On some factorizations of operators

Jorge Antezana[a,b,1], M. Laura Arias[a,c,2], Gustavo Corach[a,c,2,*]

[a] *Instituto Argentino de Matemática "Alberto P. Calderón", CONICET Saavedra 15, Piso 3 (1083), Buenos Aires, Argentina.*
[b] *Dpto. de Matemática, FCE-UNLP, La Plata, Argentina*
[c] *Dpto. de Matemática, Facultad de Ingeniería, Universidad de Buenos Aires.*

Abstract

Given two subsets $\mathcal{A}$ and $\mathcal{B}$ of the algebra of bounded linear operators on a Hilbert space $\mathcal{H}$ we denote by $\mathcal{AB} := \{AB : A \in \mathcal{A}, B \in \mathcal{B}\}$. The goal of this article is to describe $\mathcal{AB}$ if $\mathcal{A}$ and $\mathcal{B}$ denote classes of projections, partial isometries, positive (semidefinite) operators, etc. Moreover, fixed $T \in \mathcal{AB}$ we shall describe $(\mathcal{AB})_T := \{(A, B) \in \mathcal{A} \times \mathcal{B} : AB = T\}$, $p_1((\mathcal{AB})_T) := \{A \in \mathcal{A} : T = AB \text{ for some } B \in \mathcal{B}\}$ and $p_2((\mathcal{AB})_T) := \{B \in \mathcal{B} : T = AB \text{ for some } A \in \mathcal{A}\}$.

Keywords: Factorizations, polar decomposition, projections, partial isometries
2010 MSC: 47A05

1. Introduction

Let $\mathcal{H}$ be a Hilbert space and denote by $\mathcal{L}$ the algebra of bounded linear operators on $\mathcal{H}$. The main goal of this paper is the characterization of

$$\mathcal{AB} = \{AB : A \in \mathcal{A}, B \in \mathcal{B}\}, \tag{1}$$

for certain subsets $\mathcal{A}$ and $\mathcal{B}$ of $\mathcal{L}$. Moreover, for $T \in \mathcal{AB}$ we study the set:

$$(\mathcal{AB})_T := \{(A, B) \in \mathcal{A} \times \mathcal{B} : AB = T\},$$

and its natural projections

$$p_1((\mathcal{AB})_T) := \{A \in \mathcal{A} : T = AB \text{ for some } B \in \mathcal{B}\},$$

and

$$p_2((\mathcal{AB})_T) := \{B \in \mathcal{B} : T = AB \text{ for some } A \in \mathcal{A}\}.$$

Of course, it looks impossible to find methods which allow to deal with the problem for general $\mathcal{A}$ and $\mathcal{B}$. We shall show that in many concrete natural cases, the problem is not

* Corresponding author. Address: IAM-CONICET, Saavedra 15, Piso 3 (1083), Buenos Aires, Argentina.
Email addresses: antezana@mate.unlp.edu.ar (Jorge Antezana), lauraarias@conicet.gov.ar (M. Laura Arias), gcorach@fi.uba.ar (Gustavo Corach)

[1] Partially supported by CONICET-PIP 2009-435, UNLP 11X681, MTM2012-36378
[2] Partially supported by PIP 0757(CONICET) and UBACYT 20020130100637

trivial. Observe that, if we know a characterization of $\mathcal{AB}$, then for each $T \in \mathcal{AB}$ determining $(\mathcal{AB})_T$ is a kind of inverse problem. Even if $(\mathcal{AB})_T$ is determined, it looks difficult to choose, among all factorizations $AB = T$, a single one $A_0B_0 = T$ with nice properties (e.g., $||A_0 - B_0|| \le ||A - B||$ for every $(A, B) \in (\mathcal{AB})_T$, or $(A_0 - B_0)^*(A_0 - B_0) \le (A - B)^*(A - B)$ for every $(A, B) \in (\mathcal{AB})_T$). The main goals of our study will be:

(a) Characterize $\mathcal{AB}$.
(b) For $T \in \mathcal{AB}$, characterize $(\mathcal{AB})_T, p_1((\mathcal{AB})_T), p_2((\mathcal{AB})_T)$.
(c) Find a mapping $\phi : \mathcal{AB} \to \mathcal{A} \times \mathcal{B}$ such that $\phi(T)$ belongs to $(\mathcal{AB})_T$, where it has some optimal properties.

We shall restrict our study to the following cases:

i. $\mathcal{A} = \mathcal{J}$ = the set of partial isometries, $\mathcal{B} = \mathcal{L}^+$ = the cone of positive (semi-definite) operators.
ii. $\mathcal{A} = \mathcal{P}$ = the set of Hermitian projections, $\mathcal{B} = \mathcal{U}$ = the group of unitary operators.
iii. $\mathcal{A} = \mathcal{P}$, $\mathcal{B} = \mathcal{G}$ = the group of invertible operators.
iv. $\mathcal{A} = \mathcal{Q}$ = the set of all oblique (i.e., not necessarily Hermitian) projections, $\mathcal{B} = \mathcal{U}$.
v. $\mathcal{A} = \mathcal{Q}$, $\mathcal{B} = \mathcal{G}$.
vi. $\mathcal{A} = \mathcal{P}, \mathcal{B} = \mathcal{Q}$.

Case i. consists on studying general forms of polar decompositions of any $T \in \mathcal{L}$. It follows that the classic polar decomposition $T = V|T|$, with $N(V) = N(T)$, is optimal in several senses. If $\mathcal{A}$ or $\mathcal{B}$ is $\mathcal{P}$, the determination of $\mathcal{AB}$ is a particular dilation problem, where the "big" space of the dilation is fixed. The choice of the cases $\mathcal{PU}, \mathcal{PG}, \mathcal{QG}, \mathcal{QU}$ is related to some problems in frame theory. We shall explain this in the introduction of Section 3. In case i. we complete goals (a), (b), (c); in cases ii. − v. we complete goals (a), (b), in case vi. we solve (a), (c). With similar methods we achieve (a) for cases where $\mathcal{A}$ is $\mathcal{P}$ or $\mathcal{Q}$ and $\mathcal{B}$ is one of the following: $\mathcal{I}$, the set of isometries, $\mathcal{I}^*$, the set of co-isometries (i.e., $T \in \mathcal{I}^* \Leftrightarrow T^* \in \mathcal{I}$), $\mathcal{L}^h$, the real subspace of Hermitian operators, and so on.

For a nice survey (up to 1990) of factorization problems, we refer the reader to P. Y. Wu's paper [26], which also deals with sets of the type $\mathcal{A}^n = \mathcal{A} \dots \mathcal{A}$ (n times) and $\cup_{n=1}^{\infty} \mathcal{A}^n$.

We describe now a sample of known results which enter into the scheme we are following here. The classical polar decomposition of John von Neumann [19] provides us the first example of characterization of $\mathcal{AB}$. Recall that it says that for every $T \in \mathcal{L}$ there exists a unique partial isometry V_T and a unique positive operator A such that $T = V_T A$ and $N(V_T) = N(A)$, where $N(T)$ is the nullspace of $T \in \mathcal{L}$. In fact, $A = |T| = (T^*T)^{1/2}$ and $A = V_T^* T$. It also holds $T = BV_T$, where $B = (TT^*)^{1/2}$, and this is the unique pair (B, V) such that $B \in \mathcal{L}^+$, $V \in \mathcal{J}$ and $R(B) = R(T)$, where $R(T)$ denotes the range of T. In particular, it says that $\mathcal{J}\mathcal{L}^+ = \mathcal{L} = \mathcal{L}^+\mathcal{J}$. It also provides the mapping $\phi : \mathcal{L} = \mathcal{J}\mathcal{L}^+ :\to \mathcal{J} \times \mathcal{L}^+$ defined by $\phi(T) = (V_T, |T|)$, which is a good candidate for goal (c).

An invertible operator $S \in \mathcal{L}$ is called a **symmetry** if $S^{-1} = S = S^*$; the set of all symmetries is denoted by $\mathcal{S}$. Observe that $\mathcal{S} = \mathcal{U} \cap \mathcal{L}^h$. Chandler Davis [10] proved that a unitary operator U belongs to $\mathcal{SS}$ if and only if U is unitarily equivalent to U^*, i.e.,

$$\mathcal{SS} = \{U \in \mathcal{U} : U \sim_{\mathcal{U}} U^*\}.$$

The paper by H. Radjavi and J. P. Williams [21] contains several results of type (a). They prove that, if $\dim \mathcal{H} < \infty$ then

$$\begin{aligned} \mathcal{L}^h\mathcal{L}^h &= \mathcal{L}^h\mathcal{G}^h = \mathcal{G}^h\mathcal{L}^h = \{T \in \mathcal{L} : \exists A \in \mathcal{G}^h \text{ such that } A^{-1}TA = T^*\} \\ &= \{T \in \mathcal{L} : \exists G \in \mathcal{G} \text{ such that } G^{-1}TG = T^*\}. \end{aligned}$$

There is a proof of these identities in [21, Theorem 1]; in the same paper it is shown that, for an infinite dimensional $\mathcal{H}$, the first equality does not hold and it is unknown if the last equality holds. Theorem 3 of [21] characterizes

$$\mathcal{S}\mathcal{L}^h = \{T \in \mathcal{L} : T \sim_{\mathcal{U}} T^*\},$$

and Theorem 2 of the same paper proves that

$$\mathcal{G}^+\mathcal{L}^h = \{T \in \mathcal{L} : \exists B \in \mathcal{L}^h \text{ such that } T \sim B\}.$$

Radjavi and Williams also show

$$\mathcal{P}\mathcal{P} = \{T \in \mathcal{L} : TT^*T = T^2\},$$

(an unpublished theorem by T. Crimmins) and

$$\mathcal{P}\mathcal{L}^h = \{T \in \mathcal{L} : (T^*)^2T = T^*T^2\},$$

in Theorems 8 and 9 of [21]. L. G. Brown [5] proved that any contraction C can be decomposed as $C = S^*W$ for two unilateral shifts S, W. In particular, this proves

$$\mathcal{J}\mathcal{J} = \mathcal{C},$$

where $\mathcal{C} := \{T \in \mathcal{L} : ||T|| \leq 1\}$. About factorizations in idempotent matrices, C. S. Ballantine [4] proved that

$$\mathcal{Q}^k = \underbrace{\mathcal{Q}\cdots\mathcal{Q}}_{\text{k times}} = \{T \in \mathcal{L} : \operatorname{rank}(T - I) \leq k \dim N(T)\}.$$

For infinite dimensional spaces, R. J. H. Dawlings [11] proved that $T \in \mathcal{Q}^k$ for some k if and only if one of the following holds: (i) $T = I$, (ii) $\dim N(T) = \dim N(T^*) = \infty$; (iii) $0 < \dim N(T) = \dim N(T^*)$ and $\dim N(T - I)^\perp < \infty$. For partial isometries in a finite dimensional space, K. H. Kuo and P. Y. Wu [16] proved that

$$\mathcal{J}^k = \{T \in \mathcal{C} : \operatorname{rank}(I - T^*T) \leq k \dim N(T)\}.$$

As mentioned before, the survey by Wu [26] describes $\mathcal{A}\mathcal{B}$ and $\mathcal{A}^n$ for several classes of operators $\mathcal{A}$ and $\mathcal{B}$. In particular, it is proven that

$$\mathcal{L}^+\mathcal{G}^+ = \mathcal{G}^+\mathcal{L}^+ = \{T \in \mathcal{L} : \exists A \in \mathcal{L}^+ \text{ such that } T \sim A\},$$

[26, Theorem 2.9]. More recently, in [8] there is an extensive study of $\mathcal{P}\mathcal{P}$. From now on, $P_{\mathcal{S}}$ denotes the orthogonal projection in $\mathcal{L}$ with range $\mathcal{S}$. The study developed in [8] includes a characterization of $(\mathcal{PP})_T$ for every $T \in \mathcal{PP}$ and some minimality criterion. More precisely, if $T \in \mathcal{PP}$ then

$$\begin{aligned}(\mathcal{PP})_T \;=\; \{(P_{\mathcal{M}_1}, P_{\mathcal{M}_2}) &: \text{there exist closed subspaces } \mathcal{N}_i \text{ of } \mathcal{M}_i \text{ s.t. } \mathcal{M}_1 = \overline{R(T)} \oplus \mathcal{N}_1,\\ &\mathcal{M}_2 = N(T)^\perp \oplus \mathcal{N}_2,\ \mathcal{N}_1 \perp \mathcal{N}_2 \text{ and } \mathcal{N}_1 \oplus \mathcal{N}_2 \subseteq R(T)^\perp \cap N(T)\}.\end{aligned}$$

Crimmins' proofs of the characterization of $\mathcal{PP}$ shows that if $T \in \mathcal{PP}$ then $T = P_{\overline{R(T)}} P_{N(T)^\perp}$. The pair $(P_{\overline{R(T)}}, P_{N(T)^\perp}) \in (\mathcal{PP})_T$ turns to be minimal in the sense that $\overline{R(T)} \subseteq \mathcal{M}$ and $N(T) \subseteq \mathcal{N}$ and $(P_{\overline{R(T)}} - P_{N(T)^\perp})^2 \leq (P_\mathcal{M} - P_\mathcal{N})^2$ if $T = P_\mathcal{M} P_\mathcal{N}$. Of course, this implies that $||P_{\overline{R(T)}} - P_{N(T)^\perp}|| \leq ||P_\mathcal{M} - P_\mathcal{N}||$, [8, Corollary 3.9]. According to our program (a), (b), (c), paper [8] completely solves the case $\mathcal{PP}$, by providing a mapping $\phi : \mathcal{PP} \to \mathcal{P} \times \mathcal{P}$, namely $\phi(T) = (P_{\overline{R(T)}}, P_{N(T)^\perp})$, such that $\phi(T) \in (\mathcal{PP})_T$ and $\phi(T)$ is minimal in the senses mentioned above. The paper [2] contains an analogous treatment of $\mathcal{PL}^+$. It is proven that

$$\mathcal{PL}^+ = \{T \in \mathcal{L} : \exists \lambda > 0 \text{ such that } (T^*T)^2 \leq \lambda T^* T^2\},$$

[2, Theorem 3.2], and that for every $T \in \mathcal{PL}^+$ there exists a well-defined $A_T \in \mathcal{L}^+$ such that

$$\begin{aligned}(\mathcal{PL}^+)_T \;=\; \{(P, A) : R(P) = \overline{R(T)} \oplus \overline{\mathcal{M}} &\text{ with } \mathcal{M} \subseteq N(T) \text{ and}\\ A = A_T + (I - P)C(I - P) &\text{ with } C \in \mathcal{L}^+\},\end{aligned}$$

[2, Theorem 4.9]. The pair $(P_{\overline{R(T)}}, A_T)$ has also minimal properties in $(\mathcal{PL}^+)_T$. If $T \in \mathcal{PP}$ it turns out that $A_T = P_{N(T)^\perp}$. Again, paper [2] completely solves the case $\mathcal{PL}^+$, and $\phi(T) = (P_{\overline{R(T)}}, A_T)$ has minimal properties in $(\mathcal{PL}^+)_T$. For the case $\mathcal{PG}^+$, Theorem 3.3 in [2] contains the answer:

$$\mathcal{PG}^+ = \{T \in \mathcal{L}_{cr} : R(T) \dot{+} N(T) = \mathcal{H}, TP_{R(T)} \in \mathcal{L}^+\},$$

where $\mathcal{L}_{cr}$ is the set of closed range operators in $\mathcal{L}$. For $T \in \mathcal{PG}^+$ it holds $(\mathcal{PG}^+)_T = \{(P, A) : A = (((TP)^{1/2})^\dagger T)^*((TP)^{1/2})^\dagger T + (I-P)C(I-P), C \in \mathcal{L}^+ \text{ and } P = P_{R(T)}\}$, see [2, Corollary 4.5]. Here, $B^\dagger$ denotes the Moore-Penrose generalized inverse of $B \in L(\mathcal{H})$.

We describe now the results of this paper. Section 2 contains the study of the case $\mathcal{A} = \mathcal{J}$ and $\mathcal{B} = \mathcal{L}^+$, i.e. the case of (generalized) polar decompositions. For every $T \in \mathcal{L} = \mathcal{JL}^+ = \mathcal{L}^+\mathcal{J}$, the set $(\mathcal{JL}^+)_T$ is determined. Among other properties, if $(V, A) \in (\mathcal{JL}^+)_T$ then $V^*T \in \mathcal{L}^+$ and the operator A_{V^*T}, which appears in the treatment of $\mathcal{PL}^+$ mentioned before, plays a relevant role. More precisely,

$$(\mathcal{JL}^+)_T = \left\{ (V, A) \in \mathcal{J} \times \mathcal{L}^+ : \begin{array}{c} TT^* \leq \lambda TV^* \text{ for some } \lambda > 0 \\ A = A_{V^*T} + (I - V^*V)C(I - V^*V) \text{ for some C} \in \mathcal{L}^+ \end{array} \right\}.$$

As a corollary, we get descriptions of $\mathcal{PJ}$ and $\mathcal{JP}$. It is also proved that $\phi(T) = (V_T, |T|)$ has the following optimal properties: $R(V_T) \subseteq R(V)$ and $|T| \leq A$, $||V_T - |T||| \leq ||V - A||$ and $(V_T - |T|)(V_T - |T|)^* \leq (V - A)(V - A)^*$ for every $(V, A) \in (\mathcal{JL}^+)_T$. Section 3 contains the results of the case $\mathcal{A} = \mathcal{P}$ and $\mathcal{B} = \mathcal{U}$. It holds that $\mathcal{PU} = \{V \in \mathcal{J} : \dim N(V) = \dim N(V^*)\}$. Thus, using index terminology, $\mathcal{PU}$ consists of all zero index partial isometries. Moreover, for $T \in \mathcal{PU}$ it holds $(\mathcal{PU})_T = \{(P, U) \in \mathcal{P} \times \mathcal{U} : P = P_{R(T)}$ and $U = T + W$ where $W \in$

$\mathcal{J}$ with $N(W)^{\perp} = N(T)$ and $R(W) = R(T)^{\perp}\}$, $p_1((\mathcal{PU})_T) = \{P_{R(T)}\}$ and $p_2((\mathcal{PU})_T) = \{T + W : W \in \mathcal{J}$ with $N(W)^{\perp} = N(T)$ and $R(W) = R(T)^{\perp}\}$. Section 4 contains the results about $\mathcal{PG}$ and $\mathcal{QG}$. These sets coincide and consist of all $T \in \mathcal{L}_{cr}$ with index zero, i.e., $\dim N(T) = \dim N(T^*)$. In Section 5 we study the case $\mathcal{QU}$. We prove that

$$\mathcal{QU} = \left\{ T \in \mathcal{L} : \begin{array}{c} \dim N(T) = \dim N(T^*) \\ \gamma(|T^*|) > 1 \\ \dim \overline{R(TT^* - P_{R(T)})} \leq \dim R(T)^{\perp} \end{array} \right\},$$

where γ denotes the reduced minimum modulus, i.e., $\gamma(T) = \inf\{||Tx|| : x \in N(T)^{\perp}, ||x|| = 1\}$, and for $T \in \mathcal{QU}$ it holds

$$(\mathcal{QU})_T = \left\{ (Q, Q^*(TT^*)^{\dagger}T + W) : \begin{array}{c} Q \in \mathcal{Q}, \\ |Q^*| = |T^*| \\ W \in \mathcal{J} \text{ with } N(W) = N(T)^{\perp}, R(W) = N(Q) \end{array} \right\}.$$

Section 6 is devoted to $\mathcal{PQ}$. We prove that $\mathcal{PQ} = \{T \in \mathcal{L} : R(T(I - T)) \subseteq R(T(I - P))$ where $P = P_{\overline{R(T)}}\}$, and given $T \in \mathcal{PQ}$ if $Q_T := T + (T(I - P))^{\dagger}(T - T^2)$ then $Q_T \in p_2((\mathcal{PQ})_T)$ and the pair (P, Q_T) belongs to $(\mathcal{PQ})_T$ and it is minimal in the following senses: $P \leq \tilde{P}$ for all $\tilde{P} \in p_1((\mathcal{PQ})_T)$, $Q_T^* Q_T \leq \tilde{Q}^*\tilde{Q}$ for all $\tilde{Q} \in p_2((\mathcal{PQ})_T)$ and $(P - Q_T)^*(P - Q_T) \leq (\tilde{P} - \tilde{Q})^*(\tilde{P} - \tilde{Q})$ for every $(\tilde{P}, \tilde{Q}) \in (\mathcal{PQ})_T$. Finally, Section 7 contains miscellaneous results about $\mathcal{PI}^*, \mathcal{QI}^*, \mathcal{IP}, \mathcal{IQ}, \mathcal{PE}, \mathcal{QE}, \mathcal{PN}, \mathcal{QN}$, where $\mathcal{E} = \{T \in \mathcal{L} : R(T) = \mathcal{H}\}$ and $\mathcal{N} = \{T \in \mathcal{L} : N(T) = \{0\}\}$.

2. Polar decompositions: the case $\mathcal{A} = \mathcal{J}$ and $\mathcal{B} = \mathcal{L}^+$

As mentioned in the introduction, it holds $\mathcal{JL}^+ = \mathcal{L}$. In this section we characterize $(\mathcal{JL}^+)_T$ for every $T \in \mathcal{L}$ and we show some minimal properties of $(V_T, |T|)$ in $(\mathcal{JL}^+)_T$. Any factorization $T = VA$, with $V \in \mathcal{J}$ and $A \in \mathcal{L}^+$ is called a *polar decomposition* of T. Clearly, for any $V \in \mathcal{J}$ and $A \in \mathcal{L}^+$ the pair (V, A) belongs to $(\mathcal{JL}^+)_T$ for $T = VA$. However, if we return to the classical polar decomposition, for a pair (V, A) there exists $T \in \mathcal{L}$ such that $V_T = V$ and $A = |T|$ if and only if $N(V) = N(A)$. In such case, if $T = VA$ then $(V, A) \in (\mathcal{JL}^+)_T$.

We begin this section with some minimality properties of the classical polar decomposition. For this, consider on $\mathcal{L}^+$ the usual order $T_1 \leq T_2$ if $\langle T_1\xi, \xi\rangle \leq \langle T_2\xi, \xi\rangle$ for all $\xi \in \mathcal{H}$. On $\mathcal{J}$ define the Halmos order $V_1 \leq V_2$ if $V_1V_1^* \leq V_2V_2^*$, i.e., $V_1 \leq V_2$ if $R(V_1) \subseteq R(V_2)$. The next result shows that $(V_T, |T|)$ is minimal in $(\mathcal{JL}^+)_T$ if we consider the order just defined and also $(V_T, |T|) \leq (V, A)$ for all $(V, A) \in (\mathcal{JL}^+)_T$ in the sense of the statement of Proposition 2.2.

Proposition 2.1. *For any $(V, A) \in (\mathcal{JL}^+)_T$ it holds $V_T \leq V$ and $|T| \leq A$.*

Proof. Let $(V, A) \in (\mathcal{JL}^+)_T$. Since $T = VA$ then $\overline{R(T)} \subseteq R(V)$; therefore $V_TV_T^* = P_{\overline{R(T)}} \leq P_{R(V)} = VV^*$, i.e., $V_T \leq V$. On the other side, $T^*T = AV^*VA = AP_{N(V)^{\perp}}A \leq A^2$, therefore, since the square root is operator monotone (Loewner's theorem [17]), we get that $|T| \leq A$. □

Proposition 2.2. *For every* $(V,A) \in (\mathcal{J}\mathcal{L}^+)_T$ *it holds* $(V_T - |T|)(V_T^* - |T|) \leq (V-A)(V^*-A)$. *As a consequence,* $||V_T - |T||| \leq ||V - A||$ *for all* $(V,A) \in (\mathcal{J}\mathcal{L}^+)_T$.

Proof. If $(V,A) \in (\mathcal{J}\mathcal{L}^+)_T$ then $(V-A)(V^*-A) - (V_T - |T|)(V_T^* - |T|) = P_{R(V)} - P_{\overline{R(T)}} + A^2 - T^*T \geq 0$, because, by the Proposition above, it holds $P_{\overline{R(T)}} \leq P_{R(V)}$ and $T^*T \leq A^2$. □

The following result due to R. G. Douglas [12] will be useful in the sequel.

Theorem 2.3. *Let* $A, B \in \mathcal{L}$. *The following conditions are equivalent:*

1. $R(B) \subseteq R(A)$.
2. *There is a positive number* λ *such that* $BB^* \leq \lambda AA^*$.
3. *There exists* $C \in \mathcal{L}$ *such that* $AC = B$.

If one of these conditions holds then there is a unique operator $D \in \mathcal{L}$ *such that* $AD = B$ *and* $R(D) \subseteq N(A)^\perp$. *We shall call* D *the* ***reduced solution*** *of* $AX = B$. *Moreover,* $N(D) = N(B)$ *and* $||D||^2 = \inf\{\lambda > 0 : BB^* \leq \lambda AA^*\}$.

Remark 2.4. It is well known that the reduced solution of $AX = B$ is given by $A^\dagger B$. In fact, if $AX = B$ has a bounded linear solution then $R(B) \subseteq R(A)$ and therefore $A^\dagger B \in \mathcal{L}$. Put $A(A^\dagger B) = P_{\overline{R(A)}} B = B$ and $R(A^\dagger B) \subseteq R(A^\dagger) = N(A)^\perp$, thus $D = A^\dagger B$.

The next result due to Z. Sebestyén [24, pg. 300] (see also [2, Proposition 2.3]) is relevant in what follows.

Proposition 2.5. *Let* $A, B \in \mathcal{L}$.*The equation* $AX = B$ *has a positive solution if and only if* $BB^* \leq \lambda BA^*$ *for some constant* $\lambda > 0$. *In such case, there exists* $C \in \mathcal{L}^+$ *with* $N(C) = N(B)$ *such that* $AC = B$.

Proposition 2.6. *Let* $T \in \mathcal{L}$ *and* $V \in \mathcal{J}$. *The next conditions are equivalent:*

1. $V \in p_1((\mathcal{J}\mathcal{L}^+)_T)$.
2. $TT^* \leq \lambda TV^*$ *for some* $\lambda > 0$.

Proof. It is an immediate consequence of Proposition 2.5. □

For a treatment of the condition $P \in p_1((\mathcal{J}\mathcal{L}^+)_T)$ with $P \in \mathcal{P}$ we refer the reader to [2]. Observe that for $V \in p_1((\mathcal{J}\mathcal{L}^+)_T)$ it holds $\{A \in \mathcal{L}^+ : T = VA\} = \{A \in \mathcal{L}^+ : V^*T = P_{R(V^*)}A\}$. Hence, applying [2, Proposition 4.8], we get that $\{A \in \mathcal{L}^+ : T = VA\} = \{A_{V^*T} + (I - V^*V)C(I - V^*V) : C \in \mathcal{L}^+\}$ where $A_{V^*T} := (((V^*TP)^{1/2})^\dagger V^*T)^*((V^*TP)^{1/2})^\dagger V^*T$ and $P = P_{\overline{R(V^*T)}}$. Therefore, we have proved:

Theorem 2.7. *For any* $T \in \mathcal{L}$ *it holds:*

$$(\mathcal{J}\mathcal{L}^+)_T = \left\{ (V,A) \in \mathcal{J} \times \mathcal{L}^+ : \begin{array}{c} TT^* \leq \lambda TV^* \text{ for some } \lambda > 0 \\ A = A_{V^*T} + (I - V^*V)C(I - V^*V) \text{ for some C} \in \mathcal{L}^+ \end{array} \right\}.$$

Remark 2.8. Given a subspace $\mathcal{S}$ of $\mathcal{H}$ we define $\dim(\mathcal{S})$ as the cardinality of any maximal orthonormal set of $\mathcal{S}$. In the next Proposition and in many others which involve claims about $\dim \overline{R(T)}$ one should notice that $\dim R(T) = \dim \overline{R(T)}$; however this is not true, in general, for a subspace which is not an operator range (see [13]).

The next result provides a characterization of $p_2((\mathcal{J}\mathcal{L}^+)_T)$:

Proposition 2.9. *Let $T \in \mathcal{L}$ and $A \in \mathcal{L}^+$. The next conditions are equivalent:*

1. $A \in p_2((\mathcal{J}\mathcal{L}^+)_T)$, *i.e., there exists $V \in \mathcal{J}$ such that $T = VA$.*
2. $T^*T \leq A^2$ *and* $\dim \overline{R(ZZ^* - (ZZ^*)^2)} \leq \dim N(A)$ *where* $Z = A^\dagger T^*$.
3. $T^*T = AP_1A$ *for some* $P_1 \in \mathcal{P}$.

Proof. $1 \Rightarrow 2$. Let $A \in p_2((\mathcal{J}\mathcal{L}^+)_T)$. Then $T = VA$ for some $V \in \mathcal{J}$. Thus, $T^*T = AV^*VA = AP_1A \leq A^2$ where $P_1 = V^*V = P_{R(V^*)}$.As a consequence, by Theorem 2.3, the equation $T^* = AX$ has a solution in $\mathcal{L}$ and its reduced solution is $Z = A^\dagger T^* = A^\dagger AV^* = PV^*$, where $P = P_{\overline{R(A)}}$. Hence, $\overline{R(ZZ^* - (ZZ^*)^2)} = \overline{R(PP_1(I-P)P_1P)} = \overline{R(PP_1(I-P))}$ and therefore, $\dim \overline{R(ZZ^* - (ZZ^*)^2)} \leq \dim R(I-P) = \dim N(A)$.

$2 \Rightarrow 3$. Suppose that $T^*T \leq A^2$. Then, by Douglas' theorem, the equation $T^* = AX$ has a solution in $\mathcal{L}$. Let Z be the reduced solution of this equation, i.e., $T^* = AZ$ and $R(Z) \subseteq \overline{R(A)}$. Moreover, since $T^*T \leq A^2$ then $||Z|| \leq 1$. Now, define $Y := ZZ^*$. Clearly, $T^*T = AYA$ and $0 \leq Y \leq I$.

Now, as $\dim \overline{R(Y - Y^2)} \leq \dim N(A)$, then there exists a partial isometry W from $N(A)$ onto $\overline{R(Y - Y^2)}$. Then,

$$P_1 = \begin{pmatrix} Y & (Y-Y^2)^{1/2}W \\ W^*(Y-Y^2)^{1/2} & I - W^*YW \end{pmatrix} \begin{matrix} \overline{R(A)} \\ R(A)^\perp \end{matrix}$$

is an orthogonal projection. Moreover, as $A = \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix}$ then $AP_1A = AYA$, i.e., $AP_1A = T^*T$ as desired.

$3 \Rightarrow 1$. Suppose that $T^*T = AP_1A$ for some orthogonal projection $P_1 \in \mathcal{L}$. Hence, $|T| = |P_1A|$. Therefore, if $T = V_T|T|$ and $P_1A = V_{P_1A}|P_1A|$ are the polar decompositions of T and P_1A respectively, then $T = V_TV_{P_1A}^*P_1A = V_TV_{P_1A}^*A$. Define $J := V_TV_{P_1A}^*$, therefore $T = JA$ and it only remains to show that J is a partial isometry. For this, observe that $JJ^*J = V_TV_{P_1A}^*V_{P_1A}V_T^*V_TV_{P_1A}^* = V_TV_{P_1A}^*V_{P_1A}P_{\overline{R(T^*)}}V_{P_1A}^* = V_TV_{P_1A}^*V_{P_1A}V_{P_1A}^* = V_TV_{P_1A}^* = J$, where the third equality follows from the fact that $R(V_{P_1A}^*) = \overline{R((P_1A)^*)} = \overline{R(AP_1A)} = \overline{R(T^*T)} = \overline{R(T^*)}$. Therefore, $T = JA$ with $J \in \mathcal{J}$, i.e., $A \in p_2((\mathcal{J}\mathcal{L}^+)_T)$. $\square$

Remark 2.10. Clearly, $\mathcal{L} = \mathcal{J}\mathcal{L}^h$. Now, given $T \in \mathcal{L}$ and $A \in \mathcal{L}^h$ then, following the same lines as in the proof of the above proposition, there exists $V \in \mathcal{J}$ such that $T = VA$ if and only if $T^*T \leq A^2$ and $\dim \overline{R(ZZ^* - (ZZ^*)^2)} \leq \dim N(A)$, where $Z = A^\dagger T^*$ or, equivalently, $T^*T = AP_1A$ for some orthogonal projection $P_1 \in \mathcal{L}$.

The previous result allows us to describe the sets $\mathcal{P}\mathcal{J}$ and $\mathcal{J}\mathcal{P}$. The equivalence $1 \Leftrightarrow 4$ in the next corollary has been proved by Sebestyén and Magyar [25].

Corollary 2.11. *Let $T \in \mathcal{L}$ and $P = P_{\overline{R(T)}}$. The next conditions are equivalent:*

1. $T \in \mathcal{P}\mathcal{J}$.
2. $P \in p_2((\mathcal{J}\mathcal{L}^+)_{T^*})$.
3. *There exists* $P_1 \in \mathcal{P}$ *such that* $TT^* = PP_1P$.

4. $||T|| \leq 1$ *and* $\dim \overline{R(TT^* - (TT^*)^2)} \leq \dim R(T)^{\perp}$.

Briefly, it holds

$$\mathcal{PJ} = \{T \in \mathcal{C} : \dim \overline{R(TT^* - (TT^*)^2)} \leq \dim R(T)^{\perp}\},$$

and

$$\mathcal{JP} = \{T \in \mathcal{C} : \dim \overline{R(T^*T - (T^*T)^2)} \leq \dim N(T)\}.$$

The next result describes $\{A \in \mathcal{L}^+ : V_T A = T\}$ and $\{V \in \mathcal{J} : V|T| = T\}$ for $T \in \mathcal{L}$.

Proposition 2.12. *Let $T \in \mathcal{L}$ then:*

1. $\{A \in \mathcal{L}^+ : (V_T, A) \in (\mathcal{JL}^+)_T\} = \{|T| + P_{N(T)} B P_{N(T)} : B \in \mathcal{L}^+\}$.
2. $\{V \in \mathcal{J} : (V, |T|) \in (\mathcal{JL}^+)_T\} = \{V \in \mathcal{J} : VV_T^* = P_{\overline{R(T)}}\} = \{V \in \mathcal{J} : VP_{\overline{R(T^*)}} = V_T\}$.

Proof. 1. Let $A \in \mathcal{L}^+$ such that $(V_T, A) \in (\mathcal{JL}^+)_T$. Then, by Proposition 2.1, $A = |T| + B$ with $B \in \mathcal{L}^+$. Moreover, as $T = V_T A = V_T|T|$ then $V_T B = 0$. So, $R(B) \subseteq N(V_T) = N(T)$, i.e., $B = P_{N(T)}B$. Now, since $B = B^*$ then $B = P_{N(T)} B P_{N(T)}$. Hence, $A = |T| + P_{N(T)} B P_{N(T)}$ with $B \in \mathcal{L}^+$. The converse is trivial.

2. Let $(V, |T|) \in (\mathcal{JL}^+)_T$. Then, $V|T| = V_T|T| = T$. So, as $|T| = V_T^* T$ we have that $VV_T^* T = V|T| = T = P_{\overline{R(T)}} T$ or, equivalently, $T^* V_T V^* = T^* P_{\overline{R(T)}}$. Now, since $R(V_T V^*)$ and $R(P_{\overline{R(T)}})$ are both included in $N(T^*)^{\perp}$ we have that $V_T V^* = P_{\overline{R(T)}}$. For the other inclusion, if $V \in \mathcal{J}$ and $VV_T^* = P_{\overline{R(T)}}$ then $V|T| = VV_T^* T = P_{\overline{R(T)}} T = T$, i.e., $(V, |T|) \in (\mathcal{JL}^+)_T$. The second equality follows by right multiplication by V_T.

□

Remark 2.13. In [7, Proposition 3.11] it is proven that given $T \in \mathcal{L}$, $V_T|T|V_T = V_T$ and $|T|V_T|T| = |T|$ if and only if $T \in \mathcal{Q}$. Moreover, it is easy to check that if $E \in \mathcal{Q}$ then $\mathcal{T} := \{(V, A) \in (\mathcal{JL}^+)_E : VAV = V, AVA = A\} = \{(V, A) \in (\mathcal{JL}^+)_E : R(V) = R(E), N(A) = N(E)\}$. Now, since $(P_{R(E)}, E^*E) \in (\mathcal{JL}^+)_E$, $R(P_{R(E)}) = R(E)$ and $N(E^*E) = N(E)$, then $(P_{R(E)}, E^*E) \in \mathcal{T}$, i.e., $(V_E, |E|)$ is not the unique pair in $\mathcal{T}$.

Due to the uniqueness part of the classical polar decomposition, we get two mappings:

$$\begin{aligned} \alpha &: \mathcal{L} \to \mathcal{J},\ \alpha(T) = V_T, \\ \beta &: \mathcal{L} \to \mathcal{L}^+,\ \beta(T) = |T|. \end{aligned}$$

In general β is continuous but α is not. However, we are here interested in their behaviours as set mappings. The proof of the following proposition is straightforward.

Proposition 2.14. *The next identities hold:*

1. $\alpha^{-1}(\mathcal{U}) = \mathcal{N} \cap \mathcal{N}^*$. *Notice that* $\mathcal{L}_d := \{T \in \mathcal{L} : \overline{R(T)} = \mathcal{H}\} = \mathcal{N}^*$.
2. $\alpha^{-1}(\mathcal{I}) = \mathcal{N}$.
3. $\alpha^{-1}(\mathcal{I}^*) = \mathcal{N}^*$.
4. $\alpha^{-1}(\mathcal{S}) = \mathcal{SN}^+$.
5. $\alpha^{-1}(\mathcal{P}) = \mathcal{L}^+$.

6. $\beta^{-1}(\mathcal{P}) = \mathcal{J}$.
7. $\beta^{-1}(\mathcal{G}^+) = \mathcal{N} \cap \mathcal{L}_{cr}$.
8. $\beta^{-1}(\mathcal{L}^+ \cap \mathcal{N}) = \mathcal{N}$.

One can also consider the images of different classes in $\mathcal{L}$ by α and β. This has been done in [7, Theorems 5.1 and 6.1] for $\mathcal{Q}$, in [8, Theorem 5.2 and Proposition 5.5] for $\mathcal{PP}$ and in [2, Proposition 6.4] for $\mathcal{PL}^+$.

3. The case $\mathcal{A} = \mathcal{P}$ and $\mathcal{B} = \mathcal{U}$

The cases $\mathcal{PU}, \mathcal{PG}, \mathcal{QG}$ and $\mathcal{QU}$ are related to a result on frame theory. Recall that a *frame* in a (separable) Hilbert space $\mathcal{H}$ is a sequence $\{x_n\}$ of vectors of $\mathcal{H}$ for which there exist positive constants α, β such that

$$\alpha||x||^2 \leq \sum |\langle x, x_n\rangle|^2 \leq \beta||x||^2, \forall x \in \mathcal{H}. \tag{2}$$

As a consequence of Naimark's dilation theorem [18] it can be shown that if $\{x_n\}$ is a *Parseval frame* (which means that (2) holds for $\alpha = \beta = 1$) then there exist a Hilbert space $\mathcal{K} \supseteq \mathcal{H}$ and an orthonormal basis $\{e_n\}$ of $\mathcal{K}$ such that the orthogonal projection $P_{\mathcal{H}}$ maps e_n to x_n ($n \in \mathbb{N}$); see [9] for a modern proof. It should be mentioned that Han and Larson [14] rediscovered this corollary of Naimark's theorem by using a completely different approach.

Each frame $\overline{x} = \{x_n\}$ defines a bounded linear operator $T_{\overline{x}} : l^2 \to \mathcal{H}$, $T_{\overline{x}}(\{\alpha_n\}) = \sum \alpha_n x_n$ which is onto (the synthesis operator). In this context, Naimark's result says that if $\{x_n\}$ is a Parseval frame then, in a convenient extension $\mathcal{K}$ of $\mathcal{H}$, $P_{\mathcal{H}}U = T_{\overline{x}}$, where U is a unitary operator on $\mathcal{K}$ related to the orthonormal basis $\{e_n\}$. If one allows oblique projections instead of orthogonal ones, or Riesz basis instead of orthonormal ones, we are lead to equalities like $EG = T_{\overline{x}}$ for an oblique projection E and an invertible operator G. The reader is referred to [1] for more details.

This section is devoted to $\mathcal{PU}$, and the following ones deal with the other cases. However, in each section we can extract result on different factorizations $\mathcal{AB}$.

Theorem 3.1. *The set $\mathcal{PU}$ consists of all partial isomeries with zero index:*

$$\mathcal{PU} = \{V \in \mathcal{J} : \dim N(V) = \dim N(V^*)\}.$$

Moreover, fixed $T \in \mathcal{PU}$ then $(\mathcal{PU})_T = \{(P, U) \in \mathcal{P} \times \mathcal{U} : P = P_{R(T)}$ and $U = T + W$ where $W \in \mathcal{J}$ with $N(W)^\perp = N(T)$ and $R(W) = R(T)^\perp\}$, $p_1((\mathcal{PU})_T) = \{P_{R(T)}\}$ and $p_2((\mathcal{PU})_T) = \{T + W : W \in \mathcal{J}$ with $N(W)^\perp = N(T)$ and $R(W) = R(T)^\perp\}$.

Proof. If $V = PU$ for some $U \in \mathcal{U}$ and some $P \in \mathcal{P}$, then $V = VV^*V$ (i.e., V is a partial isometry). Moreover, $W := U^*(I - P)$ is a partial isometry with initial subspace $N(V^*)$ and final subspace $N(V)$. This implies that $\dim N(V) = \dim N(V^*)$. Conversely, suppose that V is a partial isometry such that $\dim N(V) = \dim N(V^*)$. Let $P = VV^*$ and let W be a partial isometry with initial subspace $N(V)$ and final subspace $N(V^*)$. Then, $V + W$ is a unitary operator and $V = P(V + W)$, i.e., $V \in \mathcal{PU}$.

On the other hand, let $T \in \mathcal{PU}$ and $P = P_{R(T)}$. Recall that T is a partial isometry. Consider $U \in \mathcal{U}$ such that $T = PU$ and let $W := U - T$. Notice that $UT^* = UU^*P = P =$

TU^* and $T^*T = U^*PU = U^*T = T^*U$. We claim that W is a partial isometry. In fact, $WW^* = (U-T)(U-T)^* = I - P - P + P = I - P$, i.e., W is a partial isometry with $R(W) = R(I-P) = R(T)^\perp$. Moreover, $W^*W = (U^*-T^*)(U-T) = I - U^*T - T^*U + T^*T = I - T^*T = I - P_{N(T)^\perp}$, i.e., $N(W) = N(T)^\perp$.

Conversely, let $U = T + W$ with W a partial isometry with $R(W) = R(T)^\perp$ and $N(W) = N(T)^\perp$. Clearly, $PU = T$. Let us show that $U \in \mathcal{U}$. Indeed, $UU^* = (T+W)(T^*+W^*) = TT^* + TW^* + WT^* + WW^* = TT^* + WW^* = P_{R(T)} + P_{R(T)^\perp} = I$. Similarly, $U^*U = I$ and the proof is finished. □

Notice that if $T \in \mathcal{PU}$ and $P = P_{R(T)}$ then $(P-U)^*(P-U) = P + I - 2Re(T)$ for all $U \in p_2((\mathcal{PU})_T)$ and, a fortiori, $||P - U|| = ||P + I - 2Re(T)||$ for all $U \in p_2((\mathcal{PU})_T)$. Hence, there is no optimal pair $(P, U) \in (\mathcal{PU})_T$ in the same sense as in Proposition 2.2.

4. The cases $\mathcal{A} = \mathcal{P}$ or $\mathcal{Q}$ and $\mathcal{B} = \mathcal{G}$

In the sequel we denote $\mathcal{R} := \{(A, B) \in \mathcal{L} \times \mathcal{L} : R(A+B) = R(A) + R(B)\}$; this set appears in [3], related to shorted operators and the Sherman-Morrison-Woodbury formula.

Theorem 4.1. *The next equality holds:*

$$\mathcal{PG} = \{T \in \mathcal{L}_{cr} : \dim N(T) = \dim N(T^*)\}.$$

Moreover, fixed $T \in \mathcal{PG}$ then $(\mathcal{PG})_T = \{(P, G) \in \mathcal{P} \times \mathcal{G} : P = P_{R(T)}$ and $G = T + W$ where $(T, W) \in \mathcal{R},\ R(W) = R(T)^\perp$ and $N(W) \cap N(T) = \{0\}\}$, $p_1((\mathcal{PG})_T) = \{P_{R(T)}\}$ and $p_2((\mathcal{PG})_T) = \{T + W : (T, W) \in \mathcal{R},\ R(W) = R(T)^\perp$ and $N(W) \cap N(T) = \{0\}\}$.

Proof. Let $T = PG$ for some $P \in \mathcal{P}$ and some $G \in \mathcal{G}$ then, clearly, $R(T) = R(P)$, i.e., T has closed range and $P = P_{R(T)}$. Moreover, the partial isometry of the polar decomposition of $G^{-1}(I - P)$ has initial subspace $N(T^*)$ and final subspace $G^{-1}(R(P)^\perp) = N(T)$. Hence, $\dim N(T) = \dim N(T^*)$. On the other side, suppose that T has closed range and $\dim N(T) = \dim N(T^*)$. Let W be a partial isometry with initial subspace $N(T)$ and final subspace $N(T^*)$, and let P be the orthogonal projection onto $R(T)$. Trivially, $T = P(T+W)$. Let us show that $T + W \in \mathcal{G}$. First, by [3, Theorem 2.10], $R(T+W) = R(T) + R(W)$ and so $R(T+W) = \mathcal{H}$. Finally, if $x \in N(T+W)$ then $Tx = -Wx \in R(T) \cap R(W) = \{0\}$. Thus, $x \in N(T) \cap N(W) = \{0\}$ and so $N(T+W) = \{0\}$ as desired. Therefore, $T = P(T+W) \in \mathcal{PG}$.

Let now $T \in \mathcal{PG}$ and consider $G \in \mathcal{G}$ such that $T = PG$. Define $W := G - T$. As $PW = 0$, then $R(W) \subseteq R(T)^\perp$. Moreover, as $\mathcal{H} = R(G) \subseteq R(T) \dot{+} R(W) \subseteq R(T) \dot{+} R(T)^\perp = \mathcal{H}$, we obtain that $(T, W) \in \mathcal{R}$ and $R(W) = R(T)^\perp$. In addition, if $x \in N(W) \cap N(T)$ then $Gx = Tx + Wx = 0$ and since G is invertible we have that $x = 0$. Conversely, let $G = T + W$ with $(T, W) \in \mathcal{R}$, $R(W) = R(T)^\perp$ and $N(W) \cap N(T) = \{0\}$. Clearly, $PG = T$. Let us show that $G \in \mathcal{G}$. First, $R(G) = R(T+W) = R(T) + R(W) = R(T)^\perp + R(T) = \mathcal{H}$. Moreover, if $x \in N(G)$ then $Gx = Tx + Wx = 0$, i.e, $Tx = -Wx \in R(T) \cap R(T)^\perp = \{0\}$. Thus, $x \in N(W) \cap N(T) = \{0\}$ and so $N(G) = \{0\}$, as desired.

□

Remark 4.2. It also holds that $\mathcal{PG} = \overline{\mathcal{G}} \cap \mathcal{L}_{cr}$. This result is due to S. Izumino [15, Theorem 3.2].

Lemma 4.3. *The following equality holds:*

$$\mathcal{QG} = \mathcal{PG}.$$

Proof. Let us show that $\mathcal{QG} = \mathcal{PG}$. For this, let us consider the matrix representation of oblique projections. Given an oblique projection Q with range $\mathcal{S}$, with respect to the 2×2 matrix decomposition induced by $\mathcal{S}$, Q has the following form

$$Q = \begin{pmatrix} I & X \\ 0 & 0 \end{pmatrix} \begin{matrix} \mathcal{S} \\ \mathcal{S}^{\perp} \end{matrix} \tag{3}$$

If P is the orthogonal projection onto $\mathcal{S}$ then $P = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}$ and, trivially,

$$Q = \begin{pmatrix} I & X \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} I & X \\ 0 & I \end{pmatrix} = PC, \tag{4}$$

where $C := \begin{pmatrix} I & X \\ 0 & I \end{pmatrix} \in \mathcal{G}$. This proves that $\mathcal{QG} \subseteq \mathcal{PG}$ and therefore the equality between these two sets because the other inclusion is trivial. □

Theorem 4.4. *Fixed* $T \in \mathcal{QG}$ *then* $(\mathcal{QG})_T = \{(Q, G) \in \mathcal{Q} \times \mathcal{G} : R(Q) = R(T)$ and $G = T + W$, where $(T, W) \in \mathcal{R}$, $R(W) = N(Q)$ and $N(W) \cap N(T) = \{0\}\}$, $p_1((\mathcal{QG})_T) = \{Q \in \mathcal{Q} : R(Q) = R(T)\}$ *and* $p_2((\mathcal{QG})_T) = \{T + W : (T, W) \in \mathcal{R},\ R(W) \dot{+} R(T) = \mathcal{H}$ and $N(W) \cap N(T) = \{0\}\}$.

Proof. For $T \in \mathcal{QG}$, let us show that $p_1((\mathcal{QG})_T) = \{Q \in \mathcal{Q} : R(Q) = R(T)\}$. For this, let $Q \in \mathcal{Q}$ with $R(Q) = R(T)$. So, as $T \in \mathcal{PG}$ there exists $G \in \mathcal{G}$ such that $T = PG$ where $P = P_{R(T)}$. Hence, using (4), $T = PG = PCC^{-1}G = QC^{-1}G$ where $C^{-1}G \in \mathcal{G}$, i.e., $Q \in p_1((\mathcal{QG})_T)$. The other inclusion is obvious.

Now, let $G \in p_2((\mathcal{QG})_T)$, i.e., $G \in \mathcal{G}$ and $T = QG$ for some $Q \in \mathcal{Q}$. Let $W := G - T$. Then the sum $R(T) + R(W)$ is direct since $R(T) = R(Q)$ and $R(W) \subseteq N(Q)$ (consequence of the fact that $QW = Q(I - Q)G = 0$). In addition, $\mathcal{H} = R(G) = R(T + W) \subseteq R(T) \dot{+} R(W)$, which shows that $R(T + W) = R(T) \dot{+} R(W) = \mathcal{H}$. Moreover, if $x \in N(W) \cap N(T)$ then $Gx = Tx + Wx = 0$ and since G is invertible we have that $x = 0$. Conversely, let $G = T + W$ with $(T, W) \in \mathcal{R}$, $R(T) \dot{+} R(W) = \mathcal{H}$ and $N(W) \cap N(T) = \{0\}$. Define $Q := Q_{R(T)//R(W)}$, i.e., $Q \in \mathcal{Q}$, $R(Q) = R(T)$ and $N(Q) = R(W)$. Clearly, $T = QG$. Let us show that G is invertible. Now, $R(G) = R(T) + R(W) = \mathcal{H}$ and if $x \in N(G)$ then $Tx = -Wx \in R(T) \cap R(W) = \{0\}$ and so $x \in N(W) \cap N(T) = \{0\}$. Therefore, $N(G) = \{0\}$ and the proof is finished.

Finally, following the proof of $p_2((\mathcal{QG})_T)$ we get the characterization of $(\mathcal{QG})_T$. □

5. The case $\mathcal{A} = \mathcal{Q}$ and $\mathcal{B} = \mathcal{U}$

As mentioned in the beginning of Section 3, the spaces $\mathcal{PG}, \mathcal{PU}, \mathcal{QG}$ and $\mathcal{QU}$ are directly related to different dilation frame problems. We devote this section to the case $\mathcal{QU}$ which is more difficult to deal with than the others.

The next result will be useful in the sequel, see [7, Theorem 6.1]. Recall that the **reduced minimum modulus** of $T \in \mathcal{L}$ is the number $\gamma(T) := \inf\{||Tx|| : x \in N(T)^{\perp}, ||x|| = 1\}$.

Theorem 5.1. *Let $B \in \mathcal{L}^+$. There exists $Q \in \mathcal{Q}$ such that $|Q| = B$ if and only if $\gamma(B) \geq 1$ and $\dim \overline{R(B^2 - P_{R(B)})} \leq \dim N(B)$.*

Theorem 5.2. *The following equality holds:*

$$\mathcal{QU} \;=\; \left\{ T \in \mathcal{L} : \begin{array}{c} \dim N(T) = \dim N(T^*), \\ \gamma(|T^*|) \geq 1, \\ \dim \overline{R(TT^* - P_{R(T)})} \leq \dim R(T)^\perp \end{array} \right\}$$

For a fixed $T \in \mathcal{QU}$ it holds

$$(\mathcal{QU})_T = \left\{ (Q, Q^*(TT^*)^\dagger T + W) : \begin{array}{c} Q \in \mathcal{Q}, \\ |Q^*| = |T^*|, \\ W \in \mathcal{J} \text{ with } R(W) = N(Q) \text{ and } N(W) = N(T)^\perp \end{array} \right\} \quad (5)$$

Proof. Suppose that $T = QU$ for some $U \in \mathcal{U}$ and $Q \in \mathcal{Q}$. Then, clearly, $|T^*| = |Q^*|$ and, by Theorem 5.1, $\gamma(|T^*|) \geq 1$ and $\dim \overline{R(TT^* - P_{R(T)})} \leq \dim R(T)^\perp$. Moreover, $\dim N(T) = \dim N(Q) = \dim N(Q^*) = \dim N(T^*)$.

Conversely, assume that $\dim N(T) = \dim N(T^*)$, $\gamma(|T^*|) \geq 1$ and $\dim \overline{R(TT^* - P_{R(T)})} \leq \dim R(T)^\perp$. Then, by Theorem 5.1, $|T^*| = |Q^*|$ for some $Q \in \mathcal{Q}$. Let $T = |T^*|V_T$ and $Q = |Q^*|V_Q$ be the polar decompositions of T and Q respectively. Observe that V_Q and V_T have the same final space, $R(T)$. As a consquence, $V_Q^*V_T(V_Q^*V_T)^*V_Q^*V_T = V_Q^*V_T$, i.e., $V_Q^*V_T \in \mathcal{J}$ and $N(V_Q^*V_T) = N(T)$, $R(V_Q^*V_T) = N(Q)^\perp$. On the other hand, $\dim N(T) = \dim N(T^*) = \dim R(T)^\perp$ and it is also equal to the dimension of any other supplement of $R(T) = R(Q)$, for instance $N(Q)$. Therefore, $\dim N(T) = \dim N(Q)$ and the partial isometry $V_Q^*V_T$ can be extended to an unitary operator U which maps $N(T)$ onto $N(Q)$. So, we get

$$T = |T^*|V_T = |Q^*|V_T = |Q^*|V_QV_Q^*V_T = Q(V_Q^*V_T) = QU,$$

which proves that $T \in \mathcal{QU}$.

Let us show equality (5). Let us first consider $U \in \mathcal{U}, Q \in \mathcal{Q}$ such that $T = QU$. Clearly, $TT^* = QQ^*$, i.e., $|T^*| = |Q^*|$. Hence, if $T = |T^*|V_T$ and $Q = |Q^*|V_Q$ are the classical polar decompositions of T and Q respectively, then $|Q^*| = QV_Q^*$ and so $QU = T = |T^*|V_T = |Q^*|V_T = QV_Q^*V_T$. Therefore, $U = V_Q^*V_T + W$ for some $W \in \mathcal{L}$ with $R(W) \subseteq N(Q)$. We claim that:

1. $W \in \mathcal{J}$ with $R(W) = N(Q)$ and $N(W) = N(T)^\perp$.
2. $V_Q^*V_T = Q^*(TT^*)^\dagger T$.

In fact,

1. Notice that $R(V_Q^*V_T) = R(V_Q^*) = N(Q)^\perp$ and since $R(W) \subseteq N(Q)$ and $\mathcal{H} = R(U) = R(V_Q^*V_T + W) = R(V_Q^*V_T) + R(W)$ we obtain that $R(W) = N(Q)$. Moreover, $I = UU^* = (V_Q^*V_T + W)(V_Q^*V_T + W)^* = V_Q^*V_TV_T^*V_Q + V_Q^*V_TW^* + WV_T^*V_Q + WW^* = P_{N(Q)^\perp} + V_Q^*V_TW^* + WV_T^*V_Q + WW^*$. Hence,

$$V_Q^*V_TW^* + WV_T^*V_Q + WW^* = P_{N(Q)}, \quad (6)$$

but since $R(W) = N(Q)$ then $R(V_Q^* V_T W^*) \subseteq N(Q) \cap R(V_Q^*) = N(Q) \cap N(Q)^\perp = \{0\}$. Therefore, $V_Q^* V_T W^* = 0$ and so, by (6), $WW^* = P_{N(Q)}$. Thus, $W \in \mathcal{J}$. It only remains to show that $N(W) = N(T)^\perp$. For this, notice that $R(V_T^* V_Q) = R(V_T^*) = N(T)^\perp$ and, as $WV_T^* V_Q = 0$ we have that $N(T)^\perp = R(V_T^* V_Q) \subseteq N(W)$ or, equivalently, $R(W^*) \subseteq N(T)$ On the other side, $\mathcal{H} = R(U^*) = R(V_T^* V_Q + W^*) = R(V_T^* V_Q) + R(W^*) = N(T)^\perp + R(W^*)$, and so $R(W^*) = N(T)$, i.e., $N(W) = N(T)^\perp$.

2. $V_Q^* V_T = Q^* |Q^*|^\dagger V_T = Q^* |T^*|^\dagger V_T = Q^* |T^*|^\dagger |T^*|^\dagger T = Q^*(TT^*)^\dagger T$.

Conversely, let $(Q, Q^*(TT^*)^\dagger T + W)$ as in (5) and let us show that this pair is in $(\mathcal{QU})_T$. First, as $R(W) = N(Q)$ and $|Q^*| = |T^*|$ we have that $Q(Q^*(TT^*)^\dagger T + W) = TT^*(TT^*)^\dagger T = P_{R(T)} T = T$. Thus, we only need to show that $Q^*(TT^*)^\dagger T + W \in \mathcal{U}$. Now, $(Q^*(TT^*)^\dagger T + W)(Q^*(TT^*)^\dagger T + W)^* = Q^*(TT^*)^\dagger T (Q^*(TT^*)^\dagger T)^* + WW^* = Q^*(TT^*)^\dagger TT^*(TT^*)^\dagger Q + WW^* = Q^*(TT^*)^\dagger Q + WW^* = Q^*(QQ^*)^\dagger Q + WW^* = V_Q^* V_Q + WW^* = P_{N(Q)^\perp} + P_{N(Q)} = I$. In a similar manner we can prove that $(Q^*(TT^*)^\dagger T + W)^*(Q^*(TT^*)^\dagger T + W) = I$, i.e., $(Q^*(TT^*)^\dagger T + W) \in \mathcal{U}$ and the proof is finished. □

6. The case $\mathcal{A} = \mathcal{P}$ and $\mathcal{B} = \mathcal{Q}$

The study of $\mathcal{PP}$ done in [8] is an invitation to consider the case $\mathcal{PQ}$. We present here a characterization of $\mathcal{PQ}$ and an example which shows that $\mathcal{PQ} \neq \mathcal{QP}$.

Theorem 6.1. *The set $\mathcal{PQ}$ can be described as:*

$$\mathcal{PQ} = \{T \in \mathcal{L} : R(T(I-T)) \subseteq R(T(I-P)) \text{ where } P = P_{\overline{R(T)}}\}. \tag{7}$$

Moreover, if $T \in \mathcal{PQ}$, $P := P_{\overline{R(T)}}$ and $Q_T := T + (T(I-P))^\dagger (T - T^2)$ then:

1. *$P \in p_1((\mathcal{PQ})_T)$ and $P \leq \tilde{P}$ for all $\tilde{P} \in p_1((\mathcal{PQ})_T)$.*
2. *$Q_T \in p_2((\mathcal{PQ})_T)$ and $Q_T^* Q_T \leq \tilde{Q}^* \tilde{Q}$ for all $\tilde{Q} \in p_2((\mathcal{PQ})_T)$.*
3. *$(P - Q_T)^*(P - Q_T) \leq (\tilde{P} - \tilde{Q})^*(\tilde{P} - \tilde{Q})$ for all $(\tilde{P}, \tilde{Q}) \in (\mathcal{PQ})_T$.*

As a consequence, $||P - Q_T|| \leq ||\tilde{P} - \tilde{Q}||$ for all $(\tilde{P}, \tilde{Q}) \in (\mathcal{PQ})_T$.

Proof. Let $T \in \mathcal{PQ}$. Then, $T = P_{\mathcal{S}} Q$ for some $P_{\mathcal{S}} \in \mathcal{P}$ and $Q \in \mathcal{Q}$. As $\overline{R(T)} \subseteq \mathcal{S}$, then $T = PT = PP_{\mathcal{S}} Q = PQ$. Thus, $P \in p_1((\mathcal{PQ})_T)$. In addition, $T - T^2 = PQ - PQPQ = PQ(I-P)Q$. Therefore, $R(T - T^2) = R(PQ(I-P)Q) \subseteq R(PQ(I-P)) = R(T(I-P))$, as claimed. Conversely, suppose that $R(T(I-T)) \subseteq R(T(I-P))$. Hence, $Q_T := T + (T(I-P))^\dagger (T - T^2) \in \mathcal{L}$ because of Remark 2.4. Moreover, notice that $R((T(I-P))^\dagger (T - T^2)) \subseteq R((T(I-P))^\dagger) = \overline{R((I-P)T^*)} \subseteq R(I-P)$. Therefore, $PQ_T = PT = T$. Finally, as $Q_T = T + (I-P)(T(I-P))^\dagger (T - T^2)$, an easy computation shows that $Q_T^2 = Q_T$, i.e. $Q_T \in \mathcal{Q}$, and (7) is proven. Furthermore, we have shown that $Q_T \in p_2((\mathcal{PQ})_T)$.

Now, let $T \in \mathcal{PQ}$, $P := P_{\overline{R(T)}}$ and $Q_T := T + (T(I-P))^\dagger (T - T^2)$. Let us prove items 1., 2. and 3.:

1. We have already proved that $P \in p_1((\mathcal{PQ})_T)$. Now, if $T = \tilde{P} Q$ for some $\tilde{P} \in \mathcal{P}$ and $Q \in \mathcal{Q}$ then $\overline{R(T)} \subseteq R(\tilde{P})$ and so $P \leq \tilde{P}$.

2. We have already proved that $Q_T \in p_2((\mathcal{PQ})_T)$. Let us prove that $Q_T^*Q_T \leq \tilde{Q}^*\tilde{Q}$ for all $\tilde{Q} \in p_2((\mathcal{PQ})_T)$. If $\tilde{Q} \in p_2((\mathcal{PQ})_T)$ then $T = P\tilde{Q}$ and so $N(\tilde{Q}) \subseteq N(T)$. On the other hand, it is straightforward that $N(Q_T) = N(T)$. Hence, $N(\tilde{Q}) \subseteq N(Q_T)$ or, equivalently, $R(Q_T^*) \subseteq R(\tilde{Q}^*)$. Then, by Douglas' theorem, $Q_T^*Q_T \leq \lambda\tilde{Q}^*\tilde{Q}$ for some positive constant λ. We claim that $\lambda \leq 1$. Applying Douglas' theorem again, we have that $||(\tilde{Q}^*)^\dagger Q_T^*||^2 = \inf\{\lambda > 0 : Q_T^*Q_T \leq \lambda\tilde{Q}^*\tilde{Q}\}$. By [20, Lemma 2.3] (see also [7, Theorem 4.1]), we have that $(\tilde{Q}^*)^\dagger = P_{R(\tilde{Q})}P_{R(\tilde{Q}^*)}$ then $||(\tilde{Q}^*)^\dagger Q_T^*|| = ||P_{R(\tilde{Q})}P_{R(\tilde{Q}^*)}Q_T^*|| = ||P_{R(\tilde{Q})}Q_T^*|| = ||Q_TP_{R(\tilde{Q})}||$ where the second equality holds because $R(Q_T^*) \subseteq R(\tilde{Q}^*)$. Now, recalling that $Q_T := T + (T(I-P))^\dagger(T - T^2)$ and replacing T by $P\tilde{Q}$ we get that

$$\begin{aligned} Q_T &= P\tilde{Q} + (P\tilde{Q}(I-P))^\dagger(P\tilde{Q} - (P\tilde{Q})^2) \\ &= P\tilde{Q} + (P\tilde{Q}(I-P))^\dagger P\tilde{Q}(I-P)\tilde{Q} \\ &= P\tilde{Q} + P_{\overline{R((I-P)\tilde{Q}^*P)}}\tilde{Q} = (P + P_{\overline{R((I-P)\tilde{Q}^*P)}})\tilde{Q} \\ &= P_{(R(P)+\overline{R((I-P)\tilde{Q}^*P)})}\tilde{Q}, \end{aligned}$$

where the last equality follows because $R(P) \perp \overline{R((I-P)\tilde{Q}^*P)}$. Therefore,

$$Q_TP_{R(\tilde{Q})} = P_{(R(P)+\overline{R((I-P)\tilde{Q}^*P)})}\tilde{Q}P_{R(\tilde{Q})} = P_{(R(P)+\overline{R((I-P)\tilde{Q}^*P)})}P_{R(\tilde{Q})}.$$

Thus, $\inf\{\lambda > 0 : Q_T^*Q_T \leq \lambda\tilde{Q}^*\tilde{Q}\} = ||Q_TP_{R(\tilde{Q})}||^2 = ||P_{(R(P)+\overline{R((I-P)\tilde{Q}^*P)})}P_{R(\tilde{Q})}||^2 \leq 1$.

3. It follows by items 1 and 2.

□

Remark 6.2. Note that $\mathcal{PQ} \neq \mathcal{QP}$: let $T = \begin{pmatrix} \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix} \in \mathcal{PQ}$. Now, $T^*(I - T^*) = \frac{1}{4}\begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \\ -1 & -1 & 0 \end{pmatrix}$ and $T^*(I - P_{R(T^*)}) = T^*P_{N(T)} = \frac{1}{2}\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$. Hence, $R(T^*(I-T^*)) = \text{gen}\{(1\ 0\ -1)^T\}$ and $R(T^*(I - P_{R(T^*)})) = \text{gen}\{(1\ 0\ 1)^T\}$, i.e., $R(T^*(I-T^*)) \nsubseteq R(T^*(I - P_{R(T^*)}))$. Therefore, by Theorem 6.1, $T^* \notin \mathcal{PQ}$ and so $T \notin \mathcal{QP}$.

Corollary 6.3. *The following equality holds:*

$$\mathcal{QP} = \{T \in \mathcal{L} : R(T^*(I - T^*)) \subseteq R(T^*P_{N(T)})\}.$$

Remark 6.4. If $T \in \mathcal{PP}$, a fortiori, $T \in \mathcal{PQ}$ so the minimal projections $P_{N(T)^\perp}$ and Q_T can be compared. It turns out that they coincide. In fact, if $T \in \mathcal{PP}$ then $T = P_{\overline{R(T)}}P_{N(T)^\perp}$ and so it suffices to replace T by $P_{\overline{R(T)}}P_{N(T)^\perp}$ in the formula of Q_T to get that $Q_T = P_{N(T)^\perp}$.

Notice that Remark 6.2 also shows that $\mathcal{QQ} \neq \mathcal{PQ}$.

7. Miscellanea

In this section we obtain some results on $\mathcal{PA}, \mathcal{AP}, \mathcal{QA}$ or $\mathcal{AQ}$ for some sets $\mathcal{A}$'s which properly contain $\mathcal{U}$ or $\mathcal{G}$. Recall that Corollary 2.11 is a characterization of $\mathcal{PJ}$.

Corollary 7.1. *The next equalities hold:*

1. $\mathcal{PI}^* = \{V \in \mathcal{J} : \dim N(V^*) \leq \dim N(V)\}$.
2. $\mathcal{IP} = \{V \in \mathcal{J} : \dim N(V) \leq \dim N(V^*)\}$.
3. $\mathcal{QI}^* = \left\{ T \in \mathcal{L} : \begin{array}{c} \dim N(T^*) \leq \dim N(T), \\ \gamma(|T^*|) \geq 1, \\ \dim \overline{R(TT^* - P_{R(T)})} \leq \dim R(T)^{\perp} \end{array} \right\}$.
4. $\mathcal{IQ} = \left\{ T \in \mathcal{L} : \begin{array}{c} \dim N(T) \leq \dim N(T^*), \\ \gamma(|T|) \geq 1, \\ \dim \overline{R(T^*T - P_{R(T^*)})} \leq \dim N(T) \end{array} \right\}$.
5. $\mathcal{PE} = \mathcal{QE} = \{T \in \mathcal{L}_{cr} : \dim N(T^*) \leq \dim N(T)\}$.
6. $\mathcal{E}^*\mathcal{P} = \mathcal{E}^*\mathcal{Q} = \{T \in \mathcal{L}_{cr} : \dim N(T) \leq \dim N(T^*)\}$.
7. $\mathcal{PN} = \mathcal{QN} = \{T \in \mathcal{L} : \dim N(T) \leq \dim N(T^*)\}$.
8. $\mathcal{L}_d\mathcal{P} = \mathcal{L}_d\mathcal{Q} = \{T \in \mathcal{L} : \dim N(T^*) \leq \dim N(T)\}$.

Proof. Items 1 and 3 can be proven following similar lines than in Theorems 3.1 and 5.2. Items 2 and 4 follows by taking adjoints in items 1 and 3, respectively.

5. If $T = PE$ with $E \in \mathcal{E}$ then $R(P) = R(T)$ and so $R(T)$ is closed. On the other hand, define $G := E^\dagger(I - P)$. Notice that $N(G) = N(I - P)$ because $E^\dagger$ is injective, and $R(G) \subseteq N(T)$ because if $y = E^\dagger(I-P)x \in R(G)$ then $Ty = PEE^\dagger(I-P)x = P(I-P)x = 0$. Therefore, $\dim N(T^*) \leq \dim N(T)$. Conversely, if $R(T) = \overline{R(T)}$ and $\dim N(T^*) \leq \dim N(T)$ then there exists a partial isometry W with initial subspace included in $N(T)$ and final subspace $N(T^*)$. Then, $T = P(T + W)$ where $P = P_{R(T)}$ and $R(T + W) = R(T) + R(W) = R(T) + N(T^*) = \mathcal{H}$ where $(T, W) \in \mathcal{R}$ by [3, Proposition 2.2]. Finally, given $T = QE$ with $Q \in \mathcal{Q}$ and $E \in \mathcal{E}$ then $T = QE = P(Q + (1 - P))E$ where $(Q + (1 - P))E \in \mathcal{E}$ because $Q + (1 - P)$ is invertible.

6. Follows by taking adjoints in item 5.

7. Let $T = PA$ with $N(A) = \{0\}$. Without loss of generality we can consider $P = P_{\overline{R(T)}}$. Then, $A|_{N(T)} : N(T) \to N(P) = N(T^*)$ is injective, so $\dim(N(T)) \leq \dim N(T^*)$. Conversely, if $\dim(N(T)) \leq \dim N(T^*)$ then there exists $B \in \mathcal{L}$ such that $B|_{N(T)} : N(T) \to N(T^*) = N(P)$ injective and $B(N(T)^\perp) = \{0\}$. Again $P = P_{\overline{R(T)}}$. Therefore, $T = P(T + B)$ and $N(T+B) = \{0\}$ because if $(T+B)x = 0$ then $Tx = -Bx \in R(T) \cap R(B) \subseteq R(T) \cap R(T)^\perp = \{0\}$, so $Bx = 0$ with $x \in N(T)$ and so $x = 0$ since $B|_{N(T)}$ is injective. The equality $\mathcal{PN} = \mathcal{QN}$ can be proved in similar way as in the previous item.

8. Follows by taking adjoints in item 7. $\square$

We show next that, in many instances, it holds $\mathcal{AB} = \mathcal{BA}$. However, at Remark 6.2 we have given an example for which $\mathcal{AB} \neq \mathcal{BA}$.

Corollary 7.2. *The next equalities hold:*

$$\mathcal{PG} = \mathcal{GP}, \ \mathcal{PU} = \mathcal{UP}, \ \mathcal{QG} = \mathcal{GQ}, \ \mathcal{QU} = \mathcal{UQ}$$

Proof. By Theorem 4.1, $T \in \mathcal{PG}$ if and only if $T^* \in \mathcal{PG}$. Taking adjoints, we get $\mathcal{PG} = \mathcal{GP}$. The same argument shows that $\mathcal{PU} = \mathcal{UP}$ and $\mathcal{QG} = \mathcal{GQ}$.

Although it is not trivial from Theorem 5.2, it also holds that $T \in \mathcal{QU}$ if and only if $T^* \in \mathcal{QU}$. In fact, if $T = QU$ with Q an oblique projection and U a unitary operator then $T^* = U^*Q^* = (U^*Q^*U)U^*$ where $U^*Q^*U \in \mathcal{Q}$ and $U^* \in \mathcal{U}$. Thus, $T^* \in \mathcal{QU}$. Now, the proof of $\mathcal{QU} = \mathcal{UQ}$ follows the same lines that above. □

Proposition 7.3. *Let $T \in \mathcal{L}$ and $P = P_{\overline{R(T)}}$. The next conditions are equivalent:*

1. $T \in \mathcal{PI}$.
2. *There exists $P_1 \in \mathcal{P}$ such that $TT^* = PP_1P$ with $\dim R(P_1) = \dim \mathcal{H}$ and $\dim N(T) \leq \dim R(P_1(I-P))$.*

Proof. $1 \Rightarrow 2$ Let $T = PZ$ with $Z \in \mathcal{I}$, then $TT^* = PZZ^*P = PP_1P$ where $P_1 = ZZ^* \in \mathcal{P}$. Since $Z \in \mathcal{I}$ then $\dim R(P_1) = \dim R(Z) = \dim \mathcal{H}$. On the other hand, let us see that $N(T) \subseteq R(Z^*(I-P))$. Indeed, if $x \in N(T)$ then $Tx = PZx = 0$, i.e, $Zx \in N(P) = R(I-P)$. Thus, $x = Z^*Zx \in R(Z^*(I-P))$. Therefore, $\dim N(T) \leq \dim R(Z^*(I-P)) = \dim ZR(Z^*(I-P)) = \dim R(P_1(I-P))$.

$2 \Rightarrow 1$ Suppose that item 2 holds. Notice that if $\dim R(P_1) = \dim \mathcal{H}$ then there exist $Z \in \mathcal{I}$ such that $ZZ^* = P_1$. Therefore, $TT^* = PZZ^*P$ for some $Z \in \mathcal{I}$. Thus, $|T^*| = |(PZ)^*|$. Now, if $T = |T^*|V_T$ and $PZ = |(PZ)^*|V_{PZ}$ are the polar decompositions of T and PZ respectively, then $T = PZV_{PZ}^*V_T$. Define $J := V_{PZ}^*V_T$. We claim that J is a partial isometry with initial space $N(T)^\perp$ and final space $N(PZ)^\perp$. In fact, $JJ^* = V_{PZ}^*V_TV_T^*V_{PZ} = V_{PZ}^*V_{PZ} = P_{N(PZ)^\perp}$, where the second equality holds because $R(V_T) = \overline{R(T)} = \overline{R(TT^*)} = \overline{R(PZZ^*P)} = \overline{R(PZ)} = R(V_{PZ})$. On the other hand, $J^*J = V_T^*V_{PZ}V_{PZ}^*V_T = V_T^*V_T = P_{N(T)^\perp}$ as desired.

Now, let F be a partial isometry with initial space $N(T)$ and a subspace of $N(PZ)$ as final space. The existence of F is guaranteed because $\dim N(PZ) = \dim Z^*(R(T)^\perp) = \dim Z^*(R(I-P)) = \dim ZR(Z^*(I-P)) = \dim R(P_1(I-P)) \geq \dim N(T)$.

Now, $J + F \in \mathcal{I}$ because $(J+F)^*(J+F) = J^*J + J^*F + F^*J + F^*F = J^*J + F^*F = P_{N(T)^\perp} + P_{N(T)} = I$ where $J^*F = 0$ because $N(J^*) = N(JJ^*) = N(PZ) \supseteq R(F)$. Thus, $Z(J+F) \in \mathcal{I}$ and $T = PZ(J+F)$, i.e., $T \in \mathcal{PI}$. □

By Theorem 4.1 and items 3 and 4 of Corollary 7.1, we get the next result.

Corollary 7.4. *The next equalities hold: $\mathcal{PE} \cap \mathcal{PN} = \mathcal{QE} \cap \mathcal{QN} = \mathcal{QG} = \mathcal{PG}$*

Remark 7.5. In spite of the results of Corollary 7.4, one cannot expect a general result of the type $\mathcal{AB} \cap \mathcal{AC} = \mathcal{A}(\mathcal{B} \cap \mathcal{C})$. In fact, consider $\mathcal{A} = \mathcal{L}^+$, $\mathcal{B} = \mathcal{J}$ and $\mathcal{C} = \mathcal{G}^+$. Then, $\mathcal{AB} = \mathcal{L}$, $\mathcal{AC} = \{T \in \mathcal{L} : \exists A \in \mathcal{L}^+ \text{ such that } T \sim A\}$ and so $\mathcal{AB} \cap \mathcal{AC} = \mathcal{AC}$. On the other hand, $\mathcal{B} \cap \mathcal{C} = \{I\}$ and so $\mathcal{A}(\mathcal{B} \cap \mathcal{C}) = \mathcal{A}$. Now, since $\mathcal{AC} = \{T \in \mathcal{L} : \exists A \in \mathcal{L}^+ \text{ such that } T \sim A\} \neq \mathcal{L}^+ = \mathcal{A}$ we get that $\mathcal{AB} \cap \mathcal{AC} \neq \mathcal{A}(\mathcal{B} \cap \mathcal{C})$.

Acknowledgments We thank the referee for a careful reading and for her/his suggestions which improved the article.

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