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# The *R*-matrix of quantum doubles of Nichols algebras of diagonal type

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Let $H$ be the quantum double of a Nichols algebra of diagonal type. We compute the $R$-matrix of 3-tuples of modules for general finite-dimensional highest weight modules over $H$. We also calculate a multiplicative formula for the universal $R$-matrix when $H$ is finite dimensional. We show the unicity of a PBW basis (or a Lusztig-type Poincaré-Birkhoff-Witt basis) with a given convex order. © *2015 AIP Publishing LLC.* [http://dx.doi.org/10.1063/1.4907379]

## I. INTRODUCTION

A remarkable property of quantum groups, introduced by Drinfeld and Jimbo in the 1980s, is the existence of an $R$-matrix for their categories of modules. This $R$-matrix is related with the existence of solutions of the Yang-Baxter equation. An explicit formula for the universal $R$-matrix of quantum groups was obtained in the 1990s[16,18,19,21] and extended to quantized enveloping superalgebras[15,27] of finite-dimensional Lie superalgebras.

We can deduce the existence of this $R$-matrix for quantized enveloping (super)algebras because they can be obtained as quotients of quantum doubles of bosonizations of the positive part by group algebras, and these quantum doubles are quasi-triangular.

A natural generalization of the positive part of these quantized enveloping (super)algebras is the Nichols algebras of diagonal type.[2] They admit a root system and a Weyl groupoid[10,11] controlling the structure of these algebras. Moreover, the classification of these Nichols algebras with finite root system includes (properly) the positive part of quantized enveloping algebras of finite dimensional contragradient Lie superalgebras and simple Lie algebras. It is natural then to ask for a formula of the $R$-matrix in this general context. We answer this question for the subfamily of finite-dimensional representations with a highest weight in a general context and obtain an explicit formula for the universal $R$-matrix when the Nichols algebra is finite-dimensional.

Although Nichols algebras appeared as an important tool for the classification of finite dimensional pointed Hopf algebras,[2] they have become very attractive for other fields of mathematics. In particular, they are related with conformal field theories. Indeed, they give place to logarithmic examples.[23–25] Starting from non-semisimple (logarithmic) conformal field theory[28] and the screening operators, we can obtain a braided Hopf algebra which is a Nichols algebra.[23] Then it becomes interesting how to make a reverse construction in order to obtain new examples of vertex operator algebras and the corresponding conformal field theories. This was the motivation to their study in mathematical physics:[24] these authors start the translation of some elements from the Nichols algebra context to the corresponding ones needed to describe the attached vertex operator algebra. They study the category of Yetter-Drinfeld modules over the Nichols algebra into a braided category, which is exactly the category of representations of the quantum double of the bosonization of this Nichols algebra by the group algebra of a finite abelian group. They complete the computation

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for a particular example,[25] describing the projective modules, and they give the $R$-matrix following the present work. The $R$-matrix encodes the $M$-matrix for the dual algebra of the corresponding quantum double, which is responsible for the monodromy in the CFT language.[7]

The organization of the paper is as follows. In Sec. II, we recall definitions and results needed for our work. They are related with quantum doubles and properties of Nichols algebras of diagonal type. We stress the importance of the Weyl groupoid and the generalized version of root systems. In Sec. III, we work over arbitrary Nichols algebras of diagonal type and compute the $R$-matrix of 3-tuples of finite-dimensional modules, generalizing the results in Ref. 26. We restrict our attention to highest weight modules, which give maybe the most important subfamily of representations. Finally in Sec. IV, we compute the universal $R$-matrix for quantum doubles of finite-dimensional Nichols algebras. The formula involves the multiplication of quantum exponentials of root vector powers, generalizing the classical ones for quantum groups.

**Notation.** We denote by $\mathbb{N}$ the set of natural numbers and by $\mathbb{N}_0$ the set of non-negative integers.

Let $\mathbf{k}$ be an algebraically closed field of characteristic zero. All the vector spaces, algebras, and tensor products are over $\mathbf{k}$. We shall use the usual notation for $q$-combinatorial numbers: for each $q \in \mathbf{k}^\times$, $n \in \mathbb{N}$, $0 \le k \le n$,

$$(n)_q = 1 + q + \ldots + q^{n-1}, \qquad (n)_q! = (1)_q(2)_q \cdots (n)_q,$$
$$\binom{n}{k}_q = \frac{(n)_q!}{(k)_q!(n-k)_q!}.$$

Let $A$ be an associative algebra. Given an element $a \in A$ such that $a^N = 0$, we define the *$q$-exponential*, for each $q$ which is not a root of unity, or it is a root of unity of order $\ge N$,

$$\exp_q(a) = \sum_{i=0}^{N-1} \frac{a^i}{(i)_q!}. \tag{1.1}$$

Let $\theta \in \mathbb{N}$. $\{\alpha_i\}_{1\le i\le\theta}$ will denote the canonical $\mathbb{Z}$-basis of $\mathbb{Z}^\theta$.

Given a Hopf algebra $H$ with coproduct $\Delta$ and antipode $\mathcal{S}$, we will use the classical Sweedler notation $\Delta(h) = h_1 \otimes h_2$, $h \in H$, and denote

$$\Delta^{(2)} := (\Delta \otimes \mathrm{id}) \circ \Delta = (\mathrm{id} \otimes \Delta) \circ \Delta.$$

A subalgebra $A$ of $H$ is a (right) coideal subalgebra $A$ if $\Delta(A) \subseteq A \otimes H$.

We denote by ${}^H_H\mathcal{YD}$ the category of (left) Yetter-Drinfeld modules over $H$; i.e., the category of $H$-modules and $H$-comodules $V$ (with coaction $\delta$) such that $\delta(h \cdot v) = h_1 v_{-1}\mathcal{S}(h_3) \otimes h_2 \cdot v_0$ for all $h \in H, v \in V$.

Given $R = \sum_i a_i \otimes b_i \in H \otimes H$, we set the elements of $H \otimes H \otimes H$

$$R^{(1,2)} = \sum_i a_i \otimes b_i \otimes 1, R^{(1,3)} = \sum_i a_i \otimes 1 \otimes b_i, R^{(2,3)} = \sum_i 1 \otimes a_i \otimes b_i.$$

## II. PRELIMINARIES

We recall some definitions and results which will be useful in the rest of this work. They are mainly related with quantum doubles of Hopf algebras and Nichols algebras of diagonal type.

### A. Skew-Hopf pairings and *R*-matrices

Let $A, B$ be two Hopf algebras. A *skew Hopf pairing* between $A$ and $B$ (see Ref. 13, Sec. 3.2.1, Ref. 17, Sec. 8.2) is a linear map $\eta : A \otimes B \to \mathbf{k}$ such that

$$\eta(xx', y) = \eta(x', y_1)(x, y_2), \qquad \eta(x, 1) = \varepsilon(x),$$
$$\eta(x, yy') = \eta(x_1, y)(x_2, y'), \qquad \eta(1, y) = \varepsilon(y),$$
$$\eta(\mathcal{S}(x), y) = \eta(x, \mathcal{S}^{-1}(y)),$$

for all $x, x' \in A$, $y, y' \in B$. In such case, $A \otimes B$ admits a unique structure of Hopf algebra, denoted by $\mathcal{D}(A,B,\eta)$ and called the *quantum double* associated to $\eta$, such that the morphisms $A \to A \otimes B$, $a \mapsto a \otimes 1$, $B \to A \otimes B$, $b \mapsto 1 \otimes b$ are Hopf algebra morphisms and

$$(a \otimes 1)(1 \otimes b) = a \otimes b, \qquad (1 \otimes b)(a \otimes 1) = \eta\left(a_1, \mathcal{S}(b_1)\right)(a_2 \otimes b_2)\eta(a_3, b_3).$$

When $A$ is finite-dimensional and $\eta$ is not degenerate, $B$ is identified with the Hopf algebra $A^*$. $\mathcal{D}(A,B,\eta) = \mathcal{D}(A)$ is the *Drinfeld double* of $A$, which admits an $R$-matrix,

$$\mathcal{R} := \sum_{i \in I} (1 \otimes b_i) \otimes (a_i \otimes 1), \tag{2.1}$$

where $\{a_i\}_{i \in I}$, $\{b_i\}_{i \in I}$ are dual bases of $A$, $B$: $\eta(a_i, b_j) = \delta_{ij}$.

## B. Weyl groupoids and convex orders on finite root systems

Fix $\theta \in \mathbb{N}$, a non-empty set $\mathcal{X} \neq \emptyset$, and $\rho : \mathbb{I} \to \mathbb{S}_{\mathcal{X}}$, where $\mathbb{I} = \{1, \ldots, \theta\}$. The pair $(\mathcal{X}, \rho)$ is a *basic datum* of rank $|\mathcal{X}|$ and type $\theta$ if $\rho_i^2 = \mathrm{id}$ for all $i \in \mathbb{I}$. We set the quiver $\mathcal{Q}_\rho$ with arrows $\{\sigma_i^x := (x, i, \rho_i(x)) : i \in \mathbb{I}, x \in \mathcal{X}\}$ over $\mathcal{X}$, with target $t(\sigma_i^x) = x$ and source $s(\sigma_i^x) = \rho_i(x)$. We adopt the convention

$$\sigma_{i_1}^x \sigma_{i_2} \cdots \sigma_{i_t} = \sigma_{i_1}^x \sigma_{i_2}^{\rho_{i_1}(x)} \cdots \sigma_{i_t}^{\rho_{i_{t-1}} \cdots \rho_{i_1}(x)}. \tag{2.2}$$

In any quotient of the free groupoid $F(\mathcal{Q}_\rho)$, i.e., the implicit superscripts are the only possible to have compositions.

Given $(\mathcal{X}, \rho)$ a basic datum of type $\mathbb{I}$, a *Coxeter datum* is a triple $(\mathcal{X}, \rho, \mathbf{M})$, where $\mathbf{M} = (\mathbf{m}^x)_{x \in \mathcal{X}}$, $\mathbf{m}^x = (m_{ij}^x)_{i,j \in \mathbb{I}}$, are Coxeter matrices such that

$$s((\sigma_i^x \sigma_j)^{m_{ij}^x}) = x, \quad i, j \in \mathbb{I}, \quad x \in \mathcal{X}. \tag{2.3}$$

The *Coxeter groupoid* $\mathcal{W}(\mathcal{X}, \rho, \mathbf{M})$[11] is the groupoid generated by $\mathcal{Q}_\rho$ with relations

$$(\sigma_i^x \sigma_j)^{m_{ij}^x} = \mathrm{id}_x, \quad i, j \in \mathbb{I}, \, x \in \mathcal{X}. \tag{2.4}$$

Notice that for $i = j$, (2.4) says that either $\sigma_i^x$ is an involution when $\rho_i(x) = x$ or else that $\sigma_i^x$ is the inverse arrow of $\sigma_i^{\rho_i}(x)$ when $\rho_i(x) \neq x$.

Given a family of generalized Cartan matrices $C = (C^x)_{x \in \mathcal{X}}$, $C^x = (c_{ij}^x)_{i,j \in \mathbb{I}}$, with *row invariance*

$$c_{ij}^x = c_{ij}^{\rho_i(x)} \quad \text{for all} \quad x \in \mathcal{X}, i, j \in \mathbb{I}, \tag{2.5}$$

set $s_i^x \in GL_\theta(\mathbb{Z})$ such that

$$s_i^x(\alpha_j) = \alpha_j - c_{ij}^x \alpha_i, \quad j \in \mathbb{I}, \quad i \in \mathbb{I}, x \in \mathcal{X}. \tag{2.6}$$

By (2.5), $s_i^x$ is the inverse of $s_i^{\rho_i(x)}$. A *generalized root system* (GRS for short) [Ref. 11, Definition 1] is a collection $\mathcal{R} := \mathcal{R}(\mathcal{X}, \rho, C, \Delta)$, where $(\mathcal{X}, \rho)$, $C$ are as above, and $\Delta = (\Delta^x)_{x \in \mathcal{X}}$ is a family of subsets $\Delta^x \subset \mathbb{Z}^{\mathbb{I}}$ such that

$$\Delta^x = \Delta_+^x \cup \Delta_-^x, \quad \Delta_+^x := \Delta^x \cap \mathbb{N}_0^I, \; \Delta_-^x := -\Delta_+^x, \tag{2.7}$$

$$\Delta^x \cap \mathbb{Z}\alpha_i = \{\pm\alpha_i\}; \tag{2.8}$$

$$s_i^x(\Delta^x) = \Delta^{\rho_i(x)}; \tag{2.9}$$

$$(\rho_i \rho_j)^{m_{ij}^x}(x) = (x), \quad m_{ij}^x := |\Delta^x \cap (\mathbb{N}_0\alpha_i + \mathbb{N}_0\alpha_j)|, \tag{2.10}$$

for all $x \in \mathcal{X}$, $i \neq j \in \mathbb{I}$. Here, $\Delta_+^x$, $\Delta_-^x$ the set of *positive*, respectively, *negative*, roots.

If $\mathbf{M} = (M^x)_{x \in \mathcal{X}}$, $M^x = (m_{ij}^x)_{i,j \in \mathbb{I}}$, then $\mathcal{W} = \mathcal{W}(\mathcal{X}, \rho, \mathbf{M})$ is the *Weyl groupoid* of $\mathcal{R}$. By Ref. 11 [Theorem 1] there exists an isomorphism of groupoids $\mathcal{W} \to \mathcal{W}(\mathcal{X}, \rho, C)$. Indeed, let $\mathcal{G} = \mathcal{X} \times GL_\theta(\mathbb{Z}) \times \mathcal{X}$, $\varsigma_i^x = (x, s_i^x, \rho_i(x))$, $i \in \mathbb{I}$, $x \in \mathcal{X}$, and $\mathcal{W}' = \mathcal{W}(\mathcal{X}, \rho, C)$ the subgroupoid of

$\mathcal{G}$ generated by all the $\varsigma_i^x$. There exists a morphism of quivers $\mathcal{Q}_\rho \to \mathcal{G}$, $\sigma_i^x \mapsto \varsigma_i^x$ with image $\mathcal{W}'$, which is the desired isomorphism.

If $w = \sigma_{i_1}^x \cdots \sigma_{i_m}$ and $\alpha \in \mathbb{Z}^\theta$, then define $w(\alpha) = s_{i_1}^x \cdots s_{i_m}(\alpha)$. Now,

$$(\Delta^{\mathrm{re}})^x = \bigcup_{y\in\mathcal{X}} \{w(\alpha_i) : i \in \mathbb{I},\ w \in \mathcal{W}(y,x)\} \tag{2.11}$$

is the set of *real roots* of $x$. The *length* of $w \in \mathcal{W}(x,\mathcal{X})$ is

$$\ell(w) = \min\{m \in \mathbb{N}_0 : \exists i_1,\dots,i_n \in \mathbb{I} \text{ such that } w = \sigma_{i_1}^x \cdots \sigma_{i_m}\}.$$

An expression $w = \sigma_{i_1}^x \cdots \sigma_{i_m}$ is *reduced* if $m = \ell(w)$.

*Proposition 2.1 (Ref. 6, Prop. 2.12). Let $w = s_{i_1}^X \cdots s_{i_m}$, $\ell(w) = m$. The roots $\beta_j = s_{i_1} \cdots s_{i_{j-1}}(\alpha_{i_j}) \in \Delta^X$ are positive and pairwise different.*

*Moreover, if $\mathcal{R}$ is finite and $w$ is an element of maximal length, then $\{\beta_j\} = \Delta_+^X$, so all the roots are real.* □

For the last part of this subsection, assume that $\mathcal{R}$ is finite.

*Definition 2.2 (Ref. 4).* Given a root system $\mathcal{R}$ and a fixed total order $<$ on $\Delta_+^X$, we say that it is *convex* if for each $\alpha,\beta \in \Delta_+^X$ such that $\alpha < \beta$ and $\alpha+\beta \in \Delta_+^X$, then $\alpha < \alpha+\beta < \beta$. It is said *strongly convex* if for each ordered subset $\alpha_1 \le \dots \le \alpha_k$ of elements of $\Delta_+^X$ such that $\alpha := \sum \alpha_i \in \Delta_+^X$, it holds that $\alpha_1 < \alpha < \alpha_k$.

**Theorem 2.3** **(Ref. 4).** *Given an order on $\Delta_+^X$, the following are equivalent:*

(1) *the order is convex,*
(2) *the order is strongly convex,*
(3) *the order is associated with a reduced expression of the longest element.* □

## C. Weyl groupoid of a Nichols algebra of diagonal type

Let $\mathbf{q} = (q_{ij}) \in (\mathbf{k}^\times)^{\theta\times\theta}$. Let $\chi : \mathbb{Z}^\theta \times \mathbb{Z}^\theta \to \mathbf{k}^\times$ be the bicharacter such that $\chi(\alpha_i,\alpha_j) = q_{ij}$. Given $1 \le i \le \theta$, we say that $\mathbf{q}$ is *$i$-finite* if for all $1 \le j \ne i \le \theta$, there exists $m \in \mathbb{N}_0$ such that $(m+1)_{q_{ii}}(1 - q_{ii}^2 q_{ij} q_{ji}) = 0$. In such case, define

$$a_{ii}^{\mathbf{q}} = 2, a_{ij}^{\mathbf{q}} = -\min\{m \in \mathbb{N}_0 | (m+1)_{q_{ii}}(1 - q_{ii}^2 q_{ji} q_{ij}) = 0\},$$

and set $s_i^{\mathbf{q}}$ as the $\mathbb{Z}$-linear automorphism of $\mathbb{Z}^\theta$ given by (??). If $\mathbf{q}$ is $i$-finite for all $i$, $A^{\mathbf{q}} = (a_{ij}^{\mathbf{q}})_{1\le i,j\le\theta}$ is the *generalized Cartan matrix* associated to $\mathbf{q}$.

Let $\mathcal{X} = (\mathbf{k}^\times)^{\theta\times\theta}$. We define $\rho_i : \mathcal{X} \to \mathcal{X}$ by $\rho_i(\mathbf{q})_{jk} = \chi(s_i^{\mathbf{q}}(\alpha_j), s_i^{\mathbf{q}}(\alpha_k))$ if $\mathbf{q}$ is $i$-finite, or $\rho_i(\mathbf{q}) = \mathbf{q}$ otherwise. Such $\rho_i$'s are involutions and $\mathcal{G}_{\mathbf{q}}$ will denote the orbit of $\mathbf{q}$ by the action of the group of bijections generated by the $\rho_i$'s.

Note that $\mathcal{C}_{\mathbf{q}} = \mathcal{C}\left(\{1,\dots,\theta\}, \mathcal{G}_{\mathbf{q}}, (\rho_i)_{1\le i\le\theta}, (C^{\mathbf{q}})_{\mathbf{q}\in\mathcal{G}_{\mathbf{q}}}\right)$ is a connected Cartan scheme, see Refs. 11 and 12. Therefore, the associated Weyl groupoid $\mathcal{W}_{\mathbf{q}}$ is called the *Weyl groupoid of* $\mathbf{q}$.

Given $V$ a vector space with a fixed basis $x_1,\dots,x_\theta$, we can consider the braided vector space $(V,c)$, where $c : V\otimes V \to V\otimes V$ is given by $c(x_i\otimes x_j) = q_{ij} x_j \otimes x_i$, $1 \le i,j \le \theta$. $(V,c)$ is *of diagonal type*. The *Nichols algebra* of $(V,c)$ is the graded braided Hopf algebra $\mathcal{B}_{\mathbf{q}} = \oplus_{n\ge0}\mathcal{B}_{\mathbf{q}}^n$ which is the quotient by the maximal homogeneous Hopf ideals of $T(V)$ with trivial intersection with $\mathbf{k}\oplus V$. A first relation with $\mathcal{W}_{\mathbf{q}}$ is the following:

$$-a_{ij}^{\mathbf{q}} = \max\{n \in \mathbb{N}_0 : (\mathrm{ad}_c x_i)^n x_j \ne 0\}, \quad i \ne j.$$

A second relation between $\mathcal{W}_{\mathbf{q}}$ and the corresponding Nichols algebras is described by Lusztig isomorphisms, as we shall see in Subsection II D.

### D. Lusztig Isomorphisms of Nichols algebras

Set $\mathbf{q}$ as in Subsection II C. $\mathcal{U}_{\mathbf{q}}$ will denote the algebra presented by generators $E_i$, $F_i$, $K_i$, $K_i^{-1}$, $L_i$, $L_i^{-1}$, $1 \le i \le \theta$, and relations

$$\begin{aligned} XY &= YX, & X, Y &\in \{K_i^{\pm 1}, L_i^{\pm 1} : 1 \le i \le \theta\}, \\ K_iK_i^{-1} &= L_iL_i^{-1} = 1, & E_iF_j - F_jE_i &= \delta_{i,j}(K_i - L_i), \\ K_iE_jK_i^{-1} &= q_{ij}E_j, & L_iE_jL_i^{-1} &= q_{ji}^{-1}E_j, \\ L_iF_jL_i^{-1} &= q_{ji}F_j, & K_iF_jK_i^{-1} &= q_{ij}^{-1}F_j. \end{aligned}$$

$\mathcal{U}_{\mathbf{q}}^{+0}$ (respectively, $\mathcal{U}_{\mathbf{q}}^{-0}$) will denote the subalgebra generated by $K_i$, $K_i^{-1}$ (respectively, $L_i$, $L_i^{-1}$), $1 \le i \le \theta$, and $\mathcal{U}_{\mathbf{q}}^{0}$ will denote the subalgebra generated by $K_i$, $K_i^{-1}$, $L_i$, and $L_i^{-1}$. Also, $\mathcal{U}_{\mathbf{q}}^{+}$ (respectively, $\mathcal{U}_{\mathbf{q}}^{-}$) will denote the subalgebra generated by $E_i$ (respectively, $F_i$), $1 \le i \le \theta$.

$\mathcal{U}_{\mathbf{q}}$ is a $\mathbb{Z}^\theta$-graded Hopf algebra, with grading determined by

$$\deg(K_i) = \deg(L_i) = 0, \quad \deg(E_i) = \alpha_i, \quad \deg(F_i) = -\alpha_i.$$

$\mathcal{U}_{\mathbf{q}}$ admits a Hopf algebra structure, with comultiplication determined by

$$\begin{aligned} \Delta(K_i) &= K_i \otimes K_i, & \Delta(E_i) &= E_i \otimes 1 + K_i \otimes E_i, \\ \Delta(L_i) &= L_i \otimes L_i, & \Delta(F_i) &= F_i \otimes L_i + 1 \otimes F_i, \end{aligned}$$

and then $\varepsilon(K_i) = \varepsilon(L_i) = 1$, $\varepsilon(E_i) = \varepsilon(F_i) = 0$.

Note that $\mathcal{U}_{\mathbf{q}}^{0}$ is isomorphic to $\mathbf{k}\mathbb{Z}^{2\theta}$ as Hopf algebras, and the subalgebra $\mathcal{U}_{\mathbf{q}}^{\ge 0}$ (respectively, $\mathcal{U}_{\mathbf{q}}^{\le 0}$) generated by $\mathcal{U}_{\mathbf{q}}^{+}$, $K_i^{\pm 1}$, $1 \le i \le \theta$ (respectively, $\mathcal{U}_{\mathbf{q}}^{-}$, $L_i^{\pm 1}$) is isomorphic to $T(V)\#\mathbf{k}\mathbb{Z}^\theta$ (respectively, $T(V^*)\#\mathbf{k}\mathbb{Z}^\theta$). $\mathcal{U}_{\mathbf{q}}$ is the associated quantum double.

Here, $\mathcal{U}_{\mathbf{q}}^{+}$ is isomorphic to $T(V)$ as braided graded Hopf algebras in ${}^{\mathbf{k}\mathbb{Z}^\theta}_{\mathbf{k}\mathbb{Z}^\theta}\mathcal{YD}$, with actions and coactions given by

$$K_i \cdot E_j = q_{ij}E_j, \qquad \delta(E_i) = K_i \otimes E_i.$$

$\underline{\Delta}(E) = E_{(1)} \otimes E_{(2)}$ denotes the braided comultiplication of $E \in \mathcal{U}_{\mathbf{q}}^{+}$. As it is $\mathbb{N}_0$-graded, we will consider $\underline{\Delta}_{n-k,k}(E)$, the component of $\underline{\Delta}(E)$ in $\mathcal{U}_{\mathbf{q}}^{+}(n-k) \otimes \mathcal{U}_{\mathbf{q}}^{+}(k)$, if $E \in \mathcal{U}_{\mathbf{q}}^{+}$ is homogeneous of degree $n$ and $k \in \{0, 1, \dots, n\}$.

By Ref. 9, Prop. 4.14, the multiplication $m : \mathcal{U}_{\mathbf{q}}^{+} \otimes \mathcal{U}_{\mathbf{q}}^{0} \otimes \mathcal{U}_{\mathbf{q}}^{-} \to \mathcal{U}_{\mathbf{q}}$ is an isomorphism of $\mathbb{Z}^\theta$-graded vector spaces.

We consider some isomorphisms involving $\mathcal{U}_{\mathbf{q}}$ [Ref. 9, Sec. 4.1].

(a) Let $\underline{a} = (a_1, \dots, a_\theta) \in (\mathbf{k}^\times)^\theta$. There exists a unique algebra automorphism $\varphi_{\underline{a}}$ of $\mathcal{U}_{\mathbf{q}}$ such that

$$\varphi_{\underline{a}}(K_i) = K_i, \quad \varphi_{\underline{a}}(L_i) = L_i, \quad \varphi_{\underline{a}}(E_i) = a_iE_i, \quad \varphi_{\underline{a}}(F_i) = a_i^{-1}F_i. \tag{2.12}$$

(b) There exists a unique algebra antiautomorphism $\Omega$ of $\mathcal{U}_{\mathbf{q}}$ such that

$$\Omega(K_i) = K_i, \quad \Omega(L_i) = L_i, \quad \Omega(E_i) = F_i, \quad \Omega(F_i) = E_i. \tag{2.13}$$

It satisfies the relation $\Omega^2 = \mathrm{id}$.

As in, Refs. 9 and 12 $\mathcal{I}_{\mathbf{q}}^{+}$ will denote the ideal of $\mathcal{U}_{\mathbf{q}}^{+}$ such that the quotient $\mathcal{U}_{\mathbf{q}}^{+}/\mathcal{I}_{\mathbf{q}}^{+}$ is isomorphic to the Nichols algebra of $V$; that is, the greatest braided Hopf ideal of $\mathcal{U}_{\mathbf{q}}^{+}$ generated by elements of degree $\ge 2$. Set

$$\mathcal{I}_{\mathbf{q}}^{-} = \Omega(\mathcal{I}_{\mathbf{q}}^{+}), \quad \mathfrak{u}_{\mathbf{q}}^{\pm} := \mathcal{U}_{\mathbf{q}}^{\pm}/\mathcal{I}_{\mathbf{q}}^{\pm}, \quad \mathfrak{u}_{\mathbf{q}} := \mathcal{U}_{\mathbf{q}}/(\mathcal{I}_{\mathbf{q}}^{-} + \mathcal{I}_{\mathbf{q}}^{+}),$$

and $\mathfrak{u}_{\mathbf{q}}^{\geq 0}$, $\mathfrak{u}_{\mathbf{q}}^{\leq 0}$ the corresponding images on the quotient. Note that $\mathfrak{u}_{\mathbf{q}}$ is the quantum double of $\mathfrak{u}_{\mathbf{q}}^{+}\#\mathbf{k}\mathbb{Z}^{\theta}$. The following result follows by Ref. 9, Lemma 6.5, Theorem 6.12.

*Proposition 2.4 (Ref. 12, Proposition 3.5, Ref. 9, Theorem 5.8). There exists a unique non-degenerate skew-Hopf pairing* $\eta : \mathfrak{u}_{\mathbf{q}}^{+} \otimes \mathfrak{u}_{\mathbf{q}}^{-}$ *such that*

$$\eta(K_i, L_j) = q_{ij}, \quad \eta(E_i, F_j) = -\delta_{ij}, \quad \eta(E_i, L_j) = \eta(K_i, F_j) = 0, \tag{2.14}$$

*for all* $1 \leq i, j \leq \theta$. *It satisfies the following condition: for all* $E \in \mathfrak{u}_{\mathbf{q}}^{+}$, $F \in \mathfrak{u}_{\mathbf{q}}^{-}$, $K \in \mathfrak{u}_{\mathbf{q}}^{+0}$, $L \in \mathfrak{u}_{\mathbf{q}}^{-0}$,

$$\eta(EK, FL) = \eta(E, F)\eta(K, L). \tag{2.15}$$

*Moreover, if* $\beta \neq \gamma \in \mathbb{N}_0^{\theta}$, *then* $\eta|_{(\mathfrak{u}_{\mathbf{q}}^{+})_{\beta} \otimes (\mathfrak{u}_{\mathbf{q}}^{-})_{-\gamma}} \cong 0.$ □

Assume that all the integers $a_{ij}^{\mathbf{q}}$ are defined, so the automorphisms $s_p^{\mathbf{q}}$ are defined. For simplicity, we denote $\underline{E}_i$, $\underline{F}_i$, $\underline{K}_i$, $\underline{L}_i$ the generators corresponding to $\mathcal{U}_{\rho_i(\mathbf{q})}$, $a_{ij} = a_{ij}^{\mathbf{q}}$, $q_{ij} = q_{ij}^{\mathbf{q}}$, $\underline{q}_{ij} = q_{ij}^{\rho_i(\mathbf{q})}$. We also define

$$\lambda_{\mathbf{q}}(i) := (-a_{pi})_{q_{pp}} \prod_{s=0}^{-a_{pi}-1} (q_{pp}^{s} q_{pi} q_{ip} - 1) \in \mathbf{k}^{\times}, \quad i \neq p. \tag{2.16}$$

Fix $p \in \{1, \ldots, \theta\}$. If $i \neq p$, we consider the elements[9]

$$E_{i,0(p)}^{+}, E_{i,0(p)}^{-} := E_i, \qquad F_{i,0(p)}^{+}, F_{i,0(p)}^{-} := F_i,$$

and recursively,

$$\begin{aligned}
E_{i,m+1(p)}^{+} &:= E_p E_{i,m(p)}^{+} - (K_p \cdot E_{i,m(p)}^{+}) E_p = (\mathrm{ad}_c E_p)^{m+1} E_i, \\
E_{i,m+1(p)}^{-} &:= E_p E_{i,m(p)}^{-} - (L_p \cdot E_{i,m(p)}^{-}) E_p, \\
F_{i,m+1(p)}^{+} &:= F_p F_{i,m(p)}^{+} - (L_p \cdot F_{i,m(p)}^{+}) F_p, \\
F_{i,m+1(p)}^{-} &:= F_p F_{i,m(p)}^{-} - (K_p \cdot F_{i,m(p)}^{-}) F_p.
\end{aligned}$$

If $p$ is explicit, we simply denote $E_{i,m(p)}^{\pm}$ by $E_{i,m}^{\pm}$. By Ref. 9, Corollary 5.4,

$$E_{i,m}^{+} F_i - F_i E_{i,m}^{+} = (m)_{q_{pp}} (q_{pp}^{m-1} q_{pi} q_{ip} - 1) L_p E_{i,m-1}^{+}. \tag{2.17}$$

**Theorem 2.5.** *There exist algebra morphisms*

$$T_p, T_p^{-} : \mathfrak{u}_{\mathbf{q}} \to \mathfrak{u}_{\rho_i(\mathbf{q})} \tag{2.18}$$

*univocally determined by the following conditions: for every* $i \neq p$,

$$\begin{aligned}
T_p(K_p) &= T_p^{-}(K_p) = \underline{K}_p^{-1}, & T_p(K_i) &= T_p^{-}(K_i) = \underline{K}_p^{m_{pi}} \underline{K}_i, \\
T_p(L_p) &= T_p^{-}(L_p) = \underline{L}_p^{-1}, & T_p(L_i) &= T_p^{-}(L_i) = \underline{L}_p^{m_{pi}} \underline{L}_i, \\
T_p(E_p) &= \underline{F}_p \underline{L}_p^{-1}, & T_p(E_i) &= \underline{E}_{i,m_{pi}}^{+}, \\
T_p(F_p) &= \underline{K}_p^{-1} \underline{E}_p, & T_p(F_i) &= \lambda_{\rho_i(\mathbf{q})}(p)^{-1} \underline{F}_{i,m_{pi}}^{+}, \\
T_p^{-}(E_p) &= \underline{K}_p^{-1} \underline{F}_p, & T_p^{-}(E_i) &= \lambda_{\rho_i(\mathbf{q})}(p)^{-1} \underline{E}_{i,m_{pi}}^{-}, \\
T_p^{-}(F_p) &= \underline{E}_p \underline{L}_p^{-1}, & T_p^{-}(F_i) &= \underline{F}_{i,m_{pi}}^{-}.
\end{aligned}$$

*Moreover,* $T_p T_p^{-} = T_p^{-} T_p = \mathrm{id}$, *and there exists* $\mu \in (\mathbf{k}^{\times})^{\theta}$ *such that*

$$T_p \circ \phi_4 = \phi_4 \circ T_p^{-} \circ \varphi_{\mu}. \tag{2.19}$$

□

By Ref. 12, Proposition 4.2, we have for all $\alpha \in \mathbb{Z}^\theta$,

$$T_p\big((\mathfrak{u}_{\mathbf{q}})_\alpha\big) = (\mathfrak{u}_{\rho_i(\mathbf{q})})_{s_p^{\mathbf{q}}(\alpha)}. \tag{2.20}$$

For $w \in \mathrm{Hom}(\mathbf{q}', \mathbf{q}) \subset \mathcal{W}_{\mathbf{q}}$ with $\ell(w) = n$ and $w = s_{i_1}^{\mathbf{q}} s_{i_2} \cdots s_{i_n}$ (a reduced expression of $w$), let $\mathfrak{u}^+_{\mathbf{q}}[w]$ be the $\mathbf{k}$-subalgebra of $\mathfrak{u}^+_{\mathbf{q}}$ generated by the elements $T^{\mathbf{q}}_{i_1} T_{i_2} \cdots T_{i_{k-1}}(E_{i_k})$, $1 \le k \le n$. Then $\mathfrak{u}^+_{\mathbf{q}}[w]$ is independent from a choice of reduced expressions of $w$, see Ref. 12, Theorem 4.8.

**Theorem 2.6 (Ref. 10).** *The correspondence $w \mapsto \mathfrak{u}^+_{\mathbf{q}}[w]$ gives a bijection from* $\mathrm{Hom}(\mathcal{W}_{\mathbf{q}}, \mathbf{q})$ *to the set of right coideal subalgebras of $\mathfrak{u}^+_{\mathbf{q}}$. Moreover for $w_1 \in \mathrm{Hom}(\mathbf{q}_1, \mathbf{q})$ and $w_2 \in \mathrm{Hom}(\mathbf{q}_2, \mathbf{q})$, $\mathfrak{u}^+_{\mathbf{q}}[w_1] \subseteq \mathfrak{u}^+_{\mathbf{q}}[w_2]$ if and only if $\ell(w_1^{-1} w_2) = \ell(w_2) - \ell(w_1)$.* □

## III. *R*-MATRIX FROM A VERSION OF A UNIVERSAL *R*-MATRIX

Most of the ideas we shall give in this section are modifications of Ref. 26, Sec. 4. Let $\mathbf{q} \in (\mathbf{k}^\times)^{\theta\times\theta}$ and $\chi : \mathbb{Z}^\theta \times \mathbb{Z}^\theta \to \mathbf{k}^\times$ the associated bicharacter. We will compute an $R$-matrix for some modules of $\mathfrak{u}_{\mathbf{q}}$ from canonical elements of $\mathfrak{u}_{\mathbf{q}}$. If $M = |\Delta^{\mathbf{q}}_+| < \infty$, the canonical elements can be obtained by Proposition 4.6.

### A. Equations for canonical elements

We recall Ref. 12, (3.18), (3.19),

$$YX = \eta(X_1, \mathcal{S}(Y_1))\eta(X_3, Y_3)X_2Y_2, \tag{3.1}$$

$$XY = \eta(X_1, Y_1)\eta(X_3, \mathcal{S}(Y_3))Y_2X_2, \quad X \in \mathfrak{u}^{\ge 0}_{\mathbf{q}},\ Y \in \mathfrak{u}^{\le 0}_{\mathbf{q}}. \tag{3.2}$$

Define the $\mathbf{k}$-linear homomorphism $\tau : \mathfrak{u}_{\mathbf{q}} \otimes \mathfrak{u}_{\mathbf{q}} \to \mathfrak{u}_{\mathbf{q}} \otimes \mathfrak{u}_{\mathbf{q}}$ by

$$\tau(X \otimes Y) := Y \otimes X.$$

Given $\mathbf{X} \in \mathfrak{u}^{\ge 0}_{\mathbf{q}}$, $\mathbf{Y} \in \mathfrak{u}^{\le 0}_{\mathbf{q}}$, we define the $\mathbf{k}$-linear homomorphisms

$$\hat\eta^{\le}_{\mathbf{X}} : \mathfrak{u}^{\le 0}_{\mathbf{q}} \to \mathbf{k}, \quad \hat\eta^{\le}_{\mathbf{X}}(Y) := \eta(\mathbf{X}, Y), \quad Y \in \mathfrak{u}^{\le 0}_{\mathbf{q}},$$
$$\hat\eta^{\ge}_{\mathbf{Y}} : \mathfrak{u}^{\ge 0}_{\mathbf{q}} \to \mathbf{k}, \quad \hat\eta^{\ge}_{\mathbf{Y}}(X) := \eta(X, \mathbf{Y}), \quad X \in \mathfrak{u}^{\ge 0}_{\mathbf{q}}.$$

*Lemma 3.1. Let $1 \le i \le \theta$ and $\beta \in \mathbb{N}_0^\theta$. Set*

$$\mathbb{N}_0^\theta(\beta; i) := \{\, \gamma \in \mathbb{N}_0^\theta - \{0, \alpha_i\} \,|\, \beta - \gamma \in \mathbb{N}_0^\theta - \{0, \alpha_i\}\}.$$

*(**i**) Let $\beta \notin \{0, \alpha_i, 2\alpha_i\}$, $Y \in \mathfrak{u}^-_{\mathbf{q}_{-\beta}}$. Set $Y', Y'' \in \mathfrak{u}^-_{\mathbf{q}_{-\beta+\alpha_i}}$ such that $[E_i, Y] = K_iY' - Y''L_i$. Then,*

$$\Delta(Y) - (Y \otimes L^\beta + 1 \otimes Y + F_i \otimes Y''L^{\alpha_i} + Y' \otimes F_iL^{\beta-\alpha_i}) \in \oplus_{\gamma\in\mathbb{N}_0^\theta(\beta;i)} \mathfrak{u}^-_{\mathbf{q}_{-\gamma}} \otimes \mathfrak{u}^-_{\mathbf{q}_{-\beta+\gamma}} L^\gamma. \tag{3.3}$$

*In particular,*

$$(\hat\eta^{\le}_{E_i} \otimes \mathrm{id})(\Delta(Y)) = -Y''L_i, \quad (\mathrm{id} \otimes \hat\eta^{\le}_{E_i})(\Delta(Y)) = -Y'. \tag{3.4}$$

*(**ii**) Let $\beta \notin \{0, \alpha_i, 2\alpha_i\}$, $X \in \mathfrak{u}^+_{\mathbf{q}_\beta}$. Set $X', X'' \in \mathfrak{u}^+_{\mathbf{q}_{\beta-\alpha_i}}$ such that $[X, F_i] = X''K_i - L_iX'$. Then,*

$$\Delta(X) - (X \otimes 1 + K^\beta \otimes X + X''K^{\alpha_i} \otimes E_i + E_iK^{\beta-\alpha_i} \otimes X') \in \oplus_{\gamma\in\mathbb{N}_0^\theta(\beta;i)} \mathfrak{u}^+_{\mathbf{q}_\gamma} K^{\beta-\gamma} \otimes \mathfrak{u}^+_{\mathbf{q}_{\beta-\gamma}}. \tag{3.5}$$

*In particular,*

$$(\mathrm{id} \otimes \hat\eta^{\ge}_{F_i})(\Delta(X)) = -X''K_i, \quad (\hat\eta^{\ge}_{F_i} \otimes \mathrm{id})(\Delta(X)) = -X'. \tag{3.6}$$

*Proof.* We prove ***(i)***; ***(ii)*** can be proved analogously. Note that

$$\Delta^{(2)}(E_i) = E_i \otimes 1 \otimes 1 + K_i \otimes E_i \otimes 1 + K_i \otimes K_i \otimes E_i.$$

Define $\bar{Y}', \bar{Y}''$ as the elements of $\mathfrak{u}^-_{\mathbf{q}\,-\beta+\alpha_i}$ satisfying the same property as (3.3) with $\bar{Y}', \bar{Y}''$ in place of $Y', Y''$. By (3.1), we have

$$\begin{aligned} YE_i &= \eta(E_i, S(F_i))\eta(1, L^\beta)\bar{Y}''L^{\alpha_i} + \eta(K_i, S(1))\eta(1, L^\beta)E_iY \\ &\quad + \eta(K_i, S(1))\eta(E_i, F_iL^{\beta-\alpha_i})K_i\bar{Y}' \\ &= \eta(E_i, -F_iL^{-\alpha_i})\eta(1, L^\beta)\bar{Y}''L^{\alpha_i} + \eta(K_i, 1)\eta(1, L^\beta)E_iY \\ &\quad + \eta(K_i, 1)\eta(E_i, F_iL^{\beta-\alpha_i})K_i\bar{Y}' \\ &= \bar{Y}''L_i + E_iY - K_i\bar{Y}', \end{aligned}$$

so the proof is complete. □

Fix $\beta \in \mathbb{N}_0^\theta$ and $m_\beta := \dim \mathfrak{u}^+_{\mathbf{q}\,\beta} = \dim \mathfrak{u}^-_{\mathbf{q}\,-\beta}$. Fix also $\{E_x^{(\beta)}\}$, $\{F_y^{(\beta)}\}$ bases of the spaces $\mathfrak{u}^+_{\mathbf{q}\,\beta}$, $\mathfrak{u}^-_{\mathbf{q}\,-\beta}$, which are dual for $\eta$. Then the matrix $[\eta(E_x^{(\beta)}, F_y^{(\beta)})]_{1\le x,y\le m_\beta}$ is invertible, we call $[b_{xy}^{(\beta)}]_{1\le x,y\le m_\beta}$ to its inverse.

*Lemma 3.2. For all $X \in \mathfrak{u}^+_{\mathbf{q}\,\beta}$, $Y \in \mathfrak{u}^-_{\mathbf{q}\,-\beta}$, it holds*

$$X = \sum_{x,y} b_{yx}^{(\beta)}\eta(X, F_y^{(\beta)})E_x^{(\beta)}, \tag{3.7}$$

$$Y = \sum_{x,y} b_{yx}^{(\beta)}\eta(E_x^{(\beta)}, Y)F_y^{(\beta)}. \tag{3.8}$$

*Proof.* We prove (3.7); the proof of (3.8) is similar. We have

$$\begin{aligned} \eta\left(\sum_{x,y} b_{yx}^{(\beta)}\eta(X, F_y^{(\beta)})E_x^{(\beta)}, F_z^{(\beta)}\right) &= \sum_{x,y} b_{yx}^{(\beta)}\eta(X, F_y^{(\beta)})\eta(E_x^{(\beta)}, F_z^{(\beta)}) \\ &= \sum_{y} \delta_{yz}\eta(X, F_y^{(\beta)}) = \eta(X, F_z^{(\beta)}), \end{aligned}$$

for all $1 \le z \le m$. (3.7) follows since $\eta_{|\mathfrak{u}^+_{\mathbf{q}\,\beta}\times\mathfrak{u}^-_{\mathbf{q}\,-\beta}}$ is non-degenerate. □

Let $C_\beta$ be the canonical element of $\mathfrak{u}^+_{\mathbf{q}\,\beta} \otimes \mathfrak{u}^-_{\mathbf{q}\,-\beta}$, i.e.,

$$C_\beta = \sum_{x,y=1}^{m_\beta} b_{yx}^{(\beta)}E_x^{(\beta)} \otimes F_y^{(\beta)}.$$

*Lemma 3.3. Let $1 \le i \le \theta$. The following identities hold:*

$$[1 \otimes E_i, C_{\beta+\alpha_i}] = C_\beta(E_i \otimes L_i) - (E_i \otimes K_i)C_\beta, \tag{3.9}$$

$$[C_{\beta+\alpha_i}, F_i \otimes 1] = (L_i \otimes F_i)C_\beta - C_\beta(K_i \otimes F_i). \tag{3.10}$$

*Proof.* We prove (3.9). Let $Y \in \mathfrak{u}^-_{\mathbf{q}\,-\beta-\alpha_i}$. Let $Y', Y'' \in \mathfrak{u}^-_{\mathbf{q}\,-\beta}$ be such that $[E_i, Y] = Y'K_i - L_iY''$. Using (3.7), we have

$$\begin{aligned} (\hat{\eta}_Y^{\geq} \otimes \mathrm{id})([1 \otimes E_i, C_{\beta+\alpha_i}]) &= \sum_{x,y} b_{yx}^{(\beta+\alpha_i)}\eta(E_x^{(\beta+\alpha_i)}, Y)[E_i, F_y^{(-\beta-\alpha_i)}] \\ &= \left[E_i, \sum_{x,y} b_{yx}^{(\beta+\alpha_i)}\eta(E_x^{(\beta+\alpha_i)}, Y)F_y^{(-\beta-\alpha_i)}\right] = [E_i, Y]. \end{aligned} \tag{3.11}$$

Now using (3.4), (3.7), and (3.11), we compute

$$
\begin{aligned}
&(\hat{\eta}_Y^{\geq}\otimes \mathrm{id})(C_\beta(E_i\otimes L_i)-(E_i\otimes K_i)C_\beta)\\
&\quad=\sum_{x,y} b_{yx}^{(\beta)}(\eta(E_x^{(\beta)}E_i,Y)F_y^{(\beta)}L_i-\eta(E_iE_x^{(\beta)},Y)K_iF_y^{(\beta)})\\
&\quad=\sum_{x,y} b_{yx}^{(\beta)}(\eta(E_i\otimes E_x^{(\beta)},\Delta(Y))F_y^{(\beta)}L_i-\eta(E_x^{(\beta)}\otimes E_i,\Delta(Y))K_iF_y^{(\beta)})\\
&\quad=\sum_{x,y} b_{yx}^{(\beta)}(-\eta(E_x^{(\beta)},Y'')F_y^{(\beta)}L_i+\eta(E_x^{(\beta)},Y')K_iF_y^{(\beta)})\\
&\quad=-Y''L_i+K_iY'=[E_i,Y]=(\hat{\eta}_Y^{\geq}\otimes\mathrm{id})([1\otimes E_i,C_{\beta+\alpha_i}]).
\end{aligned}
$$

Then (3.9) follows since $\eta_{|\mathfrak{u}^+_{\mathbf{q}\beta}\times\mathfrak{u}^-_{\mathbf{q}-\beta}}$ is non-degenerate; (3.10) is similar. □

*Lemma 3.4. Let $C'_\beta := (K^\beta\otimes 1)(\mathcal{S}\otimes\mathrm{id})(C_\beta)$. For every $\alpha\in\mathbb{N}_0^\theta$,*

$$
\sum_{\substack{\beta,\,\gamma\in\mathbb{N}_0^\theta\\ \beta+\gamma=\alpha}} C_\beta C'_\gamma=\delta_{\alpha,0}=\sum_{\substack{\beta,\,\gamma\in\mathbb{N}_0^\theta\\ \beta+\gamma=\alpha}} C'_\beta C_\gamma. \tag{3.12}
$$

*Proof.* If $\alpha=0$, (3.12) is clear. Assume $\alpha\neq 0$. We show the first equation of (3.12). Since $\eta_{|\mathfrak{u}^+_{\mathbf{q}\alpha}\times\mathfrak{u}^-_{\mathbf{q}-\alpha}}$ is non-degenerate, it suffices to show that

$$
\sum_{\substack{\beta,\,\gamma\in\mathbb{N}_0^\theta\\ \beta+\gamma=\alpha}} (\hat{\eta}_Y^{\geq}\otimes\mathrm{id}_{\mathfrak{u}_\mathbf{q}})(C_\beta C'_\gamma)=0,\quad \text{for all}\quad Y\in\mathfrak{u}^-_{\mathbf{q}-\alpha}. \tag{3.13}
$$

Write $\Delta(Y)=\sum_{\substack{\beta,\,\gamma\in\mathbb{N}_0^\theta\\ \beta+\gamma=\alpha}} Y^{(\beta,\gamma)}(1\otimes L^\beta)$, where $Y^{(\beta,\gamma)}\in\mathfrak{u}^-_{\mathbf{q}-\beta}\otimes\mathfrak{u}^-_{\mathbf{q}-\gamma}$. Further write $Y^{(\beta,\gamma)}=\sum_m Y^{(\beta,\gamma)}_{-\beta,m}\otimes Y^{(\beta,\gamma)\prime}_{-\gamma,m}$, where $Y^{(\beta,\gamma)}_{-\beta,m}\in\mathfrak{u}^-_{\mathbf{q}-\beta}$ and $Y^{(\beta,\gamma)\prime}_{-\gamma,m}\in\mathfrak{u}^-_{\mathbf{q}-\gamma}$. The left hand side of (3.13) is

$$
\begin{aligned}
&\sum_{\substack{\beta,\,\gamma\in\mathbb{N}_0^\theta\\ \beta+\gamma=\alpha}}\sum_{\substack{x,x',\\ y,y'}} b_{yx}^{(\beta)}b_{y'x'}^{(\gamma)}\eta(E_x^{(\beta)}K^\gamma\mathcal{S}(E_{x'}^{(\gamma)}),Y)F_y^{(\beta)}F_{y'}^{(\gamma)}\\
&\quad=\sum_{\substack{\beta,\,\gamma\in\mathbb{N}_0^\theta\\ \beta+\gamma=\alpha}}\sum_{\substack{m,x,y,\\ x',y'}} b_{yx}^{(\beta)}b_{y'x'}^{(\gamma)}\eta(E_x^{(\beta)},Y^{(\beta,\gamma)\prime}_{\gamma,m}L^\beta)\eta(K^\gamma\mathcal{S}(E_{x'}^{(\gamma)}),Y^{(\beta,\gamma)}_{\beta,m})F_y^{(\beta)}F_{y'}^{(\gamma)}\\
&\quad=\sum_{\substack{\beta,\,\gamma\in\mathbb{N}_0^\theta\\ \beta+\gamma=\alpha}}\sum_{\substack{m,x,y,\\ x',y'}} b_{yx}^{(\beta)}b_{y'x'}^{(\gamma)}\eta(E_x^{(\beta)},Y^{(\beta,\gamma)\prime}_{\gamma,m}L^\beta)\eta(\mathcal{S}(E_{x'}^{(\gamma)}K^\gamma),Y^{(\beta,\gamma)}_{\beta,m})F_y^{(\beta)}F_{y'}^{(\gamma)}\\
&\quad=\sum_{\substack{\beta,\,\gamma\in\mathbb{N}_0^\theta\\ \beta+\gamma=\alpha}}\sum_{\substack{m,x,y,\\ x',y'}} b_{yx}^{(\beta)}b_{y'x'}^{(\gamma)}\eta(E_x^{(\beta)},Y^{(\beta,\gamma)\prime}_{\gamma,m}L^\beta)\eta(E_{x'}^{(\gamma)}K^\gamma,\mathcal{S}^{-1}(Y^{(\beta,\gamma)}_{\beta,m}))F_y^{(\beta)}F_{y'}^{(\gamma)}\\
&\quad=\sum_{\substack{\beta,\,\gamma\in\mathbb{N}_0^\theta\\ \beta+\gamma=\alpha}}\sum_{\substack{m,x,y,\\ x',y'}} b_{yx}^{(\beta)}b_{y'x'}^{(\gamma)}\eta(E_x^{(\beta)},Y^{(\beta,\gamma)\prime}_{\gamma,m}L^\beta)\eta(E_{x'}^{(\gamma)}K^\gamma,\mathcal{S}^{-1}(Y^{(\beta,\gamma)}_{\beta,m}))L^\gamma\\
&=\sum_{\substack{\beta,\,\gamma\in\mathbb{N}_0^\theta\\ \beta+\gamma=\alpha}}\sum_{\substack{x,x',\\ y,y'}} b_{y'x'}^{(\gamma)}b_{yx}^{(\beta)}\eta(K^\gamma,L^\gamma)(\sum_m \eta(E_x^{(\beta)},Y^{(\beta,\gamma)\prime}_{\gamma,m})\eta(E_{x'}^{(\gamma)},\mathcal{S}^{-1}(Y^{(\beta,\gamma)}_{\beta,m}))L^\gamma\\
&\quad=\sum_{\substack{\beta,\,\gamma\in\mathbb{N}_0^\theta\\ \beta+\gamma=\alpha}}\sum_m \chi(\gamma,\gamma)Y^{(\beta,\gamma)\prime}_{\gamma,m}\mathcal{S}^{-1}(Y^{(\beta,\gamma)}_{\beta,m})L^\gamma\\
&\quad=\sum_{\substack{\beta,\,\gamma\in\mathbb{N}_0^\theta\\ \beta+\gamma=\alpha}}\sum_m Y^{(\beta,\gamma)\prime}_{\gamma,m}L^\gamma\mathcal{S}^{-1}(Y^{(\beta,\gamma)}_{\beta,m})=\varepsilon(Y)=0,
\end{aligned}
$$

where we use (3.8) and the grading of $\mathfrak{u}_{\mathbf{q}}$. The second equation of (3.12) is obtained in a similar way. □

*Lemma 3.5. The following identities hold:*

$$(\mathrm{id}\otimes\Delta)(C_\alpha)=\sum_{\beta+\gamma=\alpha}C_\beta^{(1,3)}C_\gamma^{(1,2)}(1\otimes 1\otimes L^\gamma), \tag{3.14}$$

$$(\Delta\otimes\mathrm{id})(C_\alpha)=\sum_{\beta+\gamma=\alpha}C_\beta^{(1,3)}C_\gamma^{(2,3)}(K^\gamma\otimes 1\otimes 1). \tag{3.15}$$

*Proof.* We show (3.14). Given $X_1\in\mathfrak{u}^+_{\mathbf{q}\gamma}$ and $X_2\in\mathfrak{u}^+_{\mathbf{q}\beta}$, we compute

$$\begin{aligned}(\mathrm{id}\otimes\hat\eta^{\leq}_{X_1}\otimes\hat\eta^{\leq}_{X_2})(\mathrm{id}\otimes\Delta)(C_\alpha)&=\sum_{x,y}b^{(\alpha)}_{yx}\eta(X_2X_1,F_y^{(-\alpha)})E_x^{(\alpha)}\\ &=X_2X_1=\sum_{x'',y'',x',y'}b^{(\beta)}_{y''x''}b^{(\gamma)}_{y'x'}\eta(X_2,F^{(\beta)}_{y''})\eta(X_1,F^{(-\gamma)}_{y'})E^{(\beta)}_{x''}E^{(\gamma)}_{x'}\\ &=(\mathrm{id}\otimes\hat\eta^{\leq}_{X_1}\otimes\hat\eta^{\leq}_{X_2})(C_\beta^{(1,3)}C_\gamma^{(1,2)}),\end{aligned}$$

where we use (3.7) twice. Since

$$(\mathrm{id}\otimes\Delta)(C_\alpha)\in\sum_{\beta+\gamma=\alpha}\mathfrak{u}^+_{\mathbf{q}\alpha}\otimes\mathfrak{u}^+_{\mathbf{q}\gamma}\otimes\mathfrak{u}^+_{\mathbf{q}\beta}L^\gamma,$$

we prove that (3.14) holds. Similarly, we obtain (3.15). □

### B. *R*-matrix for finite dimensional $\mathfrak{u}_{\mathbf{q}}$-modules

Fix $V_1$, $V_2$, $V_3$ three finite dimensional $\mathfrak{u}_{\mathbf{q}}$-modules, with associated $\mathbf{k}$-algebra homomorphisms $\rho_x:\mathfrak{u}_{\mathbf{q}}\to\mathrm{End}_{\mathbf{k}}(V_x)$, $x\in\{1,2,3\}$, such that there exist an element $v_x\in V_x$ and a $\mathbf{k}$-algebra homomorphism $\Lambda_x:\mathfrak{u}^0_{\mathbf{q}}\to\mathbf{k}$ for each $x\in\{1,2,3\}$ satisfying

$$X\cdot v_x=\Lambda_x(X)v_x \text{ for all } X\in\mathfrak{u}^0_{\mathbf{q}},\qquad V_x=\mathfrak{u}^-_{\mathbf{q}}\cdot v_x,$$
$$E_i\cdot v_x=0 \text{ for all } 1\leq i\leq\theta.$$

If $\mathcal{F}=\sum_z\mathcal{F}'_z\otimes\mathcal{F}''_z\in\mathrm{End}_{\mathbf{k}}(V_x\otimes V_y)\cong\mathrm{End}_{\mathbf{k}}(V_x)\otimes\mathrm{End}_{\mathbf{k}}(V_y)$, $1\leq x<y\leq 3$, we set $\mathcal{F}^{(x,y)}\in\mathrm{End}_{\mathbf{k}}(V_1\otimes V_2\otimes V_3)\cong\mathrm{End}_{\mathbf{k}}(V_1)\otimes\mathrm{End}_{\mathbf{k}}(V_2)\otimes\mathrm{End}_{\mathbf{k}}(V_3)$ as

$$\begin{aligned}\mathcal{F}^{(x,y)}&=\sum_z\mathcal{F}'_z\otimes\mathcal{F}''_z\otimes\mathrm{id}_{V_3}&&\text{if } x=1,y=2,\\ \mathcal{F}^{(x,y)}&=\sum_z\mathcal{F}'_z\otimes\mathrm{id}_{V_2}\otimes\mathcal{F}''_z&&\text{if } x=1,y=3,\\ \mathcal{F}^{(x,y)}&=\mathrm{id}_{V_1}\otimes\sum_z\mathcal{F}'_z\otimes\mathcal{F}''_z&&\text{if } x=2,y=3.\end{aligned}$$

Now define $f_{xy}\in GL_{\mathbf{k}}(V_x\otimes V_y)$ by

$$f_{xy}(Xv_x\otimes Yv_y):=\chi(\beta,\alpha)\Lambda_x(K^{-\beta})\Lambda_y(L^\alpha)Xv_x\otimes Yv_y$$

for $\alpha,\beta\in\mathbb{N}_0^\theta$ and $X\in\mathfrak{u}^-_{\mathbf{q}-\alpha}$, $Y\in\mathfrak{u}^-_{\mathbf{q}-\beta}$. Set also

$$C_{xy}:=\sum_{\beta\in\mathbb{N}_0^\theta}(\rho_x\otimes\rho_y)(C_\beta),\qquad R_{xy}:=C_{xy}f_{xy}^{-1}.$$

*Lemma 3.6. For each $1\leq i\leq\theta$ and $\check X\in V_x\otimes V_y$,*

$$f_{xy}((E_i\otimes 1)\check X)=(E_i\otimes L_i^{-1})f_{xy}(\check X), \tag{3.16}$$

$$f_{xy}((1\otimes E_i)\check{X}) = (K_i\otimes E_i)f_{xy}(\check{X}), \tag{3.17}$$

$$f_{xy}((F_i\otimes 1)\check{X}) = (F_i\otimes L_i)f_{xy}(\check{X}), \tag{3.18}$$

$$f_{xy}((1\otimes F_i)\check{X}) = (K_i^{-1}\otimes F_i)f_{xy}(\check{X}). \tag{3.19}$$

*Proof.* We show (3.16). For each $X\in\mathfrak{u}^-_{\mathbf{q}-\beta}$, $Y\in\mathfrak{u}^-_{\mathbf{q}-\gamma}$,

$$\begin{aligned} f_{xy}((E_i\otimes 1)Xv_x\otimes Yv_y) &= f_{xy}(E_iXv_x\otimes Yv_y)\\ &= \chi(\gamma,\beta-\alpha_i)\Lambda_x(K^{-\gamma})\Lambda_y(L^{\beta-\alpha_i})E_iXv_x\otimes Yv_y\\ &= (E_i\otimes L_i^{-1})f_{xy}(Xv_x\otimes Yv_y).\end{aligned}$$

Thus, we have (3.16). Similarly, we obtain (3.17), (3.18), and (3.19). □

Now we are ready to obtain the $R$-matrix for the modules $V_x$, $1\le x\le 3$.

**Theorem 3.7.** ***(i)*** *$C_{xy}\in \mathrm{GL}_{\mathbf{k}}(V_x\otimes V_y)$ and*

$$C_{xy}^{-1} = \sum_{\beta\in\mathbb{N}_0^\theta}(\rho_x\otimes\rho_y)(K^\beta\otimes 1)(\mathcal{S}\otimes \mathrm{id})(C_\beta). \tag{3.20}$$

***(ii)*** *For every $X\in\mathfrak{u}_{\mathbf{q}}$,*

$$R_{xy}(\rho_x\otimes\rho_y)(\Delta(X))R_{xy}^{-1} = (\rho_x\otimes\rho_y)((\tau\circ\Delta)(X)). \tag{3.21}$$

***(iii)*** *The following identities hold:*

$$\sum_{\beta\in\mathbb{N}_0^\theta}(\rho_1\otimes\rho_2\otimes\rho_3)((\Delta\otimes\mathrm{id}_{\mathfrak{u}_{\mathbf{q}}})(C_\beta)) = C_{13}^{(1,3)}(f_{13}^{(1,3)})^{-1}C_{23}^{(2,3)}f_{13}^{(1,3)}, \tag{3.22}$$

$$\sum_{\beta\in\mathbb{N}_0^\theta}(\rho_1\otimes\rho_2\otimes\rho_3)((\mathrm{id}_{\mathfrak{u}_{\mathbf{q}}}\otimes\Delta)(C_\beta)) = C_{13}^{(1,3)}(f_{13}^{(1,3)})^{-1}C_{12}^{(1,2)}f_{13}^{(1,3)}. \tag{3.23}$$

***(iv)*** *The elements $R_{xy}$ satisfy*

$$R_{12}^{(1,2)}R_{13}^{(1,3)}R_{23}^{(2,3)} = R_{23}^{(2,3)}R_{13}^{(1,3)}R_{12}^{(1,2)}. \tag{3.24}$$

*Proof.* ***(i)*** This immediately follows from (3.12).

***(ii)*** As we have algebra maps on both sides of the identity, it is enough to prove it for the generators of $\mathfrak{u}_{\mathbf{q}}$, and it follows by using Lemmata 3.3, 3.6. For example, for each $\check{X}\in V_x\otimes V_y$, by (3.16), (3.17), (3.9), we have

$$\begin{aligned}(R_{xy}\Delta(E_i) - (\tau\circ\Delta)(E_i)R_{xy})\check{X} &= (C_{xy}f_{xy}^{-1}\Delta(E_i) - (\tau\circ\Delta)(E_i)C_{xy}f_{xy}^{-1})\check{X}\\ &= \sum_{\beta\in\mathbb{N}_0^\theta}(C_\beta f_{xy}^{-1}(E_i\otimes 1 + K_i\otimes E_i) - (1\otimes E_i + E_i\otimes K_i)C_\beta f_{xy}^{-1})\check{X}\\ &= \sum_{\beta\in\mathbb{N}_0^\theta}(C_\beta(E_i\otimes L_i + 1\otimes E_i) - (1\otimes E_i + E_i\otimes K_i)C_\beta)f_{xy}^{-1}\check{X}\\ &= \sum_{\beta\in\mathbb{N}_0^\theta}([1\otimes E_i, C_{\beta+\alpha_i}] - [1\otimes E_i, C_\beta])f_{xy}^{-1}\check{X}\\ &= -\sum_{\beta\in\mathbb{N}_0^\theta,\beta-\alpha_i\notin\mathbb{N}_0^\theta}[1\otimes E_i, C_\beta]f_{xy}^{-1}\check{X} = 0.\end{aligned}$$

***(iii)*** It can be proved by using Lemmata 3.5, 3.6. In fact, we compute for each $\check{X} \in V_x \otimes V_y$,

$$\begin{aligned} C_{13}^{(1,3)}(f_{13}^{(1,3)})^{-1}C_{23}^{(2,3)}f_{13}^{(1,3)}\check{X} &= \sum_{\alpha,\, \gamma\in\mathbb{N}_0^\theta} C_\alpha^{(1,3)}(f_{13}^{(1,3)})^{-1}(C_\gamma^{(2,3)}f_{13}^{(1,3)}(\check{X})) \\ &= \sum_{\alpha,\, \gamma\in\mathbb{N}_0^\theta} C_\alpha^{(1,3)}C_\gamma^{(2,3)}(K^\gamma\otimes 1\otimes 1)\check{X} \\ &= \sum_{\beta\in\mathbb{N}_0^\theta}(\rho_1\otimes\rho_2\otimes\rho_3)((\Delta\otimes \mathrm{id}_{\mathfrak{u}_\mathbf{q}})(C_\beta))\check{X}. \end{aligned}$$

***(iv)*** In this case, the proof follows by Lemma 3.3 and the previous claims

$$\begin{aligned} R_{12}^{(1,2)}R_{13}^{(1,3)}R_{23}^{(2,3)} &= R_{12}^{(1,2)}C_{13}^{(1,3)}(f_{13}^{(1,3)})^{-1}C_{23}^{(2,3)}(f_{23}^{(2,3)})^{-1} \\ &= \sum_{\beta\in\mathbb{N}_0^\theta} R_{12}^{(1,2)}(\rho_1\otimes\rho_2\otimes\rho_3)((\Delta\otimes \mathrm{id}_{\mathfrak{u}_\mathbf{q}})(C_\beta))(f_{13}^{(1,3)})^{-1}(f_{23}^{(2,3)})^{-1} \\ &= \sum_{\beta\in\mathbb{N}_0^\theta}(\rho_1\otimes\rho_2\otimes\rho_3)((\tau\circ\Delta)\otimes \mathrm{id}_{\mathfrak{u}_\mathbf{q}})(C_\beta)R_{12}^{(1,2)}(f_{13}^{(1,3)})^{-1}(f_{23}^{(2,3)})^{-1} \\ &= C_{23}^{(2,3)}(f_{23}^{(2,3)})^{-1}C_{13}^{(1,3)}f_{23}^{(2,3)}R_{12}^{(1,2)}(f_{13}^{(1,3)})^{-1}(f_{23}^{(2,3)})^{-1} \\ &= R_{23}^{(2,3)}R_{13}^{(1,3)}f_{13}^{(1,3)}f_{23}^{(2,3)}R_{12}^{(1,2)}(f_{13}^{(1,3)})^{-1}(f_{23}^{(2,3)})^{-1} \\ &= R_{23}^{(2,3)}R_{13}^{(1,3)}R_{12}^{(1,2)}. \end{aligned}$$

□

## IV. R-MATRICES OF QUANTUM DOUBLES OF NICHOLS ALGEBRAS WITH FINITE ROOT SYSTEMS

For this section, we fix $\mathbf{q}$ such that $M = |\Delta_+^\mathbf{q}| < \infty$. First, we recall a series of results from Ref. 12, Sec. 4, which will be useful to compute explicitly the universal $R$-matrix. Then, we relate them with the chains of coideal subalgebras of Ref. 10 and compute the desired $R$-matrices of quantum doubles of Nichols algebras with finite root systems. Finally, we show some applications of the previous results to relate different PBW basis (or a Lusztig-type Poincaré-Birkhoff-Witt basis).

### A. PBW bases and Lusztig automorphisms

Set an element $w = s_{i_1}^\mathbf{q} s_{i_2}\cdots s_{i_M}$ of maximal length of $\mathcal{W}_\mathbf{q}$. Denote

$$\beta_k := s_{i_1}\cdots s_{i_{k-1}}(\alpha_{i_k}), \quad 1\le k\le M, \tag{4.1}$$

so $\beta_k \neq \beta_l$ if $k \neq l$, and $\Delta_+^\mathbf{q} = \{\beta_k | 1\le k\le M\}$. Set $q_k := \chi(\beta_k,\beta_k)$, and $N_k$ the order of $q_k$, which is possibly infinite. As in Ref. 12, Sec. 4, set

$$\begin{aligned} E_{\beta_k} &= T_{i_1}\cdots T_{i_{k-1}}(E_{i_k}) \in (\mathfrak{u}_\mathbf{q}^+)_{\beta_k}, \overline{E}_{\beta_k} = T_{i_1}^-\cdots T_{i_{k-1}}^-(E_{i_k}) \in (\mathfrak{u}_\mathbf{q}^+)_{\beta_k}, \\ F_{\beta_k} &= T_{i_1}\cdots T_{i_{k-1}}(F_{i_k}) \in (\mathfrak{u}_\mathbf{q}^-)_{\beta_k}, \overline{F}_{\beta_k} = T_{i_1}^-\cdots T_{i_{k-1}}^-(F_{i_k}) \in (\mathfrak{u}_\mathbf{q}^-)_{\beta_k}, \end{aligned}$$

for $1\le k\le M$.

**Theorem 4.1 (Ref. 12, Theorems 4.5, 4.8, 4.9).** *The sets*

$$\begin{aligned} &\{E_{\beta_M}^{a_M}E_{\beta_{M-1}}^{a_{M-1}}\cdots E_{\beta_1}^{a_1} | 0\le a_k < N_k,\ 1\le k\le M\}, \\ &\{\overline{E}_{\beta_M}^{a_M}\overline{E}_{\beta_{M-1}}^{a_{M-1}}\cdots \overline{E}_{\beta_1}^{a_1} | 0\le a_k < N_k,\ 1\le k\le M\} \end{aligned}$$

*are bases of the vector space* $\mathfrak{u}_{\mathbf{q}}^{+}$*, and the sets*

$$\{F_{\beta_M}^{a_M} F_{\beta_{M-1}}^{a_{M-1}} \cdots F_{\beta_1}^{a_1} | 0 \le a_k < N_k,\ 1 \le k \le M\},$$
$$\{\overline{F}_{\beta_M}^{a_M} \overline{F}_{\beta_{M-1}}^{a_{M-1}} \cdots \overline{F}_{\beta_1}^{a_1} | 0 \le a_k < N_k,\ 1 \le k \le M\}$$

*are bases of the vector space* $\mathfrak{u}_{\mathbf{q}}^{-}$*. Moreover, for each pair* $1 \le k < l \le M$*,*

$$E_{\beta_k} E_{\beta_l} - \chi(\beta_k, \beta_l) E_{\beta_l} E_{\beta_k} = \sum c_{a_{k+1},\dots,a_{l-1}} E_{\beta_{k+1}}^{a_{k+1}} \cdots E_{\beta_{l-1}}^{a_{l-1}} \in \mathfrak{u}_{\mathbf{q}}^{+},$$
$$\overline{E}_{\beta_k} \overline{E}_{\beta_l} - \chi^{-1}(\beta_k, \beta_l) \overline{E}_{\beta_l} \overline{E}_{\beta_k} = \sum \overline{c}_{a_{k+1},\dots,a_{l-1}} E_{\beta_{k+1}}^{a_{k+1}} \cdots E_{\beta_{l-1}}^{a_{l-1}} \in \mathfrak{u}_{\mathbf{q}}^{+},$$
$$F_{\beta_k} F_{\beta_l} - \chi(\beta_k, \beta_l) F_{\beta_l} F_{\beta_k} = \sum d_{a_{k+1},\dots,a_{l-1}} F_{\beta_{k+1}}^{a_{k+1}} \cdots F_{\beta_{l-1}}^{a_{l-1}} \in \mathfrak{u}_{\mathbf{q}}^{-},$$
$$\overline{F}_{\beta_k} \overline{F}_{\beta_l} - \chi^{-1}(\beta_k, \beta_l) \overline{F}_{\beta_l} \overline{F}_{\beta_k} = \sum \overline{d}_{a_{k+1},\dots,a_{l-1}} F_{\beta_{k+1}}^{a_{k+1}} \cdots F_{\beta_{l-1}}^{a_{l-1}} \in \mathfrak{u}_{\mathbf{q}}^{+},$$

*for some* $c_{a_{k+1},\dots,a_{l-1}}, \overline{c}_{a_{k+1},\dots,a_{l-1}}, d_{a_{k+1},\dots,a_{l-1}}, \overline{d}_{a_{k+1},\dots,a_{l-1}} \in \mathbf{k}$*.* □

Note that $E_{\beta_k} E_{\beta_l} - \chi(\beta_k, \beta_l) E_{\beta_l} E_{\beta_k} = \left[E_{\beta_k}, E_{\beta_l}\right]_c$.

Now we want to describe the coproduct of the elements of these PBW generators. First, we introduce the following subspaces of $\mathfrak{u}_{\mathbf{q}}$:

$$B_{+}^{l} := \langle \{E_{\beta_l}^{a_l} E_{\beta_{l-1}}^{a_{l-1}} \cdots E_{\beta_1}^{a_1} | 0 \le a_k < N_k\} \rangle \subseteq \mathfrak{u}_{\mathbf{q}}^{+},$$
$$C_{+}^{l} := \langle \{E_{\beta_M}^{a_M} E_{\beta_{M-1}}^{a_{M-1}} \cdots E_{\beta_1}^{a_1} | \exists j > l \text{ s.t. } a_j \ne 0\} \rangle \subseteq \mathfrak{u}_{\mathbf{q}}^{+},$$
$$D_{+}^{l} := \langle \{E_{\beta_M}^{a_M} E_{\beta_{M-1}}^{a_{M-1}} \cdots E_{\beta_1}^{a_1} | \exists j < l \text{ s.t. } a_j \ne 0\} \rangle \subseteq \mathfrak{u}_{\mathbf{q}}^{+},$$
$$B_{-}^{l} := \langle \{F_{\beta_l}^{a_l} F_{\beta_{l-1}}^{a_{l-1}} \cdots F_{\beta_1}^{a_1} | 0 \le a_k < N_k\} \rangle \subseteq \mathfrak{u}_{\mathbf{q}}^{-},$$
$$C_{-}^{l} := \langle \{F_{\beta_M}^{a_M} F_{\beta_{M-1}}^{a_{M-1}} \cdots F_{\beta_1}^{a_1} | \exists j > l \text{ s.t. } a_j \ne 0\} \rangle \subseteq \mathfrak{u}_{\mathbf{q}}^{-},$$
$$D_{-}^{l} := \langle \{F_{\beta_M}^{a_M} F_{\beta_{M-1}}^{a_{M-1}} \cdots F_{\beta_1}^{a_1} | \exists j < l \text{ s.t. } a_j \ne 0\} \rangle \subseteq \mathfrak{u}_{\mathbf{q}}^{-},$$

$1 \le l \le M$; $\langle S \rangle$ denotes the subspace spanned by a subset $S$ of $\mathfrak{u}_{\mathbf{q}}$.

*Proposition 4.2.* $B_{+}^{l}$ *(respectively,* $B_{-}^{l}$*) is a right (respectively, left) coideal subalgebra of* $\mathfrak{u}_{\mathbf{q}}^{+}$ *(respectively,* $\mathfrak{u}_{\mathbf{q}}^{-}$*).*

*Proof.* For each $1 \le l \le M$, set $w_l = s_{i_1}^{\mathbf{q}} s_{i_2} \cdots s_{i_l}$, and the corresponding right coideal subalgebra $\mathfrak{u}_{\mathbf{q}}^{+}[w_l]$ of $\mathfrak{u}_{\mathbf{q}}^{+}$ (for the braided coproduct $\underline{\Delta}$) as in Theorem 2.6; then its Hilbert series is

$$\mathcal{H}_{\mathfrak{u}_{\mathbf{q}}^{+}[w_l]} = \prod_{j=1}^{l} \mathbf{q}_{N_l}(X^{\beta_l}).$$

By the definition of $\mathfrak{u}_{\mathbf{q}}^{+}[w_l]$ in Ref. 10 (which involves the $T_j$'s), it follows that $E_{\beta_j} \in \mathfrak{u}_{\mathbf{q}}^{+}[w_l]$ for each $1 \le j \le k$. Therefore, $B_{+}^{l} \subseteq \mathfrak{u}_{\mathbf{q}}^{+}[w_l]$, because $\mathfrak{u}_{\mathbf{q}}^{+}[w_l]$ is a subalgebra. But both $N_0^{\theta}$-graded vector subspaces of $\mathfrak{u}_{\mathbf{q}}^{+}$ have the same Hilbert series by Theorem 4.1, so $B_{+}^{l} = \mathfrak{u}_{\mathbf{q}}^{+}[w_l]$ is a right coideal subalgebra.

The statement about $B_{-}^{l}$ is analogous because $\mathfrak{u}_{\mathbf{q}}^{-} \simeq \mathcal{B}_{\mathbf{q}^t}^{\mathrm{cop}}$. □

*Corollary 4.3. For each* $1 \le l \le M$*,*

$$\underline{\Delta}(E_{\beta_l}) \in E_{\beta_l} \otimes 1 + 1 \otimes E_{\beta_l} + B_{+}^{l-1} \otimes C_{+}^{l},$$
$$\underline{\Delta}(F_{\beta_l}) \in F_{\beta_l} \otimes 1 + 1 \otimes F_{\beta_l} + C_{+}^{l} \otimes B_{-}^{l-1}.$$

*Proof.* By the previous Proposition and the fact that $\mathfrak{u}_{\mathbf{q}}^{+}$ is a graded connected Hopf algebra,

$$\underline{\Delta}(E_{\beta_l}) = E_{\beta_l} \otimes 1 + 1 \otimes E_{\beta_l} + \sum E_{\beta_{l-1}}^{a_{l-1}} \cdots E_{\beta_1}^{a_1} \otimes X_{a_1,\dots,a_{l-1}},$$

for some $X_{a_1,\ldots,a_{l-1}} \in \mathfrak{u}_{\mathbf{q}}^+$. Write these elements in terms of the PBW basis

$$X_{a_1,\ldots,a_{l-1}} = \sum c_{b_m,\ldots,b_1}^{a_{l-1},\ldots,a_1} E_{\beta_M}^{b_M} E_{\beta_{M-1}}^{b_{M-1}} \cdots E_{\beta_1}^{b_1}.$$

Suppose that $c_{b_m,\ldots,b_1}^{a_{l-1},\ldots,a_1} \neq 0$. Then $\beta_l = \sum_{b_i \neq 0} b_i \beta_i + \sum_{a_j \neq 0} a_j \beta_j$, since $\mathfrak{u}_{\mathbf{q}}^+$ is $\mathbb{N}_0^\theta$-graded. As $j$ runs between 1 and $l-1$, Theorem 2.3 implies that there exists $i > l$ such that $b_i \neq 0$. The proof for $F_{\beta_l}$ is analogous. □

More generally, we can describe the coproduct of each PBW generator. In this case, we can only describe the left hand side of the tensor product.

*Proposition 4.4. For each* $1 \leq l \leq M$, $1 \leq a_l < N_l$,

$$\underline{\Delta}(E_{\beta_l}^{a_l} E_{\beta_{l-1}}^{a_{l-1}} \cdots E_{\beta_1}^{a_1}) \in \sum_{p=0}^{a_l} \binom{a_l}{p}_{q_l} E_{\beta_l}^p \otimes E_{\beta_l}^{a_l - p} E_{\beta_{l-1}}^{a_{l-1}} \cdots E_{\beta_1}^{a_1}$$
$$+ E_{\beta_l}^{a_l} \cdots E_{\beta_1}^{a_1} \otimes 1 + (D_+^l \cap B_+^l) \otimes \mathfrak{u}_{\mathbf{q}}^+,$$
$$\underline{\Delta}(F_{\beta_l}^{a_l} F_{\beta_{l-1}}^{a_{l-1}} \cdots F_{\beta_1}^{a_1}) \in \sum_{p=0}^{a_l} \binom{a_l}{p}_{q_l} F_{\beta_l}^{a_l - p} F_{\beta_{l-1}}^{a_{l-1}} \cdots F_{\beta_1}^{a_1} \otimes F_{\beta_l}^p$$
$$+ 1 \otimes F_{\beta_l}^{a_l} \cdots F_{\beta_1}^{a_1} + \mathfrak{u}_{\mathbf{q}}^- \otimes (D_-^l \cap B_-^l).$$

*Proof.* We prove the statement for the $E_{\beta_k}$'s by induction on $l$; the proof for the $F_{\beta_k}$'s is analogous. The case $l = 1$ is trivial, because $E_{\beta_1} = E_{i_1}$ is primitive, so

$$\underline{\Delta}(E_{\beta_1}^{a_1}) = \sum_{p=0}^{a_1} \binom{a_1}{p}_{q_1} E_{\beta_1}^p \otimes E_{\beta_1}^{a_1 - p}.$$

Assume that it holds for $k < l$. Now we use induction on $a_l$. If $a_l = 1$,

$$\underline{\Delta}(E_{\beta_l} E_{\beta_{l-1}}^{a_{l-1}} \cdots E_{\beta_1}^{a_1}) = \underline{\Delta}(E_{\beta_l}) \underline{\Delta}(E_{\beta_{l-1}}^{a_{l-1}} \cdots E_{\beta_1}^{a_1}).$$

Therefore, we use inductive hypothesis, Corollary 4.3, and the fact that $B_{l-1}$ is a subalgebra to conclude the proof. The inductive step on $a_l$ is completely analogous and close to the proof of results involving the coproduct of hyperletters in Ref. 14. □

## B. Explicit computation of the universal R-matrix

We will obtain now an explicit formula for the universal $R$- matrix when the Nichols algebra is finite-dimensional. By (2.1), it is enough to compute bases of $\mathfrak{u}_{\mathbf{q}}^{\geq 0}$ and $\mathfrak{u}_{\mathbf{q}}^{\leq 0}$, which are dual for $\eta$. Such bases will be those of Theorem 4.1.

The proof is similar to the one of Ref. 4, Proposition 4.2, see also Ref. 22.

*Remark 4.5.* Set for each $\alpha = (a_1, \ldots, a_\theta) \in \mathbb{Z}^\theta$

$$K^\alpha := K_1^{a_1} \cdots K_\theta^{a_\theta} \in \mathfrak{u}_{\mathbf{q}}^{+0}, \qquad L^\alpha := L_1^{a_1} \cdots L_\theta^{a_\theta} \in \mathfrak{u}_{\mathbf{q}}^{-0}.$$

For each $E \in \mathfrak{u}_{\mathbf{q}} \mathbb{Z}^\theta$-homogeneous, let $|E| \in \mathbb{Z}^\theta$ be its degree. Therefore,

$$\Delta(E) = E_{(1)} K^{|E_{(2)}|} \otimes E_{(2)}. \tag{4.2}$$

Analogously, for each homogeneous $F \in \mathfrak{u}_{\mathbf{q}}^-$,

$$\Delta(F) = F_{(1)} \otimes F_{(2)} L^{|F_{(1)}|}. \tag{4.3}$$

*Proposition 4.6. Let* $0 \leq a_i, b_i \leq N_i$, *for each* $1 \leq i \leq M$. *Then,*

$$\eta \left( E_{\beta_M}^{a_M} E_{\beta_{M-1}}^{a_{M-1}} \cdots E_{\beta_1}^{a_1}, F_{\beta_M}^{b_M} F_{\beta_{M-1}}^{b_{M-1}} \cdots F_{\beta_1}^{b_1} \right) = \prod_{i=1}^M \delta_{a_i, b_i} (a_i)_{q_i}! \eta_i^{a_i}, \tag{4.4}$$

*where* $\eta_i := \eta(E_{\beta_i}, F_{\beta_i})$ *is not zero for all* $i$.

*Proof.* We will prove (4.4) by induction on $\sum a_i, \sum b_i$; therefore, $\eta_i \neq 0$ for all $i$ because $\eta$ is a non-degenerate pairing. It is clear if $\sum a_i = 0$. If $\sum a_i = 1$, then the PBW generator is just $E_{\beta_j}$ for some $j$. For this case, we apply decreasing induction on $j$. Note that $\eta\left(E_{\beta_j}, F_{\beta_M}^{b_M} F_{\beta_{M-1}}^{b_{M-1}} \cdots F_{\beta_1}^{b_1}\right) = 0$ when $\beta_j \neq \sum_l b_l\beta_l$, by Proposition 2.4. If $\beta_j = \sum_l b_l\beta_l$ and $\beta_j$ is a simple root, the unique possibility is $b_j = 1$ and $b_l = 0$ for $l \neq j$. If $\beta_j$ is not a simple root, then either $b_j = 1$ and $b_l = 0$ for $l \neq j$ or there exists $k > j$ such that $b_k > 0$ because the order is strongly convex. In the last case,

$$\eta\left(E_{\beta_j}, F_{\beta_k}^{b_k} F_{\beta_{k-1}}^{b_{k-1}} \cdots F_{\beta_1}^{b_1}\right) = \eta\left((E_{\beta_j})_{(1)} K^{|(E_{\beta_j})_{(2)}|}, F_{\beta_k}\right)$$
$$\eta\left((E_{\beta_j})_{(2)}, F_{\beta_k}^{b_k-1} F_{\beta_{k-1}}^{b_{k-1}} \cdots F_{\beta_1}^{b_1}\right) = 0,$$

as $\eta\left((E_{\beta_j})_{(1)} K^{|(E_{\beta_j})_{(2)}|}, F_{\beta_k}\right) = 0$ by Corollary 4.3 and inductive hypothesis.

Assume that $\sum a_i, \sum b_i > 0$, and we have proved the formula for sums smaller than these two. Set $k = \max\{i : a_i \neq 0\}$, $l = \max\{j : b_j \neq 0\}$, and suppose that $k \leq l$ (otherwise the proof is analogous). By Proposition 4.4,

$$\begin{aligned}
&\eta\left(E_{\beta_k}^{a_k} E_{\beta_{k-1}}^{a_{k-1}} \cdots E_{\beta_1}^{a_1}, F_{\beta_l}^{b_l} F_{\beta_{l-1}}^{b_{l-1}} \cdots F_{\beta_1}^{b_1}\right)\\
&= \eta\left((E_{\beta_k}^{a_k} E_{\beta_{k-1}}^{a_{k-1}} \cdots E_{\beta_1}^{a_1})_{(1)} K^{|(E_{\beta_k}^{a_k} E_{\beta_{k-1}}^{a_{k-1}} \cdots E_{\beta_1}^{a_1})_{(2)}|}, F_{\beta_l}\right),\\
&\qquad \eta\left((E_{\beta_1}^{a_1} E_{\beta_{M-1}}^{a_{M-1}} \cdots E_{\beta_k}^{a_k})_{(2)}, F_{\beta_l}^{b_l-1} F_{\beta_{l-1}}^{b_{l-1}} \cdots F_{\beta_1}^{b_1}\right)\\
&= (b_l)_{q_l} \eta_l \delta_{l,k} \eta\left(E_{\beta_k}^{a_k-1} E_{\beta_{k-1}}^{a_{k-1}} \cdots E_{\beta_1}^{a_1}, F_{\beta_l}^{b_l-1} F_{\beta_{l-1}}^{b_{l-1}} \cdots F_{\beta_1}^{b_1}\right),
\end{aligned}$$

so the proof follows by inductive hypothesis. □

Now we obtain a formula for the scalars $\eta_i$. The algebras $\mathfrak{u}_{\mathbf{q}}^{\geq 0}$, $\mathfrak{u}_{\mathbf{q}}^{\leq 0}$ are canonically $\mathbb{N}_0$-graded; we denote by $d(X)$, $d(Y)$ the degree of the homogeneous elements $X \in \mathfrak{u}_{\mathbf{q}}^{\geq 0}$, $Y \in \mathfrak{u}_{\mathbf{q}}^{\leq 0}$. In fact, if $X \in (\mathfrak{u}_{\mathbf{q}}^{\geq 0})_\beta$, $Y \in (\mathfrak{u}_{\mathbf{q}}^{\leq 0})_{-\beta}$, $\beta = \sum_{i=1}^{\theta} n_i\alpha_i \in \mathbb{N}_0^\theta$, then $d(X) = d(Y) = \sum_{i=1}^{\theta} n_i$.

*Lemma 4.7. $\eta_k = (-1)^{d(E_{\beta_k})}$ for all $1 \leq k \leq M$.*

*Proof.* By induction on $k$, it is easy to prove that

$$E_{\beta_k} F_{\beta_k} - F_{\beta_k} E_{\beta_k} = K^{\beta_k} - L^{\beta_k}. \tag{4.5}$$

On the other hand, by (3.2), we have that

$$E_{\beta_k} F_{\beta_k} = \eta\left((E_{\beta_k})_1, (F_{\beta_k})_1\right) \eta\left((E_{\beta_k})_3, \mathcal{S}((F_{\beta_k})_3)\right) (F_{\beta_k})_2 (E_{\beta_k})_2. \tag{4.6}$$

Using (4.2) and the fact that $\mathfrak{u}_{\mathbf{q}}^{\geq 0}$ is $\mathbb{N}_0^\theta$-graded, we deduce that the unique term in $\Delta^{(2)}(E_{\beta_k})$ where appears $K^{\beta_k}$ in the middle is $K^{\beta_k} \otimes K^{\beta_k} \otimes E_{\beta_k}$. To compute the coefficient of this term in (4.6), it is enough to look for the term $1 \otimes 1 \otimes F_{\beta_k}$ in $\Delta^{(2)}(F_{\beta_k})$, because the components of different degrees are orthogonal for $\eta$. Using the antipode axiom and that $\mathfrak{u}_{\mathbf{q}}^{\leq 0}$ is graded, we have that $\mathcal{S}(F_{\beta_k})$ is written as $(-1)^{d(F_{\beta_k})} F_{\beta_k} L^{-\beta_k}$ plus terms of lower degree. Then the coefficient of $K^{\beta_k}$ in the right hand side of Eq. (4.6) is $(-1)^{d(F_{\beta_k})}\eta_k$, using again the orthogonality of the components of different degrees. □

We recall a generalization of Proposition 2.4. The main objective is to consider bosonizations of Nichols algebras by abelian groups, not only free abelian groups but also their quantum doubles. Similar generalizations can be found in Refs. 1 and 20 and also in Ref. 5 for finite groups.

Set $\mathbf{q}$ as above and two abelian groups $\Gamma, \Lambda$. Assume that there exists elements $g_i \in \Gamma$, $\gamma_j \in \widehat{\Gamma}$ such that $\gamma_j(g_i) = q_{ij}$, and elements $h_i \in \Lambda$, $\lambda_j \in \widehat{\Lambda}$ such that $\lambda_j(h_i) = q_{ji}$. Assume that there exists a bicharacter $\mu : \Gamma \times \Lambda \to \mathbf{k}^\times$, such that $\mu(g_i, h_j) = q_{ij}$. For example, $\Gamma = \Lambda = \mathbb{Z}^\theta$.

Set $V \in {}^{\mathbf{k}\Gamma}_{\mathbf{k}\Gamma}\mathcal{YD}$ as the vector space with a fixed basis $E_1, \ldots, E_\theta$ such that $E_i \in V_{g_i}^{\gamma_i}$, $W \in {}^{\mathbf{k}\Lambda}_{\mathbf{k}\Lambda}\mathcal{YD}$ to the vector space with a fixed basis $F_1, \ldots, F_\theta$ such that $F_i \in V_{h_i}^{\lambda_i}$. Let $\mathcal{B} = \mathcal{B}_{\mathbf{q}}\#\mathbf{k}\Gamma$ and $\mathcal{B}' = (\mathcal{B}_{\mathbf{q}^t}\#\mathbf{k}\Lambda)^{\mathrm{cop}}$.

**Theorem 4.8.** *There exists a unique skew-Hopf pairing* $\eta : \mathcal{B}\otimes\mathcal{B}' \to \mathbf{k}$ *such that for all* $1 \leq i, j \leq \theta$ *and all* $g \in \Gamma$, $h \in \Lambda$,

$$\eta(g,h) = \mu(g,h), \qquad \eta(E_i,F_j) = -\delta_{ij}, \qquad \eta(E_i,h) = \eta(g,F_j) = 0. \tag{4.7}$$

*It satisfies the following condition: for all* $E \in \mathfrak{u}_{\mathbf{q}}^+$, $F \in \mathfrak{u}_{\mathbf{q}}^-$, $g \in \mathcal{B}$, $h \in \mathcal{B}'$,

$$\eta(Eg, Fh) = \eta(E,F)\mu(g,h). \tag{4.8}$$

*The restriction* $\eta_{|\mathcal{B}_{\mathbf{q}}\otimes\mathcal{B}_{\mathbf{q}^t}}$ *coincides with the pairing in Proposition 2.4.* □

We work with the case $\Lambda = \widehat{\Gamma}$, $\Gamma$ a finite group, $\mu$ the evaluation bicharacter, and $h_i = \gamma_i, \lambda_i = g_i$ under the canonical identification of the characters of $\widehat{\Gamma}$ with $\Gamma$. In this case, $\eta$ is non-degenerate. Call $\mathfrak{u}_{\mathbf{q}}$ to the Hopf algebra corresponding to this skew-Hopf pairing, following Subsection II A, and denote $\mathcal{B} = \mathfrak{u}_{\mathbf{q}}^{\geq 0}$, $\mathcal{B}' = \mathfrak{u}_{\mathbf{q}}^{\leq 0}$ by analogy with Secs. II–III. Two dual bases for $\eta|_{\mathbf{k}\Gamma\otimes\mathbf{k}\widehat{\Gamma}}$ are $\{g\}_{g\in\Gamma}$, $\{\delta_g\}_{g\in\Gamma}$, where $\delta_g = |\Gamma|^{-1}\sum_{\gamma\in\widehat{\Gamma}}\gamma(g^{-1})\gamma$. Therefore, it has an $R$-matrix of the form

$$\mathcal{R}_1 := \sum_{g\in\Gamma}\delta_g\otimes g = \frac{1}{|\Gamma|}\sum_{g\in\Gamma,\gamma\in\widetilde{\Gamma}}\gamma(g^{-1})\,\gamma\otimes g. \tag{4.9}$$

**Theorem 4.9.** *The universal R-matrix of* $\mathfrak{u}_{\mathbf{q}}$ *is given by the formula*

$$\mathcal{R} = \Big(\prod \exp_{q_j}\Big((-1)^{d(F_{\beta_k})}F_{\beta_j}\otimes E_{\beta_j}\Big)\Big)\mathcal{R}_1, \tag{4.10}$$

*where the product is written in decreasing order.*

*Proof.* By Proposition 4.6 and Theorem 4.8, the sets

$$\{E_{\beta_M}^{a_M}\cdots E_{\beta_1}^{a_1}g : 0\leq a_i < N_i, g\in\Gamma\},$$
$$\left\{\left(\prod_{i=1}^M (a_i)_{q_i}!\eta_i^{a_i}\right)^{-1} F_{\beta_M}^{b_M}\cdots F_{\beta_1}^{b_1}\delta_g : 0\leq b_i < N_i, g\in\Gamma\right\}$$

are bases of $\mathfrak{u}_{\mathbf{q}}^{\geq 0}$, $\mathfrak{u}_{\mathbf{q}}^{\leq 0}$, respectively, which are dual for $\eta$. As in Subsection II A, a formula for the $R$-matrix is given by

$$\begin{aligned}\mathcal{R} &= \sum_{g\in\Gamma}\sum_{0\leq a_i<N_i}\left(\prod_{i=1}^M (a_i)_{q_i}!\eta_i^{a_i}\right)^{-1} F_{\beta_M}^{b_M}\cdots F_{\beta_1}^{b_1}\delta_g\otimes E_{\beta_M}^{a_M}\cdots E_{\beta_1}^{a_1}g\\ &= \left(\prod\left(\sum_{i=0}^{N_j-1}\frac{\eta_j^i}{(i)_{q_j}!}F_{\beta_j}^i\otimes E_{\beta_j}^i\right)\right)\left(\sum_{g\in\Gamma}\delta_g\otimes g\right),\end{aligned}$$

which ends the proof. □

## C. Further computations on convex PBW bases

We can refine the coproduct expression of each $E_\beta$. In consequence, we can obtain a family of left coideal subalgebras, induced by products of the same PBW generators. For each $1\leq l\leq M$, let

$$\mathbf{B}_+^l := \langle\{E_{\beta_M}^{a_M}E_{\beta_{M-1}}^{a_{M-1}}\cdots E_{\beta_l}^{a_l}\,|0\leq a_k<N_k\}\rangle\subseteq\mathfrak{u}_{\mathbf{q}}^+,$$
$$\mathbf{B}_-^l := \langle\{F_{\beta_M}^{a_M}F_{\beta_{M-1}}^{a_{M-1}}\cdots F_{\beta_l}^{a_l}\,|0\leq a_k<N_k\}\rangle\subseteq\mathfrak{u}_{\mathbf{q}}^-.$$

*Lemma 4.10. For each* $1\leq l\leq M$,

$$\underline{\Delta}(E_{\beta_l})\in E_{\beta_l}\otimes 1 + 1\otimes E_{\beta_l} + B_+^{l-1}\otimes\mathbf{B}_+^{l-1},$$
$$\underline{\Delta}(F_{\beta_l})\in F_{\beta_l}\otimes 1 + 1\otimes F_{\beta_l} + \mathbf{B}_-^{l-1}\otimes B_-^{l-1}.$$

*Proof.* Write both sides of $\underline{\Delta}(E_{\beta_l})$ as linear combinations of the elements of the PBW basis and take a term

$$E_{\beta_{l-1}}^{a_{l-1}} \cdots E_{\beta_1}^{a_1} \otimes E_{\beta_M}^{b_M} E_{\beta_{M-1}}^{b_{M-1}} \cdots E_{\beta_k}^{b_k}$$

appearing with non-zero coefficient $c$, where $k$ is such that $b_k \neq 0$. Using the orthogonality of the elements of the PBW basis,

$$\begin{aligned} 0 \neq c\, \eta\, (E_{\beta_{l-1}}^{a_{l-1}} \cdots E_{\beta_1}^{a_1} K^{|E_{\beta_M}^{b_M} E_{\beta_{M-1}}^{b_{M-1}} \cdots E_{\beta_k}^{b_k}|}, F_{\beta_{l-1}}^{a_{l-1}} \cdots F_{\beta_1}^{a_1}) \\ \eta(E_{\beta_M}^{b_M} E_{\beta_{M-1}}^{b_{M-1}} \cdots E_{\beta_k}^{b_k}, F_{\beta_M}^{b_M} F_{\beta_{M-1}}^{b_{M-1}} \cdots F_{\beta_k}^{b_k}) \\ = \eta\, (E_{\beta_l}, F_{\beta_{l-1}}^{a_{l-1}} \cdots F_{\beta_1}^{a_1} F_{\beta_M}^{b_M} F_{\beta_{M-1}}^{b_{M-1}} \cdots F_{\beta_k}^{b_k}). \end{aligned}$$

Suppose that $k < l$. Using last part of Theorem 4.1 repeatedly, we see that

$$z := F_{\beta_{l-1}}^{a_{l-1}} \cdots F_{\beta_1}^{a_1} F_{\beta_M}^{b_M} F_{\beta_{M-1}}^{b_{M-1}} \cdots F_{\beta_k}^{b_k} \in D_-^l,$$

so $\eta(E_{\beta_l}, z) = 0$, a contradiction. Then $k \geq l$, and we end the proof. □

*Proposition 4.11. $\mathbf{B}_+^l$ (respectively, $\mathbf{B}_-^l$) is a left (respectively, right) coideal subalgebra of $\mathfrak{u}_{\mathbf{q}}^+$ (respectively, $\mathfrak{u}_{\mathbf{q}}^-$).*

*Proof.* It is a consequence of Lemma 4.10 and last part of Theorem 4.1. □

For the last part of this section, we prove a result generalizing Ref. 22, Theorem 22. It establishes the uniqueness (up to scalars) of a PBW basis determining a filtration of coideal subalgebras, and it is useful to compare PBW bases coming from Lusztig isomorphisms as in the previous results and PBW bases from combinatorics as Ref. 14. Note that the first kind of PBW bases gives right and left coideal subalgebras, while some examples of the second family give left coideal subalgebras, see Ref. 4, Sec. 3.3.

**Theorem 4.12.** *Let $(\mathbf{E}_\beta)_{\beta \in \Delta_+^{\mathbf{q}}}$ be non-zero elements of $\mathfrak{u}_{\mathbf{q}}^+$, such that $\mathbf{E}_\beta \in (\mathfrak{u}_{\mathbf{q}}^+)_\beta$, and there exists an order $\beta_M > \ldots > \beta_1$ on the roots such that, for each $1 \leq k \leq M$, the elements $\mathbf{E}_{\beta_M}^{a_M} \cdots \mathbf{E}_{\beta_k}^{a_k}$, $0 \leq a_j < N_{\beta_k}$, determine a basis of a subspace $\mathbf{Y}_k$, which is a left coideal subalgebra of $\mathfrak{u}_{\mathbf{q}}^+$. Then the order on the roots is convex.*

*Moreover, if $(E_\beta)_{\beta \in \Delta_+^{\mathbf{q}}}$ denote PBW generators for the corresponding expression of the element of maximal length of $\mathcal{W}$, then there exist non-zero scalars $c_\beta$ such that $\mathbf{E}_\beta = c_\beta E_\beta$.*

*Proof.* The convexity on the order follows from the fact that the chain of coideal subalgebras $\mathbf{Y}_M \subsetneq \cdots \subsetneq \mathbf{Y}_1 = \mathcal{B}_{\mathbf{q}}$ coincides with $\mathbf{B}_+^M \subsetneq \cdots \subsetneq \mathbf{B}_+^1 = \mathcal{B}_{\mathbf{q}}$. The proof of this fact is exactly as in Ref. 4, Theorem 3.16. That is, $\mathbf{Y}_k = \mathbf{B}_+^k$ for all $1 \leq k \leq M$.

For the second statement, write $\mathbf{E}_{\beta_k} = \sum c(a_1, \cdots, a_M)\, E_{\beta_M}^{a_M} \cdots E_{\beta_1}^{a_1}$. If $c(a_1, \cdots, a_M) \neq 0$, then $\beta_k = \sum_j a_j \beta_j$, so $a_k = 1$, $a_j = 0$ for all $j \neq k$, or there exists $j < k$ such that $a_j \neq 0$. The second case is not possible because $\mathbf{E}_{\beta_k} \in \mathbf{Y}_k = \mathbf{B}_+^k$. Therefore, $\mathbf{E}_{\beta_k} = c_{\beta_k} E_{\beta_k}$ for some $c_{\beta_k} \in \mathbf{k}^\times$. □

*Example 4.13.* Let $\zeta$ be a root of unity of order 5. Let $\mathbf{q} = (q_{ij})_{1 \leq i,j \leq 2}$ be a matrix such that $q_{11} = \zeta$, $q_{22} = -1$, $q_{12}q_{21} = \zeta^2$, so its generalized Dynkin diagram is $\overset{\zeta}{\circ} \xrightarrow{\ \zeta^2\ } \overset{-1}{\circ}$, see Ref. 8. The element of maximal length on its Weyl groupoid has a reduced expression $w_0 = \mathrm{id}^{\mathbf{q}} s_1 s_2 s_1 s_2 s_1 s_2 s_1 s_2$. Then,

$$\begin{aligned} \alpha_1 < 3\alpha_1 + \alpha_2 < 2\alpha_1 + \alpha_2 < 5\alpha_1 + 3\alpha_2 \\ < 3\alpha_1 + 2\alpha_2 < 4\alpha_1 + 3\alpha_2 < \alpha_1 + \alpha_2 < \alpha_2 \end{aligned}$$

is the corresponding order on the roots. We obtain a PBW basis with generators $E_\beta$, $\beta \in \Delta_+^{\mathbf{q}}$, using the Lusztig isomorphisms. Let $\Gamma$ be a finite abelian group, $g_1, g_2 \in \Gamma$, $\gamma_1, \gamma_2 \in \widehat{\Gamma}$ such that

$\gamma_j(g_i) = q_{ij}$, so $\mathcal{B}_{\mathbf{q}}$ can be viewed as a braided Hopf algebra on the category of Yetter-Drinfeld modules of $\mathbf{k}\Gamma$. We define $\mathcal{R}_1$ as in (4.9). By Theorem 4.9,

$$\mathcal{R} = \left(\sum_{k=0}^{4} \frac{-1}{(k)_{\zeta}!} F_1 \otimes E_1\right)(1 \otimes 1 - F_{3\alpha_1+\alpha_2} \otimes E_{3\alpha_1+\alpha_2})$$
$$\left(\sum_{k=0}^{9} \frac{-1}{(k)_{-\zeta^3}!} F_{2\alpha_1+\alpha_2} \otimes E_{2\alpha_1+\alpha_2}\right)(1 \otimes 1 + F_{5\alpha_1+3\alpha_2} \otimes E_{5\alpha_1+3\alpha_2})$$
$$\left(\sum_{k=0}^{4} \frac{-1}{(k)_{\zeta}!} F_{3\alpha_1+2\alpha_2} \otimes E_{3\alpha_1+2\alpha_2}\right)(1 \otimes 1 - F_{4\alpha_1+3\alpha_2} \otimes E_{4\alpha_1+3\alpha_2})$$
$$\left(\sum_{k=0}^{9} \frac{-1}{(k)_{-\zeta^3}!} F_{\alpha_1+\alpha_2} \otimes E_{\alpha_1+\alpha_2}\right)(1 \otimes 1 - F_2 \otimes E_2)\,\mathcal{R}_1.$$

We can obtain also a PBW basis of hyperletters $\mathbf{E}_\beta = [\ell_\beta]_c$, $\beta \in \Delta_+^{\mathbf{q}}$, associated to Lyndon words $\ell_\beta$ as in Ref. 14. We compute easily the corresponding Lyndon words using Ref. 4, Corollary 3.17,

$$\begin{aligned}
&\ell_{\alpha_1} = x_1, && \ell_{3\alpha_1+\alpha_2} = x_1^3x_2, && \ell_{4\alpha_1+3\alpha_2} = x_1^2x_2x_1x_2x_2x_1x_2,\\
&\ell_{\alpha_2} = x_2, && \ell_{2\alpha_1+\alpha_2} = x_1^2x_2, && \ell_{5\alpha_1+3\alpha_2} = x_1^2x_2x_1^2x_2x_1x_2,\\
&\ell_{\alpha_1+\alpha_2} = x_1x_2, && \ell_{3\alpha_1+2\alpha_2} = x_1^2x_2x_1x_2.
\end{aligned}$$

We compute using the Shirshov decomposition, see Refs. 4 and 14 and the references there in,

$$\begin{aligned}
\mathbf{E}_{\alpha_1} &= x_1, & \mathbf{E}_{3\alpha_1+\alpha_2} &= (\mathrm{ad}_c x_1)^3 x_2,\\
\mathbf{E}_{\alpha_2} &= x_2, & \mathbf{E}_{3\alpha_1+2\alpha_2} &= [\mathbf{E}_{2\alpha_1+\alpha_2}, \mathbf{E}_{\alpha_1+\alpha_2}]_c,\\
\mathbf{E}_{\alpha_1+\alpha_2} &= (\mathrm{ad}_c x_1) x_2, & \mathbf{E}_{4\alpha_1+3\alpha_2} &= [\mathbf{E}_{3\alpha_1+2\alpha_2}, \mathbf{E}_{\alpha_1+\alpha_2}]_c,\\
\mathbf{E}_{2\alpha_1+\alpha_2} &= (\mathrm{ad}_c x_1)^2 x_2, & \mathbf{E}_{5\alpha_1+3\alpha_2} &= [\mathbf{E}_{2\alpha_1+\alpha_2}, \mathbf{E}_{3\alpha_1+2\alpha_2}]_c.
\end{aligned}$$

By the previous theorem, there exists $c_\beta \in \mathbf{k}^\times$ such that $\mathbf{E}_\beta = c_\beta E_\beta$. It can be computed as the inverse of the coefficient of $\ell_\beta$ in $E_\beta$, because $\ell_\beta$ appears with coefficient 1 in $\mathbf{E}_\beta$.

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