**}$ (see diagram (4.2)), and consequently the conclusion of Theorem 5.2 holds, without any assumption on V and W.
- For a general α, by Theorem 4.9, when V, V^*, W and W^* are locally reflexive and V^{**} and W^{**} have CMAP, the mapping Θ_α is completely isometric and hence the conclusion of Theorem 5.2 holds.

Acknowledgements. Part of this work was done during a visit of the first author to Centro de Investigación en Matemáticas (Guanajuato) in June 2013 and a visit of the second author to the Departamento de Matemática de la Universidad de San Andrés (Buenos Aires) in February 2014. We wish to thank both institutions for their warm reception and kind hospitality.

We also want to thank the referee for his/her thorough reading and for providing very useful suggestions and comments.

The first author was partially supported by CONICET PIP 0483 and ANPCyT PICT 1456. The second author was partially supported by CONACYT 182296.

References

[1] Á. Arias, *Operator Hilbert spaces without the operator approximation property*, Proc. Amer. Math. Soc. 130 (2002), 2669–2677.

[2] R. Aron, C. Boyd and Y. S. Choi, *Unique Hahn–Banach theorems for spaces of homogeneous polynomials*, J. Aust. Math. Soc. 70 (2001), 387–400.

[3] D. P. Blecher and C. Le Merdy, *Operator Algebras and Their Modules—An Operator Space Approach*, London Math. Soc. Monogr. 30, Oxford Univ. Press, Oxford, 2004.

[4] D. P. Blecher and V. I. Paulsen, *Tensor products of operator spaces* J. Funct. Anal. 99 (1991), 262–269.

[5] F. Cabello Sánchez and R. García, *The bidual of tensor product of Banach spaces*, Rev. Mat. Iberoamer. 21 (2005), 843–861.

[6] D. Dean, *The equation $L(E, X^{**}) = L(E, X)^{**}$ and the principle of local reflexivity*, Proc. Amer. Math. Soc. 40 (1973), 146–148.
[7] A. Defant and K. Floret, *Tensor Norms and Operator Ideals*, North-Holland Math. Stud. 176, North-Holland, Amsterdam, 1993.
[8] V. Dimant and M. Fernández-Unzueta, *Bilinear ideals in operator spaces*, J. Math. Anal. Appl. 429 (2015), 57–80.
[9] V. Dimant, D. Galicer and R. García, *Geometry of integral polynomials, M-ideals and unique norm preserving extensions*, J. Funct. Anal. 262 (2012), 1987–2012.
[10] E. G. Effros and Z.-J. Ruan, *On approximation properties for operator spaces*, Int. J. Math. 1 (1990), 163–187.
[11] E. G. Effros and Z.-J. Ruan, *Operator Spaces*, London Math. Soc. Monogr. 23, Oxford Univ. Press, New York, 2000.
[12] G. Godefroy, *Points de Namioka. Espaces normants. Applications à la théorie isométrique de la dualité*, Israel J. Math. 38 (1981), 209–220.
[13] R. Jain and A. Kumar, *The operator space projective tensor product: embedding into the second dual and ideal structure*, Proc. Edinburgh Math. Soc. (2) 57 (2014), 505–519.
[14] N. Kalton, *Locally complemented subspaces and $\mathcal{L}_p$-spaces for $0 < p < 1$*, Math. Nachr. 115 (1984), 71–97.
[15] A. Kumar and V. Rajpal, *Projective tensor products of C^*-algebras*, Adv. Pure Math. 4 (2014), 176–188.
[16] A. Kumar and A. M. Sinclair, *Equivalence of norms on operator space tensor products of C^*-algebras*, Trans. Amer. Math. Soc. 350 (1998), 2033–2048.
[17] J. Lindenstrauss, *Extension of compact operators*, Mem. Amer. Math. Soc. 48 (1964), 112 pp.
[18] T. Oikhberg and É. Ricard, *Operator spaces with few completely bounded maps*, Math. Ann. 328 (2004), 229–259.
[19] G. Pisier and D. Shlyakhtenko, *Grothendieck‘s theorem for operator spaces*, Invent. Math. 150 (2002), 185–217.
[20] G. Pisier, *Introduction to Operator Space Theory*, London Math. Soc. Lecture Note Ser. 294, Cambridge Univ. Press, Cambridge, 2003.
[21] H. Rosenthal, *The complete separable extension property*, J. Operator Theory 43 (2000), 329–374.

Verónica Dimant
Departamento de Matemática
Universidad de San Andrés
Vito Dumas 284
(B1644BID) Victoria
Buenos Aires, Argentina
and
CONICET
E-mail: vero@udesa.edu.ar

Maite Fernández-Unzueta
Centro de Investigación en Matemáticas (Cimat)
A.P. 402
Guanajuato, Gto., México
E-mail: maite@cimat.mx