ON THE CHARACTERIZATION OF CERTAIN STABILITY CONCEPTS FOR VOLTERRA TYPE INTEGRO – DIFFERENTIAL SYSTEMS

Paulo M. Guzmán, Juan E. Nápoles Valdes, Luciano M. Lugo Motta Bittencurt, José R. Velázquez Codina

UNNE, FaCENA, Av. Libertad 5450, Corrientes (3400), Argentina.

UNNE, Facultad de Ingeniería, Av. Las Heras 727, Resistencia (3500), Chaco, Argentina.

UNNE, FaCENA, Av. Libertad 5450, Corrientes (3400), Argentina.

UTN, FRRE, French 414, Resistencia (3500), Chaco, Argentina.

UNNE, FaCENA, Av. Libertad 5450, Corrientes (3400), Argentina.

Abstract

In this paper, we study the behavior of solutions of a perturbed and no perturbed Volterra integro – differential equation, under suitable assumptions. To clarify the results, some examples and remarks are presented.

Keywords

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INTRODUCTION

Preliminaries. The Volterra type integro – differential equations had become in a source of many works in the last 30 years. Basically these works can be grouped according the techniques used, in the following fields:

i) Those using Lyapunov's Functions and they are an extension to the integrals and integro – differentials equations, of the proceedings and techniques of the Second Lyapunov's Method for the Ordinary Differential Equations (see [3, 4, 5, 6, 18, 21, 28 – 31, 40, 42, 56, 59, 60, 54]).

ii) Those based in the use of the Bellman – Gronwall – Bihari integro – differential inequalities, and the inherent properties to the space where the unknown function is defined. (see [44 - 46, 47 - 53, 61]).

iii) Those using Functional Analysis tools in general, and Operators Theory tools in particular. (see [1, 2, 20, 22, 23, 24, 25, 27, 32, 33, 34, 35 – 37, 38, 39, 54, 57 – 58, 62, 63])

In general, these Works had been motivated for two fundamental reasons:

1) To clarify under which conditions the solutions of the system (1) fulfills specific qualitative conditions (various boundedness properties, for example) those was clarified in the Ordinary Differential Equations case in the works of Yoshizawa and Antoziewicz, but in the case we are interested are open problems.

2)   The link with the Ordinary Differential Equations, through practical applications, is one of the most outstanding features of the Integral and Integro – Differential Equations Study.

The purpose of this work is to study the qualitative behavior of the solutions of the systems:

$$x'(t) = A(t)x(t) + \int_0^t B(t,s)x(s)ds + f(t) \qquad (1)$$

$$y'(t) = A(t)y(t) + \int_0^t B(t,s)y(s)ds \qquad (2)$$

so, in the first part of this, sufficient conditions to ensure various qualitative properties of the null solution of the system (2) are obtained. In the second one sufficient conditions that guarantees the uniform bounding and / or the final uniform bounding of the solutions of the system (1). In each epigraph well known results are generalized.

The study of the systems (1) and (2) are motivated for the recent applications of this class system (in particular, see [51] and [53] where the case A constant, B convolution type and f(t) given, is studied).

Many qualitative results for type (1) equations has been obtained for constructing Lyapunov's Functionals. Such functionals requires that A(t) be negative, while in our work this condition is not used. So, our results are more general and can be aplícate to the system (1), when A is stable, identically zero or completely unstable.

In [11], Burton considers the equation (1) with A is a constant and B(t, s) = B(t, − s) and shows that the existence, uniqueness, space dimensionality and the parameter variation formula are nearly the same those in Ordinary Differential Equations Theory.

Mahfoud in [42] gave sufficient conditions that ensures that (1) have bounded Solutions, highlighting that the used method is new, allowing unify, to improve and extend previous results.

Let specify the basic concepts will be used along this work.

Definition 1. The Null Solution of the system (2) is *attractive* if for each $t_0 \geq 0$ there exists $\delta_0 \geq 0$ such that $\left| \phi_{t_0} \right| < \delta_0 \implies$ $\left| y(t, t_0, \phi) \right| \to 0$ when $t \to +\infty$.

Definition 2. The Null Solution of the system (2) is *stable* if for each $t_0 \geq 0$ there exists $\delta_0 \geq 0$ such that $\left| \phi_{t_0} \right| < \delta_0 \implies$ $\left| y(t, t_0, \phi) \right| \to 0$ when $t \to +\infty$.

Definition 3. The Null Solution of the system (2) is *asymptotically stable* if it is both stable and attractive.

Definition 4. The Null Solution of the system (2) is *asymptotically stable in a uniform way* if it is asymptotically stable, δ_0 in Definition 1 is independent of t_0 and for each $\epsilon > 0$ there exists $T > 0$ such that $\left| \phi_{t_0} \right| < \delta_0 \wedge t \geq t_0 \implies \left| y(t, t_0, \phi) \right| < \epsilon$.

Definition 5. The solutions of the system (1) are *finally uniformly bounded* by the bound B, if there exists $B > 0$ and for every $\alpha > 0$ there exists $T = T(\alpha) > 0$ such that, for $t_0 \geq 0$ and $\left| \phi_{t_0} \right| < \alpha \implies \left| x(t, t_0, \phi) \right| < B$, for all $t \geq t_0 + T(\alpha)$.

1. Non – perturbed system case.

In obtaining the results of this epigraph, the *Parameters Variation Formula* (see [26]) which we now present.

If y(t) is a solution of the system (2), with the initial function ϕ in the interval $[0, t_0]$, that solution expresses as following:

$$y(t, t_0, \phi) = h(t)h^{-1}(t_0)\phi(t_0) + \int_{t_0}^t h(t)h^{-1}(s)\int_0^u B(s,u)y(u)du\,ds$$

where h'(t) = A(t) h(t).

Now we prove a set of sufficient conditions about some different stability types for the null solution of the system (2). Those conditions are closely linked with those obtained in [29] and [30], whereby we present an abstract about the results obtained in those works.

Consider the system:

$$y'(t) = A(t)y(t) + \int_0^t G(t,s,y(s))ds \qquad (3)$$

Theorem A: Let assume that:

a) $\left|G(t,s,y(s))\right| \leq B(t,s)\left|y\right|$, where B is a continuous function for $0 \leq s \leq t$ and $\left|y\right| < H$ for some $H > 0$.

b) There exists $\beta \geq 1$ such that $\left|h(t)h^{-1}(s)\right| \leq \beta$ for $0 \leq s \leq t$.

c) $\int_0^{+\infty}\int_0^t \left|B(t,s)\right|ds\,dt < M$ for some $M > 0$

Then the null solution of the system (3) is *uniformly stable.*

Theorem B: Let assume that:

a) $\left|G(t,s,y(s))\right| \leq B(t,s)\left|y\right|$, where B is a continuous function for $0 \leq s \leq t$ and $\left|y\right| < H$ for some $H > 0$.

b) There exists $L > 0$ such that $\int_0^t \left|h(t)h^{-1}(s)\right|ds \leq L$ for $t > 0$.

c) $\operatorname*{Sup}_{t \geq 0} \int_0^t \left|B(t,s)\right|ds < \dfrac{1}{L}$

d) $\lim\limits_{s \to +\infty} \int_0^t \left|B(s,u)\right|du = 0$

Then the null solution of the system (3) is *asymptotically stable.*

Theorem C: Let assume that:

a) $\left|G(t,s,y(s))\right| \leq d(t-s)\left|y\right|$, where d is a continuous function for $0 \leq s \leq t$ and $\left|y\right| < H$ for some $H > 0$.

b) There exists $K \geq 1, \lambda > 0$ such that $\left|h(t)h^{-1}(s)\right| \leq K\exp\left[-\lambda(t-s)\right]$ for $0 \leq s \leq t$

c) $\int_0^{+\infty} d(s)ds < \dfrac{\lambda}{K}$

Then the null solution of the system (3) is *asymptotically stable in a uniform way.*

Theorem D: Let assume that:

a) $\left| G(t,s,y(s)) \right| \leq B(t,s) \left| y \right|$, where B is a continuous function for $0 \leq s \leq t$ and $\left| y \right| < H$ for some $H > 0$.

b) There exists $K \geq 1, \lambda > 0$ such that $\left| h(t)h^{-1}(s) \right| \leq K \exp\left[-\lambda(t-s) \right]$ for $0 \leq s \leq t$

c) There exists a constant $\mu > 0$ such that $\underset{t \geq 0}{\text{Sup}} \int_0^t \exp\left[\mu(t-s) \right] \left| B(t,s) \right| ds < \dfrac{\lambda}{K}$

Then the null solution of the system (3) is *asymptotically stable in a exponential way.*

Now we are ready to expose our results.

Theorem 1.1: Let suppose that the following conditions are fulfilled:

a) There exists $\alpha > 0$ such that $\int_0^t \left| h(t)h^{-1}(s) \right| ds = \alpha + r(t)$; where $\underset{t \to +\infty}{\lim} r(t) = 0.$

b) $\underset{t \geq 0}{\text{Sup}} \int_0^t \left| B(t,s) \right| ds < +\infty$

Then the null solution of the system (2) is *stable.*

Proof: As $\underset{t \to +\infty}{\lim} r(t) = 0$, there exists $K > 1$ and $t_2 \geq 0$ such that $r(t) \leq \dfrac{1}{K}$ for all $t \geq t_2$

Then, taking into account the condition (a) in addition to the above, we obtain the inequality

$$\int_{t_2}^t \left| h(t)h^{-1}(s) \right| ds \leq \alpha + \frac{1}{K} - \int_0^{t_2} \left| h(t)h^{-1}(s) \right| ds \tag{1.1}$$

Besides, there exists $L > 0$ such that

$$\alpha - \int_0^{t_2} \left| h(t)h^{-1}(s) \right| ds \leq \frac{L-1}{K} \tag{1.2}$$

Taking into account (1.1) and (1.2) we obtain

$$\int_{t_2}^t \left| h(t)h^{-1}(s) \right| ds \leq \frac{L}{K} \qquad \text{for all } t \geq t_2 \tag{$*$}$$

This means that $\lim\limits_{t \to +\infty} h(t) = 0$ (see [11]). Then it is possible to determine a constant $N > 0$ such that $h(t) \leq N$ for all $t \geq t_0 \geq t_2$ (1.3)

The value of K noted above can be chosen such that

$$\underset{t \geq 0}{\mathrm{Sup}} \int_0^t |B(t,s)| \, ds = M < \frac{L}{K} \tag{1.4}$$

Then there exists γ such that $0 < \gamma \leq M < \dfrac{L}{K}$ and $\underset{t \geq 0}{\mathrm{Sup}} \int_0^t |B(t,s)| \, ds \leq \gamma$. Let's take $\varepsilon > 0$ and define $\delta = \delta(\varepsilon, t_0) <$

$$\min\left\{ \frac{\left(1 - \frac{\gamma L}{K}\right)\varepsilon}{N|h^{-1}(t_0)|}, \varepsilon \right\} \tag{1.5}$$

Let's consider the solution of the siystem (2) such that $\left\| \phi \right\|_{t_0} < \delta$, suppose that the null solution of this system is not stable, that is to say, we can choose $t_1 > t_0$ such that

$$|y(t_1)| = \varepsilon \quad \text{and} \quad |y(t)| < \varepsilon \text{ in } [t_0, t_1) \tag{1.6}$$

But, for all $t \in [t_0, t_1)$, the *Variation of Parameters Formula* allows us to write

$$y(t) = h(t)h^{-1}(t_0)\phi(t_0) + \int_0^t h(t)h^{-1}(s)\int_0^s B(s,u)y(u)\,du\,ds$$

then

$$|y(t)| \leq |h(t)||h^{-1}(t_0)||\phi(t_0)| + \int_0^t |h(t)h^{-1}(s)|\left|\int_0^s |B(s,u)||y(u)|\,du\right|ds$$

substituing (1.3), (1.4), (1.5) and (1.6) in the last inequality, the following estimate is obtained

$$|y(t)| < N|h^{-1}(t_0)|\frac{\left(1 - \frac{\gamma L}{K}\right)\varepsilon}{N|h^{-1}(t_0)|} + \frac{L}{K}\gamma\varepsilon = \left(1 - \frac{\gamma L}{K}\right)\varepsilon + \frac{L}{K}\gamma\varepsilon = \varepsilon$$

which contradicts (1.6). Then the null solution of the system (2) is *stable.*

Theorem 1.2. Suppose that the following assertions are fulfilled

(a)   There exists $\beta \geq 1$ such that $\left| h(t)h^{-1}(s) \right| \leq \beta$ for all $t \geq s \geq 0$

(b) $\int_{t_0}^{t} \int_{0}^{s} B(s,u) du\, ds < M$; $M \in R_+^*$

Then the null solution of the system (2) is *uniformly stable*.

Proof: For $\varepsilon > 0$ let $\delta(\varepsilon) < \dfrac{\varepsilon}{\beta\, e^{N\beta}}$ and $\left\| \phi \right\|_{t_0} < \delta(\varepsilon)$ (1.7). Suppose that there exists $t_1 \geq t_0$ such that $\left| y(t_1) \right| = \varepsilon$ and $\left| y(t) \right| < \varepsilon$ in $\left[t_0, t_1 \right)$,

As $y(t) = h(t)h^{-1}(t_0)\phi(t_0) + \int_{0}^{t} h(t)h^{-1}(s) \int_{0}^{s} B(s,u)y(u)du\, ds$, using the assertions (a), (b), (1.7) and the last equality, we obtain immediately

$$\left| y(t) \right| \leq \beta \delta(\varepsilon) + \beta \int_{t_0}^{t}\int_{0}^{s} \left| B(s,u) \right| \left| y(u) \right| du\, ds \quad (1.8)$$

Let $r(t) = \underset{0 \leq s \leq t}{Sup} \left| y(s) \right|$, then

$$\left| y(t) \right| \leq r(t) \leq \beta \delta(\varepsilon) + \beta \int_{t_0}^{t}\int_{0}^{s} \left| B(s,u) \right| r(s) du\, ds \quad (1.9)$$

By the Gronwall Inequality, (1.9) becomes

$$\left| y(t) \right| \leq r(t) \leq \beta \delta(\varepsilon)\, e^{\beta \int_{t_0}^{t}\int_{0}^{s} \left| B(s,u) \right| du\, ds}$$

then

$$\left| y(t) \right| \leq r(t) \leq \beta \delta(\varepsilon)\, e^{\beta N} < \varepsilon \quad \text{in } \left[t_0, t_1 \right]$$

Thus $\left| y(t_1) \right| < \varepsilon$ which contradicts what we supposed. Hence the null solution is *stable* and as δ is independent of t_0, it is *uniformly stable*.

Remark 1.1. In [54] is studied the linear differential equation of order n:

$$y^{(n)} + \sum_{k=2}^{n} p_k(t) y^{(n-k)} = 0$$

Where the coefficients $p_k(t)$ k = 1, 2, . . . , n are continuous real functions, defined on $I = [a, +\infty)$, a $\in$ R. It demonstrated that the equation, subject to the initial conditions $y^{(k)}(t_0) = y_0^k$, k = 0, 1, . . . , n – 1, is equivalent to the Volterra Equation

$$y^{(n-1)}(t) = g(t) + \int_{t_0}^{t} A(t,s) y^{(n-1)}(s) ds \quad , t \in I \qquad (S)$$

where

$$g(t) = y_0^{(n-1)} - \sum_{j=0}^{n-2} y_0^j \sum_{k=n-j}^{n} \int_{t_0}^{t} p_k(s) \frac{(s-t_0)^{j-n+k}}{(j-n+k)!} ds$$

$$A(t,s) = -\sum_{k=2j}^{n} \int_{t_0}^{t} p_k(u) \frac{(u-s)^{k-2}}{(k-2)!} du \quad , s, t \in I$$

Then, taking into account the above and the Theorem 1.2. we can state the following result:

Corollary 1.1. If $\int_{t_0}^{t} \int_{0}^{s} |p_2(u)| du\, ds$ < M, M $\in$ R_+^* ; then the null solution of $y'' + p_2(t) y = 0$, $p_2(t)$ > 0 is *uniformly stable*.

This result completes those were presented in that work, and is consistent with the theory, in particular the case in which $p_2(t)$ is a constant (see too [41] and [43])

Theorem 1.3. Let's suppose that

(a) $\lim\limits_{s \to \infty} \int_{0}^{t} |B(s,u)| du = 0$

(b)   The hypothesis of the Theorem 1.1. are fulfilled.

Then the null solution of the system (2) is *asymptotically stable*.

Proof. The null solution of the system (2) is stable (Theorem 1.1.) we just need to prove that this solution is *attractive*.

$$\text{We know that } \underset{t \geq 0}{\text{Sup}} \int_0^t \left| B(t,s) \right| ds \leq \gamma, \text{ where } \gamma \leq M < \frac{K}{L} \tag{1.10}$$

From (1.10.) it follows that there exists $\beta > K$ such that

$$\gamma L \leq M L < \frac{K}{2 - \dfrac{K}{\beta}} < K \tag{1.11}$$

Also, there exists θ such that

$$\frac{\gamma L}{\beta} < \frac{\gamma L \left(2 - \dfrac{K}{\beta} \right)}{K} < \theta < 1 \tag{1.12}$$

Let $\varepsilon = 1$. As the null solution of the system (2) is *stable*, then we can find a number $\delta_0 = \delta(1, t_0) < 1$ such that

$$(t_0 \geq 0 \text{ and } \left\| \phi \right\|_{t_0} < \delta_0) \;\Rightarrow\; \left| y(t, t_0, \phi) \right| < 1 \tag{1.13}$$

In the following, we consider the solutions of the system (2) such that $\left\| \phi \right\|_{t_0} < \delta_0$. Between these, suppose that there exists $y(t) = y(t, t_0, \phi)$ such that

$$\underset{t \to +\infty}{\text{LimSup}} \left| y(t) \right| = \mu > 0 \tag{1.14}$$

From the conclusions (1.12) and (1. 14) it follows that there exists $t_1 \geq t_0$ such that

$$\left| y(u) \right| \leq \frac{\mu}{\theta} \text{ for all } u \geq t_1 \tag{1.15}$$

From (a), it is possible to ensure that there exists $T > t_1$ such that

$$\int_0^{t_1} \left| B(s,u) \right| du \leq \frac{K \left(\theta - \dfrac{\gamma L}{\beta} \right) \mu}{2 L \theta} \quad \text{for } s \geq T \tag{1.16}$$

Apply the *Variation of Parameters Formula*. Then we can write

$$|y(t)| \le |h(t)||h^{-1}(t_0)||\phi(t_0)| + \int_{t_0}^{T}|h(t)h^{-1}(s)|\int_{0}^{s}|B(s,u)||y(u)|du\,ds +$$

(1.17)

$$+ \int_{T}^{t}|h(t)h^{-1}(s)|\int_{0}^{t_1}|B(s,u)||y(u)|du\,ds + \int_{T}^{t}|h(t)h^{-1}(s)|\int_{t_0}^{s}|B(s,u)||y(u)|du\,ds$$

Note that by conjungating ($\ast$) (see the proof of the Theorem 1.1) (1.13) y (1.16) we obtain the following estimate:

$$\int_{T}^{t}|h(t)h^{-1}(s)|\int_{0}^{t_1}|B(s,u)||y(u)|du\,ds \le \frac{\theta - \dfrac{\gamma L}{\beta}}{2\theta}\mu$$

(1.18)

On the other hand, (1.11) and (1.15) lead us to

$$\int_{T}^{t}|h(t)h^{-1}(s)|\int_{t_1}^{s}|B(s,u)||y(u)|du\,ds \le \frac{\gamma L}{K\theta}\mu$$

(1.19)

Accordingly, from (1.13), (1.17), (1.18) and (1.19) we deduce

$$|y(t)| \le |h(t)||h^{-1}(t_0)||\phi(t_0)| + \int_{t_0}^{T}|h(t)h^{-1}(s)|\int_{0}^{s}|B(s,u)||y(u)|du\,ds + \frac{\theta - \dfrac{\gamma L}{\beta} + \dfrac{2\gamma L}{K}}{2\theta}\mu$$

Is immediately from (1.12) that $\dfrac{2\gamma L}{K} - \dfrac{\gamma L}{\beta} < \theta$, then $\theta - \dfrac{\gamma L}{\beta} + \dfrac{2\gamma L}{K} < 2\theta$

(1.20)

Also

$$\lim_{t \to +\infty}|y(t)| \le \lim_{t \to +\infty}\left[|h(t)||h^{-1}(t_0)||\phi(t_0)| + \int_{t_0}^{T}|h(t)h^{-1}(s)|\int_{0}^{s}|B(s,u)||y(u)|du\,ds +\right.$$

(1.21)

$$\left. + \frac{\theta - \dfrac{\gamma L}{\beta} + \dfrac{2\gamma L}{K}}{2\theta}\mu\right]$$

Taking into account (1.14), (1.20), (1.21) and the condition $\lim\limits_{t\to+\infty}\left|h(t)\right| = 0$ we obtain that $\mu \le \dfrac{\theta - \dfrac{\gamma L}{\beta} + \dfrac{2\gamma L}{K}}{2\theta}\,\mu < \mu,$

which is a contradiction.

So, the null solution of the system (2) is *attractive* and then, being this solution *stable*, we conclude that this solution is *asymptotically stable*.

Remark 1.2. Is easy to see that the conditions a), c) and d) in Theorem B are included in the condition a) of our Theorem. On the other hand, our condition b) is weaker than the corresponding to the result of Hara, Yoneyama and Itoh; the last thing applies also for the boundedness of $y(t)$. Thus our result, wider than the previous, is obtained with weaker conditions.

Theorem 1.4. Suppose that the following conditions are satisfied

(a)   There exists L > 0, K > 0 such that $\displaystyle\int_0^t \left|h(t)h^{-1}(s)\right|ds \le \dfrac{L}{K}$, for all $t \ge t_0$.

(b) $\displaystyle\operatorname*{Sup}_{t\ge0}\int_0^t \left|B(t,s)\right|ds = M < +\infty$

(c) $\displaystyle\lim_{t\to+\infty}\int_0^t \left|B(s,u)\right|du = 0$

Then

(i)   If $M < \dfrac{K}{L}$, then the null solution of (2) is *asymptotically stable*

(ii)   If $M \ge \dfrac{K}{L}$, and the null solution of (2) is *stable*, then it is *asymptotically stable*

Proof. (i) If $M < \dfrac{K}{L}$, it is proved that the null solution of (2) is *asymptotically stable* (see Theorem B).

(ii) If $M \ge \dfrac{K}{L}$, choose N > 1 and $\alpha > \dfrac{1}{N-1}$, or equivalently, $\dfrac{N\alpha}{\alpha+1} > 1$, so that $\dfrac{K}{L} < \gamma << \dfrac{KN\alpha}{L(\alpha+1)}$

The inequality $\dfrac{KN\alpha}{L\gamma} > \alpha + 1$ is inmediate, and accordingly $\dfrac{K}{L\gamma} - \alpha > 1$, therefore $\dfrac{K}{L} < \dfrac{\dfrac{K}{L\gamma} - \alpha\gamma}{N\alpha} =$

$\dfrac{KN\alpha}{L} - \alpha\gamma$. Thus $K < N\alpha K - \alpha\gamma L$, that is to say

$$\frac{1}{N\alpha} + \frac{L\gamma}{KN} < 1 \tag{1.22}$$

Let $\varepsilon = 1$. As the null solution of the system (2) is *stable*, then we can find a number $\delta_0 = \delta(1, t_0) < 1$ such that

$$(t_0 \geq 0 \text{ and } \|\phi\|_{t_0} < \delta_0) \Rightarrow |y(t, t_0, \phi)| < 1$$

In the following, we consider the solutions of the system (2) such that $\|\phi\|_{t_0} < \delta_0$. Between these, suppose that there exists $y(t) = y(t, t_0, \phi)$ such that

$$\underset{t \to +\infty}{\text{LimSup}} |y(t)| = \mu > 0$$

As a result of the condition (c), it is possible to find t_2 such that

$$\int_0^{t_2} |B(t,s)| ds \leq \frac{K\mu}{LN\alpha} \tag{1.23}$$

and

$$\int_{t_2}^s |B(t,s)| ds \leq \frac{\gamma}{N} \quad \text{for } s \geq T > t_2 > 0 \tag{1.24}$$

Apply the *Variation of Parameters Formula*. Then we can write

$$|y(t)| \leq |h(t)| |h^{-1}(t_0)| \delta_0 + |h(t)| \int_{t_0}^T |h^{-1}(s)| \int_0^s |B(s,u)| |y(u)| du \, ds +$$

$$\tag{1.25}$$

$$+ \int_T^t |h(t)h^{-1}(s)| \int_0^{t_2} |B(s,u)| |y(u)| du \, ds + \int_T^t |h(t)h^{-1}(s)| \int_{t_2}^s |B(s,u)| |y(u)| du \, ds$$

Hence, (a), (1.13) and (1.23) guarantees that

$$\int_T^t |h(t)h^{-1}(s)| \int_0^{t_2} |B(s,u)| |y(u)| du \, ds \leq \frac{\mu}{N\alpha} \tag{1.26}$$

The relations (a), (1.14) and (1.24) guarantees that

$$\int_{T}^{t}\left|h(t)h^{-1}(s)\right|\int_{t_2}^{s}\left|B(s,u)\right|\left|y(u)\right|du\,ds \le \frac{\gamma L\mu}{KN} \tag{1.27}$$

Finally, substituting (1.26) and (1.27) in (1.25) we obtain the estimate

$$\left|y(t)\right| \le \left|h(t)\right|\left|h^{-1}(t_0)\right|\delta_0 + \left|h(t)\right|\int_{t_0}^{T}\left|h^{-1}(s)\right|\int_{0}^{s}\left|B(s,u)\right|\left|y(u)\right|du\,ds + \frac{\mu}{N\alpha} + \frac{\gamma L\mu}{KN}$$

Therefore

$$\left|y(t)\right| \le \left|h(t)\right|\left|h^{-1}(t_0)\right|\delta_0 + \left|h(t)\right|\int_{t_0}^{T}\left|h^{-1}(s)\right|\int_{0}^{s}\left|B(s,u)\right|\left|y(u)\right|du\,ds + \left(\frac{1}{N\alpha}+\frac{\gamma L}{KN}\right)\mu$$

Then

$$\lim_{t\to+\infty}\left|y(t)\right| \le \lim_{t\to+\infty}\left[\left|h(t)\right|\left|h^{-1}(t_0)\right|\delta_0 + \left|h(t)\right|\int_{t_0}^{T}\left|h^{-1}(s)\right|\int_{0}^{s}\left|B(s,u)\right|\left|y(u)\right|du\,ds + \left(\frac{1}{N\alpha}+\frac{\gamma L}{KN}\right)\mu\right]$$

Taking into account the relation (1.22) we obtain $\mu \le \left(\dfrac{1}{N\alpha}+\dfrac{\gamma L}{KN}\right)\mu < \mu$, which contradicts what course.

Remark 1.3. It is clear that the Theorem 1.4 is a generalization of the Theorem B.

Example 1.1. Using the premises of the Theorem 1.4, we show that the null solution of the equation $y'(t) = \left[1+\dfrac{\alpha'(t)}{\alpha(t)}\right]y(t) + \dfrac{1}{3}\int_{0}^{t}\dfrac{sent}{1+(t-s)^2}y(s)ds$, where $\alpha(t)$ is a real positive and continuously differentiable function such that

$$\alpha(t) \ge 2\left[(\gamma-\alpha(0))\exp(-t)-\gamma\right], \ \alpha'(t) \ge \gamma , \ \gamma \in \mathbb{R}_{+}^{*}$$

Indeed, from the equation $h'(t) = \left[1+\dfrac{\alpha'(t)}{\alpha(t)}\right]h(t)$ we obtain $h(t) = \dfrac{1}{\alpha(t)}\exp(-t)$, hence $\int_{0}^{t}\left|h(t)\right|\left|h^{-1}(s)\right|ds = \dfrac{1}{\alpha(t)}\exp(-t)\int_{0}^{t}\alpha(s)\exp(s)ds$. But

$$\int_0^t \alpha(s)\exp(s)ds = \alpha(t)\exp(t) - \alpha(0) - \int_0^t \alpha'(s)\exp(s)ds \le \alpha(t)\exp(t) - \alpha(0) - \gamma\exp(t) + \gamma$$

we will have $\displaystyle\int_0^t |h(t)||h^{-1}(s)|ds \le \frac{1}{\alpha(t)}\exp(-t)[\alpha(t)\exp(t) - \alpha(0) - \gamma\exp(t) + \gamma]$, then

$$\int_0^t |h(t)||h^{-1}(s)|ds \le 1 + \frac{1}{\alpha(t)}[(\gamma - \alpha(0))\exp(-t) - \gamma] \le 1 + \frac{1}{2} = \frac{3}{2} = \frac{L}{K}$$

In consequence, the premise (a) is satisfied. Moreover, taking into account that

$$\underset{t\ge 0}{\text{Sup}}\int_0^t |B(t,s)|ds = \underset{t\ge 0}{\text{Sup}}\frac{1}{3}\int_0^t \left|\frac{sent}{1+(t-s)^2}\right|ds \le \frac{1}{3}\underset{t\ge 0}{\text{Sup}}\int_0^t \frac{1}{1+(t-s)^2}ds \le \frac{\pi}{6} = M$$

it follows that the premise (b) is satisfied. Also, the premise (c) is satisfied because

$$\lim_{t\to+\infty}\int_0^t |B(t,s)|ds = \lim_{t\to+\infty}\int_0^t \left|\frac{sent}{1+(t-s)^2}\right|ds \le \lim_{t\to+\infty}\int_0^t \frac{1}{1+(t-s)^2}ds = 0.$$

Theorem 1.5. Let suppose that

(a) $\left|h(t)h^{-1}(s)\right| \le K\exp(-\lambda(t-s))$ for $t \ge s \ge 0$; $K \ge 1$, $\lambda > 0$

(b) $\displaystyle\underset{t\ge 0}{\text{Sup}}\int_0^t |B(t,s)|ds < \lambda$

Then there exists $\gamma > 0$, $\beta > 0$ and $\delta \ge 0$, with $K - \gamma \ge 1$, $\beta < \dfrac{\gamma\lambda}{K}$ and $\delta \ge \dfrac{\ln\left(1 - \dfrac{\gamma}{K}\right)}{\beta - \dfrac{\gamma\lambda}{K}}$ such that the null solution of the

system (2) is *asymptotically stable in a uniform way* for $t \ge \delta$.

Proof. The condition (a) means that the null solution of the system (2) is asymptotically stable in a uniform way. (1.28)

From the condition (b) follows $\displaystyle\int_0^t |B(t,s)|ds \le N < \lambda$ for $N \in R_+^*$ (1.29)

Let N' such that N < N' < λ, then it is possible to find a continuous function g such that $\left|B(t,s)\right| \le g(t-s)$ and

$\int_0^s g(s-u)du$ < N'. Indeed, as N < N' choose α > 0 such that N > N + α. Also choose a continuous function J such that

$\int_0^t \left|J(u)\right|du$ < α, then $\left|B(t,s)\right|+\left|J(s)\right| = g(t-s)$. Then $\int_0^s g(s-u)du = \int_0^s \left(\left|B(s,u)\right|+\left|J(u)\right|\right)du$ < N + α < N'

(1.30)

Now let's compare N' with $\dfrac{\lambda}{K}$

(i)   If N' $\le \dfrac{\lambda}{K}$ then, from the estimates (1.29) and (1.30) follows that the null solution of the system (2) is *asymptotically stable in a uniform way* (see Theorem C) for t ≥ s.

(ii)   If N' > $\dfrac{\lambda}{K}$ then the Theorem will be completely proved if it is possible two constants K_1 and λ_1 such that $K_1 \ge 1$ and $\lambda_1 > 0$ for which

$$\left|h(t)h^{-1}(s)\right| \le K\exp\left(-\lambda(t-s)\right) \le \le K_1\exp\left(-\lambda_1(t-s)\right) \, , \, \frac{\lambda}{K} < N' \le \frac{\lambda_1}{K_1} \ \text{for } t \ge s + \delta$$

From the conditions imposed, there exists γ > 0, β > 0 with K − γ ≥ 1 and β < $\dfrac{\gamma\,\lambda}{K}$ such that $K\exp\left(-\lambda(t-s)\right) \le$

$$(K-\gamma)\exp\left(-\left[\frac{K-\gamma}{K}\lambda+\beta\right](t-s)\right) \text{ for } t \ge s + \delta. \text{ Indeed, if } \delta \ge \frac{\ln\left(1-\dfrac{\gamma}{K}\right)}{\beta-\dfrac{\gamma\,\lambda}{K}} \text{ then for } t - s \ge \delta \text{ it results that } t -$$

$$s \ge \frac{\ln\left(1-\dfrac{\gamma}{K}\right)}{\beta-\dfrac{\gamma\,\lambda}{K}} \text{ therefore } \left(\beta-\frac{\gamma\,\lambda}{K}\right)(t-s) \le \ln\left(1-\frac{\gamma}{K}\right) \text{ and we obtain } \exp\left[\left(\beta-\frac{\gamma\,\lambda}{K}\right)(t-s)\right] \le 1-\frac{\gamma}{K} \, ,$$

which is eqivalent to $K\exp\left[\left(\beta-\dfrac{\gamma\,\lambda}{K}+\dfrac{K\,\lambda}{K}-\lambda\right)(t-s)\right] \le K-\gamma$ and we can write

$$K\exp\left(-\lambda(t-s)\right) \le (K-\gamma)\exp\left[-\left(\beta+\frac{K-\gamma}{K}\lambda\right)(t-s)\right] \text{ for } t \ge s + \delta.$$

Now choose $\dfrac{\lambda}{K} < N' \le \dfrac{\lambda}{K} + \dfrac{\beta}{K-\gamma} = \dfrac{\dfrac{K-\gamma}{K}\lambda + \beta}{K-\gamma}$. Let $\lambda_1 = \dfrac{K-\gamma}{K}\lambda + \beta$ and $K_1 = K - \gamma$ let's check that $\dfrac{\lambda_1}{K_1} < \lambda$.

$$\dfrac{\lambda_1}{K_1} = \dfrac{\dfrac{K-\gamma}{K}\lambda + \beta}{K-\gamma} < \dfrac{\dfrac{K-\gamma}{K}\lambda + \dfrac{\gamma\lambda}{K}}{K-\gamma} = \dfrac{\lambda}{K-\gamma} < \lambda$$

Remark 1.4. We can see that the results of this theorem are obtained with weaker conditions than the conditions in Theorem C.

Theorem 1.6. Suppose that

(a) $\left| h(t)h^{-1}(s) \right| \le K \exp\left(-\lambda(t-s)\right)$ for $t \ge s \ge 0$; $K \ge 1$, $\lambda > 0$

(b) $\exists\, \mu > 0$ such that $\operatorname*{Sup}_{t \ge 0} \displaystyle\int_0^t \exp\left(\mu(t-s)\right)\left| B(t,s) \right| ds < \lambda$

Then there exists $\gamma > 0$, $\beta > 0$ and $\delta \ge 0$, with $K - \gamma \ge 1$, $\beta < \dfrac{\gamma\lambda}{K}$ and $\delta \ge \dfrac{\ln\left(1 - \dfrac{\gamma}{K}\right)}{\beta - \dfrac{\gamma\lambda}{K}}$ such that the null solution of the

system (2) is *asymptotically stable in a exponential way* for $t \ge s + \delta$.

Proof. To prove this result, is sufficient to repeat the same ideas used in the proof of the Theorem 1.5. with respect to the existence of $K_1 \ge 1$ and $\lambda_1 > 0$ such that $\left| h(t)h^{-1}(s) \right| \le K \exp\left(-\lambda(t-s)\right) \le K_1 \exp\left(-\lambda_1(t-s)\right)$ for $t \ge s$ and $N' < \dfrac{\lambda_1}{K_1} < \lambda$ and combine the above with Theorem D.

Remark 1.5. It's easy to verify that the condition (b) of this Theorem is weaker than the condition (c) of the Theorem D.

Example 1.2. This example will show the application of the preceding Theorem. Consider the scalar equation $y'(t) = -\lambda y(t) + \dfrac{1}{3}\displaystyle\int_0^t \exp\left(-C(t-s)\right)y(s)ds$, where $\lambda > 0$, $\lambda.C > 1$. It's easy to see that $\dfrac{1}{3}\displaystyle\int_0^t \exp\left(-C(t-s)\right)y(s)ds$ and from this $\left| h(t)h^{-1}(s) \right| \le \quad \le \exp\left(-\lambda(t-s)\right) \le K\exp\left(-\lambda_1(t-s)\right)$, that is the condition (a).

On the other hand, $(\lambda - C)^2 = (\lambda + C)^2 - 4(\lambda C - 1) > 0$ and $(\lambda - C)^2 + 4 < (\lambda + C)^2$ from the hypothesis $\lambda.C > 1$. Then $\lambda + C > \sqrt{(\lambda + C)^2 - 4(\lambda C - 1)}$

If we choose $t_0 = 0$ and use the Laplace's Transform to solve the given equation, we obtain

$$y(t,0,y_0) = \frac{y_0}{\alpha_1 - \alpha_2}\left[(C+\alpha_1)\exp(\alpha_1 t) - (C+\alpha_2)\exp(\alpha_2 t)\right]$$

where $\alpha_{1,2} = \dfrac{-(\lambda+C)\pm\sqrt{(\lambda+C)^2 - 4(\lambda C - 1)}}{2}$. As $\lambda.C > 1$, is possible to find $0 < \mu < C$ such that $\lambda(C-\mu)$, then

$1 - \exp[(\mu-C)t] < \lambda(C-\mu)$, thereby $\dfrac{1-\exp[(\mu-C)t]}{(C-\mu)} < \lambda$. From $B(t,s) = \exp(-C(t-s))$ we have

$$\int_0^t \exp(\mu(t-s))B(t,s)ds = \exp((\mu-C)t)\int_0^t \exp((C-\mu)s)ds = \frac{1-\exp[(\mu-C)t]}{(C-\mu)}$$

Then $\displaystyle\sup_{t\geq 0}\int_0^t \exp(\mu(t-s))|B(t,s)|ds = \dfrac{1}{C-\mu} < \lambda$ hence the premise (b) is satisfied.

As a conclusion of this epigraph, we show how is possible to use the *integro-differential inequalities* derived and applied in [44, 45]. Here are involved a set of continuous functions defined in $I = [0,+\infty)$ such that $0 < a(t) < 1$, $b(t) \geq 0$ and $\eta(t)$ is positive and monotone nondecreasing, and the constants $\alpha > 0$, $0 \leq \gamma < 1$ and $0 < p \leq 2$.

Theorem 1.7. Suppose that the following conditions are fulfilled

(a) $\quad |h(t)|\,|h^{-1}(t_0)|\,|\phi(t_0)| = \dfrac{\alpha\,a(t)}{2(1-\gamma)} \leq K\,; K \in R_+^*$

(b) $\quad |h(t)|\,|h^{-1}(s)|\displaystyle\int_0^s |B(s,u)|\,|y(u)|ds \leq \dfrac{b(s)y(s)y(t)}{1-\gamma}$

(c) $\quad \displaystyle\int_0^{+\infty} a(s)b(s)ds = M \leq \dfrac{(1-\gamma)^2 p^2}{4\,\alpha}$

Then all the solutions of the system (2) are *bounded* and the null solution thereof is *stable*.

If in addition the null solution of the linear homogeneous equation $h'(t) = A(t)h(t)$ is *uniformly stable*, then all the solutions of the system (2) are *uniformly bounded* and the null solution of this system is *uniformly stable*.

Proof. From the Variation of Parameters Formula

$$|y(t)| \leq |h(t)||h^{-1}(t_0)||\phi(t_0)| + \int_{t_0}^{t} |h(t)||h^{-1}(s)| \int_{0}^{s} |B(s,u)||y(u)|du\,ds \qquad (1.31)$$

Taking into account the premises (a) y (b), the inequality (1.31) becomes $|y(t)| \leq \dfrac{\alpha\,a(t)}{2(1-\gamma)} +$

$\dfrac{|y(t)|}{1-\gamma} \int_{0}^{+\infty} |b(s)||y(s)|ds$, and this expression matches with the inequality obtained in the proof of the Theorem 3 of

[46], for $\eta(t) = 1$. Therefore it follows that

$$|y(t)| \leq \frac{a(t)}{(1-\gamma)\left(1-\dfrac{p}{2}\right)+\sqrt{\dfrac{(1-\gamma)^2 p^2}{4} - \alpha \int_{0}^{t} |a(s)||b(s)|ds}}$$

From the last expression and the premises (a) y (c) we obtain the boundedness for $y(t)$.

We show that under the conditions imposed, the null solution of the system (2) is *stable*.

From (a) we observe that

$$\frac{2(1-\gamma)}{\alpha}|h(t)||h^{-1}(t_0)||\phi(t_0)| = a(t) \leq \frac{2(1-\gamma)}{\alpha}K = K \qquad (1.32)$$

On the other hand, For all $\varepsilon > 0$ choose $\delta = \delta(\varepsilon, t_0) > 0$ such that

$$\delta(\varepsilon, t_0) < \frac{\varepsilon}{K_1}\left[(1-\gamma)\left(1-\frac{p}{2}\right)+\sqrt{\frac{(1-\gamma)^2 (1-p)^2}{4} - M}\right] \qquad (1.33)$$

Let's consider the solution of the system (2) that satisfies the inequality

$$\delta(\varepsilon, t_0) < \|\phi\|_{t_0} \qquad (1.34)$$

Suppose that there exists $t_1 > t_0$ such that $\left| y(t_1) \right| = \varepsilon$ and $\left| y(t) \right| < \varepsilon$ en $[t_0, t_1)$. Appliying the *Variation of Parameters Formula* and using (1.31), (1.32), (1.33) and (1.34) it follows that

$$\left| y(t) \right| \leq \frac{K_1 \delta}{(1-\gamma)\left(1-\dfrac{p}{2}\right)+\sqrt{\dfrac{(1-\gamma)^2(1-p)^2}{4}}-M} < \varepsilon,$$

then $\left| y(t_1) \right| < \varepsilon$.

The contradiction obtained means that the null solution of the system (2) is *stable*.

Now, using that the null solution of the linear homogeneous equation $h'(t) = A(t)h(t)$ is *uniformly stable*, is possible to choose $L \geq 1$ such that $\left| h(t) \right| \left| h^{-1}(s) \right| \leq L$ for $t \geq s \geq 0$.

Observe that the choice of α and L is independent of t_0, therefore $\delta = \delta(\varepsilon)$, hence the *boundedness* of all the solutions, as well as the *stability* of the null solution of the system (2), is *uniform*.

Remark 1.6. Even we used the Theorem 3 on the premises of this Theorem, without difficulty we can choose others Theorems in [44, 45] in order to obtain similar results.

Remark 1.7. The obtained results in this section generalizes those of [47, 53], where they study a particular case of (1) under stronger conditions than the here considered.

STUDY OF THE PERTURBED SYSTEM

Hereinafter, we stablish a set of results that are sufficient conditions to ensure the *uniform boundedness* and the *uniform final boundedness* of all the solutions of the system (1). The fulfilling of these properties by the solutions of the system (2) is an essential part of the premises of all the theorems of this section.

Let us specify the definitions of these concepts.

Definition 6. The solutions of the system (2) are *uniformly bounded* if and only if for all $\alpha > 0$ there exists $\beta(\alpha) > 0$ such that

$$t_0 \geq 0 \ \wedge \ \left\| \phi \right\|_{t_0} < \alpha \ \wedge \ t \geq t_0 \ \Rightarrow \ y(t, t_0, \phi) \geq \beta(\alpha)$$

Definition 7. The solutions of the system (2) are *finally uniformly bounded* if and only if there exists $B > 0$ and for some $\alpha > 0$ it can find $T(\alpha) > 0$ such that

$$t_0 \geq 0 \ \wedge \ \left\| \phi \right\|_{t_0} < \alpha \ \Rightarrow \ y(t, t_0, \phi) \geq \beta(\alpha) \ ; \ \forall t \geq t_0 + T(\alpha)$$

Another result we will use is the following two Theorems obtained in [29]:

Theorem E. For the system (2) the following qualitative properties are equivalent:

(i)   The null solution is *asymptotically stable in a uniform way*.

(ii)   The null solution is *asymptotically stable in a exponential way*.

(iii) All the solutions are *uniformly bounded* and *finally uniformly bounded*.

Theorem F. Suppose that the following premises are fulfilled

(a) $\underset{t \geq 0}{\text{Sup}} \displaystyle\int_{t}^{t+1} |A(s)| ds < +\infty$

(b) $\underset{t \geq 0}{\text{Sup}} \displaystyle\int_{0}^{t} |B(t,s)| ds < +\infty$

(c)   The solutions of the system (2) are *uniformly bounded* and *finally uniformly bounded*.

(d) $\underset{t \geq 0}{\text{Sup}} \left| \exp(-t) \displaystyle\int_{0}^{t} \exp(s) f(s) ds \right| < +\infty$

Then the solutions of the system (1) are *uniformly bounded* and *finally uniformly bounded*.

Remark 2.1. From the Theorem E is inmediate that from each Theorems 1.5. and 1.6. follows the *uniform bounded* and the *final uniform bounded* of all the solutions of (2).

In what follows we will discuss our results.

Theorem 2.1. Suppose that the following conditions are fulfilled

(a)   The premises of Theorem 1.5.

(b) $|f(t)| \leq L \ ; \ L \in R_{+}^{*}$

(c) $\underset{t \geq 0}{\text{Sup}} \displaystyle\int_{t}^{t+1} |A(s)| ds < +\infty$

Then there exists $\gamma \geq 0$, $\beta \geq 0$ and $\delta \geq 0$, with $K - \gamma \geq 1$, $\beta < \dfrac{\gamma \lambda}{K}$ and $\delta \geq \dfrac{\ln\left(1 - \dfrac{\gamma}{K}\right)}{\beta - \dfrac{\gamma \lambda}{K}}$ such that the solutions of system

(1) are *uniformly bounded* and *finally uniformly bounded* for $t \geq s + \delta$.

Proof. From (a) follows that every solution $y(t)$ satisfies

$$y(t) \text{ is } \textit{uniformly and finally uniformly bounded for } t \geq \delta \tag{2.1}$$

In the other hand, one of the premises of Theorem 1.5. is $\displaystyle \sup_{t \geq 0} \int_0^t |B(t,s)| \, ds < \lambda$ \hfill (2.2)

Furthermore, from (b) follows the inequalities

$$\sup_{t \geq 0} \left| \exp(-t) \int_0^t \exp(s) f(s) \, ds \right| \leq \sup_{t \geq 0} \exp(-t) \int_0^t \exp(s) |f(s)| \, ds \leq$$

$$\tag{2.3}$$

$$\leq \sup_{t \geq 0} L\left(1 - \exp(-t)\right) = L < +\infty$$

Hence, from (c) and the conclusions (2.1), (2.2) and (2.3) the Theorem is proved. (see Theorem F)

Because of the equivalence of the premises (i), (ii) and (iii)it is possible to prove the following result.

Theorem 2.2. Suppose that the premises (b) and (c) of the Theorem 2.1. and the premises of the Theorem 1.6. are fulfilled. Then there exists $\gamma \geq 0$, $\beta \geq 0$ and $\delta \geq 0$, with $K - \gamma \geq 1$, $\beta < \dfrac{\gamma \lambda}{K}$ and $\delta \geq \dfrac{\ln\left(1 - \dfrac{\gamma}{K}\right)}{\beta - \dfrac{\gamma \lambda}{K}}$ such that all the solutions

of the system (1) are *uniformly bounded and finally uniformly bounded* for $t \geq s + \delta$. It is possible to guarantee the uniform boundedness and final uniform boundedness of all the solutions of (1) using as a part of the premises the results of [44, 45, 60], as illustrated below. In [45] it has shown the following result

Theorem G. Let $x(t)$ be a solution of the system (1) such that its derivative $x'(t)$ is a continuous function. Suppose that the following conditions are fulfilled

(a) $\quad \left| \dfrac{\partial R(t,s)}{\partial t} f(s) \right| \leq b(t) a(s) \left(|x(s)| + |x'(s)| \right)$

(b) $\quad \left| \dfrac{\partial R(t,0)}{\partial t} x(0) \right| + |f(t)| \leq a(t) |x(t)|$

(c) $\quad \displaystyle \int_0^{+\infty} \dfrac{a(t)}{1 - a(t)} \, dt < +\infty$

(d) $\quad B(t) \leq K_1 \in R_+^*$

(e) The premises of Theorem G

Then there exists $\gamma > 0$, $\beta > 0$ and $\delta \geq 0$, with $K - \gamma \geq 1$, $\beta < \dfrac{\gamma \lambda}{K}$ and $\delta \geq \dfrac{\ln\left(1 - \dfrac{\gamma}{K}\right)}{\beta - \dfrac{\gamma \lambda}{K}}$ such that all the solutions of the

system (1) are *uniformly bounded and finally uniformly bounded* for $t \geq s + \delta$.

Proof. The condition (e) ensures that the solutions $x(t)$ of the system (1) are bounded. In the other hand, from (b) of Theorem G and the boundedness of $x(t)$, it follows that $f(t)$ is bounded. If in addition it used the condition (c) and the *Variation of Parameters Formula*, it results that the solutions $y(t)$ of the system (2) are bounded. In consequence from this last conclusion and the premises (a) and (b) follows that (see Theorem 1.5.) this solutions are *bounded and finally bounded in a uniform way* (2.4)

As $f(t)$ is bounded, is inmediate that

$$\underset{t \geq 0}{\text{Sup}} \left| \exp(-t) \int_0^t \exp(s) f(s) ds \right| < +\infty \qquad (2.5)$$

Combining the premises (b) and (d) and the conclusions (2.4) and (2.5) the prove of the Theorem is obtained.

Theorem 2.4. Suppose that

(a) $\exists \, \mu > 0$ such that $\underset{t \geq 0}{\text{Sup}} \int_0^t \exp(\mu(t-s)) \|B(t,s)\| ds < \lambda$

(b) $\underset{t \geq 0}{\text{Sup}} \int_0^t \|B(t,s)\| ds < \lambda$

(c) $\underset{t \geq 0}{\text{Sup}} \int_t^{t+1} \|R(t,s)\| ds < +\infty$

(d) $\underset{t \geq 0}{\text{Sup}} \int_t^{t+1} \|A(s)\| ds < +\infty$

(e)   The premises of Theorem G.

Then there exists $\gamma > 0$, $\beta > 0$ and $\delta \geq 0$, with $K - \gamma \geq 1$, $\beta < \dfrac{\gamma \lambda}{K}$ and $\delta \geq \dfrac{\ln\left(1 - \dfrac{\gamma}{K}\right)}{\beta - \dfrac{\gamma \lambda}{K}}$ such that all the solutions of the

system (1) are *uniformly bounded and finally uniformly bounded* for $t \geq s + \delta$.

Remarks 2.2.

- If, instead of Theorem G, we consider other results of [45] (Theorems 2, 4 or 5), we obtain the same result for the Theorems 2.3. and 2.4.

- It's clear that if instead of Theorems 1, 2, 4 or 5 of [45], we choose the first corollary of these Theorems, we obtain the same conclusion.

Following the notation of [46], for all $s \in [0, +\infty)$, denote $Z(t,s)$ the square matrix of order $n \geq 1$ that satisfies the differential equation

$$y'(t) = C(t)y(t) \; ; \quad P(s)y(s) = I \tag{N}$$

where $P(t)$ is a Squire matrix of order n, derivable, bounded and nonsingular, such that $1 \leq P(t) \leq P$.

In what follows, we will need the result from [46].

Theorem H. Let $Z(t,s)$ be a solution of the equation (N) if the following inequalities are satisfied:

(a) $\left| Z(t,t_0) \right| < K_1$

(b) $\left| \int_{t_0}^{t} Z(t,u)f(u)du \right| \leq K_3$

(c) $\int_{0}^{t} \left| P(t)Z(t,s)\left[A(s)-C(s)\right] + \int_{t_0}^{t} Z(t,u)P(u)\left(B(u,s)-P'(u)Z(u,s)\left[A(s)-C(s)\right]\right)du \right| ds$

  $\leq K_2 < 1$

We will have all the solutions of the system (1) are *uniformly bounded* and the null solution of the system (2) is *uniformly stable*.

Theorem 2.5. Suppose that the following premises are satisfied

(a) $\left| h(t)h^{-1}(s) \right| \leq K \exp\left(-\lambda(t-s)\right)$ for $t \geq s \geq 0$; $K \geq 1$, $\lambda > 0$

(b) $\underset{t \geq 0}{Sup} \int_{t}^{t+1} \left| A(s) \right| ds < +\infty$

(c) $\underset{t \geq 0}{Sup} \int_{0}^{t} \left| B(t,s) \right| ds < \lambda$

(d)   the premises of Theorem H, with $Z(t,s) = K \exp\left(-(t-s)\right)$

Then there exists $\gamma > 0$, $\beta > 0$ and $\delta \geq 0$, with $K - \gamma \geq 1$, $\beta < \dfrac{\gamma \lambda}{K}$ and $\delta \geq \dfrac{\ln\left(1 - \dfrac{\gamma}{K}\right)}{\beta - \dfrac{\gamma \lambda}{K}}$ such that all the solutions of the

system (1) are *uniformly bounded and finally uniformly bounded* for $t \geq s + \delta$.

Proof. The condition (d) allows us to ensure that the solutions $x(t)$ of the system (1) are *uniformly bounded* and then, following the same process when $f(t) = 0$, we conclude that the solutions $y(t)$ of the system (2) have the same propertyFrom this last assertion and the premises (a) and (c) we arrive to the conclusion that the solutions of the system (2) are

$$\text{uniformly bounded and finally uniformly bounded} \qquad (2.6)$$

Substituting $Z(t,s) = K \exp\left(-(t-s)\right)$ in the premise (c) of Theorem H, we obtain

$$\underset{t \geq 0}{\text{Sup}} \left| \exp(-t) \int_0^t \exp(s) f(s) ds \right| < +\infty \qquad (2.5)$$

Combining the premises (b) and (c) with the assertions (2.6) and (2.7) (see Theorem F) we conclude the proof.

Theorem 2.6. Suppose that the following premises are satisfied

(a) $\left| h(t) h^{-1}(s) \right| \leq K \exp\left(-\lambda(t-s)\right)$ for $t \geq s \geq 0$; $K \geq 1$, $\lambda > 0$

(b) $\underset{t \geq 0}{\text{Sup}} \displaystyle\int_t^{t+1} \left| A(s) \right| ds < +\infty$

(c) $\exists \, \mu > 0$ such that $\underset{t \geq 0}{\text{Sup}} \left| \exp(-t) \displaystyle\int_0^t \exp(s) f(s) ds \right| < \lambda$

(d) the premises of Theorem H, with $Z(t,s) = K \exp\left(-(t-s)\right)$

Then there exists $\gamma > 0$, $\beta > 0$ and $\delta \geq 0$, with $K - \gamma \geq 1$, $\beta < \dfrac{\gamma \lambda}{K}$ and $\delta \geq \dfrac{\ln\left(1 - \dfrac{\gamma}{K}\right)}{\beta - \dfrac{\gamma \lambda}{K}}$ such that all the solutions of the

system (1) are *uniformly bounded and finally uniformly bounded* for $t \geq s + \delta$.

Remark 2.3. If, instead Theorem H, we consider one of the Theorem of [46], we obtain equivalent results to Theorems 2.5. and 2.6.

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