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Technical Note

Free vibrations of a cantilever beam with a spring–mass system attached to the free end

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Abstract

An exact solution for the title problem is obtained using the Bernoulli–Euler theory of beam vibrations. Natural frequencies are obtained for a wide range of the intervening physical parameters. The problem is of interest in naval and ocean engineering systems since in order to avoid dangerous resonance conditions the designer must be able to predict natural frequencies of the overall mechanical system: structure–motor and its elastic mounting. © 2001 Elsevier Science Ltd. All rights reserved.

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1. Introduction

Transverse vibrations of a cantilever beam carrying a concentrated mass have been studied by several researchers (Haener, 1958; Lee, 1973; Laura et al., 1974). All these studies deal with the situation where the mass is rigidly attached to the beam tip.

However, in many instances the mass is elastically attached to the structural element, the reason for this being the fact that a motor or engine is mounted on an elastic foundation or simply because the connection of the mass to the beam does possess elastic properties (Laura et al., 1977).

The present study deals with the analysis of the mechanical arrangement depicted

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in Fig. 1 and it was motivated by the preliminary design of a cantilever structure supporting a motor in the case of an offshore platform.

2. Analysis of the problem

It is assumed that the problem under consideration is governed by the classical Bernoulli–Euler theory. Accordingly the differential system is described by the partial differential equation

$$\frac{\partial^4 w}{\partial \bar{x}^4}+\frac{\rho A_0}{EI}\frac{\partial^2 w}{\partial t^2}=0 \tag{1}$$

which in the case of normal modes becomes

$$\frac{d^4 W}{d\bar{x}^4}-\frac{\rho A_0}{EI}\omega^2 W=0 \tag{2}$$

where $w(\bar{x},t)=W(\bar{x})e^{i\omega t}$.The boundary conditions at $x=L$ are

$$W(L)=\frac{dW}{dx}(L)=0 \tag{3}$$

while at $\bar{x}=0$ one has

$$\frac{d^2W}{d\bar{x}^2}(0)=0 \tag{4a}$$

and

$$-EI\frac{d^3W}{d\bar{x}^3}(L)=F \tag{4b}$$

where F is the force transmitted to the beam through the spring defined by its constant k.

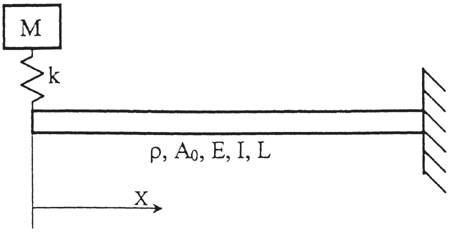

Fig. 1. Structural system under study.

Referring to Fig. 2 one determines F in the following fashion (Laura et al., 1977). Let z_1 be the displacement of the mass M and z_2 the one corresponding to the other end of the spring. Accordingly one has

$$M\frac{d^2z_1}{dt^2}=k(z_2-z_1) \tag{5}$$

Defining

$$z=z_2-z_1 \tag{6}$$

and replacing in Eq. (5) one obtains

$$M\frac{d^2z_2}{dt^2}=kz+M\frac{d^2z}{dt^2} \tag{7}$$

and since

$$z_2=W(0)\,e^{i\omega t} \tag{8}$$

substituting in Eq. (7) results in

$$M\frac{d^2z}{dt^2}+kz=-\omega^2MW(0)\,e^{i\omega t} \tag{9}$$

whose particular solution is

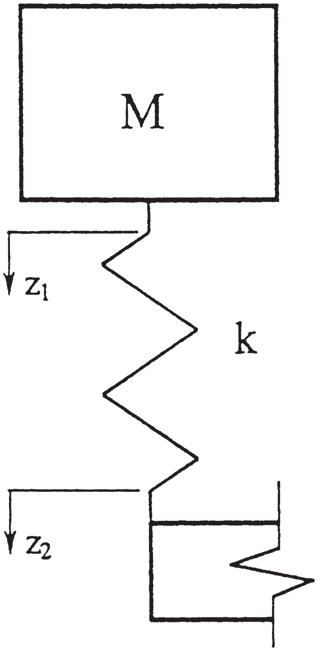

Fig. 2. Analysis of the discrete system attached to the beam tip.

$$z=\frac{\omega^2 MW(0)}{\omega^2 M-k}\,e^{i\omega t} \tag{10}$$

Accordingly

$$F=kz=\frac{\omega^2 MW(0)}{\omega^2\frac{M}{k}-1}\,e^{i\omega t} \tag{11}$$

and the boundary condition Eq. (4b) becomes

$$-EI\frac{d^3W}{dx^3}(0)=-\frac{\omega^2 MW(0)}{1-\omega^2\frac{M}{k}} \tag{12}$$

The solution of Eq. (2) is:

$$W(x)=A\cos\beta x+B\sin\beta x+C\cosh\beta x+D\sinh\beta x \tag{13}$$

where

$$x=\frac{\bar{x}}{L}\text{ and }\beta^2=\sqrt{\frac{\rho A_0}{EI}}\,\omega L^2$$

Substituting Eq. (13) in the governing boundary conditions one obtains the determinantal equation

$$\begin{vmatrix} -1 & 0 & 1 & 0 \\ \dfrac{m\beta^4}{1-\dfrac{m\beta^4}{r}} & 1 & \dfrac{m\beta}{1-\dfrac{m\beta^4}{r}} & -1 \\ \cos\beta & \sin\beta & \cosh\beta & \sinh\beta \\ -\sin\beta & \cos\beta & \sinh\beta & \cosh\beta \end{vmatrix}=0 \tag{14}$$

where

$$m=\frac{M}{\rho A_0 L}\text{ and }r=\frac{k}{\frac{EI}{L^3}}.$$

Expansion of the determinantal Eq. (14) leads to the following transcendental expression in the frequency coefficients

$$-\cos^2\beta-2\cos\beta\cosh\beta-\cosh^2\beta+\frac{2m\beta\cosh\beta\sin\beta}{1-\dfrac{m\beta^4}{r}}-$$

Table 1
Values of β_i (i=1, . . ., 6) for r=0.1

M	0.2	0.4	0.6	0.8	1.0	1.2	1.4	5	10
β_1	0.83377	0.70122	0.63366	0.58970	0.55772	0.53287	0.51273	0.37299	0.31364
β_2	1.8907	1.8904	1.8903	1.8902	1.8902	1.8902	1.8902	1.8901	1.8901
β_3	4.6951	4.6951	4.6951	4.6951	4.6951	4.6951	4.6951	4.6951	4.6951
β_4	7.8550	7.8550	7.8550	7.8550	7.8550	7.8550	7.8550	7.8550	7.8550
β_5	10.996	10.996	10.996	10.996	10.996	10.996	10.996	10.996	10.996
β_6	14.137	14.137	14.137	14.137	14.137	14.137	14.137	14.137	14.137

$$-\sin^2\beta-\frac{2m\beta\cos\beta\sinh\beta}{1-\frac{m\beta^4}{r}}+\sinh\beta=0 \tag{15}$$

3. Numerical results

The algorithmic procedure was first applied to the case of the bare cantilever beam and excellent agreement with the exact results available in the literature was achieved.

Table 1 depicts the square root of the first six frequency coefficients, $\sqrt[4]{\rho A_0/EI}\,\omega_i^{1/2}L$, for r=0.1 and m=0.2, 0.4, 0.6, 0.8, 1, 1.2, 1.4, 5 and 10. The first line of the Table contains the frequency coefficients of the discrete system altered by the presence of the beam. The remaining five lines show the frequency coefficients of the continuous system modified by the presence of the spring–mass system. As expected: the parameter m carries considerable weight on the first frequency coefficient, very little on the second and is practically negligible on the remaining.

Table 2 deals with the case r=1 for the same values of m. The value of m possesses influence now upon the first three frequency coefficients although it is very small upon the third natural frequency.

Table 2
Values of β_i (i=1, . . ., 6) for r=1

m	0.2	0.4	0.6	0.8	1.0	1.2	1.4	5	10
β_1	1.3609	1.1584	1.0505	0.97925	0.92705	0.88633	0.85322	0.62187	0.52312
β_2	2.0553	2.0305	2.0232	2.0197	2.0177	2.0164	2.0154	2.0115	2.0107
β_3	4.7039	4.7038	4.7038	4.7038	4.7038	4.7038	4.7038	4.7038	4.7038
β_4	7.8568	7.8568	7.8568	7.8568	7.8568	7.8568	7.8568	7.8568	7.8568
β_5	10.996	10.996	10.996	10.996	10.996	10.996	10.996	10.996	10.996
β_6	14.138	14.138	14.138	14.138	14.138	14.138	14.138	14.138	14.138

Table 3
Values of β_i (i=1, . . ., 6) for r=10

m	0.2	0.4	0.6	0.8	1.0	1.2	1.4	5	10
β_1	1.5907	1.4296	1.3249	1.2495	1.1914	1.1446	1.1057	0.81854	0.69069
β_2	3.0508	2.8577	2.7873	2.7510	2.7289	2.7141	2.7034	2.6571	2.6480
β_3	4.8041	4.7988	4.7971	4.7962	4.7957	4.7954	4.7952	4.7942	4.7940
β_4	7.8759	7.8758	7.8757	7.8757	7.8757	7.8757	7.8757	7.8757	7.8757
β_5	11.003	11.003	11.003	11.003	11.003	11.003	11.003	11.003	11.003
β_6	14.141	14.141	14.141	14.141	14.141	14.141	14.141	14.141	14.141

In the case of Table 3 (r=10) the value of m influences the first four frequency coefficients (admittedly: very small on the fourth frequency coefficient). The second frequency coefficient corresponds now to the spring–mass system modified by the presence of the beam.

Finally the situation where the mass is rigidly attached to the structural element is considered in Table 4. Obviously the first frequency coefficient corresponds now to the beam-concentrated mass system (the discrete system does possess, now, infinite natural frequency). The eigenvalues are in excellent agreement with those obtained in the literature (Laura et al., 1974).

Modal shapes for m=1 and r=0.1, 1 and 10 are depicted in Fig. 3. For case (a), r=0.1, the beam exhibits very small displacement amplitude in correspondence with the first mode since the discrete system vibrates considerably.

When r=1, case (b), the beam in its first mode acquires considerable vibrational behavior when compared with case (a). The beam motion is largely amplified when r=10, case (c). For this situation the discrete system exhibits minor independent dynamic performance.

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Table 4
Values of β_i (i=1, 2, . . ., 5) for $r\rightarrow\infty$

m	0.2	0.4	0.6	0.8	1.0	1.2	1.4	5	10
β_1	1.6164	1.4724	1.3757	1.3041	1.2479	1.2021	1.1636	0.87002	0.73578
β_2	4.2671	4.1444	4.0866	4.0531	4.0311	4.0157	4.0042	3.9500	3.9385
β_3	7.3184	7.2155	7.1725	7.1490	7.1341	7.1239	7.1164	7.0825	7.0756
β_4	10.402	10.318	10.285	10.267	10.257	10.249	10.244	10.220	10.215
β_5	13.507	13.437	13.410	13.396	13.388	13.382	13.378	13.359	13.355

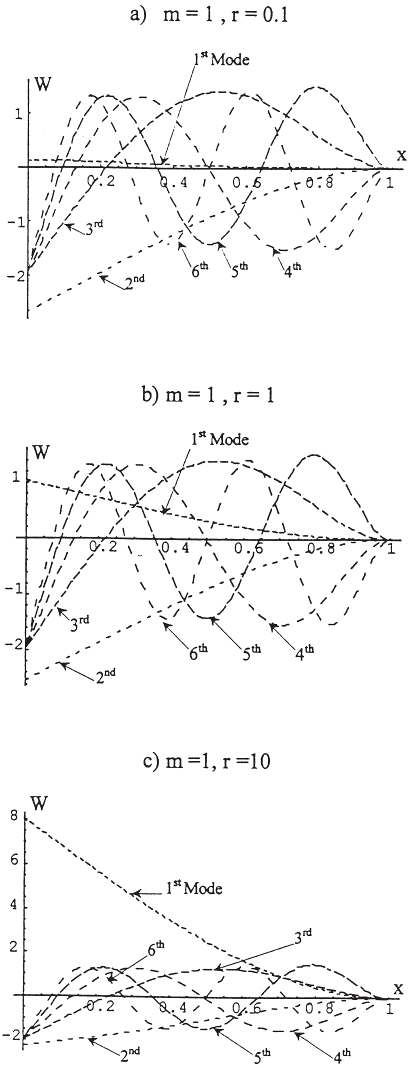

Fig. 3. Modal shapes for m=1 and r=0.1, 1 and 10.

References

Haener, J., 1958. Formulas for the frequencies including higher frequencies of uniform cantilever and free–free beams with additional masses at the ends. Journal of Applied Mechanics 25, 412.

Laura, P.A.A., Pombo, J.L., Susemihl, E.A., 1974. A note on the vibrations of a clamped–free beam with a mass at the free end. Journal of Sound and Vibration 37, 161–168.

Laura, P.A.A., Susemihl, E.A., Pombo, J.L., Luisoni, L.E., Gelos, R., 1977. On the dynamic behaviour of structural elements carrying elastically mounted concentrated masses. Applied Acoustics 10, 121–145.

Lee, T.W., 1973. Vibration frequency for a uniform beam with one end spring hinged and carrying a mass at the other free end. Journal of Applied Mechanics 95, 813–815.