The frame of fixed stars in Relational Mechanics*

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Relational mechanics is a gauge theory of classical mechanics whose laws do not govern the motion of individual particles but the evolution of the distances between particles. Its formulation gives a satisfactory answer to Leibniz's and Mach's criticisms of Newton's mechanics: relational mechanics does not rely on the idea of an *absolute space*. When describing the behavior of small subsystems with respect to the so called "fixed stars", relational mechanics basically agrees with Newtonian mechanics. However, those subsystems having huge angular momenta will deviate from the Newtonian behavior if they are described in the frame of fixed stars. Such subsystems naturally belong to the field of astronomy; they can be used to test the relational theory.

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I. INTRODUCTION

Relational mechanics is a reformulation of classical mechanics leading to dynamical equations that are valid in any frame. By extending the laws of mechanics to any frame, relational mechanics abolishes Newton's *absolute space*. So, no privileged (*inertial*) frames exist in relational mechanics because its dynamical equations obey an extended symmetry: instead of being invariant just under the Galilean group (uniform translations of frames and rigid rotations of axes) they are invariant under arbitrary time-dependent changes of orthonormal frames. In the language of field theory, the changes of orthonormal frames constitute the *gauge symmetry* of relational mechanics. Notoriously, there are frames (particular *gauge choices*) where relational mechanics makes contact with Newton's laws: in any frame where the angular momentum of the universe vanishes and the center of mass of the universe moves at a constant velocity, the dynamical equations become the Newton's laws. Nonetheless, this kind of privileged frames (we call them *Newtonian frames*) is determined not by an abstract entity like the absolute space but by the set of particles constituting the entire universe. This means that relational mechanics gives an answer to Mach's criticism of Newtonian mechanics; relational mechanics is a *Machian* theory [1].

In spite that Newton's laws can be retrieved in relational mechanics, the solutions to such equations deserves a careful understanding. In a Newtonian frame we do obtain the Keplerian orbits for the motion of planets; however, an individual evolution means nothing in relational mechanics. An individual evolution can be completely distorted by a change of frame (change of gauge). The individual positions and velocities are not *observables* but gauge dependent variables. In relational mechanics the observables (i.e., physically meaningful gauge invariant magnitudes) are the distances between particles and the angles between the straight lines joining pairs of particles. Indeed, relational equations govern just the evolution of observables magnitudes, since such evolutions are independent of the frame choice. Nevertheless, a given evolution of distances can be described through different individual evolutions in different frames (analogously, in electromagnetism one can describe a given field configuration through different gauge-related potentials). This means that the gauge symmetry endows mechanics with the essential *relational* feature claimed by Leibniz in his correspondence with Clarke [2].

So, going back to the Keplerian orbit of a planet, the statement that the planet returns to the original position in a period is not physical, unless it can be restated in terms of distances or angles. The planet position means nothing in relational mechanics. Particle positions are mere gauge-dependent mathematical tools to understand the relations between particles. Instead, a physical statement should tell about the time elapsed between successive alignments of the Sun-planet direction and some "fixed star". Notice that, since the Newtonian solution under consideration possesses non-null angular momentum, the rest of the universe would have an opposite angular momentum to cancel out the total angular momentum (as required in a Newtonian frame). Therefore the time elapsed between successive alignments of the Sun-planet direction and some "fixed star" is actually smaller than the Keplerian period, since the rest of the universe is counter-rotating in a Newtonian frame. Therefore, in a frame of fixed stars the planet go faster than expected; its velocity involves a non-Keplerian contribution that is proportional to the orbital radius. Of course, this effect is completely negligible in the case of a planetary system. In fact, the angular momentum of a planetary system is compensated by an unnoticeable rotation of the rest of the universe. Because of this reason, the "fixed stars" are really adequate to establish an *external* Newtonian frame for studying a planetary system. But, what if we are studying a galaxy or a cluster of galaxies?

The rest of the article is organized as follows. In Section II we give a brief account of the history of relational mechanics. Besides we summarize the basics of relational mechanics as they are developed in Ref. [3]. In Section III we show the dragging that affects the rotation curves of subsystems displaying a huge angular momentum, when seen in the frame of fixed stars. In Section IV we derive the relational virial theorem. In Section V we study the relational two-bodies problem. In Section VI we display the conclusions.

II. RELATIONAL MECHANICS IN BRIEF

The idea of absolute space, as a way of designating the privileged inertial frames where Newton's laws are valid, was criticized from the very beginning of the science of mechanics. In Leibniz's opinion, mechanics should describe *relations* among bodies, rather than individual evolutions relative to metaphysically defined frames [2]. Even though Newton was aware of this weakness of his formulation –in the sense that the absolute motion cannot be evidenced–, instead he thought that the absolute acceleration was a valid concept. According to Newton, the absolute acceleration is evidenced by the parabolic shape of water surface in a (absolutely) rotating bulk. However, Mach objected this idea by stating that the shape of the water only proves the rotation with respect to the rest of the universe [1], since nobody knows what would happen if the water and the bulk were the only bodies in the universe. Mach's criticism was a trigger in Einstein's route towards general relativity. Einstein baptized "Mach's principle" the idea that inertia is determined by the interaction with the rest of the universe [4–7].

The 20th century is rich in proposals to reformulate the mechanics starting from relational principles. The laws of mechanics combine potentials, which describe forces, and kinetic variables describing motion. The potentials are already relational, since they just contain the distances r_{ij} between particles:

$$V \; = \; \sum_{i<j} V_{ij}(r_{ij}) \; = \; \frac{1}{2} \sum_{i\neq j} V_{ij}(r_{ij}) \; , \tag{1}$$

($V_{ij} = V_{ji}$). Instead, the Newtonian kinetic energy is made of individual velocities; so it should be reformulated in terms of relative velocities and, possibly, distances. Early attempts of this sort can be found in References [8–11] (for a comprehensive account of these early tries see Ref. [12]). However these attempts led to anisotropies of the inertia that are not observationally supported [13, 14]. After this setback it was realized that the basic structure of the Newtonian kinetic energy should be preserved in some sense in order to keep essential features of the successful Newtonian mechanics. Noticeably, the form of the Newtonian kinetic energy is strongly linked to the Galilean transformations, the transformations between inertial frames. However, the aim of relational mechanics is putting all the frames on an equal footing, with the consequent abolition of the absolute space. For this, the Galilean *rigid* symmetry of Newton's theory should be extended to encompass any time-depending translation and rotation,

$$\mathbf{r}_i \; \longrightarrow \; \mathbf{r}_i \; + \; \xi(t) \; , \tag{2}$$

$$\mathbf{r}_i \; \longrightarrow \; \mathbf{r}_i \; + \alpha(t) \; \times \; \mathbf{r}_i \; , \tag{3}$$

what in field theory is called *gauging* the symmetry[1]. In a gauge theory each *physical state* is described by a set of *equivalent configurations*, all of them connected by gauge transformations. In our case a physical state is determined for the distances between particles, which can be read in terms of equivalent configurations of individual positions in different frames. In the language of gauge theory each set of equivalent configurations is called *orbit*, which represents a physical state. While the Newtonian kinetic energy is related to a *measure* $\sum m_i \, d\mathbf{r}_i \cdot d\mathbf{r}_i$ between near configurations $\{\mathbf{r}_i\}$ and $\{\mathbf{r}_i + d\mathbf{r}_i\}$, what is needed to build a relational kinetic energy is a measure between near orbits. Such a measure will be automatically gauge invariant. Not surprisingly, the measure between orbits can be obtained from the (Newtonian) measure between configurations. This idea was developed in Ref. [15], where the measure between near orbits was defined as the lower bound of the (Newtonian) measures between configurations representative of each orbit. The measure between near orbits leads to a gauge invariant kinetic energy, as is needed to formulate the relational mechanics. This procedure is called *best matching* [16–19].

Another way to build a gauge invariant kinetic energy involves the concept of *covariant* derivative, as is typical in gauge theory. When a rigid symmetry is gauged, the behavior of the ordinary derivative under the so extended symmetry becomes inappropriate (in a sense that will be explained below). The covariant derivative includes a term to heal this undesirable behavior. These strategy was followed in Ref. [3], whose results can be summarized as follows:

► The relational kinetic energy is built of relative positions $\mathbf{r}_{i\,j} \doteq \mathbf{r}_i - \mathbf{r}_j$ and their derivatives $\mathbf{v}_{i\,j} \doteq \dot{\mathbf{r}}_{i\,j} = \mathbf{v}_i - \mathbf{v}_j$. Both of them are invariant under time dependent translations (2). However, even though $\mathbf{r}_{i\,j}$ behaves as a vector under time dependent rotations (3), $\mathbf{v}_{i\,j}$ does not. In fact, from Eq. (3) it follows that $\mathbf{r}_{i\,j} \longrightarrow \mathbf{r}_{i\,j} + \alpha(t) \times \mathbf{r}_{i\,j}$ but

$$\dot{\mathbf{r}}_{i\,j} \; \longrightarrow \; \dot{\mathbf{r}}_{i\,j} \; + \; \alpha \times \dot{\mathbf{r}}_{i\,j} \; + \; \dot{\alpha} \times \mathbf{r}_{i\,j} \; . \tag{4}$$

So, the ordinary derivative of a vector does not behave as a vector under time-depending rotations; the last term in Eq. (4) must be healed by means of the compensating mechanism of a covariant derivative.

► For an isolated system of particles representing the entire universe, which is governed by classical interactions at a distance that are described by a potential V depending on the distances $r_{i\,j} = |\mathbf{r}_{i\,j}|$, the compensating term in the covariant derivative (the *connection*) is built of the *intrinsic* angular momentum $\mathbf{J}$ and inertia tensor $\mathbf{I}$[2]:

$$\mathbf{J} \; \doteq \; \sum_{i<j} \frac{m_i \; m_j}{M} \, \mathbf{r}_{i\,j} \times \mathbf{v}_{i\,j} \; , \tag{5}$$

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[1] $\alpha(t)$ is an infinitesimal vector directed along the axis of rotation (finite rotations require orthonormal matrices). Galileo transformations are included in the gauge group (2), (3); they are the elements having $\dot{\xi} = \mathbf{V} =$ constant, and $\dot{\alpha} = 0$.

[2] We call intrinsic those quantities of the form $\sum_{i<j} \frac{m_i m_j}{2M} \; f_{ij}(\mathbf{r}_{ij}, \mathbf{v}_{ij})$ where $f_{ij} = f_{ji}$.

$$\mathbf{I} \doteq \sum_{i<j} \frac{m_i\, m_j}{M} \left[r_{i\,j}^2\; \mathbf{1} - \mathbf{r}_{i\,j} \otimes \mathbf{r}_{i\,j}\right] \;\; . \tag{6}$$

In fact, since $\mathbf{J}$ contains relative velocities, then it behaves as a vector just under rigid rotations. But under time-dependent rotations $\delta\mathbf{J}$ gets a term proportional to $\dot{\alpha}$:

$$\begin{aligned} \delta\mathbf{J} &= \sum_{i<j} \frac{m_i\, m_j}{M}\, \mathbf{r}_{i\,j} \times \delta\mathbf{v}_{i\,j} \; + \; ... \; = \; ... \; + \sum_{i<j} \frac{m_i\, m_j}{M}\, \mathbf{r}_{i\,j} \times (\, \dot{\alpha} \times \mathbf{r}_{i\,j}) \; + \; ... \\ &= \; ... \; + \sum_{i<j} \frac{m_i\, m_j}{M} \; \left[\dot{\alpha}\; r_{i\,j}^2 - \mathbf{r}_{i\,j}\; \mathbf{r}_{i\,j} \cdot \dot{\alpha}\right] \; + \; ... \; = \; ... \; + \; \mathbf{I} \cdot \dot{\alpha} \; + \; ... \, . \end{aligned} \tag{7}$$

So $\mathbf{I}^{-1} \cdot \mathbf{J}$ is what is needed for canceling out the last term in Eq. (4). The *vectorial* relative velocity is defined as the covariant derivative of the relative position,

$$\frac{D\mathbf{r}_{i\,j}}{Dt} \doteq \frac{d\mathbf{r}_{i\,j}}{dt} \; - \; (\mathbf{I}^{-1} \cdot \mathbf{J}) \; \times \; \mathbf{r}_{i\,j} \;\; , \tag{8}$$

and the gauge invariant kinetic energy has the intrinsic form

$$T \;\; \doteq \;\; \sum_{i<j} \frac{m_i\, m_j}{2M}\, \frac{D\mathbf{r}_{i\,j}}{Dt} \cdot \frac{D\mathbf{r}_{i\,j}}{Dt} \; . \tag{9}$$

▶ The gauge invariant kinetic energy (9) can be rephrased in several ways:

$$T \;=\; \sum_{i<j} \frac{m_i\, m_j}{2M}\, \mathbf{v}_{ij} \cdot \mathbf{v}_{ij} \; - \; \frac{1}{2}\, \mathbf{J} \cdot \mathbf{I}^{-1} \cdot \mathbf{J} \; , \tag{10a}$$

$$=\; \sum_{k} \frac{m_k}{2} \left| \mathbf{v}_k - \frac{\mathbf{P}}{M} - (\mathbf{I}^{-1} \cdot \mathbf{J}) \times (\mathbf{r}_k - \mathbf{R}) \right|^2 \; , \tag{10b}$$

$$=\; \sum_{k} \frac{m_k}{2} \left| \frac{D}{Dt}(\mathbf{r}_k - \mathbf{R}) \right|^2 \; , \tag{10c}$$

where $\mathbf{R}$ is the center-of-mass position, and $\mathbf{P} \doteq \sum m_k\, \mathbf{v}_k$ is the total momentum ($\mathbf{R}$ and $\mathbf{P}$ are gauge dependent magnitudes). Equation (10c) shows that the Newtonian structure of the kinetic energy, as a sum of individual particle contributions, has been preserved. However, $\mathbf{r}_k - \mathbf{R}$ replaces $\mathbf{r}_k$ to fulfill the invariance under time-dependent translations, and the covariant derivative takes the role of the ordinary derivative to fulfill the invariance under time-dependent rotations.

The kinetic energy (10a) originally appeared in References [20–22] where, instead of gauging the symmetry, the authors obtained T by means of the minimization of the Newtonian kinetic energy with respect to translations and rotations, so making contact with the best matching ansatz of Ref. [15].

▶ The relational dynamical equations coming from the Lagrangian $L(\mathbf{r}_k,\, \mathbf{v}_k) = T - V$ are

$$m_k\, \frac{d}{dt} \left[\mathbf{v}_k - \frac{\mathbf{P}}{M} - (\mathbf{I}^{-1} \cdot \mathbf{J}) \times (\mathbf{r}_k - \mathbf{R}) \right] \;\; = \;\; -\nabla_k \left(V + \frac{1}{2}\, \mathbf{J} \cdot \mathbf{I}^{-1} \cdot \mathbf{J} \right) \; . \tag{11}$$

On the l.h.s. the expression inside brackets is $D(\mathbf{r}_k - \mathbf{R})/Dt$. Since d/dt is not a covariant derivative, then the l.h.s. is not a vector under gauge transformations; its bad behavior is compensated by the (centrifugal) gauge-dependent term on the r.h.s.

In spite of appearances, the equations (11) do not govern the dynamics of N individual particles. In fact, the number of degrees of freedom is not $3N$ but $3N - 6$ because the freedom to choose the frame involves six parameters[3]. In other words, the evolutions are determined modulo arbitrary time-dependent translations and rotations, because of the gauge invariance displayed by the Lagrangian and the dynamical equations. For instance, a configuration where the system rigidly rotates (i.e., the distances between particles remain unchanged) is equivalent to the configuration

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[3] See Ref. [3] for the structure of constraints in the Hamiltonian formulation of the theory.

of *rest*. The $3N-6$ degrees of freedom can be associated with the minimum number of distances that are needed to describe a state. For $N=2$ particles, just one distance is involved (since the rotation around the axis of symmetry is meaningless, the system has $3N-5=1$ degrees of freedom). For $N=3$ the particles form a triangle described by 3 distances or degrees of freedom. For $N>3$, each added particle must come at least with a "tripod" of 3 distances to determine its position with respect to the other particles. Thus, the minimum number of distances to describe a state is $3+3(N-3)=3N-6$, $(N>2)$. Therefore, the equations (11) govern a dynamics of distances instead of individual evolutions. As a consequence, the equations (11) remain strongly coupled even in the absence of interaction. In fact, all the particles of the system are contained in $\mathbf{R}$, $\mathbf{P}$, $\mathbf{J}$ and $\mathbf{I}$. Nevertheless, the gauge dependent magnitudes $\mathbf{P}$, $\mathbf{J}$ can be fixed by choosing an appropriate frame where $\mathbf{P}$ is constant and $\mathbf{J}$ vanishes. In such *Newtonian frames*, that are determined by the entire distribution of mass in the universe, the equations of motion become

$$m_k \, \frac{d\mathbf{v}_k}{dt} \;=\; -\nabla_k V \; . \tag{12}$$

These gauge-fixed dynamical equations could create the illusion that Newton's dynamics is got at the end of the day, since a set of individual evolutions fulfilling the Newton's laws has been obtained. However, we must keep in mind that the individual evolutions means nothing in relational mechanics. They are not observables at all but gauge-dependent magnitudes. The observables are the distances between particles. So, even if we solve the equations (12), we must analyze the meaning of such Newtonian solutions in terms of relative distances (or angles). Actually the equations (12) must be solved together with the gauge conditions $\mathbf{P} = \mathit{constant}$ and $\mathbf{J}=0$. Any Newtonian evolution (of the entire universe) makes sense if and only if the gauge conditions are fulfilled as well.

III. INTERPRETATION OF NEWTONIAN SOLUTIONS

Let us consider the idealized situation of a subsystem gravitationally isolated from the rest of the island universe. Figure 1 shows a self-gravitating subsystem composed by two equal point-like objects of mass m (the stars $\star$) sharing a circular orbit of radius a around a central object (the galaxy ʖ), surrounded by a rigid isotropic spherical shell (representing the rest of the universe $\circledcirc$). The shell has not gravitational influence on the subsystem inside it. According to Newton's second law (12), the time the stars complete their circular orbit is the Keplerian period

$$\tau_{Kepler} \;=\; 2\pi \, \sqrt{\frac{a^3}{G\,(M_{ʖ} + \frac{m_\star}{4})}} \tag{13}$$

(the contribution $m_\star/4$ comes from the gravitational interaction between the stars). Furthermore, in Newtonian mechanics the orbital solution exists irrespective of the presence of the central object and the shell. Instead, in relational mechanics any Newtonian solution comes together with the gauge condition $\mathbf{J}=0$. Since we have split the universe into three parts sharing their centers-of-mass, then the vanishing of the total intrinsic angular momentum reads[4]

$$\mathbf{J}_\star \;+\; \mathbf{J}_{ʖ} \;+\; \mathbf{J}_\circledcirc \;=0 \; , \tag{14}$$

where $\mathbf{J}_\star$, $\mathbf{J}_{ʖ}$, and $\mathbf{J}_\circledcirc$ are the intrinsic angular momentum of each part (the rigid shell rotates at a constant velocity, as required by the equations (12)). Thus the presences of the central body or the shell are essential to accomplish the requirement (14), so making sense to a Newtonian solution that displays a non-null angular momentum $\mathbf{J}_\star$[5].

Let us now focus on the meaning of τ_{Kepler} in relational mechanics. While τ_{Kepler} in Newtonian mechanics is the period of the circular motion in the privileged inertial frames, τ_{Kepler} in relational mechanics is the period between equal positions on the circle, as seen in a particular gauge fixed frame. However positions are not observables but gauge dependent magnitudes. The circular motion would look completely distorted in an (equally allowed) arbitrarily rotating frame. A physical (gauge independent) interval of time should allude only to observables. The obervables in relational mechanics are the distances between particles and the angles between lines joining particles. The statement

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[4] It is easy to verify that the intrinsic magnitudes $\mathbf{J}$ and $\mathbf{I}$ are the usual angular momentum and tensor of inertia with respect to the center of mass: $\mathbf{J} = \sum m_k \, (\mathbf{r}_k - \mathbf{R}) \times \mathbf{v}_k$, $\mathbf{I} = \sum m_k \, [|\mathbf{r}_k - \mathbf{R}|^2 \, \mathbf{1} - (\mathbf{r}_k - \mathbf{R}) \otimes (\mathbf{r}_k - \mathbf{R})]$. In general, the intrinsic magnitudes are not additive; the intrinsic angular momentum of the universe is not the sum of the intrinsic angular momentum (spin) of its parts because of orbital contributions. However, if the system is split into several parts whose centers-of-mass are coincident, as in the case of Figure 1, then $\mathbf{J}$ and $\mathbf{I}$ can be decomposed as the sum of the intrinsic quantities belonging to each part, as done in Eq. (14). In general, if the system is split into two parts A and B, then it follows that $\mathbf{J} = \mathbf{J}_A + \mathbf{J}_B + (\mathbf{R}_A - \mathbf{R}_B) \times (M_B \, \mathbf{P}_A - M_A \, \mathbf{P}_B)/M$.

[5] A system of just $N=2$ particles has only one degree of freedom (the distance between the particles). The circular motion would imply that the distance is constant. But this would only be possible in the absence of interaction. The role played by the rest of the universe as responsible of the centrifugal effect that is needed to sustain the orbital motion (and the shape of the water in the Newton's bucket as well) is analyzed in Ref. [3].

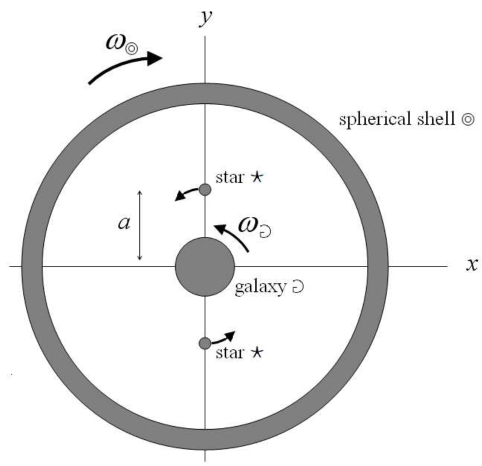

FIG. 1. Two stars orbiting a galaxy. The counter-rotating shell represents the rest of the universe in the Newtonian frame where the total intrinsic angular momentum vanishes.

that the orbit is circular is gauge invariant, because is based on the constancy of a distance. Instead, the time elapsed between successive passes through the same position of a circular orbit is a gauge dependent concept. So, to introduce a period τ defined in terms of observables, we should resort to the successive passes of the stars through the line joining the center of the circular orbit to some "fixed star" in the shell. According to Eq. (14), the shell is counter-rotating with respect to the stars-galaxy subsystem. If the stars co-rotate with the galaxy, then the interval τ between successive passes through the line is lower than τ_{Kepler}:

$$\omega_\star \tau \ = \ \omega_\circledcirc \tau \ + \ 2\pi \, \widehat{\mathbf{z}} \ \ , \tag{15}$$

where $\omega_\star = (2\pi/\tau_{Kepler})\,\widehat{\mathbf{z}}$ and $\omega_\circledcirc = \mathbf{J}_\circledcirc/I_\circledcirc = -(2\ \mathbf{J}_\star + \mathbf{J}_\eth)/I_\circledcirc \simeq -J_\eth/I_\circledcirc\,\widehat{\mathbf{z}}$. Then, it follows that the time τ is[6]

$$\tau \ \simeq \ \frac{\tau_{Kepler}}{1 + \frac{\tau_{Kepler}\ J_\eth}{2\pi\ I_\circledcirc}} \ . \tag{16}$$

So, the speed $v' = 2\pi\ a/\tau$ contains a non-Keplerian contribution:

$$v' \ = \ \frac{2\pi\ a}{\tau} \ = \ \frac{2\pi\ a}{\tau_{Kepler}} \ + \ \frac{a\ J_\eth}{I_\circledcirc} \ = \ v_{Kepler} \ + \ \frac{a\ J_\eth}{I_\circledcirc} \ . \tag{17}$$

The last term is a typical Coriolis effect. In fact, v' in Eq. (17) is the velocity in the frame of "fixed stars", which is a non-Newtonian frame. In a non-Newtonian frame, the Eq. (11) contains not only the interaction forces deriving from V but contributions associated with the mean rotation of the universe $\mathbf{I}^{-1}\cdot\mathbf{J}$ and the acceleration $\dot{\mathbf{P}}/M$. These extra contributions have the form of the inertial forces of Newtonian mechanics [3], although they are not determined by the absolute space but by the distribution of matter in the universe. Then, there exists a Coriolis effect on the velocities measured in the frame of fixed stars[7].

As a part of our Newtonian prejudices, we are used to accept that the axes of an "inertial" frame are pointed to "fixed stars". This misconception seems to be justified by experimental evidence at the scale of the solar system, where Newton's laws work very well in the frame of fixed stars. However, a frame of fixed stars could be a good approximation to a Newtonian frame only to study subsystems of negligible angular momentum; this approximation would fail when huge structures are considered.

We remark that $\mathbf{J}_\eth$ in Eqs. (14), (16) and (17) is the angular momentum of the galaxy in the Newtonian frame, which transforms to the frame of fixed stars as

$$\mathbf{J}'_\eth \ = \ \mathbf{J}_\eth - I_\eth\ \omega_\circledcirc \ \simeq \ \left(1 + \frac{I_\eth}{I_\circledcirc}\right)\ \mathbf{J}_\eth \ . \tag{18}$$

[6] In an elliptic orbit, however, τ_{Kepler} is the time elapsed between successive passes through the periastron. This is an observable, since the periastron is defined by the minimization of a distance. Notoriously the periastron suffers a cumulative shift in the frame of fixed stars because $\tau_{Kepler} > \tau$.

[7] The measurement of velocities in the universe involves the Doppler shift. Ignoring general relativity effects that are beyond this framework, the Doppler shift depends on the relative radial velocity source-observer, which is a gauge invariant magnitude (it is the change of a distance per unit of time).

IV. THE RELATIONAL VIRIAL THEOREM

By combining the equations of motion (11) we obtain equations for $\mathbf{r}_{ij}$:

$$m_i\ m_j\ \frac{d}{dt}\left[\frac{D\mathbf{r}_{ij}}{Dt}\right]\ =\ -\left(m_j\ \nabla_i - m_i\ \nabla_j\right)\left(V + \frac{1}{2}\ \mathbf{J}\cdot\mathbf{I}^{-1}\cdot\mathbf{J}\right)\ . \tag{19}$$

The intrinsic virial is a gauge invariant magnitude which can be defined for any subsystem $\mathcal{S}$:

$$G\ \doteq\ \sum_{i<j}^{N_\mathcal{S}} \frac{m_i\ m_j}{2\,M_\mathcal{S}}\ \frac{d}{dt}(r_{ij}^2)\ =\ \sum_{i<j}^{N_\mathcal{S}} \frac{m_i\ m_j}{M_\mathcal{S}}\ \mathbf{r}_{ij}\cdot\mathbf{v}_{ij}\ =\ \sum_{i<j}^{N_\mathcal{S}} \frac{m_i\ m_j}{M_\mathcal{S}}\ \mathbf{r}_{ij}\cdot\frac{D\mathbf{r}_{ij}}{Dt}\ . \tag{20}$$

Its temporal derivative is

$$\begin{aligned}
\frac{dG}{dt}\ &=\ \sum_{i<j}^{N_\mathcal{S}} \frac{m_i\ m_j}{M_\mathcal{S}}\ \left(\mathbf{v}_{ij}\cdot\frac{D\mathbf{r}_{ij}}{Dt} + \mathbf{r}_{ij}\cdot\frac{d}{dt}\left[\frac{D\mathbf{r}_{ij}}{Dt}\right]\right)\\
&=\ \sum_{i<j}^{N_\mathcal{S}} \frac{m_i\ m_j}{M_\mathcal{S}}\ \left(\mathbf{v}_{ij}\cdot\mathbf{v}_{ij} - \mathbf{v}_{ij}\cdot\ [(\mathbf{I}^{-1}\cdot\mathbf{J})\ \times\ \mathbf{r}_{ij}\ ] + \mathbf{r}_{ij}\cdot\frac{d}{dt}\left[\frac{D\mathbf{r}_{ij}}{Dt}\right]\right)\\
&=\ \sum_{k=1}^{N_\mathcal{S}} m_k\ \left(\mathbf{v}_k - \frac{\mathbf{P}_\mathcal{S}}{M_\mathcal{S}}\right)\cdot\left(\mathbf{v}_k - \frac{\mathbf{P}_\mathcal{S}}{M_\mathcal{S}}\right)\ - \mathbf{J}_\mathcal{S}\cdot\ \mathbf{I}^{-1}\cdot\mathbf{J} - \sum_{i<j}^{N_\mathcal{S}} \frac{1}{M_\mathcal{S}}\ \mathbf{r}_{ij}\cdot(m_j\ \nabla_i - m_i\ \nabla_j)\left(V + \frac{1}{2}\ \mathbf{J}\cdot\mathbf{I}^{-1}\cdot\mathbf{J}\right)
\end{aligned} \tag{21}$$

(to get the first term, replace $\sum_{i<j}$ with $(1/2)\sum_{i\neq j}$ and perform one of the involved sums; the second term results from the circular shift property of the mixed product). We will separately analyze the contributions of V and $\mathbf{J}$ to the last term of Eq. (21).

A. The potential V

The role of the potential in dG/dt does not differ from the respective one in Newton's theory. In fact, by replacing $\sum_{i<j}$ with $(1/2)\sum_{i\neq j}$ we notice that both terms $m_j\nabla_i$ and $-m_i\nabla_j$ will make the same contribution to the sum (to prove it, change $i \longleftrightarrow j$). So it follows that

$$\frac{1}{2}\sum_{i\neq j}^{N_\mathcal{S}}\ \mathbf{r}_{ij}\cdot\left(\frac{m_j}{M_\mathcal{S}}\nabla_i - \frac{m_i}{M_\mathcal{S}}\nabla_j\right)V\ =\ \sum_{i\neq j}^{N_\mathcal{S}}\ \frac{m_j}{M_\mathcal{S}}\ \mathbf{r}_{ij}\cdot\nabla_i V\ =\ \sum_{i=1}^{N_\mathcal{S}}\ (\mathbf{r}_i - \mathbf{R}_\mathcal{S})\cdot\nabla_i V\ . \tag{22}$$

If the subsystem is isolated (only internal forces act on its particles) then the total force is zero: $\sum_{i=1}^{N_\mathcal{S}} \nabla_i V\ =\ 0$. Therefore, the Newtonian result is obtained:

$$\frac{1}{2}\sum_{i\neq j}^{N_\mathcal{S}}\ \mathbf{r}_{ij}\cdot\left(\frac{m_j}{M_\mathcal{S}}\nabla_i - \frac{m_i}{M_\mathcal{S}}\nabla_j\right)V\ \ =\ \sum_{i=1}^{N_\mathcal{S}}\ \mathbf{r}_i\cdot\nabla_i V\ . \tag{23}$$

Since $V = \sum_{i<j}^{N_\mathcal{S}} V_{ij}(r_{ij})$, it follows that

$$\sum_{i=1}^{N_\mathcal{S}}\ \mathbf{r}_i\cdot\nabla_i V\ =\ \sum_{i=1}^{N_\mathcal{S}}\ \mathbf{r}_i\cdot\sum_{j=1}^{N_\mathcal{S}}\frac{\partial V_{ij}}{\partial r_{ij}}\frac{\mathbf{r}_{ij}}{r_{ij}}\ =\ \frac{1}{2}\sum_{i\neq j}^{N_\mathcal{S}}\ \mathbf{r}_{ij}\cdot\frac{\partial V_{ij}}{\partial r_{ij}}\frac{\mathbf{r}_{ij}}{r_{ij}}\ =\ \frac{1}{2}\sum_{i\neq j}^{N_\mathcal{S}}\ r_{ij}\ \frac{\partial V_{ij}}{\partial r_{ij}}\ =\ \sum_{i<j}^{N_\mathcal{S}}\ r_{ij}\ \frac{\partial V_{ij}}{\partial r_{ij}}\ . \tag{24}$$

If $V_{ij} \propto r_{ij}^\alpha$, then we obtain

$$\frac{1}{2}\sum_{i\neq j}^{N_\mathcal{S}}\ \mathbf{r}_{ij}\cdot\left(\frac{m_j}{M_\mathcal{S}}\nabla_i - \frac{m_i}{M_\mathcal{S}}\nabla_j\right)V\ \ =\ \alpha\ V_\mathcal{S}\ , \tag{25}$$

where $V_\mathcal{S}$ is the internal potential energy of the isolated subsystem.

B. The total angular momentum J

It is fulfilled that [3]

$$\nabla_i \left(\frac{1}{2}\, \mathbf{J} \cdot \mathbf{I}^{-1} \cdot \mathbf{J} \right) \;=\; -\, m_i\, (\mathbf{I}^{-1} \cdot \mathbf{J}) \times \left[\left(\mathbf{v}_i - \frac{\mathbf{P}}{M} \right) \; - (\mathbf{I}^{-1} \cdot \mathbf{J}) \times (\mathbf{r}_i - \mathbf{R}) \right] \; ; \tag{26}$$

then, the last term in Eq. (21) is

$$\begin{aligned} \mathbf{r}_{ij} \;\cdot\; \left(\frac{m_j}{M_{\mathcal{S}}} \nabla_i - \frac{m_i}{M_{\mathcal{S}}} \nabla_j \right) \left(\frac{1}{2}\, \mathbf{J} \cdot \mathbf{I}^{-1} \cdot \mathbf{J} \right) \;&=\; -\, \frac{m_i\; m_j}{M_{\mathcal{S}}}\, \mathbf{r}_{ij} \cdot \left[(\mathbf{I}^{-1} \cdot \mathbf{J}) \times \left[\mathbf{v}_{ij} \;\; - (\mathbf{I}^{-1} \cdot \mathbf{J}) \times \mathbf{r}_{ij} \right] \right] \\ =\; \frac{m_i\; m_j}{M_{\mathcal{S}}} \;\; & \left\{ (\mathbf{I}^{-1} \cdot \mathbf{J}) \cdot [\mathbf{r}_{ij} \times \mathbf{v}_{ij}] \;\; - \;\; (\mathbf{I}^{-1} \cdot \mathbf{J}) \cdot \left[\mathbf{r}_{ij} \times \left[(\mathbf{I}^{-1} \cdot \mathbf{J}) \times \mathbf{r}_{ij} \right] \right] \right\} \\ =\; \frac{m_i\; m_j}{M_{\mathcal{S}}} \;\; & \left\{ (\mathbf{I}^{-1} \cdot \mathbf{J}) \cdot [\mathbf{r}_{ij} \times \mathbf{v}_{ij}] \;\; - \;\; (\mathbf{I}^{-1} \cdot \mathbf{J}) \cdot \left[r_{ij}^2\; (\mathbf{I}^{-1} \cdot \mathbf{J}) - \mathbf{r}_{ij}\; [\mathbf{r}_{ij} \cdot (\mathbf{I}^{-1} \cdot \mathbf{J})] \right] \right\} \; , \end{aligned} \tag{27}$$

which adds to

$$\sum_{i<j}^{N_{\mathcal{S}}} \; \mathbf{r}_{ij} \cdot \left(\frac{m_j}{M_{\mathcal{S}}} \nabla_i - \frac{m_i}{M_{\mathcal{S}}} \nabla_j \right) \left(\frac{1}{2}\, \mathbf{J} \cdot \mathbf{I}^{-1} \cdot \mathbf{J} \right) \;=\; \mathbf{J}_{\mathcal{S}} \cdot \; \mathbf{I}^{-1} \cdot \mathbf{J} \; - \; (\mathbf{I}^{-1} \cdot \mathbf{J}) \cdot \mathbf{I}_{\mathcal{S}} \cdot (\mathbf{I}^{-1} \cdot \mathbf{J}) \; . \tag{28}$$

C. Summary and results

In sum, the result is

$$\frac{dG}{dt} \;\; = \sum_{k=1}^{N_{\mathcal{S}}} m_k \; \left(\mathbf{v}_k - \frac{\mathbf{P}_{\mathcal{S}}}{M_{\mathcal{S}}} \right) \cdot \left(\mathbf{v}_k - \frac{\mathbf{P}_{\mathcal{S}}}{M_{\mathcal{S}}} \right) \; - 2\; \mathbf{J}_{\mathcal{S}} \cdot \; \mathbf{I}^{-1} \cdot \mathbf{J} + (\mathbf{I}^{-1} \cdot \mathbf{J}) \cdot \mathbf{I}_{\mathcal{S}} \cdot (\mathbf{I}^{-1} \cdot \mathbf{J}) - \alpha\; V_{\mathcal{S}} \; , \tag{29}$$

where the first term is twice the Newtonian kinetic energy $T_{\mathcal{S}}^{Newton}$ in a frame where the subsystem-center-of-mass is at rest. The virial theorem is based on the assumption that

$$\frac{1}{\tau} \; \int_0^{\tau} \; \frac{dG}{dt} \; dt \;\; = \;\; \frac{G(\tau) - G(0)}{\tau} \tag{30}$$

is a vanishing quantity. This can happens both if the subsystem is periodic of period τ (the distances r_{ij} and their derivatives periodically repeat themselves) or if the subsystem remains bounded for a (going to infinity) very large time τ. In such cases it follows that

$$< 2\; T_{\mathcal{S}}^{Newton} \;\; - 2\; \mathbf{J}_{\mathcal{S}} \cdot \; \mathbf{I}^{-1} \cdot \mathbf{J} + \; (\mathbf{I}^{-1} \cdot \mathbf{J}) \cdot \mathbf{I}_{\mathcal{S}} \cdot (\mathbf{I}^{-1} \cdot \mathbf{J}) > \; = \; \alpha \; < V_{\mathcal{S}} > \; . \tag{31}$$

It is easy to check that the quantity inside the l.h.s. brackets is twice the gauge invariant kinetic energy of the subsystem (replace $\sum_{i<j}$ with $\sum_{i<j}^{N_{\mathcal{S}}}$ in Eq. (9)). The result (31) shows the influence of the rest of the universe on the evolution of an "isolated" subsystem in an arbitrary frame. In a Newtonian frame it is $\mathbf{J} = 0$, so the Newtonian form of the virial theorem is recovered. In the simple model where the rest of the universe is represented by a rigid isotropic spherical shell, the frame of fixed stars (i.e., fixed shell) is not strictly Newtonian because a non-vanishing angular momentum still remains: the angular momentum of the considered subsystem. In such a case it is $\mathbf{J}' = \mathbf{J}'_{\mathcal{S}}$; then, by replacing $\mathbf{I}_{\mathcal{S}} = \mathbf{I} - \mathbf{I}_{\circledcirc}$ (see Footnote 4) we obtain the virial theorem in the frame of fixed stars:

$$< 2\; T_{\mathcal{S}}'^{Newton} \;\; - 2\; \mathbf{J}'_{\mathcal{S}} \cdot \; \mathbf{I}^{-1} \cdot \mathbf{J}'_{\mathcal{S}} + \; (\mathbf{I}^{-1} \cdot \mathbf{J}'_{\mathcal{S}}) \cdot (\mathbf{I} - \mathbf{I}_{\circledcirc}) \cdot (\mathbf{I}^{-1} \cdot \mathbf{J}'_{\mathcal{S}}) > \; = \; \alpha \; < V_{\mathcal{S}} > , \tag{32}$$

i.e.,

$$< 2\; T_{\mathcal{S}}'^{Newton} \;\; - \mathbf{J}_{\mathcal{S}}^{'} \cdot \; \mathbf{I}^{-1} \cdot \mathbf{J}'_{\mathcal{S}} - \; (\mathbf{I}^{-1} \cdot \mathbf{J}'_{\mathcal{S}}) \cdot \mathbf{I}_{\circledcirc} \cdot (\mathbf{I}^{-1} \cdot \mathbf{J}'_{\mathcal{S}}) > \; = \; \alpha \; < V_{\mathcal{S}} > \; . \tag{33}$$

At the lowest level of approximation it is $\mathbf{I} \sim \mathbf{I}_{\circledcirc}$; thus

$$< T'^{Newton}_{\mathcal{S}} > - < \mathbf{J}'_{\mathcal{S}} \cdot \ \mathbf{I}_{\circledcirc}^{-1} \cdot \mathbf{J}'_{\mathcal{S}} > \simeq \frac{\alpha}{2} \ < V_{\mathcal{S}} > \ . \tag{34}$$

The relational virial theorem in a frame of fixed stars departs from its Newtonian version depending on how large the subsystem intrinsic angular momentum is.

If the subsystem $\mathcal{S}$ decomposes into two parts $\circlearrowleft$ and $\star$ with coincident centers-of-mass, then it follows that $\mathbf{J}_{\mathcal{S}} = \mathbf{J}_{\circlearrowleft} + \mathbf{J}_{\star}$ and $\mathbf{I}_{\mathcal{S}} = \mathbf{I}_{\circlearrowleft} + \mathbf{I}_{\star}$ (see Footnote 4); besides it is $V_{\mathcal{S}} = V_{\circlearrowleft} + V_{\star} + V_{\circlearrowleft\star}$. To exemplify, we will apply the virial theorem to the case studied in the previous Section. Since the galaxy can be considered itself as an isolated subsystem –the interaction with the stars is not relevant to its evolution– then an equation like (31) is separately valid for the galaxy as well. Thus we are left with the following relation for the stars:

$$< 2 \ T^{Newton}_{\star} \ - 2 \ \mathbf{J}_{\star} \cdot \ \mathbf{I}^{-1} \cdot \mathbf{J} + \ (\mathbf{I}^{-1} \cdot \mathbf{J}) \cdot \mathbf{I}_{\star} \cdot (\mathbf{I}^{-1} \cdot \mathbf{J}) > = \ \alpha \ < V_{\star} + V_{\circlearrowleft\star} > \ . \tag{35}$$

In the frame of fixed stars it is $\mathbf{J}' = \mathbf{J}'_{\mathcal{S}} = \mathbf{J}'_{\circlearrowleft} + \mathbf{J}'_{\star} \simeq J'_{\circlearrowleft} \ \widehat{\mathbf{z}}$. Therefore, it results

$$2 \ m \ v'^2 - 2 \ (2 \ m \ v' \ a) \ I^{-1} \ J' \ + \ (2 \ m \ a^2)(I^{-1} \ J')^2 \ \ \simeq \ \ 2 \ \frac{G \ M_{\circlearrowleft} \ m}{a} \ , \tag{36}$$

where $V_{\star}$ has been neglected. Thus

$$(v' - a \ I^{-1} \ J')^2 \ \simeq \ \frac{G \ M_{\circlearrowleft}}{a} \ = \ v^2_{Kepler} \ , \tag{37}$$

i.e.,

$$v' \ \simeq \ v_{Kepler} \ + \ a \ I^{-1} \ J' \ \simeq \ \ v_{Kepler} \ + \ \frac{a \ J_{\circlearrowleft}}{I_{\circledcirc}} \tag{38}$$

in agreement with the result (17).

V. THE RELATIONAL TWO-BODY PROBLEM

As shown in Section III, the dynamics of an “isolated” two-body system in the frame of fixed stars involves the constant $I_{\circledcirc}$, which appears as a sort of universal constant. Let us consider two bodies m_1 and m_2 interacting through a potential $V(r_{12})$ much larger than the interactions with the other particles of the universe. Like in Figure 1, we will idealize the rest of the universe as a spherical shell centered at the center-of-mass of the two-body system. Thus, no gravitational field remains inside the shell apart from the interaction $V(r_{12})$. As explained in Footnote 4, such configuration of subsystems with a common center-of-mass implies additivity: $\mathbf{J} = \mathbf{J}_{\circledcirc} + \mathbf{J}_{12}$, where $\mathbf{J}_{\circledcirc}$ stands for the intrinsic angular momentum of the rest of the universe (for $\mathbf{J}_{12}$, see Eq. (5)).

By properly combining the Eq. (11) for $k = 1, 2$ one is led to the equation of motion

$$m_1 \ m_2 \ \frac{d}{dt} \left[\mathbf{v}_{12} - (\mathbf{I}^{-1} \cdot \mathbf{J}) \times \mathbf{r}_{12}\right] \ = \ (-m_2 \ \nabla_1 + m_1 \ \nabla_2) \left(V(r_{12}) + \frac{1}{2} \ \mathbf{J} \cdot \mathbf{I}^{-1} \cdot \mathbf{J} \right) \ , \tag{39}$$

where

$$(-m_2 \ \nabla_1 + m_1 \ \nabla_2) V(r_{12}) \ = \ \left(-m_2 \ \frac{\mathbf{r}_{12}}{r_{12}} + m_1 \ \frac{\mathbf{r}_{21}}{r_{12}} \right) \ \frac{dV}{dr_{12}} \ = \ - \ (m_1 + m_2) \ \frac{dV}{dr_{12}} \ \frac{\mathbf{r}_{12}}{r_{12}} \ . \tag{40}$$

The gradient of the centrifugal term has been computed in Ref. [3]. The result is

$$\mu \ \frac{d\mathbf{v}_{12}}{dt} \ = \ - \ \frac{dV}{dr_{12}} \ \frac{\mathbf{r}_{12}}{r_{12}} + 2 \ \mu \ (\mathbf{I}^{-1} \cdot \mathbf{J}) \times \mathbf{v}_{12} - \mu \ (\mathbf{I}^{-1} \cdot \mathbf{J}) \times \left[(\mathbf{I}^{-1} \cdot \mathbf{J}) \times \mathbf{r}_{12}\right] + \mu \ \left[\frac{d}{dt}(\mathbf{I}^{-1} \cdot \mathbf{J})\right] \times \mathbf{r}_{12} \tag{41}$$

($\mu \doteq m_1 m_2/(m_1 + m_2)$ is the reduced mass), where one recognizes the Coriolis, centrifugal and Euler terms associated not with some “absolute” rotation but with the intrinsic magnitude $\mathbf{I}^{-1} \cdot \mathbf{J}$ defined by the entire universe. Tensor $\mathbf{I}$ is additive when the centers-of-mass coincide; so it is $\mathbf{I} = \ \mathbf{I}_{\circledcirc} + \mathbf{I}_{12}$. $\mathbf{I}_{\circledcirc}$ is an isotropic tensor; besides, one is free of choosing the $z-$axis along the direction of $\mathbf{J}_{12} = \mu\ \mathbf{r}_{12} \times \mathbf{v}_{12}$. Therefore, the tensor $\mathbf{I}$ and its inverse $\mathbf{I}^{-1}$ have the form

$$\mathbf{I} \ = \begin{pmatrix} \dots & \dots & 0 \\ \dots & \dots & 0 \\ 0 & 0 & I_{\circledcirc} + \mu\, r_{12}^2 \end{pmatrix} \quad , \qquad \mathbf{I}^{-1} \ = \begin{pmatrix} \dots & \dots & 0 \\ \dots & \dots & 0 \\ 0 & 0 & (I_{\circledcirc} + \mu\, r_{12}^2)^{-1} \end{pmatrix} \quad . \tag{42}$$

If one chooses the frame of fixed stars, then it is $\mathbf{J}'_{\circledcirc} = 0$; thus, it follows that

$$\mathbf{I}^{-1} \cdot \mathbf{J}' \ = \ \mathbf{I}^{-1} \cdot \mathbf{J}'_{12} \ = \ \frac{\mathbf{J}'_{12}}{I_{\circledcirc} + \mu\, r_{12}^2} \ \doteq \ I^{-1}\ \mathbf{J}'_{12} \quad , \tag{43}$$

which is independent of the choice of the $z-$axis. This result is substituted in Eq. (41) to get

$$\mu\ \left[\frac{d\mathbf{v}'_{12}}{dt}\right]' \ = \ -\frac{dV}{dr_{12}}\ \frac{\mathbf{r}'_{12}}{r_{12}} + 2\ \mu\ I^{-1}\ \mathbf{J}'_{12} \times \mathbf{v}'_{12} + \mu\ I^{-2}\ {J'_{12}}^2\ \mathbf{r}'_{12} + \mu\ \left[\frac{d}{dt}(I^{-1}\ \mathbf{J}'_{12})\right]' \times \mathbf{r}'_{12} \quad , \tag{44}$$

where we used that $\mathbf{J}'_{12}$ and $\mathbf{r}'_{12}$ are mutually perpendicular.

A. Conservation of I^{-1} $\mathbf{J}_{12}$ in the frame of fixed stars

Let us shows that the Eq. (44) leads to the conservation of $\mathbf{I}^{-1} \cdot \mathbf{J}_{12}$ in the frame of fixed stars:

$$\begin{aligned} \left[\frac{d}{dt}(I^{-1}\ \mathbf{J}'_{12})\right]' \ &= \ -I^{-2}\ 2\ \mu\ \mathbf{r}'_{12} \cdot \mathbf{v}'_{12}\ \mathbf{J}'_{12} + I^{-1}\ \mu\ \mathbf{r}'_{12} \times \left[\frac{d\mathbf{v}'_{12}}{dt}\right]' \\ &= \ -I^{-2}\ 2\ \mu\ \mathbf{r}'_{12} \cdot \mathbf{v}'_{12}\ \mathbf{J}'_{12} + I^{-1}\ \mathbf{r}'_{12} \times \left(2\ \mu\ I^{-1}\ \mathbf{J}'_{12} \times \mathbf{v}'_{12} + \mu\ \left[\frac{d}{dt}(I^{-1}\ \mathbf{J}'_{12})\right]' \times \mathbf{r}'_{12}\right) \\ &= \ I^{-1}\ \mu\ \mathbf{r}'_{12} \times \left(\left[\frac{d}{dt}(I^{-1}\ \mathbf{J}'_{12})\right]' \times \mathbf{r}'_{12}\right) \end{aligned} \tag{45}$$

Therefore one obtains

$$\left[\frac{d}{dt}(I^{-1}\ \mathbf{J}'_{12})\right]' \ = \ 0\ . \tag{46}$$

The conservation of I^{-1} $\mathbf{J}'_{12}$ reduces the equations of motion to the form

$$\mu\ \left[\frac{d\mathbf{v}'_{12}}{dt}\right]' \ = \ -\frac{dV}{dr_{12}}\ \frac{\mathbf{r}'_{12}}{r_{12}} + 2\ \mu\ I^{-1}\ \mathbf{J}'_{12} \times \mathbf{v}'_{12} + \mu\ I^{-2}\ {J'_{12}}^2\ \mathbf{r}'_{12}\ . \tag{47}$$

These equations are formally identical to those of Newton's dynamics, as regarded from a (non-inertial) frame that rotates with (absolute) constant velocity $\mathbf{\Omega} = -\ I^{-1}\ \mathbf{J}'_{12}$. Although the frame of fixed stars is non-Newtonian, the relational dynamics of the studied subsystem can be easily recovered from Newtonian dynamics by means of a simple correction involving $\mathbf{\Omega}$. However, $\mathbf{\Omega}$ is not an absolute magnitude but it is directly related to the initial conditions characterizing the solution[8].

B. Energy conservation in the frame of fixed stars

To study the energy conservation we will integrate the scalar multiplication of Eq. (47) with $\mathbf{v}'_{12}\ dt = d\mathbf{r}'_{12}$. Thus, it follows that

$$\frac{1}{2}\ \mu\ |\mathbf{v}'_{12}|^2 + V(r_{12}) - \frac{1}{2}\ \mu\ I^{-2}\ {J'_{12}}^2\ r_{12}^2 \ = \ \text{constant}\ . \tag{48}$$

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[8] The conservation of I^{-1} $\mathbf{J}'_{12}$ can be regarded as a consequence of the conservation of $\mathbf{J}_{12}$ in the Newtonian frame and the way $\mathbf{J}_{12}$ transforms under change of frame (cf. Eq. (18)).

We normally use this equation to describe the radial motion, by using the decomposition

$$|\mathbf{v}'_{12}|^2 \;=\; v_{radial}^2 + {v'}_{tangential}^{2} \;=\; v_{radial}^2 + \frac{{J'_{12}}^2}{\mu^2\ r_{12}^2}\ . \tag{49}$$

Thus, the effective potential for the radial motion is

$$V_{eff} \;=\; V(r_{12}) + \frac{(I^{-1}\ J'_{12})^2}{2\ \mu\ r_{12}^2}\ \left(I^2 - \mu^2\ r_{12}^4\right) \;=\; V(r_{12}) + \frac{(I^{-1}\ J'_{12})^2\ I_{\circledcirc}^2}{2\ \mu\ r_{12}^2} + (I^{-1}\ J'_{12})^2\ I_{\circledcirc}\ . \tag{50}$$

The radial motion is gauge invariant, since the distance between particles is an observable. This fact reflects in the effective potential (50), which keeps the form of its Newtonian version (notice that, according to Footnote 8, the conserved quantity $I_{\circledcirc}\ I^{-1}\ J'_{12}$ is equal to the value of J_{12} in the Newtonian frame).

VI. CONCLUSIONS

Newton's mechanics governs the evolutions of individual particles in the absolute space. Particle positions express themselves in Newton's laws by means of coordinates referred to a frame at rest or in uniform translation with respect to the absolute space (Galileo's symmetry). Although Galileo's symmetry implies that absolute motion is undetectable, it confers the acceleration the status of an absolute property (independent of the chosen inertial frame). Instead, relational mechanics is a theory that governs the dynamics of the distances between particles. Distances do not require a frame to manifest themselves. For practical reasons, we still use a frame to write distances in terms of particle coordinates. But the description of a configuration in terms of distances and their derivatives is completely frame-independent. Thus the (time-dependent) changes of frames constitute a gauge symmetry in relational mechanics. Therefore, the idea of absolute space as an entity that selects the allowed frames is vain in relational mechanics, since frames are reduced to the role of useful accessories. These two approaches to the laws of mechanics are conceptually very different. If a subsystem is isolated, in the sense that its interaction with particles outside the subsystem is negligible, Newton's laws describe its evolution just in terms of the (absolute) initial conditions of its own particles. Instead, relational mechanics always describes a subsystem in terms of internal and external distances; so, even if the subsystem is "isolated", its evolution will anyway depend on the relation between the subsystem and the rest of the universe. This is so because $\mathbf{I}^{-1}\cdot\mathbf{J}$ takes part in all the equations of motion; $\mathbf{I}^{-1}\cdot\mathbf{J}$ is the essential piece to guarantee the gauge invariance of the relational dynamics. However, we can exploit the gauge invariance by choosing a frame where the intrinsic angular momentum of the universe vanishes at each instant. In such frames, which are defined by the entire universe, the equations of motion become the Newton's laws (Newtonian frames). Newton's laws reappear as gauge-fixed equations of motion because the relational Lagrangian was defined by gauging the Newtonian Lagrangian. This way of recovering the Newtonian dynamics implies that a Newtonian solution for an isolated subsystem is valid if and only if it is part of a Newtonian solution with $\mathbf{J} = 0$ for the entire universe. In particular, a spinning Newtonian solution for an isolated subsystem can only work if there exists a rest of universe to make feasible the condition $\mathbf{J} = 0$. With the aim of analyzing the consequences of this statement, we have proposed a very simplified –non realistic– model where the rest of the universe is represented by a rigid isotropic shell centered at the center-of-mass of the subsystem under consideration. Thus, the subsystem remains gravitationally isolated, and $\mathbf{J}$ can be decomposed as the sum of the intrinsic angular momenta of the shell and the subsystem (see Footnote 4). This simple arrangement facilitates the building of Newtonian solutions for the entire universe; in fact, the (conserved) angular momentum of the Newtonian solution for the isolated subsystem is trivially compensated by the shell (Newtonianly) rotating at a constant velocity. In this crude model, the frame of fixed stars is the frame where the shell is at rest[9]; this is not a Newtonian frame since $\mathbf{J}$ is not null but is equal to the subsystem intrinsic angular momentum $\mathbf{J}'_S$. Then, we cannot expect to observe the Newtonian solution in the frame where the shell is at rest; instead we will observe that the Newtonian solution is dragged by a rotation $\mathbf{\Omega} = -\ I^{-1}\ \mathbf{J}'_S \simeq -\ I_{\circledcirc}^{-1}\ \mathbf{J}'_S$. As seen, the larger is $\mathbf{J}'_S$, the greater is the dragging effect (with $I_{\circledcirc}$ playing the role of a universal constant). As we have shown, the dragging effect alters the galactic rotation curves and the two-body dynamics when observed from the frame where the shell is at rest. Consequently, the virial theorem also gets terms associated with $\mathbf{J}'_S$ (see the Eq. (34) for the simpler approximate result). Notoriously, these effects of relational mechanics take part in the phenomena that led to the hypothesis of dark matter: the discrepancy between luminous masses in galaxies and clusters of galaxies and the masses inferred

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[9] The International Celestial Reference Frame (ICRF2) is defined by the positions of about 300 extragalactic sources.

from the virial theorem [23, 24], and the dynamics in galactic halos [25, 26]. Even so, it should be noticed that we have not estimated their contributions to such phenomena, which depend on the intrinsic angular momenta of galaxies or clusters of galaxies, and the unknown “universal constant” $I_{\odot}$. Anyway, it is worth mentioning that the relational dragging effect can be separated from dark matter effects: while dark matter acts in the same way whatever the direction of the rotation is, the dragging effect increases the velocities of co-rotating objects but decreases the velocities of the counter-rotating ones, since it depends on the sign of $J_{\circlearrowleft}$ in Eq. (17).

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