ABOUT A CENTER IN A LIÉNARD TYPE SYSTEM AND QUALITATIVE PROPERTIES ASSOCIATED

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Abstract

In this paper, we study the existence of local o global center of Liénard type system (1), under non-usual assumptions. The boundedness and the oscillatory nature of solutions of that system are also obtained.

INTRODUCTION

1. **Preliminares.** We consider the generalized Liénard equation:

$$x'' + f(x)x' + a(t)g(x) = 0 \tag{1}$$

various issues on the prolongabilidad, boundedness, frequency and oscillation of the solutions of the above equation, autonomous o not, have been considered in the past 50 years; many of these attempts have been made to obtain sufficient conditions on f and g for solutions of this equation possess certain qualitative properties

We recommend the reader to an extensive bibliography of the results appeared to 1962, Reissig, Sansone and Conti $[52]$, Cherkas $[8]$ and Graef $[13]$ for references to 1976, Staude $[60]$, Villari $[64]$, Zhifen $[71]$ until 1987 and recently, the works of Nagabuchi and Yamamoto $[30]$ and of Nápoles $[36]$. Additonal literatura, and others qualitative results, can be found in Burton and Townsend $[6-7]$, Elabbasy $[10]$, Furuya $[12]$ y Hricisakova $[23]$.

The Lienard equation, as natural generalization, has become a source of numerous investigations in recent years, we refer the reader to consult $[2]$, $[6-7]$, $[15-17]$, $[18-22]$, $[29]$, $[33-34]$, $[38]$, $[45]$, $[49]$, $[51]$, $[55-56]$, $[57]$, $[59]$, $[61]$, $[62-63]$, $[65]$, $[72]$ y $[73]$ and references cited therein, for a small sample of the qualitative research concerning global stability, boundedness and existence of periodic solutions.

This qualitative study of the solutions of this equation, often requires the use of appropriate Liapunov functions and community functions involved in it. To apply the direct method of Lyapunov equation (L), usually a Liapunov function is defined by

$$V(x,y) = \frac{y^2}{2} + \int_0^x g(s)ds.$$

As the derivative of V, along solutions (L) is

$$V'_{(1)} = -g(x)F(x),$$

being $F'(x) = f(x)$, requires that $g(x)F(x) > 0$ for $x \neq 0$. Furthermore, non-positivity of (1) implies that any solution with initial conditions in the region enclosed by the curve $V(x,y) = constant$, remains in this region as t increases, that is, the stability of the null solution. Moreover, the curve $V(x,y) = constant$ is exactly one orbit in the phase space of the system

$$x' = y,$$

$$y' = -g(x),$$

and the origin is a local center of such a system. If, in addition, is fulfilled $\int_0^x g(x)dx = \infty$, then the origin is a global center of the system and the solutions are bounded.

Using this idea, in our paper we study the existence of a center (local or global) of a natural generalization of the above equations, namely:

$$x' = \alpha(y) - \beta(y)F(x),$$
$$y' = -a(t)g(x), \tag{1}$$

where the functions involved, satisfy unusual conditions to be defined later, being $F(x) = \int_0^x f(s)ds$, with $f \epsilon L^1(R)$. The boundedness and the oscillation of the solutions of the system considered, will be studied in connection with the above problem.

It must be added that it is an old problem the authors establish under what conditions the system (1) has a center, unique or not (see $[35], [39-40]$ y $[58]$).

In the last 50 years, they have made multiple attempts (see [1], [3-4], [5], [9], [14-17] [19-20], [22], [24- 25], [26-27], [28], [46-47], [50], [67], [68], [69-70]) to obtain sufficient conditions for the existence and uniqueness of a center in particular cases of the system (1).

2. The existence of a local center and the oscillatory character of the solutions of the system (1).

In our paper, play a special role the following functional classes (see $[31]$):

$CS(D) := \{h \epsilon C(D): xh(x) > 0, \forall x \in D\},$

$CC(D) := \{h \epsilon C(D): h \in CS(D)\ strictly\ increasing, \forall x \in D\},$

$CP_b(D) := \{h \epsilon C(D): h(x) \geq b > 0, \forall x \in D\}, \qquad CP_0(D) = CP(D),$

$C^B(D) := \{h \epsilon C(D): h(x) \leq B, \forall x \in D\},$

defined on a real interval nondegenerate.

We consider then the system (1) subject to conditions:

a) $\alpha \in CC(R), such\ that |\alpha(y)| \geq |y|,$

b) $\beta \in CP_b^B(R),$

c) $F(0) = 0,$

d) $g \in CS(R),$

e) The functions involved in the system (1) satisfies a certain condition of uniqueness of solutions.

Under the conditions a, b, c and d, the system (1) admits the trivial solution and the only critical of this point is the origin.

It is clear that if $\alpha(y) = y, \beta(y) \equiv 1$ and $g(x) = x$, the system (1) is the "classical" Liénard:

$$x'' + f(x)x' + x = 0. \tag{2}$$

For this equation, the first sufficiently general results, the study of a local center, were obtained in [11] and [49]. The conditions of Opial are more general than Filippov, although in [11], it considers that F and g are odd and even functions, respectively, and F is class C^1. In our work, this condition is automatically hold, under the definition of the function F.

In this section, we determine conditions necessary and / or sufficient, under which all solutions of the system (1) are oscillating, additional conditions on the functions involved in the system, allowing affirm the existence of a local center at the origin are determined.

Definition 1. A critical point of a two-dimensional autonomous system is called a local center, if all the system orbits in some neighborhood of the point, are ovals (periodic solutions) surrounding the point.

Considering what is stated in the preliminaries, we assume that:

$$G(x) = \int_0^x G(r)dr \to \infty \text{ when } |x| \to \infty. \tag{3}$$

In view of the proposition, it seems reasonable to maintain the above consideration throughout the work.

Lemma 1. Suppose that:

i. $\int_0^\infty g(x)dx < \infty$ and $\sup_{x\geq 0} F(x) < \infty$.

Then there is a bounded solution of system (1).

Proof. Suppose $\exists M > 0/F(x) \leq M$ for $x \geq 0$.

Condition (1) it is obtained there $L > 0/\int_L^\infty g(x)dx < 1$.

We consider the solution $(x(t), y(t))$ the system (1) with initial conditions $(L, M+2)$ at $t = 0$.

Suppose there exists $t_1 > 0$ such that $y(t_1) = M + 1$, way we will have:

$M + 1 < y(t) \leq M + 2, 0 \leq t \leq t_1$.

Where we get that $\alpha(y(t)) > M + 1$, for $0 \leq t \leq t_1$. Hence $x(t) \geq L$, for $0 \leq t \leq t_1$ and taking $B \equiv 1$ we have $x'(t) = \alpha(y(t)) - \beta(y(t))F(x(t)) \geq M + 1 - M = 1$.

Thus

$$y(t_1) = M + 2 - \int_0^{t_1} g(x)dx > M + 2 - \int_L^\infty g(r)dr > M + 1,$$

which contradicts the initial assumption. Then we have $x'(t) \geq 1, and\ (t) > M + 1$ for all values of time in which they are defined, this shows that the solution $(x(t), y(t))$ is unbounded. ■

Remark 1. A simple, but tedious process, allows the same conclusion taking $B > 1$.

Remark 2. Similar to (i) condition, can be given to $x \leq 0$, which would extend the result to the entire plane.

Remark 3. If the condition (i) of Lemma; holds that $G(\pm\infty) < \infty$ but $F(x)$ is unbounded, is shown in Remark 10, the system (1) has a global center at the origin as is showed.

Let us admit, as in [31] (see also [53]), the existence of a function $h: R \to R$, which will allow us to divide the phase plane, in several useful regions in our study.

Lemma 2. Any solution of (1) passing through a point $P(x_0, h(x_0)), x_0 \neq 0$, on the curve $y = h(x)$ crosses the axis and two points $A(0, y_A), y_A \geq 0$, and $C(0, y_C), y_C \leq 0$. More precisely, if $x_0 > 0$, the solution $(x(t), y(t))$ of (1) of the point P at $t = 0$ or crosses the axis Oy in a finite time $-t_A < 0$ when it decreases, or tends to the origin as $t \to -\infty$, remaining in the region:

$D_1 = \{x, y: x \geq 0, y > h(x)\}$

and traverses the negative axis Oy in a finite time $t_C > 0$ when it grows or tends to the origin as $t \to +\infty$, remaining in the region:

$D_2 = \{x, y: x \geq 0, y < h(x)\}$.

Proof. We will consider only solutions in the region $x > 0$, then the argument is the same in the region $x < 0$.

Let $(x(t), y(t))$ the solution of (1) that part of the point P in $t = 0$. Suppose that $(x(t), y(t))$ does not cross the axis y. Then $(x(t), y(t))$ remains in the region D_2, hence $x'(t) < 0, y'(t) < 0$ and, then, $x(t) \leq x_0, y(t) \leq y_0$.

We consider curves:

$$W(x,y,\lambda) = \frac{(y-\lambda)^2}{2} + G(x).$$

The total derivative of W is given by:

$$W'_{(1)}(x,y,\lambda) = -[\beta(y)f(x)g(x) + g(x)(z(y)-\lambda)],$$

being $z(y) = y - \alpha(y)$. Let $M = \max_{0\le y\le y_0}|z(y)|$, then we have:

$$W'_{(1)}(x,y,-M) \le 0.$$

Therefore, the solution $(x(t),y(t))$ does not cross the curve $W(x,y,-M) = W(x_0,h(x_0),-M)$ when t increases. Thus the solution $(x(t),y(t))$ crosses the axis y in $C(0,y_C)$. As $x' = 0$ and $y' < 0$ on the characteristic curve in the region $x > 0, h(0) = 0$ implies $y_C \le 0$. Hence the solution through the negative axis and in a finite time or tends to the origin as $t \to +\infty$, since the origin is the only critical point (1). The proof of the existence of A is similar to C, replacing t by $-t$ in (1).■

In [37] we use the following alternative conditions on the function $y = h(x)$, we shall call condition (h).

(h₁) $h(x)$ has a sign not defined in a right semi neighbodhood of 0, that is, there is a decreasing sequence of numbers positive $\{x_n\}/x_n \to 0$ as $n \to \infty$ and $h(x_n) = 0$.

(h₂) $h(x)$ has a definite sign in a right semi neighbodhood of 0, that is,

$$\exists a > 0\ such\ that, for\ 0 < x \le a, |h(x)| > 0,$$

and exist γ such that $\gamma > \frac{1}{4b}$, and the inequality

$$\frac{1}{h(x)}\int_0^x \frac{g(s)}{F(s)}ds \ge \gamma, 0 < x \le a,$$

holds.

Definition 2. We will say that the system (1) is of type (F) if the condition:

$$F\left(G^{-1}(-w)\right) = F\left(G^{-1}(w)\right), for\ w \ge 0. \tag{4}$$

We are now able to study the existence of a local center.

Lemma 3. Under the conditions (h) and (4), the system (1) has a local center at the origin.

Proof. We can consider only the region $x \ge 0$. Note that $x' > 0, y' < 0$ in D_1, $x' = 0, y' < 0$ on $\{x,y: x \ge 0,\ y = h(x)\}$ and $x' < 0, y' < 0$ in D_2. From Lemma 2 we have, that any solution passes through a point $(x_0,h(x_0)), 0 < x_0 \le a$ crosses the axis y at $(0,y_C)$ and $(0,y_A)$ with $y_C < 0$ and $y_A > 0$.

Case (h₁): It is trivial.

Case (h₂): We will work only with the region D_2. If $h(x) < 0$ for $0 < x \le a$ is clear that $y_C < 0$. Suppose that $h(x) > 0$ for $0 < x \le a$ and $y_C = 0$. Then the solution $(x(t),y(t))$ of (1) passing through the point $(x_0,h(x_0))$ defines a function $y(x)$ on $0 \le x \le x_0$, which is a solution, about $0 < x < x_0$, the equation:

$$\frac{dy}{dx} = -\frac{g(x)}{\alpha(y) - \beta(y)F(x)}.$$

As $y_C = 0$ then $y(x) > 0$ for $0 < x < x_0$. For any $\varepsilon > 0$ we have:

$$y(x) - y(\varepsilon) = \int_{\varepsilon}^{x} \frac{g(\varepsilon)}{F(\varepsilon)\beta(y(\varepsilon)) - \alpha(y(\varepsilon))} d\varepsilon \geq \frac{1}{b}\int_{0}^{x} \frac{g(\varepsilon)}{F(\varepsilon)} d\varepsilon,$$

for $\varepsilon \leq x < x_0$. Thus, $y(x) \geq \frac{\gamma h(x)}{b}$, $\varepsilon \leq x < x_0$. If $\gamma \geq b$, then we have a contradiction. Suppose that $\frac{1}{4b} < \gamma < \frac{1}{b}$, hence $h(x) - y(x) \leq \gamma_1 h(x)$, where $\gamma_1 = 1 - \frac{\gamma}{b}$. Similarly we have:

$$y(x) - y(\varepsilon) = \frac{1}{b}\int_{\varepsilon}^{x} \frac{1}{\gamma_1}\frac{g(\varepsilon)}{F(\varepsilon)} d\varepsilon \geq \frac{1}{b}\gamma\frac{h(x)}{\gamma_1},$$

for $\varepsilon \leq x < x_0$. Thus, $h(x) - y(x) < \gamma_2 h(x)$ with $\gamma_2 = 1 - \frac{\gamma}{b\gamma_1}$.

By repeating this procedure, we obtain a sequence $\{\gamma_n\}$ such that $\gamma_n = 1 - \frac{\gamma}{b\gamma_{n-1}}$ and $h(x) - y(x) \leq \gamma_n h(x)$ for $\varepsilon \leq x < x_0$. If $\gamma_n > 0$, then $\{\gamma_n\}$ is decreasing, and therefore $\{\gamma_n\}$ converges to some real number λ. Moreover, $\lambda = 1 - \frac{\gamma}{\lambda}$ and $\gamma > \frac{1}{4}$ show that λ is complex, which is a contradiction. The same argument can prove that $y_A > 0$.■

Remark 4. In [26, Example 4.2], Kooij and Jianhua study the system (1), taking $\beta \equiv k > 0$ and considering that the following conditions are fulfilled:

i. $h(0) = 0, h(x)$ strictly increasing and $h(\pm\infty) = \pm\infty$,

ii. $xg(x) > 0, x \neq 0$ y $G(\pm\infty) = \infty$,

iii. $g(-x) = -g(x)$ not decreasing as x increases,

on the basis of Theorem 2.6, they affirm the existence of a center in such a system; it is clear that Lemma 3 obtained our under weaker conditions.

The same observation is valid in the case of Examples 4.3 and 4.4 of that work, where they present other particular cases of the system (1) and, on the basis of Theorems 2.7 and 3.2, obtain the existence of a local center.

The corollary 8 of [9] is applied to the system

$$x' = y - (a_3x^3 + a_2x^2 + a_1x),$$
$$y' = -x,$$

with $|a_2|$ small enough, to obtain the same conclusion of Lemma 3, this assumption is not used.

In [27], the system (1) with $\beta \equiv 1$, under very strong analyticity is studied.

It is easy to see that considerations of Lemma 3 are less demanding.

Remark 5. Odani´s results [46] are consistent with Lemma 3 as shown by the example of this work $(\mu \equiv 1)$:

$$x' = y - \left(\frac{x^3}{3} - x\right),$$
$$y' = -x\left[1 + \frac{x^2(x^2-4)}{16}\right].$$

Remark 6. The condition (h) has certain "overtones of necessity," because there are systems that do not comply, and have Unbounded solutions, let's take as an example (see [23]):

$$x' = y - \frac{3x^2}{2},$$
$$y' = -x^3.$$

Also see recommend [67] (on a result of [51]), where is presented the system

$x' = y + \sqrt{2x^2}$
$y' = -x^3,$

having the integral curve $y = -\sqrt{\frac{2x^2}{2}}$.

Remark 7. Our results are consistent with some reported in the literature (see [7] for example), and the equations

$$x'' + 2x'(2 + x^2) + 2x + \frac{x^3}{3} = ksent,$$

and

$$x'' + (x^{2n} + 1)x' + 10x = ksent$$

of this work, which are covered by our results (Figures 1, 2, 3 and 4 illustrate the above). We also recommend [50, Theorem 1], [61, Theorem 4.1] and [67, Example 3].

Lemma 4. Suppose that:

$$\int_0^\infty \frac{g(x)}{1-F_-(x)}dx = \infty, \qquad (5_-)$$

where $F_-(x) = \max(0, -F(x))$.

Let $(x(t), y)t))$ the maximal solution of (1) extended to the interval $[0,T), T \leq +\infty$, with initial conditions$(x_0, y_0) \in D_1$. If $(x(t), y)t))$ is in the region D_1 for $t \in [0,T)$, then $x(t) \to \infty, y(t) \to -\infty$, when $t \to T^-$.

Proof. Note that $x'(t) > 0, y'(t) < 0$ in the region D_1, therefore have to be $x(t)$ is monotonically increasing and $y(t)$ is monotonically decreasing. Suppose that $x(t)$ is bounded. Then $(x(t), y(t))$ remains in the region $\{x, y: 0 < x < K, y > h(x)\}$ for some $K > 0$, and therefore traverses the curve $y = h(x)$, which is a contradiction. Hence $x(t) \to \infty$ as $t \to T^-$. Suppose now that $y(t) \geq -c$ for some $c > 0$. So:

$$y(t) - y_0 = -\int_0^t g(x(s))ds = -\int_0^t \frac{g(x(s))}{\alpha(y(s)) - \beta(y(s))F(x(s))}x'(s)ds$$
$$\leq -\int_{x_0}^{x(t)} \frac{g(\varepsilon)}{\alpha(y_0) - BF_-(\varepsilon)}d\varepsilon \to -\infty,$$

when $t \to T^-$. This contradicts the assumption that $y(t) \geq -c$. ■

Similarly, it can be proved the following result.

Lemma 5. Suppose that:

$$\int_0^\infty \frac{g(x)}{1+F_+(x)}dx = \infty, \qquad (5_+)$$

with $F_+(x) = \max(0, F(x))$. Let $(x(t)), y(t))$ the maximal solution of (1) extended to the interval $(-T, 0], T \leq +\infty$, with initial conditions $(x_0, y_0) \in D_2$. If $(x(t)), y(t))$ is in the region D_2 to $t \in (-T, 0]$, then $x(-t) \to \infty, y(-t) \to -\infty$, when $t \to T^-$.

Remark 8. The condition $(5_-) - (5_+)$, which we call (5), implies $G(\infty) = \infty$.

Lemma 6. Suppose that (3) is satisfied and the condition:

$\exists c \in R\ (cte)$ and a succession $\{x_n\}$ such that $x_n \to \infty$ when $n \to \infty$ and $h(x_n) = 0$ (6)

is fulfilled. Then every sequence $(x(t)), y(t))$ of (1) that part of a point $(x_0, y_0) \in D_1$ (D_2) crosses the curve $y = h(x)$, when t increases (decreases).

Proof. By Lemma 4, if $(x_0, y_0) \in D_1$ and the solution in D_1 for all finite time, then $y(t) \to -\infty$. Therefore it follows immediately from (6) that the solution crosses the curve $y = h(x)$ when t increases. Similarly, by Lemma 5, if $(x_0, y_0) \in D_2$ and remains in D_2 for all finite time, then the solution crosses the curve $y = h(x)$. ■

For our next result, we need to return to the condition (F) given in [37].

(F) $|F(x)| > 0$ for x large enough and for any $b > 0$, exist a number b_1, such that the inequality

$$\frac{1}{F(x)} \int_0^x \frac{g(s)}{F(s)} ds \geq K,$$

is satisfied for $x \geq b_1$, where $K > \frac{1}{4}$ and it does not depend on b_1.

Lemma 7. Suppose that (3) and (F) holds. Then every solution $(x(t)), y(t))$ (1) that part of a point $(x_0, y_0) \in D_1$ (D_2) crosses the curve $y = h(x)$ when t decreases (increases).

Proof. Consider only the case where $h(x) > 0$ for sufficiently large x. The proof in the other case is essentially the same. Now consider $(x_0, y_0) \in D_2$. Suppose that there is a solution $(x(t)), y(t))$ the (1) passing through a certain point (x_0, y_0) which does not cross the curve $y = h(x)$. Then the trajectory of this solution can be considered a function $y(x)$ which is solution of the equation (4). By Lemma 5, we have $x(t) \to \infty$, and $y(t) \to +\infty$ when t decreases and $y(t) \to +\infty$, therefore exist $K > 0$ such that $y(x) \geq 0$ for all $x \geq k$, so we have $h(x) - y(x) \leq h(x)$.

As $y(x)$ it is a solution of (4) follows from (F) exists $k_1 > K$ such that:

$$y(x) - y(K) = \int_K^x \frac{g(\varepsilon)}{F(\varepsilon)\beta(y(\varepsilon)) - \alpha(y(\varepsilon))} d\varepsilon \geq \frac{1}{b} \int_K^x \frac{g(\varepsilon)}{F(\varepsilon)} d\varepsilon \geq \frac{1}{b} N h(x),$$

for $x \geq k_1$. Therefore, $h(x) - y(x) \leq N_1 h(x)$ for $x \leq k_1$, where $N_1 = 1 - \frac{N}{b}$. By a similar argument, exist $k_2 > k_1$ such that:

$$y(x) - y(k_1) = \int_{k_1}^x \frac{g(\varepsilon)}{F(\varepsilon)\beta(y(\varepsilon)) - \alpha(y(\varepsilon))} d\varepsilon \geq \frac{1}{b} \int_{k_1}^x \frac{g(\varepsilon)}{F(\varepsilon)} d\varepsilon \geq \frac{1}{b} \frac{N}{N_1} h(x),$$

for $x \geq k_2$.

Thus, $h(x) - y(x) \leq N_2 h(x)$ for $x \geq k_3$, where $N_2 = 1 - \frac{N}{N_1 b}$. Repeating this procedure, we obtain two sequences $\{k_n\}$ and $\{N_n\}$ such that $N_n = 1 - \frac{N}{N_{n-1} b}$ and $h(x) - y(x) \leq N_n h(x)$, for all $x \geq k_n$.

As in the proof of Lemma 3, $N > \frac{1}{4b}$, implies $N_n \leq 0$ for some n, and this is a contradiction.

Similarly in case $(x_0, y_0) \in D_1$, we can show that the trajectory that passes through the point (x_0, y_0) intersects the curve $y = h(x)$. This completes the proof. ■

2.1 Oscillation of solutions. In this section we study the oscillation of the solutions of the system (1), make use of the Second Method of Liapunov.

Let H_1 is the set of functions $h(x)$ defined on $x \geq 0$ with $h(0) = 0$ such that:

$(i)_1$ $h(x)$ satisfies (h_1),

$(ii)_1$ $\exists a > 0/h(x) < 0$ for $0 < x \leq a$,

or

$(iii)_1$ $\exists a > 0/h(x) < 0$ for $0 < x \leq a$ and to $\gamma > \frac{1}{4b}, \frac{1}{h(x)} \int_0^x \frac{g(s)}{F(s)} ds \geq \gamma$.

Let H_2 is the set of functions $h(x)$ defined on $x \geq 0$ with $h(0) = 0$ such that:

$(i)_2$ $h(x)$ satisfies (6),

$(ii)_2$ $h(x) > 0$ for x large,

or

$(iii)_2$ $h(x) < 0$ for x large and $h(x)$ satisfies the condition

$$h(x) \in H_1 \text{ y } -h(-x) \in H_1, x \geq 0. \tag{7}$$

In addition to the above condition we consider that:

$$h(x) \in H_2 \text{ and } -h(-x) \in H_2, x \geq 0. \tag{8}$$

Note that ((h)) and ((6) F)) correspond to (7) and (8), respectively.

Remark 9. Prolongability for solutions, we refer the reader to [42-43] (F has a definite sign) and [32-33] (when F has a behavior any).

Theorem 1. Under considerations (3), (7) and (8) all solutions of the system (1) are oscillating.

If we omit (3) and (8) we obtain:

Theorem 2. Under consideration (7), suppose:

Exist sequences $\{x_n\} \to +\infty$ y $\{x'_n\} \to -\infty$ such that $h(x_n) \to +\infty$ y $h(x'_n) \to -\infty$, when $n \to \infty$ (9)

then all the solutions of (1) are oscillating.

For the proof of these theorems, we take:

$\{x_n\} \to +\infty$ and $\{x'_n\} \to -\infty$ such that $h(x_n) + G(x_n) \to +\infty$ and $h('x_n) + G(x'_n) \to -\infty$, when $n \to \infty$ (10)

so we get our final result.

Definition 3. A critical point of a two-dimensional autonomous system is called a global center, if all the system orbits are ovals (periodic solutions) surrounding the point.

We will use again the curve $y = h(x)$ and the regions D_1 and D_2.

Remark 11. Let us return to the condition (h). Suppose $\exists a > 0$ and $\gamma > \frac{1}{4}$ such that:

$0 < (F(x))^2 \leq 2\gamma^{-1} G(x)$, for $0 < |x| \leq a$. Then (h) is satisfied. In particular, consider the case that the system (1) is linear (see [19]), putting:

$$\alpha(y) = y, \beta \equiv 1, F(x) = \mu|x| \text{ y } g(x) = \lambda x.$$

We see that (h) is only true if $\mu^2 < 4\lambda$ and the origin is a local and global center if and only if the condition (h) hold (see Lemma 3). You can see examples of less "trivial" local or global in, [38-39] Let the functions $F^*(u)$ y $\phi(x)$ defined above for R:

$$F^*(u) = \begin{cases} F\left(G^{-1}\left(\frac{u^2}{2}\right)\right), & u \geq 0, \\ F\left(G^{-1}\left(-\frac{u^2}{2}\right)\right), & u < 0, \end{cases} \qquad \phi(x) = \begin{cases} \sqrt{2G(x)}, & x \geq 0, \\ -\sqrt{-2G(x)}, & x < 0, \end{cases}$$

and application $\Phi: (x, y) \rightarrow (u, v)$ by $\Phi(x, y) = (\phi(x), y)$.

Then $F^*(u), \phi(x)$ *and* $\Phi(x, y)$ are continuous. We should note that $F^*(u)$ is an odd function if (11) holds.

We consider the system:

$$u' = \alpha(v) - \beta(v)F^*(u),$$

(11)

$$v' = -u.$$

In [31] the author recommended, given the topological similarity of behavior of the trajectories of the system (1) and Liénard Equation (2), finding an isomorphism between them, which would be a result of undoubted theoretical and practical interest.

Lemma 8. If (4) holds, Φ is an isomorphism of the plane (x, y) on the plane (u, v) which is a one to one correspondence between all orbits (1) and those of (11). Without the condition (4), Φ is a correspondence between all orbits in a neighborhood of the plane origin (x, y) and those of the plane (u, v).

Proof. It is obvious that Φ is a homeomorphism on the entire plane and a diffeomorphism half-plane (x, y) right (left) on the (u, v) right half-plane (left).

Consider an orbit $T(x, y)$ of (1) and curve $C(u, v)$ which is the image of T by the application Φ. We show that C is an orbit (11). In the region $u > 0$, if a point (u, v) belongs to the curve C, then:

$$\frac{du}{dv} = -\frac{g(x)}{\sqrt{2G(x)}} \frac{\alpha(y) - \beta(y)F(x)}{-g(x)} = \frac{\alpha(v) - \beta(v)F^*(u)}{-u}.$$

A similar result is obtained in the region $u < 0$. Therefore the curve C in the region $u > 0, u < 0$ is on an orbit solution (11). If the orbit crosses the axis y in $(0, y_1)$, then the curve C also crosses the axis v in $(0, y_1)$. Let $C'(u, v)$ of the orbit (11) passing through the point $(0, y_1)$ and $T'(x, y)$ of the inverse image the C' by the application Φ. Since the solutions of (1) are unique, T and T' coincide and C and C' therefore also coincide by injectivity of Φ. Thus Φ applies injective orbits (1) those of (11). ■

Some of the results of the previous section, we can rewrite as follows.

Lemma 9. Suppose that conditions (3) and (4) hold. If $y_A > 0$ and $y_C < 0$ are satisfied as in the previous lemma and if:

The functions α and β are odd and pair functions, respectively, then all orbit (1) passing through the curve $y = h(x)$ is an oval surrounding the origin. (12)

Remark 12. If consideration (3) is omitted, this result is true in a neighborhood of the origin.

Proof. By Lemma 8, we can assume $g(x) = x$ in the system (1). Thus, (4) shows that $F(x)$ is an odd function. Therefore, if $(x(t), y(t))$ is a solution of (1), $(-x(-t), -y(-t))$ so is, that is, the orbits defined by (1) have symmetry about the axis Y. Thus, every orbit is an oval surrounding the origin. ■

The following result is a consequence of the results of the previous section and Lemma 9.

Lemma 10. Under the considerations (3), (4), (h) and (12), all orbit the system (1) passing through the curve $y = h(x)$, is an oval surrounding the origin.

Lemma 11. Suppose (5) and (6) are fulfilled. Then, any solution $(x(t), y(t))$ of (1) with initial conditions $(x_0, y_0) \in D_1$ crosses the curve $y = h(x)$ when t increases (decreases).

Proof. By Lemma 10, if $(x_0, y_0) \in D_1$ and the solution remains in D_1 then $y(t) \to -\infty$ as t increases. Therefore it follows from (5) that the solution crosses the curve $y = h(x)$ as t increases. Similarly, by Lemma A, if $(x_0, y_0) \in D_2$ then the solution crosses the curve $y = h(x)$ when t decreases.■

Theorem 4. Under considerations (3), (4), (h), (5), (12), the system (1) has a global center at the origin.

Proof. This theorem is an immediate consequence of the previous two theorems.■

Remark 13. It is sufficient to take $\alpha(y) = y, \beta \equiv 1, F(x) = -x^4|senx|$ and $g(x) = x$ (see [19]), to ensure that the condition (12) in the previous theorem, can not be weakened.

Theorem 4 shows that the system (1) has a global center at the origin if F (x) is "oscillating". We will now discuss the case when F (x) keeps the sign on an interval not bounded. We must return some results proved in [37].

Lemma A (B). Suppose that $(5_-)((5_+))$ hold. Let $(x(t), y(t))$ the maximal solution of (1) extended to the interval $[0,T)$ $((-T,0]), T \le +\infty$, with initial conditions $(x_0, y_0) \in D_1$ (D_2). If $(x(t), y(t))$ it is in the region D_1 (D_2) for $t \in [0,T)$ $((-T,0])$, then $x(t) \to \infty, y(t) \to \infty$ $(x(-t) \to \infty, y(-t) \to \infty)$ when $t \to T^-$ (T^+).

Lemma C. Let $H(x,y)$ and $H_1(x,y)$ continuous in $D \subset R^2$. Suppose the solution of the equations

$$\frac{dy}{dx} = H_1(x,y) \qquad (13_1)$$

and

$$\frac{dy}{dx} = H_2(x,y) \qquad (13_2)$$

are unique to the right and $H_1(x,y) \le H_2(x,y)$ for all $(x,y) \in D$. Let $y_1(x), y_2(x)$ solutions (13_1) and (13_2) respectively, defined on $[x_1, x_2]$, with $y_1(x_1) \le y_2(x_1)$. Then $y_1(x) \le y_2(x)$ for all $x \in [x_1, x_2]$.

Lemma 12. Suppose (5) and (F) are satisfied. Then, any solution $(x(t), y(t))$ of (1) with initial condition $(x_0, y_0) \in D_2(D_1))$ crosses the curve $y = h(x)$ when t decreases (increases).

Proof. We consider only the case $F(x) > 0$ for x sufficiently large. The proof in the other case, is essentially the same. Take $(x_0, y_0) \in D_2$. Suppose there is a solution $(x(t), y(t))$ of (1), with such initial conditions, which does not cross the curve $y = h(x)$. Then the orbit of such a solution may be considered a function $y(x)$, solution of the equation:

$$\frac{dy}{dx} = -\frac{g(x)}{\alpha(y) - \beta(y)F(x)} \qquad (14)$$

By Lemma B, we have $x(t) \to \infty, y(t) \to -\infty$ as t decreases and $y(x) \to +\infty$ when $x \to +\infty$.

Therefore, exist $b > 0/y(x) \geq 0$ to $x \geq b$, so we have que $\beta(y)F(x) - \alpha(y) < \beta(y)F(x)$.

Since $y(x)$ is a solution of (14), it follows from (F) there $b_1 > b$ such that:

$$y(x) - y(b) = \int_b^x \frac{g(r)}{\beta(y(r))F(r) - \alpha(y(r))} dr \geq \frac{1}{B} \int_b^x \frac{g(r)}{F(r)} dr \geq \gamma \frac{F(x)}{B},$$

for $x \geq b_1$. Thus, $\beta(y)F(x) - \alpha(y) \leq \gamma_1 F(x)$ for $x \leq b_1$, where $\gamma_1 = 1 - \frac{\gamma}{B}$. n a similar way we get there $b_2 > b_1$ such that:

$$y(x) - y(b_1) = \int_{b_1}^x \frac{g(r)}{\beta(y(r))F(r) - \alpha(y(r))} dr \geq \frac{1}{B} \int_{b_1}^x \frac{g(r)}{F(r)} dr \geq \gamma_1 \frac{F(x)}{B},$$

for $x \geq b_2$. Thus, $\beta(y)F(x) - \alpha(y) \leq \gamma_2 F(x)$ for $x \leq b_2$, where $\gamma_2 = 1 - \frac{\gamma_1}{B}$. By repeating this process, we obtain two sequences $\{b_n\}$ y $\{\gamma_n\}$ such that $\gamma_n = 1 - \frac{\gamma}{\gamma_{n-1}}$ y , $\beta(y)F(x) - \alpha(y) \leq \gamma_n F(x)$ for all $x \geq b_n$.

As in the proof of Lemma 3, $\gamma > \frac{1}{4}$ implies $\gamma_n \leq 0$ for some n and this is a contradiction. Similarly, in case $(x_0, y_0) \in D_1$, we can show that the orbit of (1) with these initial conditions, crosses the curve $y = h(x)$. This completes the proof. ■

Theorem 5. Under the considerations (3), (4), (h), (5) and (F), the system (1) has an overall center at the origin.

Proof. This result is an immediate consequence of Lemmas 10 and 11. ■

If assumption (5) is no considered then we have.

Theorem 6. Under the considerations (3), (4), (h), (13) and

$$\exists \{x_n\} \to +\infty \; y \; \{x'_n\} \to -\infty \; such \; that \; F(x_n) \to +\infty \; y \; F(x'_n) \to -\infty \; when \; n \to \infty \quad (15)$$

the system (1) has a global center at the origin.

Proof. It is clear that the orbit of (1) with initial conditions (x_0, y_0) cuts the curve $y = h(x)$. The use of Lemma 10 completes the proof of the theorem. ■

Remark 14. If conditions (3) and (4) are omitted in this theorem, but F and g are odd, then the system (1) has a local center at the origin. This can be demonstrated, using Lemmas 2 and 3, as in the proof of the previous theorem (see [35] for another demonstration). We recommend consulting [42,54,58] for other results in this direction.

2.1 Stability and bounded. The definitions required for this section can be found in [66].

Consider the system:

$$x' = \alpha(y) - \beta(y)F_1(x), \qquad (16)$$
$$y' = -a(t)g(x)$$

where F_1 is a continuous function that satisfies condition initial (c).

Denote by $T_{(16)}(P)$ orbit system (16) having as an initial condition point P. If this orbit is an oval surrounding the origin, denote by $R_{(16)}(P)$ in the closed region contained orbit.

The following result, will play a key role in this section.

Lemma 13. Suppose that $R_{(16)}(P)$ to a P (different origin) is defined and

$$F_1(x) \leq F(x) \ for\ all\ x \geq 0. \tag{17}$$

Then for any $(x_0, y_0) \in R_{(16)}(P)$ with $x_0 \geq 0$, the solution $(x(t), y(t))$ does not cut the curve $T_{(16)}(P)$ in the half-plane $x \geq 0$.

Proof. Consider the following cases.

1. $x_0 = 0, y_0 < 0$.

The proof is trivial.

2. $x_0 > 0$ or $x_0 = 0, y_0 > 0$.

We assume that the Lemma is invalid and let $t_1 = inf\{t: (x(t), y(t)) \notin R_{(16)}(P),\ x(t) \geq 0\}$.

Then $(x(t), y(t)) \notin T_{(16)}(P)$ and for some η will have:

$$(x(t), y(t)) \notin R_{(16)}(P)\ and\ x(t) > 0\ for\ all\ t \in (t_1, t_1 + \eta]. \tag{18}$$

Is $(x_1(t), y_1(t))$ the solution of (6) having initial condition in (P) such that

$$(x_1(t_1), y_1(t_1)) = (x_1(t), y_1(t)).$$

Consider first the case $(x_1(t_1), y_1(t_1)) \in D_1$.

Then $(x(t), y(t))$ and $(x_1(t), y_1(t))$ defining solutions $y(x), y_1(x)$ of equations defined on D_1:

$$\frac{dy}{dx} = \frac{g(x)}{\beta(y)F(x) - \alpha(y)}, \tag{19}$$

and

$$\frac{dy}{dx} = \frac{g(x)}{\beta(y)F_1(x) - \alpha(y)}, \tag{20}$$

which exist on $[x_1, x_1 + a]$ for some $a > 0$ and satisfy:

$$y(x) > y_1(x)\ para\ x \in (x_1, x_1 + a], \tag{21}$$

where $x_1 = x(t_1)$.

Conversely, for a solution $y(x)$ of (19) on $[x_1, x_1 + a]$, the solution $x(t)$ of the equation $x' = \alpha(y) - \beta(y)F(x)$ with $x(0) = x_1$ and the solution $y(t)$ of the equation $y' = -g(x(t))$ with $y(0) = y(x_1)$ define a solution (1). Therefore, the uniqueness of solutions of (1) allows us to affirm the uniqueness of the solutions of (20) on D_1. Similarly, solutions (20) are unique. Thus, it follows from the C and (17):

$$y(x) \leq y_1(x)\ for\ x \in [x_1, x_1 + a],$$

which contradicts (21). Therefore, in this subcase, $(x(t), y(t))$ not short $T_{(16)}(P)$. Consider now that $(x(t_1), y(t_1)) \in D_2$. The proof this subcase, is performed similar to the above, it suffices to consider the equations:

$$\frac{dx}{dy} = \frac{\beta(y)F(x) - \alpha(y)}{g(x)} \text{ and } \frac{dx}{dy} = \frac{\beta(y)F(x) - \alpha(y)}{g(x)}.$$

This completes the proof. ■

In [43] and [53], we provide results on stability of the null solution (1) and the boundedness of solutions, using the Second Method of Lyapunov. In such cases, if considered the following assumptions:

$$g(x)F(x) > 0\ for\ x \neq 0, \tag{22}$$

because use of a certain Lyapunov function whose derivative along the system (1) was defined negative under the condition becomes (22).

In this section, we are interested and study the system (1) when the condition (22) is not fulfilled. Note, moreover, that there $k > 0$ such that $G^{-1}(w)$ is defined for $|w| < k$, provided that (22) is not met.

Theorem 7. Suppose $-F(-x)$ and $-g(-x)$ satisfy (h) for $x \geq 0$ and

$$F\left(G^{-1}(-w)\right) \leq F\left(G^{-1}(w)\right) \; for \; 0 \leq w \leq k. \tag{23}$$

Then the null solution of (1) is uniformly stable.

Proof. Define the function $\Gamma(x)$ as $\Gamma(x) = F(x)$ for $x \leq 0$ and $\Gamma\left(G^{-1}(x)\right) = F(G^{-1}(x))$ for $w \geq 0$. Then Lemma 3 shows that the source is a local center system:

$$x' = \alpha(y) - \beta(y)\Gamma(x),$$

(24)

$$y' = -g(x).$$

Therefore, there is a neighborhood V of the origin such that for any $(x_0, y_0) \in V$, orbit $T_{(24)}(x_0, y_0)$ of the solution (24) is an oval that surrounds the origin. Let $A(0, y_A)$ and $C(0, y_C)$ points in which $T_{(24)}(x_0, y_0)$ crosses the y-axis, with $y_A > 0, y_C < 0$. For him Lema C and (22) any solution $\left(x(t), y(t)\right)$ of (1) with initial conditions $R_{(24)}(x_0, y_0)$ and $x(0) \geq 0$ not cross the curve $T_{(24)}(x_0, y_0)$ in the right half plane and therefore the solution intersects the axis and at a certain instant $t_1 > 0$ or tends to the origin as $t \to \infty$. In these cases, $\left(x(t), y(t)\right)$ is a solution of (24) for $t > t_1$ and therefore intersects the positive axis and at a certain instant $t_2 > t_1$. The uniqueness of solutions of (24) implies that $y_A \geq y(t_2)$. Thus, the solution $\left(x(t), y(t)\right)$ is defined for all later time and remains in the region $R_{(24)}(x_0, y_0)$ for all $t \geq 0$. By the same argument, it is clear that any solution $\left(x(t), y(t)\right)$ of (1) with initial conditions $R_{(24)}(x_0, y_0)$ and $x(0) < 0$ remains in $R_{(24)}(x_0, y_0)$. Since the choice of the neighborhood is arbitrary, the null solution of (1) is uniformly stable. ■

Corollary 1. Suppose $F(-x) \leq 0$ for $0 \leq x \leq k$, $F(x)$ and $g(x)$ satisfy (11) for $0 < x < k$ and conditions (12) and (23) holds. Then the null solution of (1) is uniformly stable.

Proof. Through the transformation $(u, v) = (-x, -y)$, the system (1) becomes the system:

$$x' = \alpha_1(v) - \beta_1(v)F_1(u),$$

(25)

$$y' = -g_1(u),$$

where $\alpha_1(v) = -\alpha_1(-v),\ \beta_1(v) = \beta_1(-v), F_1(u) = -F_1(-u)$ y $g_1(u) = -g_1(-u)$. It is easy to verify that the system (25) satisfies the conditions of Theorem 7. This completes the proof. ■

The following results are referred to the boundedness of the solutions of the system (1).

Theorem 8. Suppose $-F(-x)$ and $-g(-x)$ satisfy (5) and ((6) or (F)) for $x \geq 0$ and also satisfy the following condition:

$$F\left(G^{-1}(-w)\right) \leq F\left(G^{-1}(w)\right) \; for \; all \; w \geq 0. \tag{26}$$

Then the solutions of (1) are uniformly bounded.

Proof. Define the function $\Gamma(x)$ as $\Gamma(x) = F(x)$ for $x \leq 0$ and $\Gamma\left(G^{-1}(x)\right) = F(G^{-1}(x))$ for $w \geq 0$ and consider the system (24). Note that (4) holds for this system.

First we show, that for any $x_0 > 0$ large enough, there exist $y_0 \in R$ such that the orbit $T_{(24)}(x_0, y_0)$ encircles the origin. By Lemmas 11 and 12, for any $y_0 \in R$, $T_{(24)}(x_0, y_0)$ intersects the curve $y = h(x)$. Therefore, under Lemas 4 and 11, it is sufficient to show that $y_A > 0, y_C < 0$ for some $y_0 \in R$.

But this is obviously the case that $\Gamma(x)$ is oscillating, that is, there is a sequence $\{x_n\}$, tending to ∞, such that $\Gamma(x_n) = 0$.

Suppose there is $K > 0$ such that $\Gamma(x) < 0$ $(\Gamma(x) > 0)$ for $x > K$.

Choose $y_0 > 0$ such that $(x_0, y_0) \in D_1$. Then it is easy to obtain that $y_A > 0, y_C < 0$. Thus in any case, $T_{(24)}(x_0, y_0)$ is an oval, for certain $y_0 \in R$.

Of the latter, easily get there $M > 0$ such that for any (x_0, y_0) with $x_0^2 + y_0^2 > M, T_{(24)}(x_0, y_0)$ is an oval that surrounds the origin.

The rest of the show is similar to the Theorem 8. ■

Corollary 2. Suppose $F(-x) \leq 0$ for $x > 0$ large, F (x) and g (x) satisfy (5) and ((12) or (F)) to $x \geq 0$ and that the condition (26) is met. Then the solutions of (1) are uniformly bounded.

Proof. By transforming $(x, y) \to (-x, -y)$, this reduces to Theorem 8. ■

3. **Concluding remarks.** Take the system

$$x' = \frac{y}{5} - x(x-1)(x+1.1),$$
$$y' = -x^3,$$

it is easy to see that the condition $|\alpha(y)| \geq |y|$ it fails, however it has a center at the origin (see [26]).

On the other hand, if the system (1) we $g(x) = F(x)$, you can easily check the condition (h) is not always true.

Thus, the question arises:

Under what additional conditions, we obtain the existence of a center (local or global) in these systems?.

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