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A family of nonlinear Schrödinger equations admitting q-plane wave solutions

F.D. Nobre [a], A.R. Plastino [b,*]

[a] Centro Brasileiro de Pesquisas Físicas and National Institute of Science and Technology for Complex Systems, Rua Xavier Sigaud 150, 22290-180, Rio de Janeiro, RJ, Brazil

[b] CeBio y Secretaría de Investigación, Universidad Nacional del Noroeste de la Prov. de Buenos Aires, UNNOBA-Conicet, Roque Saenz Peña 456, Junin, Argentina

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ABSTRACT

Nonlinear Schrödinger equations with power-law nonlinearities have attracted considerable attention recently. Two previous proposals for these types of equations, corresponding respectively to the Gross–Pitaievsky equation and to the one associated with nonextensive statistical mechanics, are here unified into a single, parameterized family of nonlinear Schrödinger equations. Power-law nonlinear terms characterized by exponents depending on a real index q, typical of nonextensive statistical mechanics, are considered in such a way that the Gross–Pitaievsky equation is recovered in the limit $q \to 1$. A classical field theory shows that, due to these nonlinearities, an extra field $\Phi(\vec{x}, t)$ (besides the usual one $\Psi(\vec{x}, t)$) must be introduced for consistency. The new field can be identified with $\Psi^*(\vec{x}, t)$ only when $q \to 1$. For $q \neq 1$ one has a pair of coupled nonlinear wave equations governing the joint evolution of the complex valued fields $\Psi(\vec{x}, t)$ and $\Phi(\vec{x}, t)$. These equations reduce to the usual pair of complex-conjugate ones only in the $q \to 1$ limit. Interestingly, the nonlinear equations obeyed by $\Psi(\vec{x}, t)$ and $\Phi(\vec{x}, t)$ exhibit a common, soliton-like, traveling solution, which is expressible in terms of the q-exponential function that naturally emerges within nonextensive statistical mechanics.

1. Introduction

A wide variety of phenomena within the realm of complex systems are nowadays studied by means of nonlinear (NL) partial differential evolution equations [1,2]. The computing resources provided by modern technology have stimulated greatly (and allowed for substantial advances in) the investigation of models based on this kind of equations, which are rarely analytically tractable. Particularly in physics, many areas have benefited from these developments in the study of NL equations, like nonlinear optics, superconductivity, plasma physics, and nonequilibrium statistical mechanics, since many physical situations in these areas are described in terms of these equations.

The NL equations are usually introduced in the literature through generalizations of linear ones, so that the later may be recovered in certain limit cases. One of the procedures for doing this concerns the addition of extra NL terms to a linear equation; as an example one has the Gross–Pitaievsky equation (GPE) [3,4]. In a three-dimensional space, the GPE is given by

$$i\hbar \frac{\partial \Psi(\vec{x}, t)}{\partial t} = -\frac{\hbar^2}{2m} \nabla^2 \Psi(\vec{x}, t) + K |\Psi(\vec{x}, t)|^{\nu} \, \Psi(\vec{x}, t) \, , \qquad (1)$$

where K and ν ($\nu > 0$) are real numbers; this equation recovers the linear Schrödinger equation, for a free particle of mass m, by taking $K = 0$. One should notice that, by recourse to the quantum mechanical probability density,

$$\rho(\vec{x}, t) = |\Psi(\vec{x}, t)|^2 \, , \qquad (2)$$

Eq. (1) may be recast as

$$i\hbar \frac{\partial \Psi(\vec{x}, t)}{\partial t} = -\frac{\hbar^2}{2m} \nabla^2 \Psi(\vec{x}, t) + \tilde{K} [\rho(\vec{x}, t)]^{\nu/2} \, \Psi(\vec{x}, t) \, . \qquad (3)$$

The simplest solution of Eq. (1) is the well-known plane wave,

$$\Psi(\vec{x}, t) = \Psi_0 \exp\left[i(\vec{k} \cdot \vec{x} - \omega t) \right] \, . \qquad (4)$$

Some of the solutions of Eq. (1) have an associated time-independent probability density [1,3,4]. In these cases, the GPE can be formally regarded as a linear Schrödinger equation in the presence of the effective potential $V(x) = \tilde{K}[\rho(\vec{x})]^{\nu/2}$. As an example, for the plane-wave solution of Eq. (4), the GPE becomes the linear Schrödinger equation in the presence of a constant potential.

* Corresponding author.
E-mail addresses: fdnobre@cbpf.br (F.D. Nobre), arplastino@unnoba.edu.ar (A.R. Plastino).

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Another common procedure for introducing NL equations consists in modifying exponents of existing linear terms. This recipe has been employed, e.g., for NL Fokker–Planck equations [2]; in particular, the introduction of a power $(2-q)$ in the probability of the diffusion term [5–7] has been used within the framework of nonextensive statistical mechanics [8]. This type of equation has led to the possibility of explaining many interesting physical phenomena related to anomalous diffusion. Along these lines, and inspired on nonextensive thermostatistics, the following NL Schrödinger equation for a free particle of mass m in a three-dimensional space has been recently advanced [9],

$$i\hbar\frac{\partial}{\partial t}\left[\frac{\Psi(\vec{x},t)}{\Psi_0}\right]=-\frac{1}{2-q}\,\frac{\hbar^2}{2m}\nabla^2\left[\frac{\Psi(\vec{x},t)}{\Psi_0}\right]^{2-q}. \quad (5)$$

The real parameter q characterizes the nonlinear aspects of the equation, with the linear case being recovered in the particular limit $q=1$. The index q above is inspired in the entropic index that appears in the definition of Tsallis entropy [10]. Here we shall restrict our considerations to the range of q-values $1\le q<3$, for which the one dimensional q-plane waves have a finite norm (in higher dimensional spaces one still has the q-plane wave solutions, but they are localized only in the direction of propagation). One significant way in which equation (1) differs from equation (5) concerns the amplitude Ψ_0, which has to be incorporated to the later proposal because the wave function $\Psi(\vec{x},t)$ is raised to a real power, in contrast to what happens in (1). The $q=2$ case of equation (5) deserves a clarification. A cursory glance at (5) may suggest that its right hand side is ill defined for that value of the q parameter. However, it can be verified that for $q=2$ equation (5) is equivalent to $i\hbar\frac{\partial}{\partial t}\left[\frac{\Psi(\vec{x},t)}{\Psi_0}\right]=-\frac{\hbar^2}{2m}\nabla\left[\frac{\Psi_0}{\Psi(\vec{x},t)}\nabla\left(\frac{\Psi(\vec{x},t)}{\Psi_0}\right)\right]$.

It is worth mentioning here that (5) is not the only quantum evolution law inspired on the nonextensive thermostatistical formalism that has been proposed. We have, for instance, the approaches advanced in [11,12], where the linear and unitary character of quantum evolution is preserved at the level of pure states, but a nonlinearity is introduced in the equation of motion of the time evolution operator [11], or at the level of von Neumann's equation of motion for mixed quantum states [12]. The q-logarithmic nonlinear Schrödinger equation associated with anisentropic gas dynamics recently studied by Rogers and Ruggeri [13] constitute another notable example of a nonlinear quantum-like evolution equation related to the nonextensive thermostatistical formalism.

A considerable research effort has been devoted in recent years to investigate the properties and solutions of Eq. (5). Various types of solutions have been explored, both for free-particle dynamics [9,14–16], and for particles in the presence of different potentials [14,15,17,18]. This equation has been analyzed from the point of view of the pilot-wave representation of quantum mechanics [19], and it has also been extended to two particle systems, in order to investigate its entanglement-related features [18]. Moreover, it was shown to be closely related to the hypergeometric differential equation [20]. The evolution equation (5) can be embedded within a classical field theory based upon an appropriate Lagrangian [21]. However, to implement this Lagrangian formulation in a consistent way, one has to introduce a new field $\Phi(\vec{x},t)$, besides the original field $\Psi(\vec{x},t)$. The new field can be identified with $\Psi^*(\vec{x},t)$ only when $q\to 1$. Consequently, for $q\neq 1$, one has a pair of wave equations by means of which the two fields $\Psi(\vec{x},t)$ and $\Phi(\vec{x},t)$ evolve and interact, whereas in the limit $q\to 1$ these two equations "collapse" into the usual pair of complex-conjugate equations, respectively governing $\Psi(\vec{x},t)$ and $\Psi^*(\vec{x},t)$ [21]. Similar procedures yielded nonlinear extensions of the basic relativistic wave equations, that is, the Dirac [9], the Klein–Gordon [9,22], and the Proca [23] equations. In the last two instances, the classical field theoretical formulation also highlighted the necessity of incorporating extra fields [23], analogously to what happened in connection with Eq. (5).

The main purpose of the present work is to unify the above-mentioned previous proposals for nonlinear Schrödinger equations [Eqs. (1) and (5)], in view of a solution typical of nonextensive statistical mechanics, the so-called q-plane wave [9]. In the next section we review the main properties of this solution, as well as the classical field theory applied to Eq. (5). Based on this, we propose a classical Lagrangian density in order to derive the pair of equations that unify Eqs. (1) and (5); these equations relate two fields, $\Psi(\vec{x},t)$ and $\Phi(\vec{x},t)$, and it is shown that the later field becomes $\Psi^*(\vec{x},t)$ when $q\to 1$. Then, the q-plane wave solution is analyzed and particularly, how the famous de Broglie and Planck relations get modified by the contribution $K\neq 0$ of Eq. (1). Then, in Section 3 we present our main conclusions.

2. General nonlinear Schrödinger equation

Generalizations of three important equations of quantum physics were proposed in Ref. [9], by introducing nonlinear terms, through modifications of exponents of existing linear terms; these equations were the Schrödinger of Eq. (5), the Klein–Gordon, and Dirac ones. All these generalizations presented a common solution, given by the q-plane wave,

$$\Psi(\vec{x},t)=\Psi_0\exp_q\left[i(\vec{k}\cdot\vec{x}-\omega t)\right], \quad (6)$$

expressed in terms of the q-exponential function $\exp_q(u)$ that emerges in nonextensive statistical mechanics [8]. The mathematical apparatus associated with the nonextensive thermostatistical formalism has been recently applied to the study of a surprising variety of phenomena including, among others, nuclear matter [24–26], quantum chromodynamics [27], and self-gravitating systems [28]. The q-exponential function is at the core of the mathematical structure associated with the nonextensive thermostatistical formalism. In the case of a pure imaginary argument ia, where a is a real number, the q-exponential is defined by,

$$\exp_q(ia)=[1+(1-q)ia]^{\frac{1}{1-q}}. \quad (7)$$

By recourse to the well known relation $\lim_{\varepsilon\to 0}(1+\varepsilon)^{1/\varepsilon}=e$ it can be verified that, in the limit $q\to 1$, the q-exponential reduces to the standard exponential function,

$$\exp_1(ia)\equiv\exp(ia). \quad (8)$$

We provide now a brief summary of the main properties of the q-exponential function that we are going to need in the present work (a detailed, systematic analysis of the q-exponential and related functions was given by Borges in [29]). Let us consider the real and imaginary parts of a q-exponential function having pure imaginary argument,

$$\exp_q(\pm ia)=\cos_q(a)\pm i\sin_q(a). \quad (9)$$

These real and imaginary components can be conveniently expressed in terms of the $\cos_q$ and $\sin_q$ functions, which are defined as,

$$\cos_q(a)=r_q(a)\cos\left\{\frac{1}{q-1}\arctan[(q-1)a]\right\}, \quad (10)$$

and

$$\sin_q(a)=r_q(a)\sin\left\{\frac{1}{q-1}\arctan[(q-1)a]\right\}, \quad (11)$$

where

$$r_q(a) = \left[1 + (1-q)^2 a^2\right]^{1/[2(1-q)]}. \tag{12}$$

It is clear from (12) that, for $q > 1$, $r_q(a)$ is a monotonously increasing function of a. It also transpires from (10) and (11) that the functions $\cos_q(a)$ and $\sin_q(a)$ can not vanish simultaneously. Therefore, one always has $\exp_q(\pm ia) \neq 0$. However, the moduli of both $\cos_q(a)$ and $\sin_q(a)$ (and, consequently, $r_q(a)$ as well) tend to zero in the limits $a \to \pm\infty$. Finally, notice that for $q > 1$ the squared modulus of the q-exponential (7), which is given by $r_q^2(a) = \left[1 + (1-q)^2 a^2\right]^{1/[(1-q)]}$, has the shape of a q-Gaussian density. The q-Gaussian densities are proportional to $\left[1 - (1-q)\beta a^2\right]^{\frac{1}{1-q}}$, with β a real positive constant. For reasons not yet fully understood, q-Gaussians seem to be ubiquitous, appearing in a surprisingly diverse set of scenarios in physics, biology, economics, among other fields. The standard Gaussian distribution corresponds to the $q \to 1$ limit of the q-Gaussian family.

Exponential plane waves and Gaussian distributions play a central role in physics. As already mentioned, these mathematical objects admit non-trivial generalizations within the mathematical formalism of nonextensive thermostatistics. Consequently, the nonlinear Schrödinger equation (5), admitting q-plane wave solutions, is of interest from the point of view of mathematical physics. Moreover, due to the peculiar features of the q-plane waves, these solutions might be relevant for the description of complex phenomena related to dark matter, nonlinear quantum optics, plasma physics, and others.

The q-plane wave solutions have some basic mathematical properties that are physically appealing. It follows from equations (9)–(11) that the q-plane waves share the oscillatory character typical of the standard exponential plane waves. On the other hand, it also transpires from (12) that, in contrast with exponential plane waves, the q-plane waves are, for $q \neq 1$, localized structures, with a degree of localization depending on the value of q. Finally, the q-plane waves behave in a soliton-like fashion. Indeed, in the one-dimensional case one has a soliton-like structure traveling at a constant speed $v = \omega/k$, resembling nonlinear excitations with a shape constant in time.

The evolution equation (5), together with the associated complex conjugate equation for $\Psi^*(\vec{x}, t)$, do not seem to admit a Lagrangian formulation on their own (that is, a formulation based on a Lagrangian involving solely the fields Ψ and Ψ^* and their space-time derivatives). Nor do the fields Ψ and Ψ^* lead to a clear probabilistic interpretation. In fact, the norm $\int |\Psi|^2 d\mathbf{x}$ is not preserved for general solutions of (5) (see, for instance, [14,16]). The norm is, however, preserved for important particular solutions, such as the q-plane waves. These difficulties can be ameliorated by introducing a second field $\Phi(\vec{x}, t)$ (and its complex conjugate). It was proved in [21] that there is an appropriate Lagrangian density

$$\mathcal{L} = \mathcal{L}\left[\Psi, \dot{\Psi}, \vec{\nabla}\Psi, \Phi, \dot{\Phi}, \vec{\nabla}\Phi\right], \tag{13}$$

involving the fields Ψ and Φ (and their complex conjugates) and the corresponding space-time derivatives, such that the concomitant Euler–Lagrange equation for Φ leads precisely to the nonlinear Schrödinger equation (5). On the other hand, the Euler–Lagrange equation for Ψ yields the dynamic law governing the evolution of the new field,

$$i\hbar \frac{\partial \Phi(\vec{x}, t)}{\partial t} = \frac{\hbar^2}{2m} \left[\frac{\Psi(\vec{x}, t)}{\Psi_0}\right]^{1-q} \nabla^2 \Phi(\vec{x}, t)\,. \tag{14}$$

The form of the Lagrangian density (13) is such that, for $q = 1$, and making the identification $\Phi(\vec{x}, t) = \Psi^*(\vec{x}, t)$, it reduces to the standard Lagrangian for the linear Schrödinger equation. In this case the evolution equation for Φ coincides with the usual, linear, evolution equation for Ψ^* (that is, the complex conjugate of the standard Schrödinger equation). For $q \neq 1$ there isn't, in general, such a simple and direct relation between the fields Ψ and Φ (or, at least, such a relation is not known). Each of them have to be determined as solution of their respective evolution equations, which means that the Lagrangian formulation leading to the nonlinear Schrödinger equation (5) entails, for $q \neq 1$, an extra level of kinematical and dynamical complexity. This feature can be highlighted by rewriting (5) as

$$i\hbar \frac{\partial}{\partial t}\left[\frac{\Psi(\vec{x}, t)}{\Psi_0}\right] = -\frac{\hbar^2}{2m} \vec{\nabla} \cdot \left[\left(\frac{\Psi(\vec{x}, t)}{\Psi_0}\right)^{1-q} \nabla \left(\frac{\Psi(\vec{x}, t)}{\Psi_0}\right)\right], \tag{15}$$

and considering a small increment of q around $q = 1$, i.e., $q = 1 + \epsilon$ ($0 < \epsilon \ll 1$). One then obtains,

$$i\hbar \frac{\partial}{\partial t}\left[\frac{\Psi(\vec{x}, t)}{\Psi_0}\right] = -\frac{\hbar^2}{2m}\left\{ -\epsilon \left[\frac{\Psi(\vec{x}, t)}{\Psi_0}\right]^{-\epsilon-1} \left[\nabla \left(\frac{\Psi(\vec{x}, t)}{\Psi_0}\right)\right]^2 + \left[\frac{\Psi(\vec{x}, t)}{\Psi_0}\right]^{-\epsilon} \nabla^2 \left(\frac{\Psi(\vec{x}, t)}{\Psi_0}\right)\right\}, \tag{16}$$

whereas (14) becomes

$$i\hbar \frac{\partial \Phi(\vec{x}, t)}{\partial t} = \frac{\hbar^2}{2m} \left[\frac{\Psi(\vec{x}, t)}{\Psi_0}\right]^{-\epsilon} \nabla^2 \Phi(\vec{x}, t)\,. \tag{17}$$

One notices that (17) becomes the complex conjugate of (16) in the limit $\epsilon \to 0$, where the factor multiplying the Laplacian terms in both equations yields $\lim_{\epsilon \to 0}[\Psi(\vec{x}, t)/\Psi_0]^{\epsilon} = 1$. An important aspect of (5), which becomes very clear when this equation is cast as (16), is that its nonlinearity has a very different structure than the one corresponding to the GPE (1). In the GPE case, the nonlinear contribution $K|\Psi(\vec{x}, t)|^{\nu}$ is introduced in order to take into account a type of mean-field potential due to other particles in the system [3,4], while the nonlinear contributions in (16) are expected to describe other types of physical phenomena, like, e.g., dark matter [21]. Hence, a unification of these two quantum equations should cover wider physical phenomena than those covered by the GPE alone.

As already explained, no general relation between the fields Ψ and Φ, beyond the one just given by the evolution equation (14), is known. However, in the special case of the q-plane wave solutions, such a connection does exist. After substituting the q-plane wave solution (6) into the evolution equation (14), it can be verified that

$$\frac{\Phi(\vec{x}, t)}{\Phi_0} = \left\{\exp_q\left[i(\vec{k}\cdot\vec{x} - \omega t)\right]\right\}^{-q} = \left[\frac{\Psi(\vec{x}, t)}{\Psi_0}\right]^{-q}, \tag{18}$$

constitutes a solution of (14). We shall refer to the joint pair of solutions (6) and (18), for the coupled evolution equations governing Ψ and Φ, as the "q-plane wave solutions" for these two fields. An important property of these q-plane wave solutions is the fact that they are compatible with the de Broglie and Planck relations. Indeed, if the expressions (6) and (18) satisfy the alluded evolution equations, one necessarily has $\hbar\omega = \frac{\hbar^2 k^2}{2m}$, which, by recourse to the identifications $\vec{p} = \hbar\vec{k}$ and $E = \frac{p^2}{2m}$, leads (for all values of q) to the standard relation between the moment and the energy of a non-relativistic free particle, $E = \frac{p^2}{2m}$ [9]. The above explained formulation, based on the two fields Ψ and Φ, suggests to generalize Eq. (2) by introducing the probability density,

$$\rho(\vec{x}, t) = \frac{1}{2\Omega\Psi_0\Phi_0}[\Psi(\vec{x}, t)\Phi(\vec{x}, t) + \Psi^*(\vec{x}, t)\Phi^*(\vec{x}, t)], \tag{19}$$

where Ω is an appropriate constant with dimensions of volume corresponding, for q-plane waves, to a probability density normalized within a box of volume Ω. Indeed, in the case of the q-plane wave solutions associated with free particles, this proposal leads to $\rho(\vec{x},t) = 1/\Omega$, immediately yielding $[\partial\rho(\vec{x},t)/\partial t] = 0$, as well as the nonintegrability feature typical of the usual exponential plane-wave solutions of the linear Schrödinger equation.

With the purpose of unifying both Eqs. (1) and (5), herein we will consider the following Lagrangian density,

$$\mathcal{L} = \frac{A'}{\Phi_0\Psi_0}\Bigg\{ i\hbar\, \Phi(\vec{x},t)\dot{\Psi}(\vec{x},t) - \frac{\hbar^2}{2m}\left[\frac{\Psi(\vec{x},t)}{\Psi_0}\right]^{1-q} \cdot [\vec{\nabla}\Phi(\vec{x},t)]\cdot[\vec{\nabla}\Psi(\vec{x},t)] - b\Psi(\vec{x},t)\Phi(\vec{x},t)\left[\frac{\Psi(\vec{x},t)}{\Psi_0}\frac{\Phi(\vec{x},t)}{\Phi_0}\right]^{\delta-1}\left[\frac{\Psi(\vec{x},t)}{\Psi_0}\right]^{\alpha} - b\Psi^*(\vec{x},t)\Phi^*(\vec{x},t)\left[\frac{\Psi^*(\vec{x},t)}{\Psi_0}\frac{\Phi^*(\vec{x},t)}{\Phi_0}\right]^{\delta-1}\left[\frac{\Psi^*(\vec{x},t)}{\Psi_0}\right]^{\alpha} - i\hbar\, \Phi^*(\vec{x},t)\dot{\Psi}^*(\vec{x},t) - \frac{\hbar^2}{2m}\left[\frac{\Psi^*(\vec{x},t)}{\Psi_0}\right]^{1-q} \cdot [\vec{\nabla}\Phi^*(\vec{x},t)]\cdot[\vec{\nabla}\Psi^*(\vec{x},t)]\Bigg\}, \tag{20}$$

where A' is a constant, whereas b, δ, and α represent real parameters. In this way, the Euler–Lagrange equation for the field Φ yields

$$i\hbar\frac{\partial}{\partial t}\left[\frac{\Psi(\vec{x},t)}{\Psi_0}\right] = -\frac{1}{2-q}\frac{\hbar^2}{2m}\nabla^2\left[\frac{\Psi(\vec{x},t)}{\Psi_0}\right]^{2-q} + b\delta\left[\frac{\Psi(\vec{x},t)}{\Psi_0}\frac{\Phi(\vec{x},t)}{\Phi_0}\right]^{\delta-1}\left[\frac{\Psi(\vec{x},t)}{\Psi_0}\right]^{\alpha+1}, \tag{21}$$

whereas the Euler–Lagrange equation for the field Ψ leads to

$$i\hbar\frac{\partial}{\partial t}\left[\frac{\Phi(\vec{x},t)}{\Phi_0}\right] = \frac{\hbar^2}{2m}\left[\frac{\Psi(\vec{x},t)}{\Psi_0}\right]^{1-q}\nabla^2\left[\frac{\Phi(\vec{x},t)}{\Phi_0}\right] - b(\delta+\alpha)\left[\frac{\Psi(\vec{x},t)}{\Psi_0}\frac{\Phi(\vec{x},t)}{\Phi_0}\right]^{\delta}\left[\frac{\Psi(\vec{x},t)}{\Psi_0}\right]^{\alpha-1}. \tag{22}$$

The q-plane wave solution of Eq. (6), together with Eq. (18), are still solutions of Eqs. (21) and (22) provided that the real parameter α satisfies $\alpha = \delta(q-1)$. In this case, the pair of equations above become

$$i\hbar\frac{\partial}{\partial t}\left[\frac{\Psi(\vec{x},t)}{\Psi_0}\right] = -\frac{1}{2-q}\frac{\hbar^2}{2m}\nabla^2\left[\frac{\Psi(\vec{x},t)}{\Psi_0}\right]^{2-q} + K\left[\frac{\Psi(\vec{x},t)}{\Psi_0}\frac{\Phi(\vec{x},t)}{\Phi_0}\right]^{\delta-1}\left[\frac{\Psi(\vec{x},t)}{\Psi_0}\right]^{[\delta(q-1)+1]}, \tag{23}$$

whereas the Euler–Lagrange equation for the field Ψ leads to

$$i\hbar\frac{\partial}{\partial t}\left[\frac{\Phi(\vec{x},t)}{\Phi_0}\right] = \frac{\hbar^2}{2m}\left[\frac{\Psi(\vec{x},t)}{\Psi_0}\right]^{1-q}\nabla^2\left[\frac{\Phi(\vec{x},t)}{\Phi_0}\right] - Kq\left[\frac{\Psi(\vec{x},t)}{\Psi_0}\frac{\Phi(\vec{x},t)}{\Phi_0}\right]^{\delta}\left[\frac{\Psi(\vec{x},t)}{\Psi_0}\right]^{[\delta(q-1)-1]}, \tag{24}$$

where we have defined $K = b\delta$. One sees that the equations above recover Eqs. (5) and (14) respectively, by setting $K = 0$. Moreover, in the particular limit $q = 1$ one has $\Phi(\vec{x},t) = \Psi^*(\vec{x},t)$ [cf. Eq. (18)] so that Eqs. (23) and (24) recover Eq. (1) and its complex conjugate by identifying $\nu \leftrightarrow (\delta - 1)$.

Notice that the general structure of the pair of evolution equations (23)–(24) is considerably more complex than the structure of the pair (5) and (14). Indeed, one sees that equation (5) is autonomous, involving only the original field Ψ, while equation (14) involves both fields Ψ and Φ, being linear in the second field Φ. On the other hand, both evolution equations (23)–(24) depend on the two fields Ψ and Φ, and, moreover, both equations depend nonlinearly on each of these two fields. The autonomous nature of the field equation (5) (which is one of the two field equations corresponding to the case $K = 0$) allows for the dynamics of the field Ψ to be studied independently of the dynamics of the field Φ. In other words, in this case the evolution given by equation (5) constitutes a subject worth of investigation on its own right. Indeed, the wave equation (5) for Ψ was first introduced without any reference to the field Φ [9]. The situation is completely different in the general, $K \neq 0$ scenario. In this case the evolution of the two fields is more intertwined and nonlinear than in the $K = 0$ case.

One readily sees that by substituting the q-plane wave solution of Eq. (6) [together with Eq. (18)] in Eqs. (23) and (24), one obtains the following dispersion relation

$$\hbar\omega = \frac{\hbar^2 k^2}{2m} + K\,; \quad \Rightarrow \quad E = \frac{p^2}{2m} + K\,; \quad (\forall q), \tag{25}$$

where in the last equality we have made the identifications $\vec{k} \to \vec{p}/\hbar$ and $\omega \to E/\hbar$. Hence, by means of the q-plane wave solution, the energy of the free particle increases by a constant value K, so that one may define modified quantities, like momentum $\tilde{\vec{p}}$ and wave vector $\tilde{\vec{k}}$,

$$\tilde{p}^2 = p^2 + 2mK\,; \qquad \tilde{k}^2 = k^2 + \frac{2mK}{\hbar^2}\,; \quad (\forall q), \tag{26}$$

in order to keep the form of de Broglie relation, $\tilde{\vec{p}} = \hbar\tilde{\vec{k}}$. Moreover, inserting these solutions in Eq. (19), one gets $\rho(\vec{x},t) = 1/\Omega$ (and, of course, $[\partial\rho(\vec{x},t)/\partial t] = 0$) with the continuity equation clearly satisfied. It is remarkable that the q-plane waves constitute exact solutions of the coupled and highly nonlinear pair of field equations (23)–(24).

It is worth to consider in some detail the behavior of equation (23) and its q-plane wave solutions for q-values $q = 1 + \epsilon$ close to 1 (continuing with the discussion started in connection with equation (16)). Writing $\Psi = \exp(A)$ and $\Phi = \exp(B)$ one can recast (in one dimension) equation (23), to first order in ϵ, as,

$$i\hbar\frac{\partial A}{\partial t} = -\frac{\hbar^2}{2m}\left[\left(\frac{\partial A}{\partial x}\right)^2 + \frac{\partial^2 A}{\partial x^2}\right] + \epsilon\frac{\hbar^2}{2m}\left[\left(\frac{\partial A}{\partial x}\right)^2 + A\left(\frac{\partial A}{\partial x}\right)^2 + A\frac{\partial^2 A}{\partial x^2}\right] + K(1+\epsilon\delta A)\exp[(\delta-1)(A+B)]. \tag{27}$$

The solution to this equation corresponding to a first order expansion of the q-plane wave solution around the plane wave solution associated with $q = 1$ is,

$$A = i(kx - \omega t) - \frac{1}{2}\epsilon(kx-\omega t)^2,$$
$$B = -i(kx - \omega t) - i\epsilon(kx-\omega t) + \frac{1}{2}\epsilon(kx-\omega t)^2. \tag{28}$$

These solutions correspond to

$$\Psi = \exp\left[i(kx-\omega t) - \frac{1}{2}\epsilon(kx-\omega t)^2\right]. \tag{29}$$

Interestingly, we see that the first order expansion of the q-plane wave solution has the form of a Gaussian wave packet that travels keeping a constant width.

So far we have considered the unified nonlinear evolution equations in the absence of an external potential $V(\vec{x})$. We shall now briefly discuss a possible extension of the field equations (23)–(24) that incorporates the effects of an external potential. Let us consider the wave equations,

$$i\hbar\frac{\partial}{\partial t}\left[\frac{\Psi(\vec{x},t)}{\Psi_0}\right] = -\frac{1}{2-q}\,\frac{\hbar^2}{2m}\nabla^2\left[\frac{\Psi(\vec{x},t)}{\Psi_0}\right]^{2-q} + K\left[\frac{\Psi(\vec{x},t)}{\Psi_0}\frac{\Phi(\vec{x},t)}{\Phi_0}\right]^{\delta-1}\left[\frac{\Psi(\vec{x},t)}{\Psi_0}\right]^{[\delta(q-1)+1]} + V(\vec{x})\left[\frac{\Psi(\vec{x},t)}{\Psi_0}\right]^{q}, \quad (30)$$

and

$$i\hbar\frac{\partial}{\partial t}\left[\frac{\Phi(\vec{x},t)}{\Phi_0}\right] = \frac{\hbar^2}{2m}\left[\frac{\Psi(\vec{x},t)}{\Psi_0}\right]^{1-q}\nabla^2\left[\frac{\Phi(\vec{x},t)}{\Phi_0}\right] - Kq\left[\frac{\Psi(\vec{x},t)}{\Psi_0}\frac{\Phi(\vec{x},t)}{\Phi_0}\right]^{\delta}\left[\frac{\Psi(\vec{x},t)}{\Psi_0}\right]^{[\delta(q-1)-1]} + V(\vec{x})\left[\frac{\Psi(\vec{x},t)}{\Psi_0}\right]^{q-1}\frac{\Phi(\vec{x},t)}{\Phi_0}. \quad (31)$$

There are several reasons indicating that the evolution equations (30)–(31) have the appropriate structure in order to take into account an external potential $V(\vec{x})$. First of all, in the $K \to 0$ limit the equations (30)–(31) reduce to the appropriate ones in the presence of an external potential corresponding to the equations (5) and (14) (see a discussion on this issue in [15]). It is physically reasonable that the effect of an external potential is given by an additive term whose structure does not depend on the Gross–Pitaievsky-like nonlinearity given by the second terms on the right hand sides of (23)–(24). If one considers the particular case of a constant potential $V(\vec{x}) = V_0$, it can be verified after some algebra that the q-plane waves are still solutions, provided that the following relation is satisfied,

$$\hbar\omega = \frac{\hbar^2 k^2}{2m} + K + V_0\,; \quad \Rightarrow \quad E = \frac{p^2}{2m} + K + V_0\,; \quad (\forall q). \quad (32)$$

The above relation has an immediate physical interpretation: the constant potential gives rise to an additive contribution to the energy.

The field equations (30)–(31) look quite formidable. It seems uniquely that exact analytical can be obtained. However, it is possible that such solutions exist for some particular potentials. Here we shall consider, as an illustration, one particular, stationary solution of a shifted delta potential in one spatial dimension. Let us consider the following shifted attractive delta potential,

$$V(x) = V_0 - D\delta(x), \quad (33)$$

where V_0 and D are positive constants and $\delta(x)$ is Dirac's delta function. It can be verified that the wave functions

$$\frac{\Psi}{\Psi_0} = \exp_q(-\beta|x|),$$
$$\frac{\Phi}{\Phi_0} = \exp_q^{-q}(-\beta|x|), \quad (34)$$

where β is a real, positive parameter, constitute stationary solutions of (30)–(31), provided that the following relations are satisfied,

$$D = \frac{\hbar^2\beta}{m}, \quad (35)$$
$$V_0 + K = \frac{\hbar^2\beta^2}{2m}. \quad (36)$$

The wave functions Ψ and Φ given by (34) satisfy the equations obtained by setting in (30)–(31) the time derivatives equal to zero. In the $q \to 1$ limit with $K = 0$, this solution corresponds to the ground state of a shifted delta potential whose ground state energy vanishes (consequently, the time dependent phase associated with the stationary solution is zero, and the solution is strictly time independent). The solution (34) is not, strictly speaking, a q-plane wave. But formally it is closely related to q-plane waves. It can be regarded as a q-plane wave with a vanishing frequency ($\omega = 0$) and an imaginary wave number $k = i\beta$. The degree of localization of the solution (34) is given by the parameter β. Increasing values of β correspond to increasing localization. It follows then from (35) that the degree of localization increases both with the strength of the attractive delta well (given by the parameter D) and with the mass m of the particle. It is interesting that the relations (35)–(36) do not depend on the parameter q.

3. Conclusions

We have developed an exact classical field theory with the purpose of unifying two previous proposals for nonlinear Schrödinger equations, corresponding respectively, to the Gross–Pitaievsky equation and to the one associated with nonextensive statistical mechanics. We have shown that, due to the nonlinear aspects of this later contribution, besides the usual $\Psi(\vec{x},t)$, an extra field $\Phi(\vec{x},t)$ must be introduced for consistency; this later field becomes $\Psi^*(\vec{x},t)$ only when $q \to 1$. Consequently, for $q \neq 1$, we have derived a pair of quantum equations by means of which the two fields $\Psi(\vec{x},t)$ and $\Phi(\vec{x},t)$ are related, whereas in the limit $q \to 1$ they become the usual pair of complex-conjugate equations.

These equations present a common, soliton-like, traveling solution, which is written in terms of the q-exponential function that naturally emerges within nonextensive statistical mechanics. By considering this q-plane wave solution, we have shown that the total energy of the free particle gets increased by a constant value, so that the de Broglie and Planck relations are preserved for all values of q. We also advanced a pair of nonlinear evolution equations for the fields $\Psi(\vec{x},t)$ and $\Phi(\vec{x},t)$ that incorporate the effects of an external potential $V(\vec{x})$.

There is observational and theoretical evidence indicating that nonextensive thermostatistics may be relevant for the description of dark matter [30,31]. On the other hand, it has been suggested that the dark matter consists of Bose–Einstein condensates [32,33]. This raises the intriguing possibility that the nonlinear Schrödinger equations advanced here, that unifies the Gross–Pitaievskii equation and the nonlinear one related to the q-thermostatistical formalism, might describe dark matter particles. The possible connection between the nonlinear Schrödinger equations associated with the q-statistics [9] and dark matter has also been pointed out in [21], on the basis of formal properties of this equation. In order to appropriately explore this possibility it would be necessary to develop techniques to obtain more general solutions of the nonlinear equations explored in this work, beyond the analytical ones presented here. Given the fact that the nature of dark matter constitutes one of the most pressing open questions in contemporary physics and astronomy [34], this speculation may deserve to be further explored.

As a final comment, notice that the $q \to 1$ limit of the nonlinear Schrödinger equation considered here is itself nonlinear. This suggests the possibility that other types of nonlinear wave equations may also admit sensible generalizations within the context of the nonextensive thermostatistical theory. Unfortunately, no general procedure or algorithm is known to obtain these extensions. Consequently, these kind of generalizations have to be investigated individually, on a case-by-case basis. Possible venues of exploration that may be worth pursuing would be to investigate the existence of extensions, based or inspired on nonextensive thermostatistical mechanics, of the nonlinear, nonlocal Schrödinger wave equations proposed in [35], or of Schrödinger equations with logarithmic nonlinearities [36,37]. It would also be interesting to investigate if the nonlinear extensions inspired on the nonextensive thermostatistical formalism have a connection with the deformed dynamics advanced in [38]. Any further elucidation of these or related issues would be very welcome.

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