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Entanglement and quantum logical gates. Part I.

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Abstract Is it possible to give a *logical characterization* of entanglement and of entanglement-measures in terms of the probabilistic behavior of some gates? This question admits different (positive or negative) answers in the case of different systems of gates and in the case of different classes of density operators. In the first part of this article we investigate possible relations between entanglement-measures and the probabilistic behavior of quantum computational conjunctions.

1 Introduction

Entanglement, the characteristic feature of quantum theory often described as mysterious and potentially paradoxical, represents an essential resource in quantum information (for quantum computers, teleportation, quantum cryptography, semantic applications, etc.).

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As it is well known, quantum computations are performed by (quantum logical) gates: unitary quantum operations that transform pieces of quantum information (mathematically represented as density operators of convenient Hilbert spaces) in a reversible way. We investigate the following question: is it possible to give a *logical characterization* of entanglement and of entanglement-measures in terms of the probabilistic behavior of some gates? This question admits different (positive or negative) answers in the case of different systems of gates and in the case of different classes of density operators. We will first discuss this problem by referring to the behavior of *quantum computational conjunctions*, defined in terms of the Toffoli-gate. Other examples of gates will be considered in the second part of this article.

2 States and gates

Let us first recall some basic definitions. As is well known, the general mathematical environment for quantum computation is the Hilbert space $\mathcal{H}^{(n)} := \underbrace{\mathbb{C}^2 \otimes \ldots \otimes \mathbb{C}^2}_{n-times}$. Any piece of quantum information is represented by a density operator ρ of a space $\mathcal{H}^{(n)}$. A *quregister* (which is a pure state) is represented by a a unit-vector $|\psi\rangle$ of a space $\mathcal{H}^{(n)}$ or, equivalently, by the corresponding density operator $P_{|\psi\rangle}$ (the projection-operator that projects over the closed subspace determined by $|\psi\rangle$). A *qubit* (or *qubit-state*) is a quregister of the space $\mathbb{C}^2$. A *register* (which represents a *certain* piece of information) is an element $|x_1, \ldots, x_n\rangle$ of the canonical orthonormal basis of a space $\mathcal{H}^{(n)}$ (where $x_i \in \{0, 1\}$); a *bit* is a register of $\mathbb{C}^2$. Following a standard convention, we assume that the bit $|1\rangle$ represents the truth-value *Truth*, while the bit $|0\rangle$ represents the truth-value *Falsity*. On this basis, we can identify, in each space $\mathcal{H}^{(n)}$, two special projection-operators $P_1^{(n)}$ and $P_0^{(n)}$ that represent, respectively, the *truth-property* and the *falsity-property*. The truth-property $P_1^{(n)}$ is the projection-operator that projects over the closed subspace spanned by the set of all registers $|x_1, \ldots, x_{n-1}, 1\rangle$; while the falsity-property $P_0^{(n)}$ is the projection-operator that projects over the closed subspace spanned by the set of all registers $|x_1, \ldots, x_{n-1}, 0\rangle$. In this way, truth and falsity are dealt with as mathematical representatives of possible physical properties. Accordingly, by applying the Born-rule, one can naturally define the probability-value $\mathtt{p}(\rho)$ of any density operator ρ of $\mathcal{H}^{(n)}$ as follows:

$$\mathtt{p}(\rho) := \mathrm{tr}(P_1^{(n)}\rho), \text{ where tr is the trace-functional.}$$

Hence, $\mathtt{p}(\rho)$ represents the probability that the information ρ is true [6,5].

We will denote by $\mathfrak{D}(\mathcal{H}^{(n)})$ the set of all density operators of $\mathcal{H}^{(n)}$, while $\mathfrak{D}$ will represent the union $\bigcup_n \{\mathfrak{D}(\mathcal{H}^{(n)})\}$.

Pure pieces of quantum information are processed by *quantum logical gates* (briefly, *gates*): unitary operators that transform quregisters into quregisters in a reversible way. It is expedient to recall the definition of some gates that play a special role both from the computational and from the logical point of view.

Definition 1 *(The Toffoli-gate)*
For any $m, n, p \geq 1$, the *Toffoli-gate* (defined on $\mathcal{H}^{(m+n+p)}$) is the linear operator $\mathtt{T}^{(m,n,p)}$ such that, for every element $|x_1, \ldots, x_m\rangle \otimes |y_1, \ldots, y_n\rangle \otimes |z_1, \ldots, z_p\rangle$ of the canonical basis,

$$\mathtt{T}^{(m,n,p)}|x_1, \ldots, x_m, y_1, \ldots, y_n, z_1, \ldots, z_p\rangle$$
$$= |x_1, \ldots, x_m, y_1, \ldots, y_n, z_1, \ldots, z_{p-1}\rangle \otimes |x_m y_n \widehat{+} z_p\rangle,$$

where $\widehat{+}$ represents the addition modulo 2.

Definition 2 *(The swap-gate)*
For any $m, n \geq 1$, the *swap-gate* (defined on $\mathcal{H}^{(m,n)}$) is the linear operator $\mathtt{SW}^{(m,n)}$ such that, for every element $|x_1, \ldots, x_n\rangle \otimes |y_1, \ldots, y_n\rangle$ of the canonical basis,

$$\mathtt{SW}^{(m,n)}|x_1, \ldots, x_m, y_1, \ldots, y_n\rangle = |y_1, \ldots, y_n, x_1, \ldots, x_m\rangle.$$

All gates can be canonically extended to the set $\mathfrak{D}$ of all density operators. Let $\mathtt{G}$ be any gate defined on $\mathcal{H}^{(n)}$. The corresponding *density-operator gate* (also called *unitary quantum operation*) ${}^{\mathfrak{D}}\mathtt{G}$ is defined as follows for any $\rho \in \mathfrak{D}(\mathcal{H}^{(n)})$:

$${}^{\mathfrak{D}}\mathtt{G}\rho = \mathtt{G}\,\rho\,\mathtt{G}^{\dagger} \ \text{ where } \ \mathtt{G}^{\dagger} \ \text{ is the adjoint of } \ \mathtt{G}.$$

For the sake simplicity (and following the notation introduced in [1]), also the operations ${}^{\mathfrak{D}}\mathtt{G}$ will be briefly called *gates*.

3 Entanglement and entanglement-measures

Consider the product-space $\mathcal{H}^{(m+n)} = \mathcal{H}^{(m)} \otimes \mathcal{H}^{(n)}$. Any $\rho \in \mathfrak{D}(\mathcal{H}^{(m+n)})$ represents a possible state for a composite physical system $S = S_1 + S_2$ (consisting of two subsystems). According to the quantum formalism, ρ determines the two reduced states $Red^{(1)}_{[m,n]}(\rho)$ and $Red^{(2)}_{[m,n]}(\rho)$ that represent the states of S_1 and of S_2 (in the context ρ)[1]. In such a case, we say that ρ is a *a bipartite state* (with respect to the decomposition (m, n)). It may happen that ρ is a pure state, while $Red^{(1)}_{[m,n]}(\rho)$ and $Red^{(2)}_{[m,n]}(\rho)$ are proper mixtures. In this case the information about the whole system is more precise than the pieces of information about its parts.

Definition 3 (*Factorizability, separability and entanglement*)
Let ρ be a bipartite state of $\mathcal{H}^{(m+n)}$ (with respect to the decomposition (m, n)).

1) ρ is called a *(bipartite) factorized state* of $\mathcal{H}^{(m+n)}$ iff $\rho = \rho_1 \otimes \rho_2$, where $\rho_1 \in \mathfrak{D}(\mathcal{H}^{(m)})$ and $\rho_2 \in \mathfrak{D}(\mathcal{H}^{(n)})$;

[1] For an operation definition of reduced state, see [9, Lemma 2.2].

2) ρ is called a *(bipartite) separable state* of $\mathcal{H}^{(m+n)}$ iff $\rho = \sum_i w_i \rho_i$, where each ρ_i is a bipartite factorized state of $\mathcal{H}^{(m+n)}$, $w_i \in [0,1]$ and $\sum_i w_i = 1$;
3) ρ is called a *(bipartite) entangled state* of $\mathcal{H}^{(m+n)}$ iff ρ is not separable.

Definition 4 (*Maximally entangled states and maximally mixed states*)

1) A pure bipartite state ρ of $\mathcal{H}^{(m+n)}$ is called *maximally entangled* iff $Red^{(1)}_{[m,n]} = \frac{1}{2^m}\mathtt{I}^{(m)}$ or $Red^{(2)}_{[m,n]} = \frac{1}{2^n}\mathtt{I}^{(n)}$, where $\mathtt{I}^{(m)}$ and $\mathtt{I}^{(n)}$ are the identity operators of the spaces $\mathcal{H}^{(m)}$ and $\mathcal{H}^{(n)}$, respectively.
2) A state ρ of $\mathcal{H}^{(m+n)}$ is called a *maximally mixed state* of $\mathcal{H}^{(m+n)}$ iff $\rho = \frac{1}{2^{(m+n)}}\mathtt{I}^{(m+n)}$.

How to measure the "entanglement-degree" of a given state? As is well known, different definitions for the concept of *entanglement-measure* (which quantify different aspects of entanglement) have been proposed in the literature.[8] Any normalized entanglement-measure EM defined on the set of all bipartite states of $\mathcal{H}^{(m+n)}$ is supposed to satisfy the following minimal conditions:

1. $EM(\rho) \in [0,1]$;
2. $EM(\rho) = 0$, if ρ is separable;
3. $EM(\rho) = 1$, if ρ is a maximally entangled pure state;
4. $EM(\rho)$ is *invariant* under *locally unitary maps*. This means that: $EM(\rho) = EM((U_1^{(m)} \otimes U_2^{(n)})\rho(U_1^{(m)} \otimes U_2^{(n)})^\dagger)$, for any unitary operators $U_1^{(m)}$ of $\mathcal{H}^{(m)}$ and $U_2^{(n)}$ of $\mathcal{H}^{(n)}$.

We will use here a particular definition of entanglement-measure represented by the notion of *concurrence*.

Definition 5 (*The concurrence of a bipartite state*)

1) Let $P_{|\psi\rangle}$ be a bipartite pure state of $\mathcal{H}^{(m+n)}$. The *concurrence* of $P_{|\psi\rangle}$ is defined as follows:
$$\mathcal{C}(P_{|\psi\rangle}) = \sqrt{2\Big(1 - \sum_i \lambda_i^2\Big)},$$
where the numbers λ_i are eigenvalues of $Red^{(1)}_{[m,n]}(P_{|\psi\rangle})$ (or, equivalently, of $Red^{(2)}_{[m,n]}(P_{|\psi\rangle})$).
2) Let ρ be a bipartite mixed state of $\mathcal{H}^{(m+n)}$. The *concurrence* of ρ is defined as follows:
$$\mathcal{C}(\rho) = \inf\left\{\sum_i w_i \mathcal{C}(P_{|\psi_i\rangle}) : \rho = \sum_i w_i P_{|\psi_i\rangle}\right\}.$$

One can easily show that the concurrence $\mathcal{C}$ satisfies the minimal conditions required for the general notion of entanglement-measure. We have:
$$\mathcal{C}(P_{|\psi\rangle}) = \sqrt{2(1 - \mathrm{tr}(Red^{(1)}_{[m,n]}(P_{|\psi\rangle})Red^{(1)}_{[m,n]}(P_{|\psi\rangle})))}.$$

In the study of entanglement-phenomena it is interesting to isolate a special class of bipartite states represented by the *Werner states*. This notion has been introduced in [12] to show that entangled bipartite states do not necessarily exhibit non-local correlations. It is true that any bipartite state must be entangled in order to produce non-local correlations; at the same time there are examples of entangled Werner states whose correlations satisfy some simple instances of Bell-inequalities.

Definition 6 *(Werner state)*
A *Werner state* is a bipartite state ρ of a space $\mathcal{H}^{(n)} \otimes \mathcal{H}^{(n)}$ that satisfies the following condition for any unitary operator U of $\mathcal{H}^{(n)}$:

$$(U \otimes U)(\rho)(U \otimes U)^{\dagger} = \rho.$$

Hence, any Werner state is invariant under local unitary transformations.

Interestingly enough, one can prove that the class of all Werner states of $\mathcal{H}^{(n)} \otimes \mathcal{H}^{(n)}$ can be represented as a one-parameter manifold of states.

Lemma 1 [9]
Any Werner state of the space $\mathcal{H}^{(2n)}$ can be represented as follows:

$$\rho_w^{(2n)} = \frac{1}{2^{2n}-1}\Big[\left(1-\frac{w}{2^n}\right)\mathtt{I}^{(2n)} + \left(w-\frac{1}{2^n}\right)\mathtt{SW}^{(n,n)}\Big],$$

where $-1 \leq w \leq 1$ (while $\mathtt{I}^{(2n)}$ and $\mathtt{SW}^{(n,n)}$ are the identity operator and the swap-gate of the space $\mathcal{H}^{(2n)}$).

A deep and simple correlation connects the concurrence of a Werner state $\rho_w^{(2n)}$ with the parameter w.

Theorem 1 [2, Eq.33]
$\mathcal{C}(\rho_w^{(2n)}) = \begin{cases} -w, & \textit{if}\, w < 0; \\ 0, & \textit{otherwise}. \end{cases}$

As a consequence, one can easily show that:

1. $\rho_w^{(2n)}$ is separable iff $w \geq 0$.
2. $\rho_w^{(2n)}$ is maximally mixed iff $w = \frac{1}{2^n}$.
3. $\rho_w^{(2n)}$ is pure iff $n = 1$ and $w = -1$;
4. If $\rho_w^{(2n)}$ is pure, then $\rho_w^{(2n)}$ is maximally entangled.

Consider now the case of *two-qubit Werner states* $\rho_w^{(2)}$, which live in the space $\mathbb{C}^2 \otimes \mathbb{C}^2$ and consequently have the following form:

$$\rho_w^{(2)} = \frac{1}{3}\Big[\left(1-\frac{1}{2}w\right)\mathtt{I}^{(2)} + \left(w-\frac{1}{2}\right)\mathtt{SW}^{(1,1)}\Big].$$

We obtain:

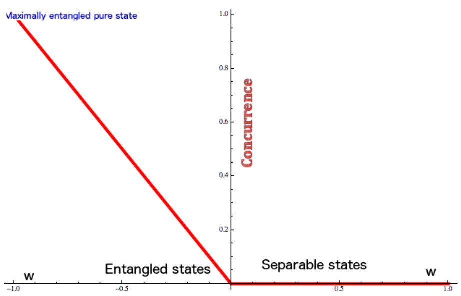

Fig. 1

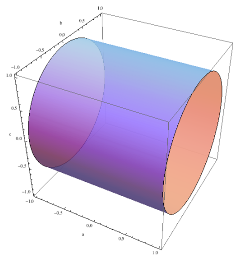

Fig. 2 The class of all $\rho_{(abc)}$ states

1. $\rho_w^{(2)}$ is separable iff $w \geq 0$;
2. $\rho_w^{(2)}$ is maximally mixed iff $w = \frac{1}{2}$ iff $\rho_w^{(2)}$ is factorized;
3. $\rho_w^{(2)}$ is a maximally entangled pure state iff $w = -1$ iff $\rho_w^{(2)}$ is the *Bell state* $P_{\frac{1}{\sqrt{2}}(|0,1\rangle - |1,0\rangle)}$.

The concurrence of an arbitrary two-qubit Werner state in terms of the parameter w is depicted in Figure 1.

Another interesting subclass of the class of all two-qubit states is the class of all *three-parameter qubit states* $\rho_{_{(abc)}}$, defined as follows:

$$\rho_{_{(abc)}} = \frac{1}{4}\begin{pmatrix} 1+a & 0 & 0 & 0 \\ 0 & 1-b & ic & 0 \\ 0 & -ic & 1+b & 0 \\ 0 & 0 & 0 & 1-a \end{pmatrix},$$

where a, b, c are three real numbers such that $a^2 \leq 1$ and $b^2 + c^2 \leq 1$. One can prove that a state $\rho_{_{(abc)}}$ is separable iff $a^2 + c^2 \leq 1$.

4 Entanglement and quantum computational conjunctions

As is well known, the Toffoli-gate allows us to define a quantum computational conjunction that behaves according to classical logic, whenever it is applied

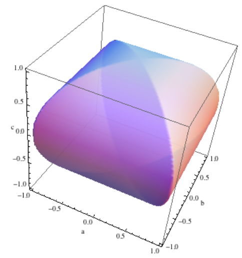

Fig. 3 The class of all **separable** $\rho_{(abc)}$ states

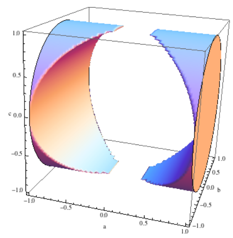

Fig. 4 The class of all **entangled** $\rho_{(abc)}$ states

to certain pieces of information (bits and registers). At the same time this conjunction may have some deeply anti-classical features when the inputs are uncertain pieces of quantum information. The most peculiar property is represented by a *holistic* behavior: generally the conjunction defined on a global piece of quantum information (represented by a given density operator) cannot be described as a function of its separate parts. This is the reason why the quantum computational conjunction is called "holistic".

Definition 7 (*The holistic conjunction*)
For any $m, n \geq 1$ and for any density operator $\rho \in \mathfrak{D}(\mathcal{H}^{(m+n)})$, the *holistic conjunction* $\mathtt{AND}^{(m,n)}$ is defined as follows:

$$\mathtt{AND}^{(m,n)}(\rho) = {}^{\mathfrak{D}}\mathtt{T}^{(m,n,1)}(\rho \otimes P_0^{(1)}).$$

Clearly, the falsity-property $P_0^{(1)}$ plays, in this definition, the role of an *ancilla.* Generally we have:

$$\mathtt{AND}^{(m,n)}(\rho) \neq \mathtt{AND}^{(m,n)}(Red^{(1)}_{[m,n]}(\rho) \otimes Red^{(2)}_{[m,n]}(\rho)).$$

Hence, the holistic conjunction defined on a global information consisting of two parts does not generally coincide with the conjunction of the two separate parts. As an example, consider the following density operator (which corresponds to a *maximally entangled pure state*):

$$\rho = P_{\frac{1}{\sqrt{2}}(|0,0\rangle + |1,1\rangle)}.$$

We have:

$$\mathtt{AND}^{(1,1)}(\rho) = P_{\frac{1}{\sqrt{2}}(|0,0,0\rangle+|1,1,1\rangle)} \ ,$$

which also represents a *maximally entangled pure state*. At the same time we have:

$$\mathtt{AND}^{(1,1)}(Red^{(1)}_{[1,1]}(\rho) \otimes Red^{(2)}_{[1,1]}(\rho)) = \mathtt{AND}^{(1,1)}(\frac{1}{2}\mathtt{I}^{(1)} \otimes \frac{1}{2}\mathtt{I}^{(1)}),$$

which is a proper mixture.

The following theorem sums up the main probabilistic properties of the holistic conjunction.

Theorem 2 [3, § 3]
For any $\rho \in \mathfrak{D}(\mathcal{H}^{(m+n)})$:

1) $\mathrm{p}(\mathtt{AND}^{(m,n)}(\rho)) = \mathrm{tr}((P_1^{(m)} \otimes P_1^{(n)})\rho)$.
2) $\mathrm{p}(\mathtt{AND}^{(m,n)}(\rho)) \leq \mathrm{p}(Red^{(1)}_{[m,n]}(\rho))$ *and* $\mathrm{p}(\mathtt{AND}^{(m,n)}(\rho)) \leq \mathrm{p}(Red^{(2)}_{[m,n]}(\rho))$.

Consequently, for any factorized density operator $\rho = \rho_1 \otimes \rho_2$ of the space $\mathcal{H}^{(m+n)}$ (with $\rho_1 \in \mathfrak{D}(\mathcal{H}^{(m)})$ and $\rho_2 \in \mathfrak{D}(\mathcal{H}^{(n)})$) we have:

$$\mathrm{p}(\mathtt{AND}^{(m,n)}(\rho)) = \mathrm{p}(Red^{(1)}_{[m,n]}(\rho))\mathrm{p}(Red^{(2)}_{[m,n]}(\rho)) = \mathrm{p}(\rho_1)\mathrm{p}(\rho_2).$$

Definition 8 (*Probabilistic factorizability*)
A conjunction $\mathtt{AND}^{(m,n)}(\rho)$ is called *probabilistically factorizable* iff $\mathrm{p}(\mathtt{AND}^{(m,n)}(\rho)) = \mathrm{p}(Red^{(1)}_{[m,n]}(\rho))\mathrm{p}(Red^{(2)}_{[m,n]}(\rho))$.

Is there any interesting correlation between the probabilistic factorizability of $\mathtt{AND}^{(m,n)}(\rho)$, the factorizability of ρ and the separability of ρ?

Of course, we have:

$$\rho \text{ is factorized } \implies \mathtt{AND}^{(m,n)}(\rho) \text{ is probabilistically factorizable.}$$

The following questions arise:

1. $\mathtt{AND}^{(m,n)}(\rho)$ is probabilistically factorizable $\Rightarrow$ ρ is factorized?
2. $\mathtt{AND}^{(m,n)}(\rho)$ is probabilistically factorizable $\Rightarrow$ ρ is separable?
3. ρ is separable $\Rightarrow$ $\mathtt{AND}^{(m,n)}(\rho)$ is probabilistically factorizable?

These three questions have a negative answer. As to our third question, consider the following counterexample: take a Werner state $\rho_w^{(2)}$ such that $0 \leq w \leq 1$ and $w \neq \frac{1}{2}$. Thus, $\rho_w^{(2)}$ is a separable, non-factorized state since the only factorized Werner state is $\rho_{1/2}^{(2)}$. At the same time, $\mathtt{AND}^{(m,n)}(\rho_w^{(2)})$ is not probabilistically factorizable, because:

$$\mathrm{p}(\mathtt{AND}^{(m,n)}(\rho_w^{(2)})) = \frac{1+w}{6} \neq \frac{1}{4} = \mathrm{p}(Red^{(1)}_{[m,n]}(\rho_w^{(2)}))\mathrm{p}(Red^{(2)}_{[m,n]}(\rho_w^{(2)})).$$

Negative answers to our first and to our second question can be obtained as a consequence of the following general theorem.

Theorem 3
For any $\epsilon \in (0,1] \subset \mathbb{R}$ and for any $m, n \geq 1$, there exists a bipartite pure state $P_{|\psi_\epsilon\rangle}$ of $\mathcal{H}^{(m+n)}$ such that:
1) $P_{|\psi_\epsilon\rangle}$ is entangled;
2) $\mathcal{C}(P_{|\psi_\epsilon\rangle}) = \epsilon$;
3) $\mathrm{p}(P_{|\psi_\epsilon\rangle}) = \frac{1}{2}$;
4) $\mathrm{p}(\mathtt{AND}^{(m,n)}(P_{|\psi_\epsilon\rangle})) = \mathrm{p}(Red^{(1)}_{[m,n]}(P_{|\psi_\epsilon\rangle}))\mathrm{p}(Red^{(2)}_{[m,n]}(P_{|\psi_\epsilon\rangle}))$.

Proof
1)-2) Let $\epsilon \in (0,1]$ and let $k_\epsilon := \sqrt{1-\epsilon^2}$. Define the numbers a_{00} and a_{11} as follows:

$$a_{00} := \frac{1}{\sqrt{2}}\sqrt{1-k_\epsilon} \qquad \text{and} \qquad a_{11} := \frac{1}{\sqrt{2}}\sqrt{1+k_\epsilon}.$$

We have: $k_\epsilon \in [0,1)$ and $a_{00}, a_{11} \in (0,1)$. Consider the following pure state:

$$|\varphi_\epsilon\rangle = a_{00}|\underbrace{0,\ldots,0}_{m+n}\rangle + a_{11}|\underbrace{0,\ldots,0,1}_{m},\underbrace{0,\ldots,0,1}_{n}\rangle.$$

One can easily show that:

$$\mathcal{C}(P_{|\varphi_\epsilon\rangle}) = 2|a_{00}a_{11}| = \epsilon.$$

Since $0 < \epsilon \leq 1$, the state $|\varphi_\epsilon\rangle$ turns out to be entangled. Consider now the state

$$|\psi_\epsilon\rangle := (\mathtt{I}^{(m+n-1)} \otimes \sqrt{\mathtt{I}}^{(1)})|\varphi_\epsilon\rangle.$$

An easy computation shows that:

$$|\psi_\epsilon\rangle := \tfrac{1}{2}\sqrt{1-k_\epsilon}|\underbrace{0,\ldots,0}_{m},\underbrace{0,\ldots,0}_{n}\rangle + \tfrac{1}{2}\sqrt{1-k_\epsilon}|\underbrace{0,\ldots,0}_{m},\underbrace{0,\ldots,0,1}_{n}\rangle$$
$$+ \tfrac{1}{2}\sqrt{1+k_\epsilon}|\underbrace{0,\ldots,0,1}_{m},\underbrace{0,\ldots,0}_{n}\rangle - \tfrac{1}{2}\sqrt{1+k_\epsilon}|\underbrace{0,\ldots,0,1}_{m},\underbrace{0,\ldots,0,1}_{n}\rangle.$$

Since $\mathtt{I}^{(m+n-1)} \otimes \sqrt{\mathtt{I}}^{(1)}$ is a locally unitary map and concurrence is invariant under locally unitary maps, the state $|\psi_\epsilon\rangle$ is entangled and its concurrence is ϵ.
3) An easy computation shows that:

$$\mathrm{p}(P_{|\psi_\epsilon\rangle})) = \frac{1}{4}(1-k_\epsilon+1+k_\epsilon) = \frac{1}{2}.$$

(4) By Theorem 2,

$$\mathrm{p}(\mathtt{AND}^{(m,n)}(P_{|\psi_\epsilon\rangle})) = \mathrm{tr}((P_1^{(m)} \otimes P_1^{(n)})P_{|\psi_\epsilon\rangle}) = \frac{1}{4}(1+k_\epsilon).$$

It turns out that:

$$\mathrm{p}(Red^{(1)}_{[m,n]}(P_{|\psi_\epsilon\rangle})) = \frac{1}{2}(1+k_\epsilon) \quad \text{and} \quad \mathrm{p}(Red^{(2)}_{[m,n]}(P_{|\psi_\epsilon\rangle})) = \frac{1}{2}.$$

Hence,

$$\mathrm{p}(\mathtt{AND}^{(m,n)}(P_{|\psi_\epsilon\rangle})) = \mathrm{p}(Red^{(1)}_{[m,n]}(P_{|\psi_\epsilon\rangle}))\mathrm{p}(Red^{(2)}_{[1,1]}(P_{|\psi_\epsilon\rangle})).$$

Consequently, there are infinitely many entangled pure states $P_{|\psi_\epsilon\rangle}$, whose holistic conjunction is probabilistically factorizable. On this basis we can conclude that the probabilistic behavior of holistic conjunctions cannot characterize either entanglement or entanglement-measures.

In spite of these general negative results, some interesting correlations between entanglement, entanglement-measures and the probabilistic behavior of holistic conjunctions can be found in the case of Werner states and in the case of three-parameter qubit states.

Theorem 4
Let $\rho_w^{(2n)}$ be a Werner state of $\mathcal{H}^{(2n)}$.

1) $\mathrm{p}(\rho_w^{(2n)}) = \frac{1}{2}$*;*

2) $\mathrm{p}(\mathtt{AND}^{(n,n)}(\rho_w^{(2n)})) = \frac{2^{2n-1}+2^{n-1}w-1}{2(2^{2n}-1)}$*;*

3) $\mathrm{p}(Red^{(1)}_{[n,n]}(\rho_w^{(2n)})) = \mathrm{p}(Red^{(2)}_{[n,n]}(\rho_w^{(2n)})) = \frac{1}{2}$*;*

Proof
1) $\mathrm{p}(\rho_w^{(2n)}) = \mathrm{tr}(P^{(2n)}\rho_w^{(2n)}) = \frac{1}{2}\Big[\sum_{i=1}^{2^{2n-1}}\left(1-\frac{w}{2^n}\right) + \sum_{i=1}^{2^{n-1}}\left(w-\frac{1}{2^n}\right)\Big] = \frac{1}{2}$.
2) $\mathrm{p}(\mathtt{AND}^{(n,n)}(\rho_w^{(2n)})) = \mathrm{tr}(P^{(2n+1)}\,{}^{\mathfrak{D}}\mathtt{T}^{(n,n,1)}(\rho_w \otimes P_0^{(1)})) = \mathrm{tr}((P_1^{(n)} \otimes P_1^{(n)})\rho_w) = \frac{1}{2^{2n}-1}\Big[\sum_{i=1}^{2^{2(n-1)}}\left(1-\frac{w}{2^n}\right) + \sum_{i=1}^{2^{n-1}}\left(w-\frac{1}{2^n}\right)\Big] = \frac{2^{2n-1}+2^{n-1}w-1}{2(2^{2n}-1)}$.
3) In a similar way.

As a consequence, we obtain:

$$\mathtt{AND}^{(n,n)}(\rho_w^{(2n)}) \text{ is probabilistically factorizable iff}$$

$$\mathrm{p}(\mathtt{AND}^{(n,n)}(\rho_w^{(2n)})) = \frac{1}{4} \text{ iff } w = \frac{1}{2^n} \text{ iff } \rho_w^{(2n)} = \frac{1}{2^{2n}}\mathtt{I}^{(2n)}.$$

Hence, the holistic conjunction of a Werner state $\rho_w^{(2n)}$ is probabilistically factorizable iff $\rho_w^{(2n)}$ is the maximally mixed state $\frac{1}{2^{2n}}\mathtt{I}^{(2n)}$. Accordingly, the class of all Werner states $\rho_w^{(2n)}$ for which $\mathtt{AND}^{(n,n)}(\rho_w^{(2n)})$ is probabilistically factorizable coincides with the singleton of the maximally mixed state. Whenever $\rho_w^{(2n)}$ is a non-factorized, separable Werner state, its holistic conjunction $\mathtt{AND}^{(n,n)}(\rho_w^{(2n)})$ cannot be probabilistically factorizable (because $\frac{1}{2^{2n}}\mathtt{I}^{(2n)}$ is the only factorized Werner state of $\mathcal{H}^{(2n)}$).

A different situation arises in the case of three-parameter two-qubit states $\rho_{(abc)}$. As we already know, any $\rho_{(abc)}$ satisfies the following condition:

$$\rho_{(abc)} \text{ is separable iff } a^2 + c^2 \leq 1.$$

Furthermore we have:

1. $\mathrm{p}(\rho_{(abc)}) = \frac{1-a}{4}$;
2. $\mathrm{p}(Red^{(1)}_{[1,1]}(\rho_{(abc)})) = \frac{1}{2} - \frac{a+b}{4}$;

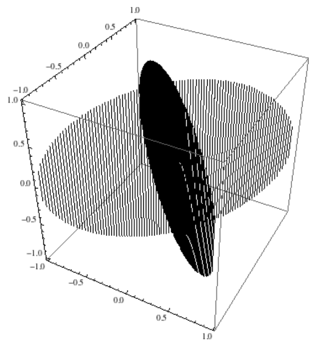

Fig. 5 The class of all **probabilistically factorizable** $\mathtt{AND}^{(1,1)}\rho_{(abc)}$'s

3. $\mathrm{p}(Red^{(2)}_{[1,1]}(\rho_{(abc)})) = \frac{1}{2} - \frac{a-b}{4}$.

Theorem 5

(1) $\mathtt{AND}^{(1,1)}(\rho_{(abc)})$ *is probabilistically factorizable iff* $a^2 = b^2$ *.*

(2) If $\mathtt{AND}^{(1,1)}(\rho_{(abc)})$ *is probabilistically factorizable, then* $\rho_{(abc)}$ *is separable.*

(3) If $\mathtt{AND}^{(1,1)}(\rho_{(abc)})$ *is probabilistically factorizable and* $c \neq 0$*, then* $\rho_{(abc)}$ *is separable but not factorized.*

Proof
(1) $\mathtt{AND}^{(1,1)}(\rho_{(abc)})$ is probabilistically factorizable iff
$\frac{1-a}{4} = \mathrm{p}(\rho_{(abc)}) = \mathrm{p}(Red^{(1)}_{[1,1]}(\rho_{(abc)}))\mathrm{p}(Red^{(2)}_{[1,1]}(\rho_{(abc)})) =$
$\dfrac{1}{16}(-2+a-b)(-2+a+b))$ iff $a^2 = b^2$.
(2) If $\mathtt{AND}^{(1,1)}(\rho_{(abc)})$ is probabilistically factorizable, then, by (1), $a^2 = b^2$. By definition of $\rho_{(abc)}$, $b^2 + c^2 \leq 1$. Thus, $a^2 + c^2 \leq 1$ and therefore $\rho_{(abc)}$ is separable.
(3) Suppose that $\mathtt{AND}^{(1,1)}(\rho_{(abc)})$ is probabilistically factorizable. By (2), $\rho_{(abc)}$ is separable. By hypothesis, $c \neq 0$. Thus, $\rho_{(abc)}$ cannot be factorized since every factorized $\rho_{(abc)}$ is a diagonal matrix (with $c = 0$).

By Theorem 5(3), we can conclude that the class of all $\rho_{(abc)}$'s for which $\mathtt{AND}^{(1,1)}(\rho_{(abc)})$ is probabilistically factorizable contains separable states that are not factorized.

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