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A Note on Bounded Solutions of a Generalized Lienard Type System

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Abstract

In this paper we study the boundedness of solutions of some generalized Liénard type system under non usual conditions on evolved functions using the Second Method of Lyapunov.

Keywords: Boundedness; asymptotic behavior; Liénard equation Type.

1. Introduction

The Liapunov´s Second Method (o Direct) has long been recognized as the most general method for the study of the stability of equilibrium points of systems described by differential equations. The method was first presented in his classical memoir, which appeared in Russian in 1892 and was translated into French in 1907 and English in 1949 [1]. Statements and proofs of mathematical results underlying the method and numerous examples and references can be found in the books of Antosiewicz [2], Barbashin and Krasovskii [3], Cesari [4], Demidovich [5], Hahn [6] and Yoshizawa [7] and bibliography listed therein. His method is a powerful tool because his simplicity for the research of the stability.

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For instance, if within a neighborhood of the equilibrium point an appropriate "energy" function is always decreasing we expect the equilibrium to be asymptotically stable. The Second Method generalizes this idea, if such function $\boldsymbol{V}\,(\boldsymbol{t};\boldsymbol{x})$, called Liapunov Function, can be constructed for the system (or equation) in a neighborhood of the equilibrium point and if in that neighborhood $\boldsymbol{V'}(\boldsymbol{t},\boldsymbol{x}) \leq \boldsymbol{0}$ for $\boldsymbol{x} \neq \boldsymbol{0}$ being $\boldsymbol{V}\,(\boldsymbol{t};\ \boldsymbol{x})$ positive defined, then the equilibrium point is asymptotically stable. If one knows only that $\boldsymbol{V'}(\boldsymbol{t},\boldsymbol{x}) \leq \boldsymbol{0}$ for $\boldsymbol{x} \neq \boldsymbol{0}$, then, in general, one can conclude only the origin is stable.

The above need some clarifications. $\boldsymbol{V}\,(\boldsymbol{t};\ \boldsymbol{x})$ denote an arbitrary Liapunov`s Function defined on an open set $\boldsymbol{S} \subset \boldsymbol{IxR^m}$ with continuous partial derivatives with respect to all arguments, corresponding to $\boldsymbol{V}\,(\boldsymbol{t};\ \boldsymbol{x})$; and we define the function

$$V'_{(1)}(t,x) \coloneqq \lim_{h\to 0^+} sup \frac{V\big(t+h, x+hf(t,x)\big) - V(t,x)}{h},$$

called the total derivative of $V\ (t;\ x)$ for system (1). Under the above conditions,

$$V'_{(1)}(t,x) = \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(t,x).$$

The main difficulty is to construct a suitable $V(t,x)$, this requires experience and technique.

The classical Liénard equation

$$x'' + f(x)x' + g(x) = 0,$$

Appears as simplified model in many domains in science and engineering [8]. It was intensively studied during the first half of 20th century as it can be used to model oscillating circuits or simple pendulums. In the simple pendulum case, f and g represents the friction and acceleration terms. One of the first models where this equation appears was introduced by Balthasar Van del Pol [9,10], considering the equation

$$x'' + \mu(x^2 - 1)x' + x = 0,$$

for modeling the oscillations of a triode vacuum tube. The Liénard equation, which is often taken as the typical example of nonlinear self-excited vibration problem, can be used to model resistor-inductor-capacitor circuits with nonlinear circuit elements. It can also be used to model certain mechanical systems which contain the nonlinear damping coefficients and the restoring force or stiffness. Moreover, some nonlinear evolution equations (such as the Burguers - Korteweg - de Vries equation) which arise from various physical phenomena can also be transformed to equation (3). Therefore, the study of equation (3) is of physical significance.

We recommend [11] for other references about more applications.

The main purpose of this note is to present two results about the existence of bounded solutions of system

$$x' = \alpha(y) - \beta(y)F(x), \tag{1}$$

$$y' = -g(x),$$

where $\alpha(y), \beta(y), f(x)$ y $g(x)$ are continuous functions that satisfies the following conditions:

$$xf(x) > 0, xg(x) > 0, y\alpha(y) > 0, y\beta(y) > 0 \tag{2}$$

for all $x, y \neq 0$ and $F(x) = \int_0^x f(s)ds$.

Clearly classical Liénard equation is a particular case of system (1).

2. Results

Theorem 1. Let the condition (2) be satisfied. Then every solution $(x(t), y(t))$ of (1), with both components positives in $t\epsilon[T-\delta, T)$ is continuable to the right of T.

Proof. Now we consider the following function

$$V(x, y) = A(y) + G(x)$$

with $A(y) = \int_0^y \alpha(y)ds,\ G(x) = \int_0^x g(s)ds$. By the conditions imposed on $\alpha(y)$ and $f(x)$ we obtain $V(x, y) \geq 0, \forall x, y \neq 0$. To verify that V is the required Lyapunov function we derive throughout the system, and we have:

$$V'_{(1)}(t, x) = \alpha(y)y' + g(x)x' - \beta(y)g(x)F(x) < 0.$$

For $t\epsilon[T-\delta, T)$. By integration we get

$$V(t) = V(T-\delta) - \int_{T-\delta}^{t} \beta\big(y(s)\big)g\big(x(s)\big)F\big(x(s)\big)ds \leq V(T-\delta).$$

Then

$$0 < A(y) + G(x) \leq V(t-\delta) \Longrightarrow |y| \leq k, siendo\ k = cte.$$

Integrating the first equation of the system (1.1), we get:

$$x(t) = x(T-\delta) + \int_{T-\delta}^{t} \alpha(s)ds - \int_{T-\delta}^{t} \beta\big(y(s)\big)F\big(x(s)\big)ds,$$

with $\int_{T-\delta}^{t} \alpha(s)ds \leq c, (c = cte)$ because y is bounded. Therefore,

$$0 < x(t) = h - \int_{T-\delta}^{t} \beta(y(s))F(x(s))ds \le h - b\int_{T-\delta}^{t} F(x(s))ds,$$

with $h = k + c$.

If $\int_{T-\delta}^{t} F(x(s))ds = \infty$ we have $x(t) < 0$ in $[T-\delta, T)$. Which contradicts the assumption that $x(t)$ is positive in $[T-\delta, T)$. Hence x is bounded. The proof of the Theorem is complete.■

Theorem 2. Let the condition (2) be satisfied and suppose that $\alpha'(y) \in C^1(R)$ with $\alpha'(y) > 0, |\beta(y)| \le M$ for all y, $g(x)$ is a bounded function on R and $F(-\infty) < \infty$. Then every solution $(x(t), y(t))$ of (1) which exists and is negative on $[T-\delta, T)$, $T < 0$, is continuable to the right of T.

Proof. Let $x(t)$ be a solution with negative values in $[T-\delta, T)$. Hence

$$\left(x'(t) + \beta(y)F(x)\right)' = \left(\alpha(y)\right)' = \alpha'(y)y' = -\alpha'(y)g(x) > 0.$$

Then $x'(t) + \beta(y)F(x)$ is increasing in $[T-\delta, T)$. Therefore

$$x'(t_0) + \beta(y(t_0))F(x(t_0)) < x'(t) + \beta(y(t))F(x(t)) \le x'(t) + \beta(y(t))K < x'(t)$$

In $t_0 \le t < T$ because $F(x(t)) \le K < \infty$. So, we have

$$(x'(t_0) + \beta(y(t_0))F(x(t_0)) < x'(t)$$

which implies that $x'(t)$ is bounded from below on $[T-\delta, T)$. Integrating the last inequality, we have:

$$(x'(t_0) + \beta(y(t_0))F(x(t_0))(t - t_0) \le x(t) - x(t_0)$$

then, we conclude

$$(x'(t_0) + \beta(y(t_0))F(x(t_0))(t - t_0) + x(t_0) \le x(t) < 0$$

hence $\lim_{t\to T^-} x(t)$ exists and is finite. In addition

$$\lim_{t\to T^-} x'(t) = \lim_{t\to T^-}\left[x'(t) + \beta(y(t))F(x(t))\right] - \lim_{t\to T^-}\left[\beta(y(t))F(x(t))\right] < \infty.$$

If we suppose that $|g(x)| \le M_2$, it follows immediately that $y'(t)$ is bounded, that is to say

$$-M_2 \le y'(t) \le M_2.$$

Integrating the last inequality, we have

$-M_2(t-t_0) \leq y(t) - y(t_0) \leq M_2(t-t_0)$

in $[t_0,T)$. From which is obtained

$-M_2(t-t_0) + y(t_0) \leq y(t) \leq M_2(t-t_0) + y(t_0)$.

Then the proof is complete.■

Corollary. Under assumptions on Theorems 1 and 2 every solution $(x(t), y(t))$ of (1) is continuable to the right and can be defined on an infinite interval $[t_0,\infty)$.

3. Conclusion

In this section we present some observations in conclusion.

Remark 1. *If in (1) we have*

$x' = y + 2x,$

(3)

$y' = -x.$

the Theorem 1 is very clear, because this system has an unbounded solution

$\big(x(t), y(t)\big) = (e^t, -e^t)$.

Remark 2. *Our results contains as particular case, the Theorem 2.1 and 2.6 of* [12].

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