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On the shape of possible counterexamples to the Jacobian Conjecture

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We improve the algebraic methods of Abhyankar for the Jacobian Conjecture in dimension two and describe the shape of possible counterexamples. We give an elementary proof of the result of Heitmann in [5], which states that $\gcd(\deg(P), \deg(Q)) \geq 16$ for any counterexample (P, Q). We also prove that $\gcd(\deg(P), \deg(Q)) \neq 2p$ for any prime p.

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Introduction

Let K be a field of characteristic zero. The Jacobian Conjecture (JC) in dimension two, stated by Keller in [7], says that any pair of polynomials $P, Q \in L := K[x, y]$ with $[P, Q] := \partial_x P \partial_y Q - \partial_x Q \partial_y P \in K^\times$ defines an automorphism of $K[x, y]$.

In this paper we improve the algebraic methods of Abhyankar describing the shape of the support of possible counterexamples. We use elementary algebraic methods combined with basic discrete analytic geometry on the plane, i.e. on the points $\mathbb{N}_0 \times \mathbb{N}_0$ in the case of $L = K[x, y]$ and in $\frac{1}{l}\mathbb{Z} \times \mathbb{N}_0$ in the case of $L^{(l)} := K[x^{\pm\frac{1}{l}}, y]$.

The first innovation is a definition of the directions and an order relation on them, based on the crossed product of vectors, which simplifies substantively the treatment of consecutive directions associated with the Newton polygon of Jacobian pairs. It is related to [5, Lemma 1.15] and enables us to simplify substantially the treatment of the Newton polygon and its edges (compare with [2, 7.4.14]).

The second innovation lies in the use of the polynomial F with $[F, \ell_{\rho,\sigma}(P)] = \ell_{\rho,\sigma}(P)$, obtained in Theorem 2.6 for a given Jacobian pair (P, Q). This element can be traced back to 1975 in [6]. There also appears the element $G_0 \in K[P, Q]$, which becomes important in the proof of our Proposition 7.1. The polynomial F mentioned above is well known and used by many authors, see for example [6,10] and [11, 10.2.8] (together with [11, 10.2.17 i)]). In Theorem 2.6, we add some geometric statements on the shape of the supports, especially about the endpoints (called st and en) associated to an edge of the Newton Polygon. In [5, Proposition 1.3] some of these statements, presented in an algebraic form, can be found.

We will apply different endomorphisms in order to deform the support of a Jacobian pair. Opposed to most of the authors working in this area [5,12,9], we remain all the time in L (or $L^{(l)}$). In order to do this we use the following very simple expression of the change of the Jacobian under an endomorphism $\varphi\colon L \to L$ (or $L \to L^{(l)}$, or $L^{(l)} \to L^{(l)}$):

$$[\varphi(P), \varphi(Q)] = \varphi([P, Q])[\varphi(x), \varphi(y)].$$

Another key ingredient is the concept of regular corners and its classification, which we present in Section 5. The geometric fact that certain edges can be cut above the diagonal, Proposition 5.16, was already known to Joseph and used in [6, Theorem 4.2], in order to prove the polarization theorem.

In Section 6 we give an elementary proof of a result of [5]: If

$$B := \begin{cases} \infty & \text{if the jacobian conjecture is true,} \\ \min\bigl(\gcd(v_{1,1}(P), v_{1,1}(Q))\bigr) & \text{if it is false, where } (P,Q) \text{ runs} \\ & \text{on the counterexamples,} \end{cases}$$

then $B \geq 16$. In spite of Heitmann's assertion "Nothing like this appears in the literature but results of this type are known by Abhyankar and Moh and are easily inferred from their published work", referring to his result, we do not know how to do this, and we did not find anything like this in the literature till now. For example, in the survey papers [2] and [3], this result is not mentioned, although in [3, Corollary 8.9] it is proven that $B \geq 9$.

In Section 7 we present our main new results: Propositions 7.1 and 7.3 and Corollaries 7.2 and 7.4. At first sight they look rather technical, but are related to the fact that for a Jacobian pair (P, Q) in $K[x, y]$ we know that P and Q are star symmetric. Propositions 7.1 and 7.3 yield partial star symmetries between elements in $K[P, Q]$ and P, whereas Corollaries 7.2 and 7.4 guarantee that the leading forms of P associated with certain directions can be written as powers of certain polynomials. This allows us to establish a very strong divisibility criterion for the possible regular corners, Theorem 7.6, which enables us to prove that $B \neq 2p$ for all prime p. This result is announced to be proven by Abhyankar in a remark after [5, Theorem 1.16], and it is said that it can be proven similarly to [5, Proposition 2.21]. However, we could not translate the proof of [5, Proposition 2.21] to our setting nor modify it to give $B \neq 2p$. Once again in the survey articles [2] and [3], this result is not mentioned, although in [2, Lemma 6.1] it is proven that $\gcd(\deg(P), \deg(Q)) \neq p$. We also found [12, Theorem 4.12] from which $B \neq 2p$ follows. But the proof relies on [12, Lemma 4.10], which has a gap, since it claims without proof that $I_2 \subset \frac{1}{m}\Gamma(f_2)$, an assertion which cannot be proven to be true. The same article claims to have proven that $B > 16$, and the author claims to have verified that $B > 33$, but it relies on the same flawed argument, so $B \geq 16$ remains up to the moment the best lower limit for B.

One part of our strategy is described by [5]: "The underlying strategy is the minimal counterexample approach. We assume the Jacobian conjecture is false and derive properties which a minimal counterexample must satisfy. The ultimate goal is either a contradiction (proving the conjecture) or an actual counterexample." Actually this is the strategy followed by Moh in [9], who succeeded in proving that for a counterexample (P, Q), $\max(\deg(P), \deg(Q)) > 100$. The trouble of this strategy is that the number of equations and variables one has to solve in order to discard the possible counterexamples, grows rapidly, and the brute force approach with computers gives no conceptual progress, although it allows us to increase the lower bound for $\max(\deg(P), \deg(Q))$.

The approach followed in [5] is more promising, since every possible B ruled out actually eliminates a whole infinite family of possible counterexamples and cannot be achieved by computer power.

Using the classification of regular corners we can produce the algebraic data corresponding to a resolution at infinity, and these data are strongly related to the shape of a possible counterexample. It would be interesting to describe thoroughly the relation between the algebraic and topological methods used in the different approaches mentioned above.

The results in the first six sections of this paper are analogous to those established for the one dimensional Dixmier conjecture in [4]. The first section is just a reminder of definitions and properties from [4]. In Section 2 we give an improved version of the analogous results in that paper, the main difference being the proof of the existence of G_0 in Theorem 2.6 and Proposition 2.11(5). In section 3 we recall some of the results of [4] about the order of directions. At the beginning of Section 4 we introduce the concept of a minimal pair and prove that a minimal pair can be assumed to have a trapezoidal shape.

The results corresponding to Proposition 5.3 of [4] now are distributed along various propositions that classify regular corners in section 5. In section 6 we obtain the fact that $B \geq 16$, in the same way as the corresponding result in [4]. The rest of the results in this paper are new.

We point out that the proof that $B \neq 2p$ for any prime number p can be adapted easily to the case of the Dixmier conjecture.

1. Preliminaries

We recall some notations and properties from [4]. For each $l \in \mathbb{N}$, we consider the commutative K-algebra $L^{(l)}$, generated by variables $x^{\frac{1}{l}}$, $x^{\frac{-1}{l}}$ and y, subject to the relation $x^{\frac{1}{l}} x^{\frac{-1}{l}} = 1$. In other words $L^{(l)} := K[x^{\frac{1}{l}}, x^{\frac{-1}{l}}, y]$. Obviously, there is a canonical inclusion $L^{(l)} \subseteq L^{(h)}$, for each $l, h \in \mathbb{N}$ such that $l|h$. We define the set of directions by

$$\mathfrak{V} := \{(\rho, \sigma) \in \mathbb{Z}^2 : \gcd(\rho, \sigma) = 1\}.$$

We also define

$$\mathfrak{V}_{\geq 0} := \{(\rho, \sigma) \in \mathfrak{V} : \rho + \sigma \geq 0\},$$
$$\mathfrak{V}_{>0} := \{(\rho, \sigma) \in \mathfrak{V} : \rho + \sigma > 0\}$$

and

$$\mathfrak{V}^0 := \{(\rho, \sigma) \in \mathfrak{V} : \rho + \sigma > 0 \text{ and } \rho > 0\}.$$

Note that $\mathfrak{V}_{\geq 0} = \mathfrak{V}_{>0} \cup \{(1, -1), (-1, 1)\}$.

Definition 1.1. For all $(\rho, \sigma) \in \mathfrak{V}$ and $(i/l, j) \in \frac{1}{l}\mathbb{Z} \times \mathbb{Z}$ we write $v_{\rho,\sigma}(i/l, j) := \rho i/l + \sigma j$.

Definition 1.2. Let $(\rho, \sigma) \in \mathfrak{V}$. For $P = \sum a_{\frac{i}{l},j} x^{\frac{i}{l}} y^j \in L^{(l)} \setminus \{0\}$, we define:

- The *support* of P as $\operatorname{Supp}(P) := \left\{(i/l, j) : a_{\frac{i}{l},j} \neq 0\right\}$.
- The (ρ, σ)-*degree* of P as $v_{\rho,\sigma}(P) := \max\left\{v_{\rho,\sigma}(i/l, j) : a_{\frac{i}{l},j} \neq 0\right\}$.
- The (ρ, σ)-*leading term* of P as $\ell_{\rho,\sigma}(P) := \displaystyle\sum_{\{\rho\frac{i}{l} + \sigma j = v_{\rho,\sigma}(P)\}} a_{\frac{i}{l},j} x^{\frac{i}{l}} y^j$.

Remark 1.3. To abbreviate expressions we set $v_{\rho,\sigma}(0) := -\infty$ and $\ell_{\rho,\sigma}(0) := 0$, for all $(\rho,\sigma) \in \mathfrak{V}$. Moreover, instead of $\operatorname{Supp}(P) = \{a\}$ we will write $\operatorname{Supp}(P) = a$.

Definition 1.4. We say that $P \in L^{(l)}$ is (ρ,σ)-*homogeneous* if $P = \ell_{\rho,\sigma}(P)$.

Definition 1.5. We assign to each direction its corresponding unit vector in S^1, and we define an *interval* in $\mathfrak{V}$ as the preimage under this map of an arc of S^1 that is not the whole circle. We consider each interval endowed with the order that increases counterclockwise.

For each $P \in L^{(l)} \setminus \{0\}$, we let $\mathrm{H}(P)$ denote the convex hull of the support of P. As it is well known, $\mathrm{H}(P)$ is a polygon, called the *Newton polygon of* P, and it is evident that each one of its edges is the convex hull of the support of $\ell_{\rho,\sigma}(P)$, where (ρ,σ) is orthogonal to the given edge and points outside of $\mathrm{H}(P)$.

Notation 1.6. Let $(\rho,\sigma) \in \mathfrak{V}$ arbitrary. We let $\mathrm{st}_{\rho,\sigma}(P)$ and $\mathrm{en}_{\rho,\sigma}(P)$ denote the first and the last point that we find on $H(\ell_{\rho,\sigma}(P))$ when we run counterclockwise along the boundary of $H(P)$. Note that these points coincide when $\ell_{\rho,\sigma}(P)$ is a monomial.

The *cross product* of two vectors $A = (a_1, a_2)$ and $B = (b_1, b_2)$ in $\mathbb{R}^2$ is $A \times B := \det\left(\begin{smallmatrix} a_1 & a_2 \\ b_1 & b_2 \end{smallmatrix}\right)$.

Remark 1.7. Note that if $\ell_{\rho,\sigma}(P)$ is not a monomial, then $(\rho,\sigma) \times (\mathrm{en}_{\rho,\sigma}(P) - \mathrm{st}_{\rho,\sigma}(P)) > 0$.

Remark 1.8. If $(\rho_0,\sigma_0) < (\rho,\sigma) < (-\rho_0,-\sigma_0)$, then $v_{\rho_0,\sigma_0}(\mathrm{en}_{\rho,\sigma}(P)) \le v_{\rho_0,\sigma_0}(\mathrm{st}_{\rho,\sigma}(P))$, while if $(\rho_0,\sigma_0) > (\rho,\sigma) > (-\rho_0,-\sigma_0)$, then $v_{\rho_0,\sigma_0}(\mathrm{en}_{\rho,\sigma}(P)) \ge v_{\rho_0,\sigma_0}(\mathrm{st}_{\rho,\sigma}(P))$, with equality in both cases only if $\ell_{\rho,\sigma}(P)$ is a monomial. Moreover, in the first case

$$\mathrm{st}_{\rho,\sigma}(P) = \operatorname{Supp}(\ell_{\rho_0,\sigma_0}(\ell_{\rho,\sigma}(P))) \qquad \text{and} \qquad \mathrm{en}_{\rho,\sigma}(P) = \operatorname{Supp}(\ell_{-\rho_0,-\sigma_0}(\ell_{\rho,\sigma}(P))).$$

Hence, if $(\rho,\sigma) \in \mathfrak{V}_{>0}$, then

$$\mathrm{st}_{\rho,\sigma}(P) = \operatorname{Supp}(\ell_{1,-1}(\ell_{\rho,\sigma}(P))) \qquad \text{and} \qquad \mathrm{en}_{\rho,\sigma}(P) = \operatorname{Supp}(\ell_{-1,1}(\ell_{\rho,\sigma}(P))),$$

and, if $\rho + \sigma < 0$, then

$$\mathrm{st}_{\rho,\sigma}(P) = \operatorname{Supp}(\ell_{-1,1}(\ell_{\rho,\sigma}(P))) \qquad \text{and} \qquad \mathrm{en}_{\rho,\sigma}(P) = \operatorname{Supp}(\ell_{1,-1}(\ell_{\rho,\sigma}(P))).$$

Remark 1.9. Let $P, Q \in L^{(l)} \setminus \{0\}$ and $(\rho,\sigma) \in \mathfrak{V}$. The following assertions hold:

(1) $\ell_{\rho,\sigma}(PQ) = \ell_{\rho,\sigma}(P)\ell_{\rho,\sigma}(Q)$.
(2) If $P = \sum_i P_i$, $v_{\rho,\sigma}(P_i) = v_{\rho,\sigma}(P)$ and $\sum_i \ell_{\rho,\sigma}(P_i) \neq 0$, then $\ell_{\rho,\sigma}(P) = \sum_i \ell_{\rho,\sigma}(P_i)$.

(3) $v_{\rho,\sigma}(PQ) = v_{\rho,\sigma}(P) + v_{\rho,\sigma}(Q)$.
(4) $\mathrm{st}_{\rho,\sigma}(PQ) = \mathrm{st}_{\rho,\sigma}(P) + \mathrm{st}_{\rho,\sigma}(Q)$.
(5) $\mathrm{en}_{\rho,\sigma}(PQ) = \mathrm{en}_{\rho,\sigma}(P) + \mathrm{en}_{\rho,\sigma}(Q)$.
(6) $-v_{-\rho,-\sigma}(P) \le v_{\rho,\sigma}(P)$.

We will use freely these facts throughout the article.

Notation 1.10. For $P, Q \in L^{(l)}$ we write $[P, Q] := \det \mathrm{J}(P, Q)$, where $\mathrm{J}(P, Q)$ is the jacobian matrix of (P, Q).

Definition 1.11. Let $P, Q \in L^{(l)}$. We say that (P, Q) is a *Jacobian pair* if $[P, Q] \in K^{\times}$.

Remark 1.12. Let $P, Q \in L^{(l)} \setminus \{0\}$ and let $(\rho, \sigma) \in \mathfrak{V}$. We have:

(1) If P and Q are (ρ, σ)-homogeneous, then $[P, Q]$ is also. If moreover $[P, Q] \neq 0$, then

$$v_{\rho,\sigma}([P, Q]) = v_{\rho,\sigma}(P) + v_{\rho,\sigma}(Q) - (\rho + \sigma).$$

(2) If $P = \sum_i P_i$ and $Q = \sum_j Q_j$ are the (ρ, σ)-homogeneous decompositions of P and Q, then the (ρ, σ)-homogeneous decomposition $[P, Q] = \sum_k [P, Q]_k$ is given by

$$[P, Q]_k = \sum_{i+j=k+\rho+\sigma} [P_i, Q_j]. \tag{1.1}$$

(3) If $[P, Q] = 0$, then $[\ell_{\rho,\sigma}(P), \ell_{\rho,\sigma}(Q)] = 0$.

Proposition 1.13. *Let $P, Q \in L^{(l)} \setminus \{0\}$ and $(\rho, \sigma) \in \mathfrak{V}$. We have*

$$v_{\rho,\sigma}([P, Q]) \le v_{\rho,\sigma}(P) + v_{\rho,\sigma}(Q) - (\rho + \sigma). \tag{1.2}$$

Moreover

$$v_{\rho,\sigma}([P, Q]) = v_{\rho,\sigma}(P) + v_{\rho,\sigma}(Q) - (\rho + \sigma) \iff [\ell_{\rho,\sigma}(P), \ell_{\rho,\sigma}(Q)] \neq 0$$

and in this case $[\ell_{\rho,\sigma}(P), \ell_{\rho,\sigma}(Q)] = \ell_{\rho,\sigma}([P, Q])$.

Proof. It follows directly from the decomposition (1.1). □

Definition 1.14. We say that two vectors $A, B \in \mathbb{R}^2$ are *aligned* and write $A \sim B$, if $A \times B = 0$.

Remark 1.15. Note that the restriction of $\sim$ to $\mathbb{R}^2 \setminus \{0\}$ is an equivalence relation. Note also that if $A \in \mathbb{R} \times \mathbb{R}_{>0}$, $B \in \mathbb{R} \times \mathbb{R}_{\ge 0}$ and $A \sim B$, then $B = \lambda A$ for some $\lambda \ge 0$.

2. Shape of Jacobian pairs

The results in this section appear in several papers, for instance [1,5] and [6], but we need to establish them in a slightly different form, including the geometric information about the shape of the support.

Proposition 2.1. *Let $(\rho,\sigma) \in \mathfrak{V}$ and let $P, Q \in L^{(l)} \setminus \{0\}$ be two (ρ,σ)-homogeneous elements. Set $\tau := v_{\rho,\sigma}(P)$ and $\mu := v_{\rho,\sigma}(Q)$.*

(1) *If $\tau = \mu = 0$, then $[P, Q] = 0$.*
(2) *Assume that $[P, Q] = 0$ and $(\mu, \tau) \neq (0, 0)$. Let $m, n \in \mathbb{Z}$ with $\gcd(m, n) = 1$ and $n\tau = m\mu$. Then*
   a) *There exists $\alpha \in K^\times$ such that $P^n = \alpha Q^m$.*
   b) *There exist $R \in L^{(l)}$ and $\lambda_P, \lambda_Q \in K^\times$, such that*

$$P = \lambda_P R^m \quad \text{and} \quad Q = \lambda_Q R^n. \tag{2.1}$$

   Moreover
   – *if $\mu\tau < 0$, then $P, Q \in K[x^{1/l}, x^{-1/l}]$,*
   – *if $\mu\tau \geq 0$, then we can choose $m, n \in \mathbb{N}_0$,*
   – *if $P, Q \in L$, then $R \in L$.*

Proof. (1) If $\rho = 0$, then $P, Q \in K[x^{1/l}, x^{-1/l}]$ and if $\rho \neq 0$, then $P, Q \in K[z]$ where $z := x^{-\sigma/\rho} y$. In both cases, $[P, Q] = 0$ follows easily.

(2a) This is [6, Proposition 2.1(2)].

(2b) Assume first that $\mu\tau < 0$ and take $n, m \in \mathbb{Z}$ coprime with $n\tau = m\mu$. By statement (a), there exists $\alpha \in K^\times$ such that $P^n = \alpha Q^m$. Since $mn < 0$, necessarily $P, Q \in K[x^{1/l}, x^{-1/l}]$ and $\rho \neq 0$. Moreover, since P and Q are (ρ,σ)-homogeneous,

$$P = \lambda_P x^{\frac{r}{l}} \quad \text{and} \quad Q = \lambda_Q x^{\frac{u}{l}},$$

for some $\lambda_P, \lambda_Q \in K^\times$ and $r, u \in \mathbb{Z}$ with $rn = um$. Clearly $R := x^{\frac{r}{lm}} = x^{\frac{u}{ln}}$ satisfies (2.1). In order to finish the proof we only must note that, since m and n are coprime, $R \in L^{(l)}$.

Assume now $\mu\tau \geq 0$ and let $m, n \in \mathbb{N}_0$ be such that $n\tau = m\mu$ and $\gcd(m, n) = 1$. Set

$$z := \begin{cases} x^{-\frac{\sigma}{\rho}} y & \text{if } \rho \neq 0, \\ x^{\frac{1}{l}} & \text{if } \rho = 0, \end{cases}$$

and write

$$P = x^{\frac{r}{l}} y^s f(z) \quad \text{and} \quad Q = x^{\frac{u}{l}} y^v g(z),$$

where $f, g \in K[z]$ with $f(0) \neq 0 \neq g(0)$. By statement (2a) there exists $\alpha \in K^\times$ such that

$$x^{\frac{nr}{l}} y^{ns} f^n(z) = P^n = \alpha Q^m = \alpha x^{\frac{mu}{l}} y^{mv} g^m(z),$$

from which we obtain

$$(nr/l, ns) = (mu/l, mv) \quad \text{and} \quad f^n = \alpha g^m.$$

Since $\gcd(m, n) = 1$, by the second equality there exist $h \in K[z]$ and $\lambda_P, \lambda_Q \in K^\times$ such that

$$f = \lambda_P h^m(z) \quad \text{and} \quad g = \lambda_Q h^n(z) \tag{2.2}$$

Take $c, d \in \mathbb{Z}$ such that $cm + dn = 1$ and define $(a/l, b) := c(r/l, s) + d(u/l, v)$. Since

$$m(a/l, b) = (r/l, s) \quad \text{and} \quad n(a/l, b) = (u/l, v),$$

it follows from (2.2), that $R := x^{\frac{a}{l}} y^b h(z)$ satisfies (2.1), as desired.

Finally, if $P, Q \in L$, then $v_{-1,0}(R) = \frac{1}{m} v_{-1,0}(P) \leq 0$, which combined with the fact that $R \in L^{(1)}$ implies that $R \in L$. □

Lemma 2.2. *Let $(\rho, \sigma) \in \mathfrak{V}$ and let $P, Q \in L^{(l)} \setminus \{0\}$ be such that $[P, Q] \in K^\times$. If $v_{\rho,\sigma}(P) \neq 0$, then there exists $G_0 \in K[P, Q]$ such that*

$$[\ell_{\rho,\sigma}(G_0), \ell_{\rho,\sigma}(P)] \neq 0 \quad \textit{and} \quad [[\ell_{\rho,\sigma}(G_0), \ell_{\rho,\sigma}(P)], \ell_{\rho,\sigma}(P)] = 0.$$

Moreover, if we define recursively $G_i := [G_{i-1}, P]$, then $[\ell_{\rho,\sigma}(G_i), \ell_{\rho,\sigma}(P)] = 0$ for $i \geq 1$.

Proof. Let $t \in \mathbb{N}$ and set

$$M(t) := \operatorname{linspan}\{P^i Q^j : i, j = 0, \dots, t\}.$$

Since $\{P^i Q^j\}$ is linearly independent, we have

$$\dim M(t) = (t + 1)^2. \tag{2.3}$$

On the other hand, a direct computation shows that

$$mt \leq -v_{-\rho,-\sigma}(z) \leq v_{\rho,\sigma}(z) \leq Mt \quad \text{for each } z \in M(t),$$

where

$$m := \min\{0, -v_{-\rho,-\sigma}(P), -v_{-\rho,-\sigma}(Q), -v_{-\rho,-\sigma}(P) - v_{-\rho,-\sigma}(Q)\}$$

and

$$M := \max\{0, v_{\rho,\sigma}(P), v_{\rho,\sigma}(Q), v_{\rho,\sigma}(P) + v_{\rho,\sigma}(Q)\}.$$

Consequently,

$$J := v_{\rho,\sigma}(M(t) \setminus \{0\}) \subseteq \frac{1}{l}\mathbb{Z} \cap [mt, Mt].$$

For each $\beta \in J$ we take a $z_\beta \in M(t)$ with $v_{\rho,\sigma}(z_\beta) = \beta$. We first prove that there exist $t \in \mathbb{N}$ and $H \in M(t)$, such that

$$[\ell_{\rho,\sigma}(P), \ell_{\rho,\sigma}(H)] \neq 0. \tag{2.4}$$

Assume by contradiction that

$$[\ell_{\rho,\sigma}(P), \ell_{\rho,\sigma}(H)] = 0 \qquad \text{for all } H \in M(t) \text{ and all } t \in \mathbb{N}.$$

We claim that then $M(t) = \operatorname{linspan}\{z_\beta : \beta \in J\}$. In fact, suppose this equality is false and take $z \in M(t) \setminus \operatorname{linspan}\{z_\beta : \beta \in J\}$ with $\beta := v_{\rho,\sigma}(z)$ minimum. By assumption

$$[\ell_{\rho,\sigma}(P), \ell_{\rho,\sigma}(z)] = 0 = [\ell_{\rho,\sigma}(P), \ell_{\rho,\sigma}(z_\beta)].$$

Since $v_{\rho,\sigma}(P) \neq 0$ and $v_{\rho,\sigma}(z) = v_{\rho,\sigma}(z_\beta)$, by Proposition 2.1(2b) there exist $R \in L^{(l)}$, $\lambda, \lambda_\beta \in K^\times$ and $n \in \mathbb{Z}$, such that

$$\ell_{\rho,\sigma}(z) = \lambda R^n \quad \text{and} \quad \ell_{\rho,\sigma}(z_\beta) = \lambda_\beta R^n.$$

Hence $v_{\rho,\sigma}(z - \lambda\lambda_\beta^{-1} z_\beta) < v_{\rho,\sigma}(z)$, which contradicts the choice of z, finishing the proof of the claim. Consequently $\dim M(t) \leq l(M - m)t$, which contradicts (2.3) if we take $t \geq l(M - m)$. Thus we can find $H \in K[P, Q]$ such that (2.4) is satisfied.

We now define recursively $(H_j)_{j\geq 0}$ by setting

$$H_0 := H, \quad \text{and} \quad H_{j+1} := [H_j, P].$$

Since $H_0 \in K[P, Q]$, eventually $H_n = 0$. Let k be the largest index for which $H_k \neq 0$. By Remark 1.12(3) we know that $[\ell_{\rho,\sigma}(H_k), \ell_{\rho,\sigma}(P)] = 0$. But we also have $[\ell_{\rho,\sigma}(H_0), \ell_{\rho,\sigma}(P)] \neq 0$ and hence there exists a largest j such that $[\ell_{\rho,\sigma}(H_j), \ell_{\rho,\sigma}(P)] \neq 0$. By Proposition 1.13 we have

$$[\ell_{\rho,\sigma}(H_j), \ell_{\rho,\sigma}(P)] = \ell_{\rho,\sigma}(H_{j+1}),$$

and so $G_0 := H_j$ satisfies the required conditions. □

Proposition 2.3. *Let $P, Q \in L^{(l)} \setminus \{0\}$ and $(\rho,\sigma) \in \mathfrak{V}$. If $[\ell_{\rho,\sigma}(P), \ell_{\rho,\sigma}(Q)] = 0$, then*

$$\mathrm{st}_{\rho,\sigma}(P) \sim \mathrm{st}_{\rho,\sigma}(Q) \quad \textit{and} \quad \mathrm{en}_{\rho,\sigma}(P) \sim \mathrm{en}_{\rho,\sigma}(Q).$$

Proof. Consider (ρ_0, σ_0) such that $(\rho_0, \sigma_0) < (\rho, \sigma) < (-\rho_0, -\sigma_0)$. By Remark 1.12(3),

$$0 = [\ell_{(\rho_0,\sigma_0)}(\ell_{\rho,\sigma}(P)), \ell_{(\rho_0,\sigma_0)}(\ell_{\rho,\sigma}(Q))].$$

On the other hand, by Remark 1.8 there exist $\mu_P, \mu_Q \in K^\times$ such that

$$\ell_{\rho_0,\sigma_0}(\ell_{\rho,\sigma}(P)) = \mu_P x^{\frac{r}{l}} y^s \quad \text{and} \quad \ell_{\rho_0,\sigma_0}(\ell_{\rho,\sigma}(Q)) = \mu_Q x^{\frac{u}{l}} y^v,$$

where $(r/l, s) = \mathrm{st}_{\rho,\sigma}(P)$ and $(u/l, v) = \mathrm{st}_{\rho,\sigma}(Q)$. Clearly

$$0 = [\ell_{\rho_0,\sigma_0}(\ell_{\rho,\sigma}(P)), \ell_{\rho_0,\sigma_0}(\ell_{\rho,\sigma}(Q))] = \mu_P \mu_Q \left(\frac{rv}{l} - \frac{us}{l}\right) x^{\frac{r+u}{l}-1} y^{s+v-1},$$

from which $\mathrm{st}_{\rho,\sigma}(P) \sim \mathrm{st}_{\rho,\sigma}(Q)$ follows. Similar arguments yield $\mathrm{en}_{\rho,\sigma}(P) \sim \mathrm{en}_{\rho,\sigma}(Q)$, finishing the proof. □

Proposition 2.4. *Let $P, Q, R \in L^{(l)} \setminus \{0\}$ be such that*

$$[\ell_{\rho,\sigma}(P), \ell_{\rho,\sigma}(Q)] = \ell_{\rho,\sigma}(R),$$

where $(\rho,\sigma) \in \mathfrak{V}$. We have:

(1) $\mathrm{st}_{\rho,\sigma}(P) \nsim \mathrm{st}_{\rho,\sigma}(Q)$ *if and only if* $\mathrm{st}_{\rho,\sigma}(P) + \mathrm{st}_{\rho,\sigma}(Q) - (1,1) = \mathrm{st}_{\rho,\sigma}(R)$.
(2) $\mathrm{en}_{\rho,\sigma}(P) \nsim \mathrm{en}_{\rho,\sigma}(Q)$ *if and only if* $\mathrm{en}_{\rho,\sigma}(P) + \mathrm{en}_{\rho,\sigma}(Q) - (1,1) = \mathrm{en}_{\rho,\sigma}(R)$.

Proof. (1) It is enough to prove it when P, Q and R are (ρ,σ)-homogeneous, so we will assume it. Choose $(\rho_0,\sigma_0) \in \mathfrak{V}$ such that $(\rho_0, \sigma_0) < (\rho, \sigma) < (-\rho_0, -\sigma_0)$. By Remark 1.8

$$\ell_{\rho_0,\sigma_0}(P) = \mu_P x^{\frac{r}{l}} y^s, \quad \ell_{\rho_0,\sigma_0}(Q) = \mu_Q x^{\frac{u}{l}} y^v \quad \text{and} \quad \ell_{\rho_0,\sigma_0}(R) = \mu_R x^{\frac{a}{l}} y^b, \tag{2.5}$$

where

$$\mu_P, \mu_Q, \mu_R \in K^\times, \quad \left(\frac{r}{l}, s\right) := \mathrm{st}_{\rho,\sigma}(P), \quad \left(\frac{u}{l}, v\right) := \mathrm{st}_{\rho,\sigma}(Q) \quad \text{and}$$
$$\left(\frac{a}{l}, b\right) := \mathrm{st}_{\rho,\sigma}(R).$$

Clearly

$$[\ell_{\rho_0,\sigma_0}(P), \ell_{\rho_0,\sigma_0}(Q)] = \mu_P \mu_Q \left(\frac{rv}{l} - \frac{us}{l}\right) x^{\frac{r+u}{l}-1} y^{s+v-1}$$

and hence, by Proposition 1.13,

$$\begin{aligned}&\operatorname{st}_{\rho,\sigma}(P) \nsim \operatorname{st}_{\rho,\sigma}(Q) \iff [\ell_{\rho_0,\sigma_0}(P), \ell_{\rho_0,\sigma_0}(Q)] \neq 0 \iff \\ &\ell_{\rho_0,\sigma_0}(R) = [\ell_{\rho_0,\sigma_0}(P), \ell_{\rho_0,\sigma_0}(Q)].\end{aligned} \tag{2.6}$$

Consequently if $\operatorname{st}_{\rho,\sigma}(P) \nsim \operatorname{st}_{\rho,\sigma}(Q)$, then

$$\mu_R x^{\frac{a}{l}} y^b = \mu_P \mu_P \left(\frac{rv}{l} - \frac{us}{l}\right) x^{\frac{r+u}{l}-1} y^{s+v-1},$$

which evidently implies that

$$\operatorname{st}_{\rho,\sigma}(P) + \operatorname{st}_{\rho,\sigma}(Q) - (1,1) = \operatorname{st}_{\rho,\sigma}(R).$$

Reciprocally if this last equation holds, then by (2.5)

$$v_{\rho_0,\sigma_0}(P) + v_{\rho_0,\sigma_0}(Q) - (\rho_0 + \sigma_0) = v_{\rho_0,\sigma_0}(R),$$

and so, again by Proposition 1.13,

$$[\ell_{\rho_0,\sigma_0}(P), \ell_{\rho_0,\sigma_0}(Q)] \neq 0,$$

which by (2.6) implies that $\operatorname{st}_{\rho,\sigma}(P) \nsim \operatorname{st}_{\rho,\sigma}(Q)$.

(2) It is similar to the proof of statement (1). □

Remark 2.5. Let $(\rho,\sigma) \in \mathfrak{V}$ and let $P, F \in L^{(l)}$ be (ρ,σ)-homogeneous such that $[F, P] = P$. If F is a monomial, then $F = \lambda xy$ with $\lambda \in K^{\times}$, and, either $\rho + \sigma = 0$ or P is also a monomial.

Theorem 2.6. *Let $P \in L^{(l)}$ and let $(\rho,\sigma) \in \mathfrak{V}_{>0}$ be such that $v_{\rho,\sigma}(P) > 0$. If $[P,Q] \in K^{\times}$ for some $Q \in L^{(l)}$, then there exists $G_0 \in K[P,Q]\backslash\{0\}$ and a (ρ,σ)-homogeneous element $F \in L^{(l)}$ such that*

$$\begin{aligned}&v_{\rho,\sigma}(F) = \rho + \sigma, \quad [F, \ell_{\rho,\sigma}(P)] = \ell_{\rho,\sigma}(P) \quad \textit{and} \\ &[\ell_{\rho,\sigma}(G_0), \ell_{\rho,\sigma}(P)]F = \ell_{\rho,\sigma}(G_0)\ell_{\rho,\sigma}(P).\end{aligned} \tag{2.7}$$

Moreover, we have

(1) *If $P, Q \in L$, then we can take $F \in L$.*
(2) $\operatorname{st}_{\rho,\sigma}(P) \sim \operatorname{st}_{\rho,\sigma}(F)$ *or* $\operatorname{st}_{\rho,\sigma}(F) = (1,1)$.
(3) $\operatorname{en}_{\rho,\sigma}(P) \sim \operatorname{en}_{\rho,\sigma}(F)$ *or* $\operatorname{en}_{\rho,\sigma}(F) = (1,1)$.
(4) $\operatorname{st}_{\rho,\sigma}(P) \nsim (1,1) \nsim \operatorname{en}_{\rho,\sigma}(P)$.
(5) *If we define recursively $G_i := [G_{i-1}, P]$, then $[\ell_{\rho,\sigma}(G_i), \ell_{\rho,\sigma}(P)] = 0$ for $i \geq 1$.*

Proof. From Lemma 2.2 we obtain G_0 such that the hypotheses of Lemma 2.2 of [6] are satisfied. Hence, by this lemma,

$$F := \frac{\ell_{\rho,\sigma}(G_0)\ell_{\rho,\sigma}(P)}{[\ell_{\rho,\sigma}(G_0), \ell_{\rho,\sigma}(P)]} \in L^{(l)}$$

and if P and Q are in L, then $F \in L$. Hence statement (1) is true. Furthermore an easy computation shows that statement (5) is also true and equalities (2.7) are satisfied. Statements (2) and (3) follow from Proposition 2.4. For statement (4), assume that $\mathrm{st}_{\rho,\sigma}(P) \sim (1,1)$. We claim that this implies that $\mathrm{st}_{\rho,\sigma}(F) = (1,1)$. Otherwise, by statement (2) we have

$$\mathrm{st}_{\rho,\sigma}(F) \sim \mathrm{st}_{\rho,\sigma}(P) \sim (1,1),$$

which implies $\mathrm{st}_{\rho,\sigma}(F) \sim (1,1)$, since $\mathrm{st}_{\rho,\sigma}(F) \neq (0,0) \neq \mathrm{st}_{\rho,\sigma}(P)$. So there exists $\lambda \in \mathbb{Q} \setminus \{1\}$ such that $\mathrm{st}_{\rho,\sigma}(F) = \lambda(1,1)$. But this is impossible because $v_{\rho,\sigma}(F) = \rho + \sigma$ implies $\lambda = 1$. Hence the claim is true, and so

$$\mathrm{st}_{\rho,\sigma}(P) + \mathrm{st}_{\rho,\sigma}(F) - (1,1) = \mathrm{st}_{\rho,\sigma}(P),$$

which by Proposition 2.4(1) leads to the contradiction

$$\mathrm{st}_{\rho,\sigma}(P) \nsim \mathrm{st}_{\rho,\sigma}(F) = (1,1).$$

Similarly $\mathrm{en}_{\rho,\sigma}(P) \nsim (1,1)$. $\square$

Remark 2.7. In general, the conclusions of Theorem 2.6 do not hold if $\rho + \sigma < 0$. For instance, consider the following pair in $L^{(1)}$:

$$P = x^{-1} + x^3y(2 + 18x^2y + 36x^4y^2) + x^9y^3(8 + 72x^2y + 216x^4y^2 + 216x^6y^3)$$

and

$$Q = x^2y + x^6y^2(1 + 6x^2y + 9x^4y^2).$$

Clearly $[P,Q] = -1$ and $v_{1,-2}(P) = 3 > 0$. However, one can show that there is no $F \in L^{(1)}$ such that $[F, \ell_{1,-2}(P)] = \ell_{1,-2}(P)$.

Remark 2.8. Let $P \in L^{(l)} \setminus \{0\}$ and $(\rho,\sigma) \in \mathfrak{V}$ with $\rho > 0$. If $\ell_{\rho,\sigma}(P) = x^{\frac{r}{l}} y^s p(x^{-\frac{\sigma}{\rho}} y)$, where

$$p := \sum_{i=0}^{\gamma} a_i x^i \in K[x] \quad \text{with } a_0 \neq 0 \text{ and } a_\gamma \neq 0,$$

then, by Remark 1.8 with $(\rho_0, \sigma_0) = (0, -1)$,

$$\mathrm{st}_{\rho,\sigma}(P) = \Big(\frac{r}{l}, s\Big) \quad \text{and} \quad \mathrm{en}_{\rho,\sigma}(P) = \Big(\frac{r}{l} - \frac{\gamma\sigma}{\rho}, s+\gamma\Big). \tag{2.8}$$

Definition 2.9. Let $P \in L^{(l)} \setminus \{0\}$. We define the set of *directions associated with* P as

$$\mathrm{Dir}(P) := \{(\rho,\sigma) \in \mathfrak{V} : \#\,\mathrm{Supp}(\ell_{\rho,\sigma}(P)) > 1\}.$$

Remark 2.10. Note that if $P \in L^{(l)} \setminus \{0\}$ is a monomial, then $\mathrm{Dir}(P) = \emptyset$ and that if $P \in L^{(l)} \setminus \{0\}$ is (ρ,σ)-homogeneous, but is not a monomial, then $\mathrm{Dir}(P) = \{(\rho,\sigma), (-\rho,-\sigma)\}$. Furthermore, if $P \in L^{(l)} \setminus \{0\}$ is not homogeneous, then any two consecutive directions of P are separated by less than 180°.

Proposition 2.11. *Let $(\rho,\sigma) \in \mathfrak{V}^0$ and $P, F \in L^{(l)} \setminus \{0\}$. Assume that F is (ρ,σ)-homogeneous, $v_{\rho,\sigma}(P) > 0$ and*

$$[F, \ell_{\rho,\sigma}(P)] = \ell_{\rho,\sigma}(P). \tag{2.9}$$

Write

$$F = x^{\frac{u}{l}} y^v f(z) \quad \textit{and} \quad \ell_{\rho,\sigma}(P) = x^{\frac{r}{l}} y^s p(z) \quad \textit{with } z := x^{-\frac{\sigma}{\rho}} y \textit{ and } p(0) \neq 0 \neq f(0).$$

Then

(1) *f is separable and every irreducible factor of p divides f.*
(2) *If $(\rho,\sigma) \in \mathrm{Dir}(P)$, then $v_{0,1}(\mathrm{st}_{\rho,\sigma}(F)) < v_{0,1}(\mathrm{en}_{\rho,\sigma}(F))$.*
(3) *Suppose that $p, f \in K[z^k]$ for some $k \in \mathbb{N}$ and let $\overline{p}$ and $\overline{f}$ denote the univariate polynomials defined by $p(z) = \overline{p}(z^k)$ and $f(z) = \overline{f}(z^k)$. Then $\overline{f}$ is separable and every irreducible factor of $\overline{p}$ divides $\overline{f}$.*
(4) *If $P, F \in L$ and $v_{0,1}(\mathrm{en}_{\rho,\sigma}(F)) - v_{0,1}(\mathrm{st}_{\rho,\sigma}(F)) = \rho$, then the multiplicity of each linear factor (in an algebraic closure of K) of p is equal to*

$$\frac{1}{\rho}\deg(p) = \frac{1}{\rho}\big(v_{0,1}(\mathrm{en}_{\rho,\sigma}(P)) - v_{0,1}(\mathrm{st}_{\rho,\sigma}(P))\big).$$

(5) *Assume that $(\rho,\sigma) \in \mathrm{Dir}(P)$. If $s > 0$ or $\#\,\mathrm{factors}(p) > 1$, then there exist no (ρ,σ)-homogeneous element $R \in L^{(l)}$ such that*

$$v_{\rho,\sigma}(R) = \rho + \sigma \quad \textit{and} \quad [R, \ell_{\rho,\sigma}(P)] = 0. \tag{2.10}$$

Consequently, in this case F satisfying (2.9) is unique.

Proof. Note that, since $[-,-]$ is a derivation in both variables, we have

$$\begin{aligned}[F, \ell_{\rho,\sigma}(P)] &= \big[x^{\frac{u}{l}} y^v f(z), x^{\frac{r}{l}} y^s p(z)\big] \\ &= x^{\frac{u+r}{l}-1} y^{v+s-1}\big(cf(z)p(z) + azf(z)p'(z) - bzf'(z)p(z)\big),\end{aligned}$$

where

$$c := \begin{pmatrix} \frac{u}{l} \\ v \end{pmatrix} \times \begin{pmatrix} \frac{r}{l} \\ s \end{pmatrix}, \quad a := \begin{pmatrix} \frac{u}{l} \\ v \end{pmatrix} \times \begin{pmatrix} -\frac{\sigma}{\rho} \\ 1 \end{pmatrix} = \frac{1}{\rho} v_{\rho,\sigma}(F) \quad \text{and}$$

$$b := \begin{pmatrix} \frac{r}{l} \\ s \end{pmatrix} \times \begin{pmatrix} -\frac{\sigma}{\rho} \\ 1 \end{pmatrix} = \frac{1}{\rho} v_{\rho,\sigma}(P).$$

Hence, by equality (2.9) there exists $h \in \mathbb{N}_0$ such that

$$z^h p = cpf + azp'f - bzf'p. \tag{2.11}$$

Let g be a linear factor of p in an algebraic closure of K, with multiplicity m. Write $p = p_1 g^m$ and $f = f_1 g^n$, where $n \geq 0$ is the multiplicity of g in f. Since

$$p' = p_1 m g^{m-1} g' + p_1' g^m \quad \text{and} \quad f' = f_1 n g^{n-1} g' + f_1' g^n,$$

equality (2.11) can be written

$$z^h p_1 g^m = g^{m+n-1}\big(g(cp_1 f_1 + azf_1 p_1' - bzf_1' p_1) + (am - bn)zf_1 p_1 g'\big),$$

which implies $n \leq 1$. But $n = 0$ is impossible since $a, m > 0$. So, statement (1) follows.

Assume now $(\rho, \sigma) \in \operatorname{Dir}(P)$. Then $\deg p > 0$, and so, by statement (1) we have $\deg f > 0$. Consequently statement (2) follows from Remark 2.8. Using now that

$$zp'(z) = kt\overline{p}'(t) \quad \text{and} \quad zf'(z) = kt\overline{f}'(t) \qquad \text{where } t := z^k,$$

we deduce from (2.11) the equality

$$z^h \overline{p}(t) = c\overline{p}(t)\overline{f}(t) + at\overline{p}'\overline{f}(t) - bt\overline{f}'(t)\overline{p}(t).$$

The same procedure as above, but using this last equality instead of (2.11), yields statement (3).

Now we prove statement (4). Write

$$F = \sum_{i=0}^{\alpha} b_i x^{u-i\sigma} y^{v+i\rho} \quad \text{and} \quad \ell_{\rho,\sigma}(P) = \sum_{i=0}^{\gamma} c_i x^{r-i\sigma} y^{s+i\rho}$$

with $b_0 \neq 0$, $b_\alpha \neq 0$, $c_0 \neq 0$ and $c_\gamma \neq 0$. By definition

$$f = \sum_{i=0}^{\alpha} b_i z^{i\rho} \quad \text{and} \quad p = \sum_{i=0}^{\gamma} c_i z^{i\rho}.$$

Moreover, since by (2.8),

$$\alpha\rho = v_{0,1}(\mathrm{en}_{\rho,\sigma}(F)) - v_{0,1}(\mathrm{st}_{\rho,\sigma}(F)),$$

it follows from the hypothesis that $\alpha = 1$. Hence

$$f(z) = b_0 + b_1 z^\rho = \mu(z^\rho - \mu') = \overline{f}(z^\rho),$$

where $\mu := b_1$ and $\mu' := b_0/b_1$. Consequently, by statement (3), there exists $\mu_P \in K^\times$ such that

$$p(z) = \mu_P(z^\rho - \mu')^\gamma,$$

from which statement (4) follows easily. Finally we prove statement (5). For this we first prove (2.12) below, and then we prove that for any R satisfying (2.10) there exists $\lambda \in K^\times$ such that $F_\lambda := F - \lambda R$ satisfies (2.9) and $\mathrm{en}_{\rho,\sigma}(P) \nsim \mathrm{en}_{\rho,\sigma}(F_\lambda)$, which is a contradiction.

Assume that $\#\,\mathrm{factors}(p) > 1$ or that $s > 0$. We claim that $\mathrm{en}_{\rho,\sigma}(F) \neq (1,1)$. If the first inequality holds, then, by statement (1), we have $\deg(f) > 1$. Consequently, by Remark 2.8, it is impossible that $\mathrm{en}_{\rho,\sigma}(F) = (1,1)$. Assume that $s > 0$. By Proposition 2.4(1), either

$$\mathrm{st}_{\rho,\sigma}(F) = (1,1) \quad \text{or} \quad \mathrm{st}_{\rho,\sigma}(F) \sim \mathrm{st}_{\rho,\sigma}(P) = (r/l, s).$$

In the first case $v_{0,1}(\mathrm{st}_{\rho,\sigma}(F)) = 1$, while in the second one, since by Remark 1.12(1) we know that $\mathrm{st}_{\rho,\sigma}(F) \neq (0,0)$, there exists $\lambda > 0$ such that $\mathrm{st}_{\rho,\sigma}(F) = \lambda\,\mathrm{st}_{\rho,\sigma}(P)$. So $v_{0,1}(\mathrm{st}_{\rho,\sigma}(F)) = \lambda s > 0$. In both cases, by statement (2),

$$v_{0,1}(\mathrm{en}_{\rho,\sigma}(F)) > v_{0,1}(\mathrm{st}_{\rho,\sigma}(F)) \geq 1,$$

which clearly implies $\mathrm{en}_{\rho,\sigma}(F) \neq (1,1)$, as desired. Thus, by Proposition 2.4(2) we conclude that, if $\#\,\mathrm{factors}(p) > 1$ or $s > 0$, then

$$[F, \ell_{\rho,\sigma}(P)] = \ell_{\rho,\sigma}(P) \Longrightarrow \mathrm{en}_{\rho,\sigma}(P) \sim \mathrm{en}_{\rho,\sigma}(F). \tag{2.12}$$

Suppose that $R \in L^{(l)}$ is a (ρ,σ)-homogeneous element that satisfies condition (2.10). By Proposition 2.3 we know that $\mathrm{en}_{\rho,\sigma}(P) \sim \mathrm{en}_{\rho,\sigma}(R)$ and so $\mathrm{en}_{\rho,\sigma}(F) \sim \mathrm{en}_{\rho,\sigma}(R)$. Since by Remark 1.12(1)

$$v_{\rho,\sigma}(F) = \rho + \sigma = v_{\rho,\sigma}(R), \tag{2.13}$$

this implies that

$$\mathrm{en}_{\rho,\sigma}(F) = \mathrm{en}_{\rho,\sigma}(R). \tag{2.14}$$

Let $\bar{r}$ be an univariate polynomial such that $\bar{r}(0) \neq 0$ and $R = x^{\frac{h}{l}} y^k \bar{r}(z)$. We have

$$F = x^{\frac{\overline{u}}{l}}\mathfrak{f}(z) \quad \text{and} \quad R = x^{\frac{\overline{h}}{l}}\mathfrak{r}(z),$$

where $\mathfrak{f}(z) := z^v f(z)$, $\mathfrak{r}(z) := z^k \bar{r}(z)$, $\overline{u} := u + v\sigma l/\rho$ and $\overline{h} := h + k\sigma l/\rho$. By Remark 2.8 and equality (2.14)

$$\deg(\mathfrak{f}) = \deg(f) + v = v_{0,1}(\mathrm{en}_{\rho,\sigma}(F)) = v_{0,1}(\mathrm{en}_{\rho,\sigma}(R)) = \deg(\bar{r}) + k = \deg(\mathfrak{r}).$$

Moreover $\overline{u} = \overline{h}$ since, by equality (2.13),

$$\rho\frac{\overline{u}}{l} = v_{\rho,\sigma}(F) = v_{\rho,\sigma}(R) = \rho\frac{\overline{h}}{l}.$$

Let $\lambda \in K^{\times}$ be such that $\deg(\mathfrak{f} - \lambda\mathfrak{r}) < \deg(\mathfrak{f})$ and let

$$F_\lambda := F - \lambda R = x^{\frac{\overline{u}}{l}}\big(\mathfrak{f}(z) - \lambda\mathfrak{r}(z)\big).$$

Again by Remark 2.8

$$\mathrm{en}_{\rho,\sigma}(F_\lambda) = \mathrm{en}_{\rho,\sigma}(F) - t(-\sigma,\rho) \qquad \text{where } t := \frac{\deg(\mathfrak{f}) - \deg(\mathfrak{f} - \lambda\mathfrak{r})}{\rho} > 0.$$

Hence

$$\mathrm{en}_{\rho,\sigma}(P) \times \mathrm{en}_{\rho,\sigma}(F_\lambda) = -t(\mathrm{en}_{\rho,\sigma}(P) \times (-\sigma,\rho)) = -tv_{\rho,\sigma}(P) < 0,$$

and so $\mathrm{en}_{\rho,\sigma}(P) \nsim \mathrm{en}_{\rho,\sigma}(F_\lambda)$. But, since

$$[F_\lambda, \ell_{\rho,\sigma}(P)] = [F - \lambda R, \ell_{\rho,\sigma}(P)] = [F, \ell_{\rho,\sigma}(P)] - \lambda[R, \ell_{\rho,\sigma}(P)] = [F, \ell_{\rho,\sigma}(P)] = \ell_{\rho,\sigma}(P),$$

this contradicts (2.12), and hence, such an R cannot exist. Clearly the uniqueness of F follows, since any other F' satisfying (2.9) yields $R := F - F'$ which satisfies (2.10). □

3. More on the order on directions

In this section we consider the same order on directions as other authors, e.g. [1] and [5], but we profit from the following characterization of this order in small intervals: If I is an interval in $\mathfrak{V}$ and if there is no closed half circle contained in I, which means that there is no $(\rho,\sigma) \in I$ with $(-\rho,-\sigma) \in I$, then for $(\rho,\sigma), (\rho'\sigma') \in I$ we have

$$(\rho,\sigma) < (\rho',\sigma') \iff (\rho,\sigma) \times (\rho',\sigma') > 0. \tag{3.1}$$

We also present in Proposition 3.10 the chain rule for Jacobians in a convenient way.

Remark 3.1. Let $(\rho,\sigma) \in \mathfrak{V}$ and let $P, Q \in L^{(l)}$. If

$$v_{\rho,\sigma}(P) > 0, \quad v_{\rho,\sigma}(Q) \geq 0 \quad \text{and} \quad [\ell_{\rho,\sigma}(P), \ell_{\rho,\sigma}(Q)] = 0,$$

then by Proposition 2.1(2) we know that there exist $\lambda_P, \lambda_Q \in K^{\times}$, $m, n \in \mathbb{N}_0$ coprime and a (ρ,σ)-homogeneous element $R \in L^{(l)}$, with $R \in L$ if $P, Q \in L$, such that

$$\ell_{\rho,\sigma}(P) = \lambda_P R^m \qquad \text{and} \qquad \ell_{\rho,\sigma}(Q) = \lambda_Q R^n.$$

Note that $v_{\rho,\sigma}(P) > 0$ implies $m \in \mathbb{N}$. Consequently, we have

(1) $\frac{n}{m} = \frac{v_{\rho,\sigma}(Q)}{v_{\rho,\sigma}(P)}$,
(2) $\mathrm{st}_{\rho,\sigma}(Q) = \frac{n}{m}\,\mathrm{st}_{\rho,\sigma}(P)$,
(3) $\mathrm{en}_{\rho,\sigma}(Q) = \frac{n}{m}\,\mathrm{en}_{\rho,\sigma}(P)$,

and, if moreover $v_{\rho,\sigma}(Q) > 0$, then

$$\begin{aligned}(\rho,\sigma) \in \mathrm{Dir}(P) &\Leftrightarrow \ell_{\rho,\sigma}(P) \text{ is not a monomial}\\ &\Leftrightarrow R \text{ is not a monomial}\\ &\Leftrightarrow \ell_{\rho,\sigma}(Q) \text{ is not a monomial}\\ &\Leftrightarrow (\rho,\sigma) \in \mathrm{Dir}(Q).\end{aligned}$$

By Proposition 1.13 the condition $[\ell_{\rho,\sigma}(P), \ell_{\rho,\sigma}(Q)] = 0$ can be replaced by

$$v_{\rho,\sigma}([P,Q]) < v_{\rho,\sigma}(P) + v_{\rho,\sigma}(Q) - (\rho+\sigma).$$

We will use freely this fact.

For each $(r/l, s) \in \frac{1}{l}\mathbb{Z} \times \mathbb{Z} \setminus \mathbb{Z}(1,1)$ there exists a unique $(\rho,\sigma) \in \mathfrak{V}_{>0}$, denoted by $\mathrm{dir}(r/l, s)$, such that $v_{\rho,\sigma}(r/l, s) = 0$. In fact clearly

$$(\rho,\sigma) = \begin{cases} (-ls/d, r/d) & \text{if } r - ls > 0,\\ (ls/d, -r/d) & \text{if } r - ls < 0,\end{cases} \tag{3.2}$$

where $d := \gcd(r, ls)$, satisfies the required condition, and the uniqueness is evident.

Remark 3.2. Note that if $(\rho,\sigma) \in \mathfrak{V}_{>0}$, $(r/l, s) \neq (r'/l, s')$ and $v_{\rho,\sigma}(r/l, s) = v_{\rho,\sigma}(r'/l, s')$ then

$$(\rho,\sigma) = \mathrm{dir}\left(\left(\frac{r}{l}, s\right) - \left(\frac{r'}{l}, s'\right)\right).$$

In particular

$$(\rho,\sigma)=\operatorname{dir}\big(\operatorname{en}_{\rho,\sigma}(P)-\operatorname{st}_{\rho,\sigma}(P)\big)\qquad \text{for all } P\in L^{(l)}\setminus\{0\} \text{ and } (\rho,\sigma)\in \operatorname{Dir}(P)\cap\mathfrak{V}_{>0}.$$

Remark 3.3. Let $(a/l,b)\in\frac{1}{l}\mathbb{Z}\times\mathbb{N}$ and set

$$(\overline{\rho},\overline{\sigma}):=\frac{1}{d}(bl,-a),\quad \text{where } d:=\gcd(bl,a).$$

Then, for any $(\rho,\sigma)\in\mathfrak{V}$, we have

$$v_{\rho,\sigma}(a/l,b)>0 \iff (\overline{\rho},\overline{\sigma})\times(\rho,\sigma)>0 \iff (\overline{\rho},\overline{\sigma})<(\rho,\sigma)<(-\overline{\rho},-\overline{\sigma}).$$

Definition 3.4. Let $P\in L^{(l)}\setminus\{0\}$ which is not a monomial and $(\rho,\sigma)\in\mathfrak{V}$. We define the *successor* $\operatorname{Succ}_P(\rho,\sigma)$ of (ρ,σ) to be the first element of $\operatorname{Dir}(P)$ that one encounters starting from (ρ,σ) and running counterclockwise, and the *predecessor* $\operatorname{Pred}_P(\rho,\sigma)$, to be the first one, if we run clockwise.

Note that $\mathfrak{V}_{>0}$ is the interval $](1,-1),(-1,1)[$ and the order on $\mathfrak{V}_{>0}$ is given by (3.1).

Lemma 3.5. *Let $(a/l,b),(c/l,d)\in\frac{1}{l}\mathbb{Z}\times\mathbb{Z}$ and $(\rho,\sigma)\in\mathfrak{V}_{>0}$. If $v_{1,-1}(a/l,b)>v_{1,-1}(c/l,d)$, then*

$$v_{\rho,\sigma}\left(\frac{a}{l},b\right)>v_{\rho,\sigma}\left(\frac{c}{l},d\right)\iff \operatorname{dir}\left(\left(\frac{a}{l},b\right)-\left(\frac{c}{l},d\right)\right)>(\rho,\sigma)$$

and

$$v_{\rho,\sigma}\left(\frac{a}{l},b\right)<v_{\rho,\sigma}\left(\frac{c}{l},d\right)\iff \operatorname{dir}\left(\left(\frac{a}{l},b\right)-\left(\frac{c}{l},d\right)\right)<(\rho,\sigma).$$

Proof. Let

$$(\rho',\sigma'):=\operatorname{dir}\big((a/l,b)-(c/l,d)\big)\quad\text{and}\quad g:=\gcd(bl-dl,a-c).$$

Since $v_{1,-1}(a/l,b)>v_{1,-1}(c/l,d)$ implies $a-c>bl-dl$, we have

$$(\rho',\sigma')=\left(\frac{dl-bl}{g},\frac{a-c}{g}\right).$$

Consequently

$$v_{\rho,\sigma}\left(\frac{a}{l},b\right)-v_{\rho,\sigma}\left(\frac{c}{l},d\right)=\frac{g}{l}\left(\rho\frac{a-c}{g}-\sigma\frac{dl-bl}{g}\right)=\frac{g}{l}(\rho\sigma'-\rho'\sigma)=\frac{g}{l}(\rho,\sigma)\times(\rho',\sigma'),$$

and so, the result follows immediately from (3.1). □

Corollary 3.6. *Let $(a/l,b),(c/l,d)\in\frac{1}{l}\mathbb{Z}\times\mathbb{Z}$ and $(\rho,\sigma)<(\rho',\sigma')$ in $\mathfrak{V}_{>0}$. If*

$$v_{1,-1}(a/l,b)>v_{1,-1}(c/l,d),$$

then

$$v_{\rho',\sigma'}\left(\frac{a}{l},b\right)\ge v_{\rho',\sigma'}\left(\frac{c}{l},d\right)\Longrightarrow v_{\rho,\sigma}\left(\frac{a}{l},b\right)>v_{\rho,\sigma}\left(\frac{c}{l},d\right)$$

and

$$v_{\rho,\sigma}\left(\frac{a}{l},b\right)\le v_{\rho,\sigma}\left(\frac{c}{l},d\right)\Longrightarrow v_{\rho',\sigma'}\left(\frac{a}{l},b\right)<v_{\rho',\sigma'}\left(\frac{c}{l},d\right).$$

Proof. It follows easily from Lemma 3.5. □

The next two propositions are completely clear. The first one asserts that if you have two consecutive edges of a Newton polygon, then all that is between them is the common vertex. The second one asserts that if the end point of an edge coincides with the starting point of another edge, then they are consecutive.

Proposition 3.7. *Let $P\in L^{(l)}\setminus\{0\}$ and let (ρ_1,σ_1) and (ρ_2,σ_2) be consecutive elements in $\operatorname{Dir}(P)$. If $(\rho_1,\sigma_1)<(\rho,\sigma)<(\rho_2,\sigma_2)$, then $\operatorname{en}_{\rho_1,\sigma_1}(P)=\operatorname{Supp}(\ell_{\rho,\sigma}(P))=\operatorname{st}_{\rho_2,\sigma_2}(P)$.*

Proposition 3.8. *Let $P\in L^{(l)}\setminus\{0\}$ and let $(\rho,\sigma),(\rho',\sigma')\in\mathfrak{V}$. If $\operatorname{en}_{\rho,\sigma}(P)=\operatorname{st}_{\rho',\sigma'}(P)$, then there is no $(\rho'',\sigma'')\in\operatorname{Dir}(P)$ such that $(\rho,\sigma)<(\rho'',\sigma'')<(\rho',\sigma')$.*

Proposition 3.9. *For $k\in\mathbb{Z}$ consider the automorphism of $L^{(l)}$ defined by*

$$\varphi(x^{\frac{1}{l}}):=x^{\frac{1}{l}}\qquad\text{and}\qquad\varphi(y):=y+\lambda x^{\frac{k}{l}}.$$

Let (ρ,σ) be the direction defined by $\rho>0$ and $\frac{\sigma}{\rho}=\frac{k}{l}$. We have

$$\ell_{\rho,\sigma}(\varphi(P))=\varphi(\ell_{\rho,\sigma}(P)),\quad \ell_{-\rho,-\sigma}(\varphi(P))=\varphi(\ell_{-\rho,-\sigma}(P))\quad\text{and}$$
$$\ell_{\rho_1,\sigma_1}(\varphi(P))=\ell_{\rho_1,\sigma_1}(P),$$

for all $P\in L^{(l)}\setminus\{0\}$ and all $(\rho,\sigma)<(\rho_1,\sigma_1)<(-\rho,-\sigma)$. Moreover $\operatorname{en}_{\rho,\sigma}(\varphi(P))=\operatorname{en}_{\rho,\sigma}(P)$.

Proof. Take $d:=\gcd(k,l)>0$, $\rho:=l/d$ and $\sigma:=k/d$. Clearly $\frac{\sigma}{\rho}=\frac{k}{l}$. Moreover, since φ is (ρ,σ)-homogeneous it is also clear that

$$\ell_{\rho,\sigma}(\varphi(P))=\varphi(\ell_{\rho,\sigma}(P))\qquad\text{and}\qquad\ell_{-\rho,-\sigma}(\varphi(P))=\varphi(\ell_{-\rho,-\sigma}(P))$$

for all $P\in L^{(l)}\setminus\{0\}$. Now we prove that the last equality is also true. By the hypothesis about (ρ_1,σ_1) we have $\rho_1\sigma<\rho\sigma_1$. Thus

$$\ell_{\rho_1,\sigma_1}\left(y+\lambda x^{\frac{\sigma}{\rho}}\right)=y,$$

since $\rho>0$. Consequently

$$\ell_{\rho_1,\sigma_1}\left(\varphi(x^{\frac{i}{l}}y^j)\right)=\ell_{\rho_1,\sigma_1}\left(x^{\frac{i}{l}}(y+\lambda x^{\frac{\sigma}{\rho}})^j\right)=x^{\frac{i}{l}}y^j,$$

from which

$$\ell_{\rho_1,\sigma_1}(\varphi(P))=\ell_{\rho_1,\sigma_1}(\varphi(\ell_{\rho_1,\sigma_1}(P)))=\ell_{\rho_1,\sigma_1}(P),$$

follows. The last assertion follows from the second equality in (2.8) and the fact that the monomials of greatest degree in y of $\ell_{\rho,\sigma}(\varphi(P))$ and $\ell_{\rho,\sigma}(P)$ coincide. □

Proposition 3.10. *Let $R_0,R_1\in\{L,L^{(l)}\}$, $P,Q\in R_0$ and $\varphi\colon R_0\to R_1$ an algebra morphism. Then*

$$[\varphi(P),\varphi(Q)]=\varphi([P,Q])[\varphi(x),\varphi(y)]. \tag{3.3}$$

Proof. Recall the (formal) Jacobian chain rule (see for example [2, (1.7), p. 1160]) which generalizes the (formal) derivative chain rule and which says that given any 2-variable rational functions $f_1(x,y),f_2(x,y),g_1(x,y),g_2(x,y)\in K(x,y)$, we have

$$J_{(x,y)}(h_1,h_2)=J_{(f_1,f_2)}(g_1,g_2)J_{(x,y)}(f_1,f_2),$$

where by definition $h_i(x,y):=g_i(f_1(x,y),f_2(x,y))$, and

$$J_{(f_1,f_2)}(g_1,g_2):=j(f_1(x,y),f_2(x,y))\quad\text{with}\quad j(x,y):=J_{(x,y)}(g_1,g_2). \tag{3.4}$$

Assume first that $l=1$. Then equality (3.3) follows applying equality (3.4) with

$$g_1:=P,\quad g_2:=Q,\quad f_1:=\varphi(x)\quad\text{and}\quad f_2:=\varphi(y),$$

since $\varphi([P,Q])=j(\varphi(x),\varphi(y))$, where $j(x,y):=[P,Q]\in L^{(1)}\subseteq K(x,y)$. Assume now that l is arbitrary. Identifying $L^{(l)}$ with $K[z,z^{-1},y]$ via $z=x^{1/l}$, we obtain

$$[P,Q]=(P_zQ_y-P_yQ_z)\frac{1}{lz^{l-1}},\quad\text{for } P,Q\in L^{(l)}.$$

Consequently equality (3.3) is valid for $R_0,R_1\in\{L,L^{(l)}\}$. □

4. Minimal pairs and (m,n)-pairs

Our next aim is to determine a lower bound for

$$B := \begin{cases} \infty & \text{if the jacobian conjecture is true,} \\ \min\bigl(\gcd(v_{1,1}(P), v_{1,1}(Q))\bigr) & \text{if JC is false, where } (P,Q) \text{ runs} \\ & \text{on the counterexamples.} \end{cases}$$

A *minimal pair* is a counterexample (P,Q) to JC such that $B = \gcd(v_{1,1}(P), v_{1,1}(Q))$.

An (m,n)-pair is a Jacobian pair (P,Q) with $P,Q \in L^{(l)}$ for some l, that satisfies certain conditions (see Definition 4.3).

In this section we prove that if $B < \infty$, then there exists a minimal pair that is also an (m,n)-pair for some $m,n \in \mathbb{N}$. We could prove the result using only our previous results, but we prefer to use the well known fact that a counterexample to JC can be brought into a subrectangular shape, following an argument communicated by Leonid Makar-Limanov.

Proposition 4.1. *Let $(\rho,\sigma) \in \mathfrak{V}$ be such that $(1,0) \le (\rho,\sigma) \le (0,1)$. If (P,Q) is a counterexample to JC, then*

$$v_{\rho,\sigma}(P) > 0, \qquad v_{\rho,\sigma}(Q) > 0 \qquad \textit{and} \qquad v_{\rho,\sigma}(P) + v_{\rho,\sigma}(Q) - (\rho+\sigma) > 0.$$

Proof. Note that if $(1,0) \le (\rho,\sigma) \le (1,1)$, then $\rho \ge \sigma \ge 0$, while if $(1,1) \le (\rho,\sigma) \le (0,1)$, then $\sigma \ge \rho \ge 0$. In the case $\rho \ge \sigma \ge 0$ it is enough to prove that $v_{\rho,\sigma}(P), v_{\rho,\sigma}(Q) > \rho$. Assume for example that $v_{\rho,\sigma}(P) \le \rho$, then

$$(i,j) \in \operatorname{Supp}(P) \Longrightarrow i\rho + j\sigma \le \rho \Longrightarrow i = 0, \text{ or } i = 1 \text{ and } j = 0,$$

which means that $P = \mu x + f(y)$ for some $\mu \in K$ and $f \in K[y]$, and obviously (P,Q) can not be a counterexample to JC. The case $\sigma \ge \rho \ge 0$ is similar. □

Remark 4.2. If (P,Q) is a minimal pair, then neither $v_{1,1}(P)$ divides $v_{1,1}(Q)$ nor $v_{1,1}(Q)$ divides $v_{1,1}(P)$. This fact can be proven using a classical argument given for example in the proof of [11, Theorem 10.2.23].

Definition 4.3. Let $m,n \in \mathbb{N}$ be coprime with $n,m > 1$. A pair (P,Q) of elements $P,Q \in L^{(l)}$ (respectively $P,Q \in L$) is called an (m,n)-*pair* in $L^{(l)}$ (respectively in L), if

$$[P,Q] \in K^{\times}, \quad \frac{v_{1,1}(P)}{v_{1,1}(Q)} = \frac{v_{1,0}(P)}{v_{1,0}(Q)} = \frac{m}{n} \quad \text{and} \quad v_{1,-1}(\mathrm{en}_{1,0}(P)) < 0.$$

An (m,n)-pair (P,Q) is called *a standard (m,n)-pair* if $P,Q \in L^{(1)}$ and $v_{1,-1}(\mathrm{st}_{1,0}(P)) < 0$.

Lemma 4.4. *Let $(\rho,\sigma),(\rho',\sigma') \in \mathfrak{V}$ and $A,B \in \frac{1}{l}\mathbb{Z} \times \mathbb{N}_0$ such that*

$$v_{\rho,\sigma}(A)v_{\rho',\sigma'}(B) = v_{\rho,\sigma}(B)v_{\rho',\sigma'}(A) \quad \textit{and} \quad (\rho,\sigma) \times (\rho',\sigma') \neq 0.$$

Then $A \sim B$.

Proof. Write $A = (a_1, a_2)$ and $B = (b_1, b_2)$. The Lemma follows immediately from the equality

$$\begin{pmatrix} \rho & \sigma \\ \rho' & \sigma' \end{pmatrix} \begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix} = \begin{pmatrix} v_{\rho,\sigma}(A) & v_{\rho,\sigma}(B) \\ v_{\rho',\sigma'}(A) & v_{\rho',\sigma'}(B) \end{pmatrix},$$

taking determinants. □

Remark 4.5. Let $P, Q \in L^{(l)}$ and $(\rho, \sigma) \in \mathfrak{V}$. Assume that $v_{\rho,\sigma}(P) \neq 0$, $\mathrm{st}_{\rho,\sigma}(P) \sim \mathrm{st}_{\rho,\sigma}(Q)$ and $\mathrm{en}_{\rho,\sigma}(P) \sim \mathrm{en}_{\rho,\sigma}(Q)$. Then

$$\mathrm{en}_{\rho,\sigma}(Q) = \lambda\, \mathrm{en}_{\rho,\sigma}(P) \quad \text{and} \quad \mathrm{st}_{\rho,\sigma}(Q) = \lambda\, \mathrm{st}_{\rho,\sigma}(P), \quad \text{with} \quad \lambda := \frac{v_{\rho,\sigma}(Q)}{v_{\rho,\sigma}(P)}.$$

If (P, Q) is an (m, n)-pair, then (Q, P) is an (n, m)-pair, as is shown by the following proposition.

Proposition 4.6. *Let (P, Q) be an (m, n)-pair. Then the following properties hold:*

(1) $v_{1,0}(P), v_{1,0}(Q) > 0$.
(2) $\mathrm{en}_{1,0}(Q) \sim \mathrm{en}_{1,0}(P)$ *and* $\mathrm{en}_{1,0}(Q) = \frac{n}{m}\, \mathrm{en}_{1,0}(P)$.
(3) $\frac{1}{m}\, \mathrm{en}_{1,0}(P) = \frac{1}{n}\, \mathrm{en}_{1,0}(Q) \in \frac{1}{l}\mathbb{N} \times \mathbb{N}$ *and* $v_{1,-1}(\mathrm{en}_{1,0}(Q)) < 0$.
(4) $v_{0,-1}(\mathrm{en}_{1,0}(P)) < -1$ *and* $v_{0,-1}(\mathrm{en}_{1,0}(Q)) < -1$.
(5) *Neither P nor Q are monomials.*

Proof. Item (1) follows from inequality (1.2), since $v_{10}(P) < 0$ implies $v_{10}(Q) < 0$. Now we prove item (2). Assume by contradiction that $\mathrm{en}_{1,0}(Q) \nsim \mathrm{en}_{1,0}(P)$. By Propositions 1.13, 2.3, and 2.4(2) we have

$$\mathrm{en}_{1,0}(Q) + \mathrm{en}_{1,0}(P) = (1, 1), \tag{4.1}$$

which combined with the fact that $v_{1,-1}(\mathrm{en}_{1,0}(P)) < 0$ and $v_{1,0}(P) > 0$ implies that there exists $0 < r < l$ with $\mathrm{en}_{1,0}(P) = (r/l, 1)$ and $\mathrm{en}_{1,0}(Q) = ((l - r)/l, 0)$. Set

$$M := \{(1, 1)\} \cup ((\mathrm{Dir}(P) \cup \mathrm{Dir}(Q)) \cap](1, 0), (1, 1)[) = \{(\rho_0, \sigma_0) < \cdots < (\rho_k, \sigma_k) = (1, 1)\}.$$

We claim that

$$v_{0,1}(\mathrm{st}_{\rho_j,\sigma_j}(P)) + v_{0,1}(\mathrm{st}_{\rho_j,\sigma_j}(Q)) > 1, \quad \text{for } j > 0. \tag{4.2}$$

In fact, if $k = 0$ this is trivial. Otherwise, by Proposition 3.7 and Remark 1.8, we have

$$v_{0,1}(\mathrm{st}_{\rho_j,\sigma_j}(P)) \leq v_{0,1}(\mathrm{en}_{\rho_j,\sigma_j}(P)) = v_{0,1}(\mathrm{st}_{\rho_{j+1},\sigma_{j+1}}(P)), \quad \text{for } 0 \leq j < k,$$

with strict inequality if $(\rho_j,\sigma_j)\in\mathrm{Dir}(P)$, and the same is true for Q. The claim follows immediately from these facts, since

$$(\rho_0,\sigma_0)\in\mathrm{Dir}(P)\cup\mathrm{Dir}(Q)\quad\text{and}\quad v_{0,1}(\mathrm{st}_{\rho_0,\sigma_0}(P))+v_{0,1}(\mathrm{st}_{\rho_0,\sigma_0}(Q))=1,$$

where the equality follows from Proposition 3.7 and equality (4.1).

Inequality (4.2) implies that $\mathrm{st}_{\rho_j,\sigma_j}(P)+\mathrm{st}_{\rho_j,\sigma_j}(Q)\neq(1,1)$ for $j>0$, and so Proposition 2.4(1) and Proposition 3.7 yield

$$\mathrm{en}_{\rho_j,\sigma_j}(P)=\mathrm{st}_{\rho_{j+1},\sigma_{j+1}}(P)\sim\mathrm{st}_{\rho_{j+1},\sigma_{j+1}}(Q)=\mathrm{en}_{\rho_j,\sigma_j}(Q)\quad\text{for } 0\le j<k. \tag{4.3}$$

On the other hand, $\rho_j>0$ because $(\rho_j,\sigma_j)\in\,](1,0),(1,1)]$, and hence,

$$v_{\rho_j,\sigma_j}(Q)\ge v_{\rho_j,\sigma_j}(\mathrm{en}_{1,0}(Q))=\frac{l-r}{l}>0.$$

This allows us to use Remark 4.5 combined with (4.3), in order to prove inductively that

$$\frac{v_{\rho_0,\sigma_0}(P)}{v_{\rho_0,\sigma_0}(Q)}=\frac{v_{\rho_k,\sigma_k}(P)}{v_{\rho_k,\sigma_k}(Q)}=\frac{m}{n}=\frac{v_{1,0}(P)}{v_{1,0}(Q)}. \tag{4.4}$$

Set $A:=\mathrm{en}_{1,0}(P)$ and $B:=\mathrm{en}_{1,0}(Q)$. By Proposition 3.7, we have $v_{\rho_0,\sigma_0}(A)=v_{\rho_0,\sigma_0}(P)$ and $v_{\rho_0,\sigma_0}(B)=v_{\rho_0,\sigma_0}(Q)$. Consequently, by (4.4),

$$v_{1,0}(A)v_{\rho_0,\sigma_0}(B)-v_{1,0}(B)v_{\rho_0,\sigma_0}(A)=v_{1,0}(P)v_{\rho_0,\sigma_0}(Q)-v_{1,0}(Q)v_{\rho_0,\sigma_0}(P)=0,$$

which, by Lemma 4.4 with $(\rho,\sigma)=(1,0)$ and $(\rho',\sigma')=(\rho_0,\sigma_0)$, leads to $A\sim B$, contradicting the assumption that $\mathrm{en}_{1,0}(Q)\nsim\mathrm{en}_{1,0}(P)$ and proving item (2).

From item (2) we obtain

$$\frac{1}{m}\,\mathrm{en}_{1,0}(P)=\frac{1}{n}\,\mathrm{en}_{1,0}(Q)\in\frac{1}{l}\mathbb{N}\times\mathbb{N}_0\quad\text{and}\quad v_{1,-1}(\mathrm{en}_{1,0}(Q))<0.$$

But $v_{0,1}(\mathrm{en}_{1,0}(P)),v_{0,1}(\mathrm{en}_{1,0}(Q))>0$, since $v_{1,-1}(\mathrm{en}_{1,0}(P)),v_{1,-1}(\mathrm{en}_{1,0}(Q))<0$, and so item (3) holds. Thus $v_{0,-1}(\mathrm{en}_{1,0}(P))<-1$ and $v_{0,-1}(\mathrm{en}_{1,0}(Q))<-1$, which is item (4). In order to check item (5), assume for instance that P is a monomial. Then, by item (4),

$$v_{0,-1}(P)+v_{0,-1}(Q)=v_{0,-1}(\mathrm{en}_{1,0}(P))+v_{0,-1}(Q)<-1+0,$$

which contradicts inequality (1.2). □

Proposition 4.7. *Let (P,Q) be a minimal pair. Then there exist $m,n\in\mathbb{N}$ which are coprime, and $\varphi\in\mathrm{Aut}(L)$ such that $(\varphi(P),\varphi(Q))$ is an (m,n)-pair satisfying $v_{1,1}(\varphi(P))=v_{1,1}(P)$ and $v_{1,1}(\varphi(Q))=v_{1,1}(Q)$. Moreover, (Fig. 1)*

$$(-1,1)<\mathrm{Succ}_{\varphi(P)}(1,0),\mathrm{Succ}_{\varphi(Q)}(1,0)<(-1,0). \tag{4.5}$$

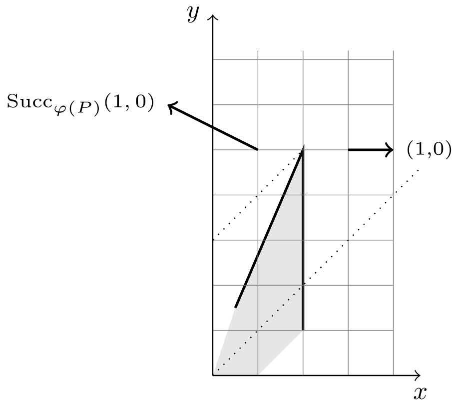

Fig. 1. The shape of $\varphi(P)$ according to Proposition 4.7.

Proof. Since (P,Q) is a counterexample to JC, by [11, Corollary 10.2.21] there exists an automorphism φ of L and integers $1 \le a \le b$ such that

$$(a,b) \in \mathrm{Supp}(\varphi(P)) \subseteq \{(i,j) : 0 \le i \le a,\ 0 \le j \le b\}. \tag{4.6}$$

We can also achieve that inequality (4.5) is satisfied. This is a well known fact (see for instance [10, p. 8] or [8, discussion at 1.12]). We claim that there exist $m,n \in \mathbb{N}$ such that $(\overline{P},\overline{Q}) := (\varphi(P),\varphi(Q))$ is an (m,n)-pair. Clearly

$$\mathrm{en}_{1,0}(\overline{P}) = (a,b) = \mathrm{st}_{1,1}(\overline{P}). \tag{4.7}$$

Moreover $v_{1,-1}(\mathrm{en}_{1,0}(\overline{P})) = a-b < 0$, since by Theorem 2.6(4), we have $a<b$. Now we prove that there exist $m,n \in \mathbb{N}$ coprime, such that

$$\frac{v_{1,1}(\overline{P})}{v_{1,1}(\overline{Q})} = \frac{m}{n} = \frac{v_{1,0}(\overline{P})}{v_{1,0}(\overline{Q})}. \tag{4.8}$$

By Proposition 4.1 the hypotheses of Remark 3.1 are satisfied for $(\overline{P},\overline{Q})$ and all $(\rho,\sigma) \in \mathfrak{V}$ such that $(1,0) \le (\rho,\sigma) \le (1,1)$. Hence there exists $m,n \in \mathbb{N}$ coprime such that $\frac{v_{11}(\overline{P})}{v_{11}(\overline{Q})} = \frac{m}{n}$,

$$(a,b) = \mathrm{st}_{1,1}(\overline{P}) = \frac{m}{n}\,\mathrm{st}_{1,1}(\overline{Q}) \tag{4.9}$$

and

$$\mathrm{Dir}(\overline{Q}) \cap](1,0),(1,1)[= \mathrm{Dir}(\overline{P}) \cap](1,0),(1,1)[= \emptyset,$$

where the last equality follows from (4.7) and Proposition 3.8. Hence by Proposition 3.7 we have $\mathrm{en}_{1,0}(\overline{Q}) = \mathrm{st}_{1,1}(\overline{Q})$ which, combined with (4.7) and (4.9), gives

$$\mathrm{en}_{1,0}(\overline{P}) = \frac{m}{n}\,\mathrm{en}_{1,0}(\overline{Q}).$$

This yields equality (4.8).

Next we prove that

$$v_{1,1}(\overline{P}) = v_{1,1}(P) \qquad \text{and} \qquad v_{1,1}(\overline{Q}) = v_{1,1}(Q). \tag{4.10}$$

For this consider the inverse $\psi := \varphi^{-1}$. Set $M := v_{1,1}(\psi(x))$ and $N := v_{1,1}(\psi(y))$. By [6] and [11, Corollary 5.1.6(a)], we know that either $N|M$ or $M|N$. If $M = N = 1$ then clearly ψ and φ preserve $v_{1,1}$, as desired.

We assert that the case $M|N$ and $N > 1$, and the case $N|M$ and $M > 1$, are impossible. Assume for example $M|N$ and $N > 1$ and set $R := \ell_{1,1}(\psi(x))$. Since

$$v_{1,1}([\psi(x), \psi(y)]) = 0 < M + N - 2 = v_{1,1}(\psi(x)) + v_{1,1}(\psi(y)) - 2,$$

it follows from Proposition 1.13, that $[\ell_{1,1}(\psi(x)), \ell_{1,1}(\psi(y))] = 0$. Hence, by Proposition 2.1,

$$\ell_{1,1}(\psi(y)) = \lambda R^k \qquad \text{for some } \lambda \in K^\times \text{ and } k \in \mathbb{N}.$$

By (4.6) we know that $\ell_{1,1}(\overline{P}) = \lambda_P x^a y^b$ for some $\lambda_P \in K^\times$, and that

$$i \le a, \quad j \le b \quad \text{and} \quad i + j < a + b \qquad \text{for all } (i,j) \in \mathrm{Supp}(\overline{P}) \setminus \{(a,b)\}.$$

Hence, for all such (i, j), we have

$$v_{1,1}(\psi(x^i y^j)) = iv_{1,1}(R) + jv_{1,1}(R^k) < av_{1,1}(R) + bv_{1,1}(R^k) = v_{1,1}(\psi(x^a y^b)),$$

and so

$$v_{1,1}(\psi(\overline{P})) = v_{1,1}(\psi(x^a y^b)) = v_{1,1}(R)(a + kb). \tag{4.11}$$

On the other hand by equality (4.9) we can write $a = \bar{a}m$ and $b = \bar{b}m$ with $\bar{a}, \bar{b} \in \mathbb{N}$. Hence equality (4.11) can be written as

$$v_{1,1}(\psi(\overline{P})) = mv_{1,1}(R)(\bar{a} + k\bar{b}).$$

By (4.9) we have $\mathrm{st}_{1,1}(\overline{Q}) = \frac{n}{m}(a, b) = n(\bar{a}, \bar{b})$. So, by Proposition 4.1 and Remark 3.1,

$$(n\bar{a}, n\bar{b}) \in \mathrm{Supp}(\overline{Q}) \subseteq \{(i,j) : 0 \le i \le n\bar{a},\ 0 \le j \le n\bar{b}\}. \tag{4.12}$$

A similar computation as above, but using (4.12) instead of (4.6), shows that

$$v_{1,1}(\psi(\overline{Q})) = nv_{1,1}(R)(\bar{a} + k\bar{b}).$$

Consequently

$$\gcd(v_{1,1}(\psi(\overline{P}))v_{1,1}(\psi(\overline{Q}))) = v_{1,1}(R)(\bar{a}+k\bar{b}) \geq \bar{a}+\bar{b} = \gcd(v_{1,1}(\overline{P}), v_{1,1}(\overline{Q})),$$

where the last equality follows from equality (4.9).

Since $(\psi(\overline{P}), \psi(\overline{Q}))) = (P,Q)$ is a minimal pair, equality must hold, and so we have $k=1$ and $v_{1,1}(R)=1$, which contradicts $kv_{1,1}(R) = v_{1,1}(\ell_{1,1}(\psi(y))) = N > 1$.

Similarly one discards the case $N|M$ and $M>1$, which finishes the proof of (4.10).

Hence $(\overline{P}, \overline{Q})$ is minimal pair and so, by Remark 4.2, we have $m,n>1$. □

5. Regular corners of (m,n)-pairs

It is known (see e.g. [11, Theorem 10.2.1]) that the Newton polygons of a Jacobian pair (P,Q) in L are similar. The same is not true in $L^{(l)}$, but it is almost true. One of the basic geometric reasons for this difference is the fact that, by Propositions 1.13, 2.3 and 2.4, if two corners of P and Q are not aligned, then they must sum to $(1,1)$. In L this is only possible for $(1,0)$ and $(0,1)$, but in $L^{(l)}$ this happens for all $(k/l,0)$ and $(1-k/l,1)$ if $k\in\mathbb{Z}\setminus\{0\}$ (see Case I.b), equality (5.5)).

We will analyze the edges and corners of the Newton polygons of an (m,n)-pair, corresponding to the directions in

$$I :=](1,-1),(1,0)] = \{(\rho,\sigma)\in\mathfrak{V} : (1,-1) < (\rho,\sigma) \leq (1,0)\}.$$

Note that for $(\rho,\sigma)\in I$ we have $\rho+\sigma>0$, $\sigma\leq 0$ and $\rho>0$. In particular we will analyze what we call regular corners (see Definition 5.5). The conditions we will find on regular corners will allow us to discard many "small" cases in Sections 6 and 7, and to obtain lower bounds for B.

From now on we assume that K is algebraically closed unless otherwise stated.

Lemma 5.1. *Let (P,Q) be an (m,n)-pair in $L^{(l)}$ and let $(\rho,\sigma)\in I$. If $\mathrm{en}_{\rho,\sigma}(P) = \frac{m}{n}\mathrm{en}_{\rho,\sigma}(Q)$, then $v_{\rho,\sigma}(P)>0$ and $v_{\rho,\sigma}(Q)>0$. Moreover, if $v_{0,-1}(\mathrm{st}_{\rho,\sigma}(P)) < -1$ or $v_{0,-1}(\mathrm{st}_{\rho,\sigma}(Q)) < -1$, then $[\ell_{\rho,\sigma}(P), \ell_{\rho,\sigma}(Q)] = 0$.*

Proof. Assume by contradiction that $v_{\rho,\sigma}(P)\leq 0$. Then $v_{\rho,\sigma}(Q) = \frac{n}{m}v_{\rho,\sigma}(P)\leq 0$. But then, since $\rho+\sigma>0$, we have

$$v_{\rho,\sigma}(P)+v_{\rho,\sigma}(Q)-(\rho+\sigma) < 0 = v_{\rho,\sigma}([P,Q]),$$

which contradicts (1.2) and proves $v_{\rho,\sigma}(P)>0$. The same argument proves that $v_{\rho,\sigma}(Q)>0$.

Now assume for instance that

$$v_{0,-1}(\mathrm{st}_{\rho,\sigma}(P)) < -1 \qquad \text{and} \qquad [\ell_{\rho,\sigma}(P), \ell_{\rho,\sigma}(Q)] \neq 0.$$

Since $(0,-1) < (\rho,\sigma) < (0,1)$, by Remark 1.8 we have

$$v_{0,-1}(\ell_{\rho,\sigma}(P)) = v_{0,-1}(\ell_{0,-1}(\ell_{\rho,\sigma}(P))) = v_{0,-1}(\mathrm{st}_{\rho,\sigma}(P)) < -1,$$

and so, we obtain

$$v_{0,-1}(\ell_{\rho,\sigma}(P)) + v_{0,-1}(\ell_{\rho,\sigma}(Q)) - (-1+0) < 0 = v_{0,-1}([\ell_{\rho,\sigma}(P), \ell_{\rho,\sigma}(Q)]),$$

which contradicts inequality (1.2), proving $[\ell_{\rho,\sigma}(P), \ell_{\rho,\sigma}(Q)] = 0$. Similar arguments apply to the case $v_{0,-1}(\mathrm{st}_{\rho,\sigma}(Q)) < -1$. □

For $P \in L^{(l)} \setminus \{0\}$ we set

$$A(P) := \{(\rho,\sigma) \in \mathrm{Dir}(P) \cap I : v_{0,-1}(\mathrm{st}_{\rho,\sigma}(P)) < -1 \text{ and } v_{1,-1}(\mathrm{st}_{\rho,\sigma}(P)) < 0\}.$$

Proposition 5.2. *Let $(\rho,\sigma) \in \mathrm{Dir}(P) \cap I$. If $(\rho',\sigma') < (\rho,\sigma) \leq (1,0)$ for some $(\rho',\sigma') \in A(P)$, then $(\rho,\sigma) \in A(P)$.*

Proof. It suffices to prove the result in the case in which $\mathrm{en}_{\rho',\sigma'}(P) = \mathrm{st}_{\rho,\sigma}(P)$. In this case, since $(0,-1) < (\rho',\sigma') < (0,1)$ and $(1,-1) < (\rho',\sigma') < (-1,1)$, it follows from Remark 1.8 that

$$v_{0,-1}(\mathrm{st}_{\rho,\sigma}(P)) = v_{0,-1}(\mathrm{en}_{\rho',\sigma'}(P)) < v_{0,-1}(\mathrm{st}_{\rho',\sigma'}(P)) < -1$$

and

$$v_{1,-1}(\mathrm{st}_{\rho,\sigma}(P)) = v_{1,-1}(\mathrm{en}_{\rho',\sigma'}(P)) < v_{1,-1}(\mathrm{st}_{\rho',\sigma'}(P)) < 0,$$

which implies that $(\rho,\sigma) \in A(P)$. □

Proposition 5.3. *Let (P,Q) be an (m,n)-pair and $(\overline{\rho},\overline{\sigma}) := \max(A(P))$. Then*

$$](\overline{\rho},\overline{\sigma}),(1,0)] \cap \mathrm{Dir}(Q) = \emptyset.$$

Proof. Assume that the statement is false and take

$$(\rho,\sigma) := \max\bigl(](\overline{\rho},\overline{\sigma}),(1,0)] \cap \mathrm{Dir}(Q)\bigr).$$

By Proposition 5.2 we know that $](\overline{\rho},\overline{\sigma}),(1,0)] \cap \mathrm{Dir}(P) = \emptyset$. Hence, by Proposition 3.7,

$$\mathrm{st}_{\rho,\sigma}(P) = \mathrm{en}_{\rho,\sigma}(P) = \mathrm{en}_{1,0}(P), \tag{5.1}$$

and so, by Proposition 4.6(4),

$$v_{0,-1}(\mathrm{st}_{\rho,\sigma}(P)) = v_{0,-1}(\mathrm{en}_{1,0}(P)) < -1.$$

On the other hand,

$$\mathrm{en}_{\rho,\sigma}(P) = \mathrm{en}_{1,0}(P) = \frac{m}{n}\,\mathrm{en}_{1,0}(Q) = \frac{m}{n}\,\mathrm{en}_{\rho,\sigma}(Q),$$

where the first equality follows from (5.1), the second on from the definition of (m,n)-pair, and the third one, from the fact that $](\rho,\sigma),(1,0)] \cap \mathrm{Dir}(Q) = \emptyset$ and Proposition 3.7. Hence, by Lemma 5.1 and Remark 3.1, we conclude that $(\rho,\sigma) \in \mathrm{Dir}(P)$, which is a contradiction. $\square$

Proposition 5.4. *If (P,Q) is an (m,n)-pair and $(\rho,\sigma) \in A(P)$, then*

(1) $\mathrm{en}_{\rho,\sigma}(P) = \frac{m}{n}\,\mathrm{en}_{\rho,\sigma}(Q)$,
(2) $\mathrm{st}_{\rho,\sigma}(P) = \frac{m}{n}\,\mathrm{st}_{\rho,\sigma}(Q)$,
(3) $(\rho,\sigma) \in \mathrm{Dir}(Q)$.

Moreover $A(Q) = A(P)$ and, if we set

$$(\rho_1,\sigma_1) := \begin{cases} \min(A(P)) & \text{if } A(P) \neq \emptyset, \\ \min(\mathrm{Succ}_P(1,0), \mathrm{Succ}_Q(1,0)) & \text{if } A(P) = \emptyset, \end{cases}$$

then $\mathrm{Pred}_P(\rho_1,\sigma_1) = \mathrm{Pred}_Q(\rho_1,\sigma_1) \in I$.

Proof. Assume $A(P) \neq \emptyset$ and write $A(P) = \{(\rho_1,\sigma_1) < (\rho_2,\sigma_2) < \cdots < (\rho_k,\sigma_k)\}$, where we are considering the order of I. We will prove inductively statements (1), (2) and (3) for (ρ_j,σ_j), starting from $j = k$. Let $(\overline{\rho},\overline{\sigma}) := \max(A(P) \cup A(Q))$. We have

$$\mathrm{en}_{\overline{\rho},\overline{\sigma}}(P) = \mathrm{en}_{1,0}(P) = \frac{m}{n}\,\mathrm{en}_{1,0}(Q) = \frac{m}{n}\,\mathrm{en}_{\overline{\rho},\overline{\sigma}}(Q),$$

where the first equality follows from Propositions 3.7 and 5.2, the second one, from Proposition 4.6(2) and the third one, from Propositions 3.7, since $](\overline{\rho},\overline{\sigma}),(1,0)] \cap \mathrm{Dir}(Q) = \emptyset$ by Propositions 5.2 and 5.3. Hence, by Lemma 5.1 and Remark 3.1, we have

$$(\overline{\rho},\overline{\sigma}) \in \mathrm{Dir}(P) \cap \mathrm{Dir}(Q) \qquad \text{and} \qquad \mathrm{st}_{\overline{\rho},\overline{\sigma}}(P) = \frac{m}{n}\,\mathrm{st}_{\overline{\rho},\overline{\sigma}}(Q).$$

On the other hand by Proposition 5.2, we have $(\overline{\rho},\overline{\sigma}) = (\rho_k,\sigma_k)$, and so statements (1), (2) and (3) hold for (ρ_k,σ_k).

Let now $j \geq 1$, assume that statements (1), (2) and (3) hold for $(\rho_{j+1},\sigma_{j+1})$ and set

$$(\tilde{\rho},\tilde{\sigma}) = \max\{\mathrm{Pred}_P(\rho_{j+1},\sigma_{j+1}), \mathrm{Pred}_Q(\rho_{j+1},\sigma_{j+1})\}.$$

Then

$$\mathrm{en}_{\tilde{\rho},\tilde{\sigma}}(P) = \mathrm{st}_{\rho_{j+1},\sigma_{j+1}}(P) = \frac{m}{n}\,\mathrm{st}_{\rho_{j+1},\sigma_{j+1}}(Q) = \frac{m}{n}\,\mathrm{en}_{\tilde{\rho},\tilde{\sigma}}(Q),$$

where the second equality holds by condition 2) for $(\rho_{j+1},\sigma_{j+1})$. Moreover by Propositions 5.2 and 3.7,

$$(\tilde\rho,\tilde\sigma)=(\rho_j,\sigma_j) \qquad \text{or} \qquad \mathrm{st}_{\tilde\rho,\tilde\sigma}(P)=\mathrm{en}_{\tilde\rho,\tilde\sigma}(P)=\mathrm{st}_{\rho_{j+1},\sigma_{j+1}}(P),$$

and so $v_{0,-1}(\mathrm{st}_{\tilde\rho,\tilde\sigma}(P))<-1$. Hence, by Lemma 5.1 and Remark 3.1, we have

$$(\tilde\rho,\tilde\sigma)\in \mathrm{Dir}(P)\cap \mathrm{Dir}(Q) \qquad \text{and} \qquad \mathrm{st}_{\tilde\rho,\tilde\sigma}(P)=\frac{m}{n}\,\mathrm{st}_{\tilde\rho,\tilde\sigma}(Q).$$

On the other hand, by Proposition 5.2 we have $(\tilde\rho,\tilde\sigma)=(\rho_j,\sigma_j)$, and so statements (1), (2) and (3) hold for (ρ_j,σ_j).

Now we will prove that $A(P)=A(Q)$. By symmetry it suffices to prove that $A(P)\subseteq A(Q)$. Let $(\rho,\sigma)\in A(P)$. By statement (3) we already know $(\rho,\sigma)\in \mathrm{Dir}(Q)$. So we have to prove only that

$$v_{0,-1}(\mathrm{st}_{\rho,\sigma}(Q))<-1 \quad \text{and} \quad v_{1,-1}(\mathrm{st}_{\rho,\sigma}(Q))<0.$$

By statement (2)

$$v_{1,-1}(\mathrm{st}_{\rho,\sigma}(Q))=\frac{n}{m}v_{1,-1}(\mathrm{st}_{\rho,\sigma}(P))<0.$$

Note now that again by statement (2)

$$\frac{1}{m}v_{0,-1}(\mathrm{st}_{\rho,\sigma}(P))\in\mathbb{Z},$$

and so

$$\frac{1}{m}v_{0,-1}(\mathrm{st}_{\rho,\sigma}(P))\le -1,$$

since $v_{0,-1}(\mathrm{st}_{\rho,\sigma}(P))<0$. Hence, once again by statement (2),

$$v_{0,-1}(\mathrm{st}_{\rho,\sigma}(Q))=\frac{n}{m}v_{0,-1}(\mathrm{st}_{\rho,\sigma}(P))\le -n<-1,$$

which proves $A(P)\subseteq A(Q)$, as desired.

Now we prove

$$(\rho_0,\sigma_0):=\mathrm{Pred}_P(\rho_1,\sigma_1)=\mathrm{Pred}_Q(\rho_1,\sigma_1)\in I.$$

Set $(\hat\rho,\hat\sigma):=\max\{\mathrm{Pred}_P(\rho_1,\sigma_1),\mathrm{Pred}_Q(\rho_1,\sigma_1)\}$. We first prove that

$$(1,-1)<(\hat\rho,\hat\sigma)<(\rho_1,\sigma_1).$$

Assume by contradiction that $(-\rho_1,-\sigma_1)\le(\hat\rho,\hat\sigma)\le(1,-1)$, which, by Proposition 3.7, implies that

$$\mathrm{en}_{1,-1}(P)=\mathrm{st}_{\rho_1,\sigma_1}(P)\quad\text{and}\quad \mathrm{en}_{1,-1}(Q)=\mathrm{st}_{\rho_1,\sigma_1}(Q). \tag{5.2}$$

If $(\rho_1,\sigma_1)\in A(P)\cap A(Q)$, then this implies

$$v_{1,-1}(P)=v_{1,-1}(\mathrm{st}_{\rho_1,\sigma_1}(P))<0\quad\text{and}\quad v_{1,-1}(Q)=v_{1,-1}(\mathrm{st}_{\rho_1,\sigma_1}(Q))<0,$$

and so, by inequality (1.2), we have $v_{1,-1}([P,Q])<0$, which contradicts that $[P,Q]\in K^{\times}$. Hence we can suppose that $A(P)=\emptyset$, which by Proposition 3.7, implies that

$$\mathrm{st}_{\rho_1,\sigma_1}(P)=\mathrm{en}_{1,0}(P)\quad\text{and}\quad \mathrm{st}_{\rho_1,\sigma_1}(Q)=\mathrm{en}_{1,0}(Q).$$

Consequently, by (5.2),

$$\mathrm{en}_{1,-1}(P)=\mathrm{en}_{1,0}(P)\quad\text{and}\quad \mathrm{en}_{1,-1}(Q)=\mathrm{en}_{1,0}(Q).$$

By the definition of (m,n)-pair and Proposition 4.6(3), this implies that

$$v_{1,-1}(P)=v_{1,-1}(\mathrm{en}_{1,0}(P))<0\quad\text{and}\quad v_{1,-1}(Q)=v_{1,-1}(\mathrm{en}_{1,0}(Q))<0,$$

and so, again by inequality (1.2), we have $v_{1,-1}([P,Q])<0$, which contradicts that $[P,Q]\in K^{\times}$.

In order to conclude the proof, we must show that $\mathrm{Pred}_P(\rho_1,\sigma_1)=\mathrm{Pred}_Q(\rho_1,\sigma_1)$. Assume this is false and suppose for example that $\mathrm{Pred}_P(\rho_1,\sigma_1)<\mathrm{Pred}_Q(\rho_1,\sigma_1)$, which implies

$$\mathrm{st}_{\hat\rho,\hat\sigma}(P)=\mathrm{en}_{\hat\rho,\hat\sigma}(P)=\mathrm{st}_{\rho_1,\sigma_1}(P). \tag{5.3}$$

If $A(P)\neq\emptyset$, then by Lemma 5.1, the conditions of Remark 3.1 are satisfied for $(\hat\rho,\hat\sigma)$. Consequently, by this remark, $(\hat\rho,\hat\sigma)\in\mathrm{Dir}(P)$, contradicting (5.3). Assume now $A(P)=\emptyset$, which implies that $(\hat\rho,\hat\sigma)\le(1,0)<(\rho_1,\sigma_1)$. Hence, by (5.3), Proposition 3.7 and Proposition 4.6(2),

$$\mathrm{st}_{\hat\rho,\hat\sigma}(P)=\mathrm{en}_{\hat\rho,\hat\sigma}(P)=\mathrm{en}_{1,0}(P)=\frac{n}{m}\,\mathrm{en}_{1,0}(Q)=\frac{n}{m}\,\mathrm{en}_{\hat\rho,\hat\sigma}(Q)$$

and

$$v_{0,-1}(\mathrm{st}_{\hat\rho,\hat\sigma}(P))=v_{0,-1}(\mathrm{en}_{1,0}(P))<-1.$$

Hence again by Lemma 5.1, the conditions of Remark 3.1 are satisfied for $(\hat\rho,\hat\sigma)$, and therefore $(\hat\rho,\hat\sigma)\in\mathrm{Dir}(P)$, which contradicts (5.3). The case $\mathrm{Pred}_Q(\rho_1,\sigma_1)<\mathrm{Pred}_P(\rho_1,\sigma_1)$, can be discarded using a similar argument. □

Definition 5.5. A *regular corner* of an (m,n)-pair (P,Q) in $L^{(l)}$, is a pair $(A,(\rho,\sigma))$, where $A=(a/l,b)\in \frac{1}{l}\mathbb{Z}\times \mathbb{N}_0$ and $(\rho,\sigma)\in I$ such that

(1) $b\ge 1$ and $b>a/l$,
(2) $(\rho,\sigma)\in \operatorname{Dir}(P)$,
(3) $\left(\frac{a}{l},b\right)=\frac{1}{m}\operatorname{en}_{\rho,\sigma}(P)$.

A regular corner $(A,(\rho,\sigma))$ is said to be *at* the point A.

Proposition 5.6. *If $(A,(\rho,\sigma))$ is a regular corner of an (m,n)-pair (P,Q), then at least one of the following three facts is true:*

(a) $(\rho,\sigma)\in A(P)$,
(b) $\operatorname{en}_{\rho,\sigma}(P)=\operatorname{en}_{1,0}(P)$,
(c) $(\rho,\sigma)=\operatorname{Pred}_P(\rho_1,\sigma_1)$, *where* $(\rho_1,\sigma_1):=\min(A(P))$.

Moreover, there exists exactly one regular corner $(A,(\rho,\sigma))$ such that $(\rho,\sigma)\notin A(P)$.

Proof. Assume that $(\rho,\sigma)\notin A(P)$ and define $(\rho_1,\sigma_1):=\operatorname{Succ}_P(\rho,\sigma)$. If $(\rho,\sigma)<(\rho_1,\sigma_1)\le(1,0)$, then $(\rho_1,\sigma_1)\in A(P)$, which implies that $(\rho_1,\sigma_1)=\min(A(P))$ by Proposition 5.2, and so item (c) holds. Otherwise $(\rho,\sigma)\le(1,0)<(\rho_1,\sigma_1)$ and, by Proposition 3.7, we conclude that $\operatorname{en}_{\rho,\sigma}(P)=\operatorname{en}_{1,0}(P)$. □

Corollary 5.7. *If $(A,(\rho,\sigma))$ is a regular corner of an (m,n)-pair (P,Q), then*

(1) $v_{\rho,\sigma}(P)>0$ *and* $v_{\rho,\sigma}(Q)>0$,
(2) $\operatorname{en}_{\rho,\sigma}(P)=\frac{m}{n}\operatorname{en}_{\rho,\sigma}(Q)$,
(3) $(\rho,\sigma)\in\operatorname{Dir}(Q)$.

Proof. If $(\rho,\sigma)\in A(P)$, or $(\rho,\sigma)=\operatorname{Pred}_P(\rho_1,\sigma_1)$, where $(\rho_1,\sigma_1):=\min(A(P))$, then Lemma 5.1 and Proposition 5.4 yield the result. Hence, by Proposition 5.6, it suffices to prove the assertions when $\operatorname{en}_{\rho,\sigma}(P)=\operatorname{en}_{1,0}(P)$ and $(\rho,\sigma)\notin A(P)$. We claim that $A(P)=\emptyset$. In fact, if there exists $(\rho',\sigma')\in A(P)$ with $(\rho',\sigma')<(\rho,\sigma)$, then $(\rho,\sigma)\in A(P)$ by Proposition 5.2. On the other hand, if there exists $(\rho',\sigma')\in A(P)$ with $(\rho',\sigma')>(\rho,\sigma)$, then

$$(\rho',\sigma')\ge \operatorname{Succ}_P(\rho,\sigma)=\operatorname{Succ}_P(1,0)>(1,0),$$

which contradicts $(\rho',\sigma')\in I$. Now we set

$$(\rho_1',\sigma_1'):=\min(\operatorname{Succ}_P(1,0),\operatorname{Succ}_Q(1,0)).$$

Then $(\rho,\sigma)=\operatorname{Pred}_P(\rho_1',\sigma_1')$ and from Proposition 5.4 we obtain

$$(\rho,\sigma)=\mathrm{Pred}_Q(\rho_1',\sigma_1')\in \mathrm{Dir}(Q).$$

Since $(\rho,\sigma)\le(1,0)<(\rho_1',\sigma_1')$, by Proposition 3.7 this implies $\mathrm{en}_{\rho,\sigma}(Q)=\mathrm{en}_{1,0}(Q)$. Consequently, by Proposition 4.6(2),

$$\mathrm{en}_{\rho,\sigma}(P)=\mathrm{en}_{1,0}(P)=\frac{m}{n}\,\mathrm{en}_{1,0}(Q)=\frac{m}{n}\,\mathrm{en}_{\rho,\sigma}(Q),$$

and Lemma 5.1 concludes the proof. □

Remark 5.8. If $((a/l,b),(\rho,\sigma))$ is a regular corner of an (m,n)-pair (P,Q) in $L^{(l)}$, then $a>0$.

Let $(A,(\rho,\sigma))$ be a regular corner of an (m,n)-pair (P,Q) in $L^{(l)}$. Write

$$\ell_{\rho,\sigma}(P)=x^{k/l}\mathfrak{p}(z)\quad \text{where } z:=x^{-\sigma/\rho}y \text{ and } \mathfrak{p}(z)\in K[z]. \tag{5.4}$$

Since $(\rho,\sigma)\in \mathrm{Dir}(P)$ the polynomial $\mathfrak{p}(z)$ is not a constant. Moreover by Corollary 5.7(1) and Theorem 2.6(4) we know that $v_{1,-1}(\mathrm{st}_{\rho,\sigma}(P))\neq 0$. Hence, one of the following five mutually excluding conditions is true:

- I.a) $[\ell_{\rho,\sigma}(P),\ell_{\rho,\sigma}(Q)]\neq 0$ and $\mathrm{st}_{\rho,\sigma}(P)\sim \mathrm{st}_{\rho,\sigma}(Q)$.
- I.b) $[\ell_{\rho,\sigma}(P),\ell_{\rho,\sigma}(Q)]\neq 0$ and $\mathrm{st}_{\rho,\sigma}(P)\nsim \mathrm{st}_{\rho,\sigma}(Q)$.
- II.a) $[\ell_{\rho,\sigma}(P),\ell_{\rho,\sigma}(Q)]=0$, $\#\,\mathrm{factors}(\mathfrak{p}(z))>1$ and $v_{1,-1}(\mathrm{st}_{\rho,\sigma}(P))<0$.
- II.b) $[\ell_{\rho,\sigma}(P),\ell_{\rho,\sigma}(Q)]=0$, $\#\,\mathrm{factors}(\mathfrak{p}(z))>1$ and $v_{1,-1}(\mathrm{st}_{\rho,\sigma}(P))>0$.
- III) $[\ell_{\rho,\sigma}(P),\ell_{\rho,\sigma}(Q)]=0$ and $\mathfrak{p}(z)=\mu(z-\lambda)^r$ for some $\mu,\lambda\in K^\times$ and $r\in\mathbb{N}$.

Remark 5.9. Let (P,Q) be an (m,n)-pair in $L^{(l)}$ and let $(\rho,\sigma)\in \mathrm{Dir}(P)\cap I$. If $(A,(\rho',\sigma'))$ is a regular corner and $(\rho',\sigma')<(\rho,\sigma)\le(1,0)$, then $(\rho,\sigma)\in A(P)$. In fact, by Proposition 5.2, it suffices to consider the case in which $\mathrm{en}_{\rho',\sigma'}(P)=\mathrm{st}_{\rho,\sigma}(P)$, and in that case it follows easily from the definition of $A(P)$ and Definition 5.5.

Remark 5.10. Let (P,Q) be an (m,n)-pair in $L^{(l)}$ and let $(\rho,\sigma)\in\mathfrak{V}$. If $(\rho,\sigma)\in A(P)$ then $\left(\frac{1}{m}\,\mathrm{en}_{\rho,\sigma}(P),(\rho,\sigma)\right)$ is a regular corner and we are in the Case II.a).

Remark 5.11. In the Case II.a), if we set $(\rho',\sigma'):=\mathrm{Pred}_P(\rho,\sigma)$, then $\left(\frac{1}{m}\,\mathrm{st}_{\rho,\sigma}(P),(\rho',\sigma')\right)$ is a regular corner of (P,Q).

Remark 5.12. If (P,Q) is an (m,n)-pair, then $\frac{1}{m}\,\mathrm{en}_{1,0}(P)$ is the first component of a regular corner of (P,Q).

Proposition 5.13 (Cases I.a) and I.b)). *Let (P,Q) be an (m,n)-pair in $L^{(l)}$, and $((a/l,b),(\rho,\sigma))$ a regular corner of (P,Q). Assume $[\ell_{\rho,\sigma}(P),\ell_{\rho,\sigma}(Q)]\neq 0$. Then $l-a/b>1$ and the following assertions hold:*

a) *If* $\mathrm{st}_{\rho,\sigma}(P) \sim \mathrm{st}_{\rho,\sigma}(Q)$, *then*

$$\frac{1}{m}\,\mathrm{st}_{\rho,\sigma}(P) \in \frac{1}{l}\mathbb{Z} \times \mathbb{N}_0 \quad \textit{and} \quad \mathrm{st}_{\rho,\sigma}(P) \sim (1,0).$$

b) *If* $\mathrm{st}_{\rho,\sigma}(P) \nsim \mathrm{st}_{\rho,\sigma}(Q)$, *then there exists* $k \in \mathbb{N}$, *with* $k < l - \frac{a}{b}$, *such that*

$$\{\mathrm{st}_{\rho,\sigma}(P), \mathrm{st}_{\rho,\sigma}(Q)\} = \left\{ \left(\frac{k}{l}, 0\right), \left(1 - \frac{k}{l}, 1\right) \right\}. \tag{5.5}$$

Proof. a) Since $\mathrm{st}_{\rho,\sigma}(P) \sim \mathrm{st}_{\rho,\sigma}(Q)$, it follows from Corollary 5.7, that

$$\frac{1}{m}\,\mathrm{st}_{\rho,\sigma}(P) = \frac{1}{n}\,\mathrm{st}_{\rho,\sigma}(Q),$$

and so

$$A' := \frac{1}{m}\,\mathrm{st}_{\rho,\sigma}(P) \in \frac{1}{l}\mathbb{Z} \times \mathbb{N}_0, \tag{5.6}$$

because m and n are coprime. Hence $A' = (a'/l, b')$ with $a' \in \mathbb{Z}$ and $b' \in \mathbb{N}_0$. Now we prove that $\mathrm{st}_{\rho,\sigma}(P) \sim (1,0)$ or, equivalently, that $b' = 0$. Assume by contradiction that $b' > 0$. By Remark 2.8 we can write

$$\ell_{\rho,\sigma}(P) = x^{\frac{ma'}{l}} y^{mb'} f(z) \quad \text{and} \quad \ell_{\rho,\sigma}(Q) = x^{\frac{na'}{l}} y^{nb'} g(z),$$

where $z := x^{-\frac{\sigma}{\rho}} y$ and $f(z), g(z) \in K[z]$. Since $nb', mb' \geq 1$, the term y divides both $\ell_{\rho,\sigma}(P)$ and $\ell_{\rho,\sigma}(Q)$. Consequently y is a factor of $[\ell_{\rho,\sigma}(P), \ell_{\rho,\sigma}(Q)]$. Since by Proposition 1.13, we know that $[\ell_{\rho,\sigma}(P), \ell_{\rho,\sigma}(Q)] = \ell_{\rho,\sigma}([P,Q]) \in K^{\times}$, this is a contradiction which proves that $b' = 0$.

We next prove $l - a/b > 1$ in this case. Since, by Corollary 5.7(1),

$$a' = \frac{l}{\rho} v_{\rho,\sigma}\left(\frac{a'}{l}, 0\right) = \frac{l}{\rho m} v_{\rho,\sigma}(\mathrm{en}_{\rho,\sigma}(P)) > 0,$$

it suffices to show that $l - a/b > a'$. Assume that this is false. Then $1 - \frac{a'}{l} \leq \frac{a}{bl}$, and so

$$v_{\rho,\sigma}\left(1 - \frac{a'}{l}, 1\right) \leq \frac{1}{b} v_{\rho,\sigma}\left(\frac{a}{l}, b\right) = \frac{1}{bm} v_{\rho,\sigma}(\mathrm{en}_{\rho,\sigma}(P)),$$

since $\rho > 0$. Moreover,

$$v_{\rho,\sigma}\left(\frac{a'}{l}, 0\right) = \frac{1}{m} v_{\rho,\sigma}(\mathrm{st}_{\rho,\sigma}(P)),$$

and so, by Proposition 1.13,

$$v_{\rho,\sigma}(P)+v_{\rho,\sigma}(Q)=\rho+\sigma=v_{\rho,\sigma}\left(1-\frac{a'}{l},1\right)+v_{\rho,\sigma}\left(\frac{a'}{l},0\right)\le\left(\frac{1}{bm}+\frac{1}{m}\right)v_{\rho,\sigma}(P)$$
$$\le v_{\rho,\sigma}(P).$$

But this is impossible since $v_{\rho,\sigma}(Q)>0$ by Corollary 5.7(1).

b) By Proposition 1.13,

$$[\ell_{\rho,\sigma}(P),\ell_{\rho,\sigma}(Q)]=\ell_{\rho,\sigma}([P,Q])\in K^{\times},$$

and consequently, by Proposition 2.4(1),

$$\mathrm{st}_{\rho,\sigma}(P)+\mathrm{st}_{\rho,\sigma}(Q)=(1,1).$$

Therefore equality (5.5) is true for some $k\in\mathbb{Z}$. Applying $v_{\rho,\sigma}$ we obtain

$$\left\{\rho\frac{k}{l},\rho\left(1-\frac{k}{l}\right)+\sigma\right\}=\{v_{\rho,\sigma}(P),v_{\rho,\sigma}(Q)\},$$

which by Corollary 5.7(1), implies $k>0$. Assume that $\mathrm{st}_{\rho,\sigma}(Q)=\left(1-\frac{k}{l},1\right)$. By Corollary 5.7(2),

$$n\left(\rho\frac{a}{l}+\sigma b\right)=v_{\rho,\sigma}(\mathrm{en}_{\rho,\sigma}(Q))=v_{\rho,\sigma}\left(1-\frac{k}{l},1\right)=\rho-\rho\frac{k}{l}+\sigma,$$

and so

$$k=l-na+\frac{\sigma l}{\rho}(1-bn). \tag{5.7}$$

On the other hand, since $v_{\rho,\sigma}(P)>0$ and $\frac{l}{\rho bm}>0$, we have

$$\frac{l\sigma}{\rho}+\frac{a}{b}=\frac{l}{\rho b}\left(\rho\frac{a}{l}+\sigma b\right)=\frac{l}{\rho bm}v_{\rho,\sigma}(\mathrm{en}_{\rho,\sigma}(P))>0.$$

Multiplying this inequality by $bn-1>0$, we obtain

$$\frac{\sigma l}{\rho}(1-bn)<\frac{a}{b}(bn-1).$$

Combining this with equality (5.7) we conclude that

$$k=l-na+\frac{\sigma l}{\rho}(1-bn)<l-na+\frac{a}{b}(bn-1)=l-\frac{a}{b},$$

as desired. In the case $\mathrm{st}_{\rho,\sigma}(P)=\left(1-\frac{k}{l},1\right)$ the proof of $k<l-\frac{a}{b}$ is similar. Since $k\ge 1$ we also obtain $l-a/b>1$ in the case b). □

Proposition 5.14 *(Case II). Let (P,Q) be an (m,n)-pair in $L^{(l)}$, let $((a/l,b),(\rho,\sigma))$ be a regular corner of (P,Q) and let F be as in Theorem 2.6. Assume that $[\ell_{\rho,\sigma}(P),\ell_{\rho,\sigma}(Q)]=0$ and write $\ell_{\rho,\sigma}(P)=x^{k/l}\mathfrak{p}(z)$, where $z:=x^{-\sigma/\rho}y$ and $\mathfrak{p}(z)\in K[z]$. If $\#\operatorname{factors}(\mathfrak{p}(z))>1$, then*

(1) $\operatorname{en}_{\rho,\sigma}(F)\sim(a/l,b)$.
(2) $\rho/\gcd(\rho,l)\le b$.

Set $d:=\gcd(a,b)$, $\overline{a}:=a/d$, $\overline{b}:=b/d$ and write $\operatorname{en}_{\rho,\sigma}(F)=\mu(\overline{a}/l,\overline{b})$. We have:

(3) $(\rho,\sigma)=\operatorname{dir}(\operatorname{en}_{\rho,\sigma}(F)-(1,1))=\operatorname{dir}(\mu\overline{a}-l,\mu\overline{b}l-l)$.
(4) $\mu\in\mathbb{N}$, $\mu\le l(bl-a)+1/\overline{b}$, $d\nmid\mu$ and $d>1$.

Proof. Write $F=x^{1+\sigma/\rho}\mathfrak{f}(z)$, where $\mathfrak{f}(z)\in K[z]$. Note that $\mathfrak{p}(z)=z^s p(z)$ and $\mathfrak{f}(z)=z^v f(z)$, where p, f, s and v are the same as in Proposition 2.11. Moreover $s>0$ implies $v>0$ by Remark 2.8 and Theorem 2.6(2), and so, by Proposition 2.11(1), each irreducible factor of $\mathfrak{p}$ divides $\mathfrak{f}$. Since $\#\operatorname{factors}(\mathfrak{p}(z))>1$, we have $\deg(\mathfrak{f})\ge 2$. Hence $\operatorname{en}_{\rho,\sigma}(F)\ne(1,1)$ by Remark 2.8. Consequently, by Theorem 2.6(3), we have $\operatorname{en}_{\rho,\sigma}(F)\sim\operatorname{en}_{\rho,\sigma}(P)$ which yields statement (1).

Now we prove statement (2). Let $A_1':=\frac{1}{m}\operatorname{st}_{\rho,\sigma}(P)$. By Remark 3.1(2) we have $A_1'\in\frac{1}{l}\mathbb{Z}\times\mathbb{N}_0$. Write $A_1'=(a'/l,b')$. Since $b'<b$ by Remark 2.8, and $v_{\rho,\sigma}(a/l,b)=v_{\rho,\sigma}(a'/l,b')$, there exists $h\in\mathbb{N}$, such that

$$\left(-\frac{\sigma hl/\rho}{l},h\right)=h\left(-\frac{\sigma}{\rho},1\right)=\left(\frac{a}{l},b\right)-\left(\frac{a'}{l},b'\right)\in\frac{1}{l}\mathbb{Z}\times\mathbb{N}_0.$$

Hence ρ divides σhl. Set

$$\overline{\rho}:=\frac{\rho}{\gcd(\rho,l)}\quad\text{and}\quad\overline{l}:=\frac{l}{\gcd(\rho,l)}.$$

Clearly $\overline{\rho}$ divides $h\sigma\overline{l}$, and so $\overline{\rho}\mid h=b-b'$, which implies $\overline{\rho}\le b$, as desired.

Statement (3) follows from Remark 3.2 and the fact that $\operatorname{en}_{\rho,\sigma}(F)\ne(1,1)$.

It remains to prove statement (4). First note that $\mu\in\mathbb{N}$, since $\overline{b}\in\mathbb{N}$, $\mu\overline{b}\in\mathbb{N}$, $\mu\overline{a}\in\mathbb{Z}$ and $\gcd(\overline{a},\overline{b})=1$. On the other hand, by Remark 3.1 we know that there exists $\lambda_P,\lambda_Q\in K^\times$ and a (ρ,σ)-homogeneous element $R\in L^{(l)}$, such that

$$\ell_{\rho,\sigma}(P)=\lambda_P R^m\qquad\text{and}\qquad\ell_{\rho,\sigma}(Q)=\lambda_Q R^n,$$

which implies

$$\operatorname{en}_{\rho,\sigma}(R)=(a/l,b)=d(\overline{a}/l,\overline{b})=\frac{d}{\mu}\operatorname{en}_{\rho,\sigma}(F).$$

Next we prove that $d \nmid \mu$. In fact, if we assume that $d|\mu$, then we have

$$v_{\rho,\sigma}(R^{\mu/d}) = v_{\rho,\sigma}(F) = \rho + \sigma \qquad \text{and} \qquad [R^{\mu/d}, \ell_{\rho,\sigma}(P)] = 0,$$

where the last equality follows from the fact that $[-,-]$ is a Poisson bracket and $[R^n, \ell_{\rho,\sigma}(P)] = 0$. But this contradicts Proposition 2.11(5) and proves $d \nmid \mu$. From this it follows immediately that $d > 1$. Finally we prove that $\mu \le l(bl - a) + 1/\bar{b}$. Since

$$(\mu\bar{a} - l) - (\mu\bar{b}l - l) = \mu(\bar{a} - \bar{b}l) = \frac{\mu}{d}(a - bl) < 0,$$

from equalities (3.2) and statement (3) it follows that

$$\rho = \frac{(\mu\bar{b} - 1)l}{d_1}, \quad \text{where } d_1 := \gcd(\mu\bar{a} - l, \mu\bar{b}l - l).$$

Now note that d_1 divides $\bar{b}l(\mu\bar{a} - l) - \bar{a}(\mu\bar{b}l - l) = l(\bar{a} - \bar{b}l)$, and therefore

$$d_1 \le l(\bar{b}l - \bar{a}),$$

since $\bar{b}l - \bar{a} > 0$. Hence, by statement (2),

$$b \ge \frac{\rho}{\gcd(\rho, l)} \ge \frac{\rho}{l} = \frac{(\mu\bar{b} - 1)}{d_1} \ge \frac{(\mu\bar{b} - 1)}{l(\bar{b}l - \bar{a})},$$

which implies $\mu\bar{b} - 1 \le bl(\bar{b}l - \bar{a}) = \bar{b}l(bl - a)$, as desired. □

Remark 5.15. Let $((a/l, b), (\rho, \sigma))$ be a regular corner of an (m, n)-pair (P, Q) and let L be the straight line that includes $\operatorname{Supp}(\ell_{\rho,\sigma}(P))$. The intersection of L with the diagonal $x = y$ is the point

$$\lambda(1, 1), \quad \text{where} \quad \lambda = \frac{m}{l}\left(\frac{a\rho + bl\sigma}{\rho + \sigma}\right) \quad \text{(Fig. 2)}.$$

In fact

$$\lambda(\rho + \sigma) = v_{\rho,\sigma}\big(\lambda(1, 1)\big) = v_{\rho,\sigma}(\ell_{\rho,\sigma}(P)) = v_{\rho,\sigma}(m(a/l, b)) = \frac{m}{l}(a\rho + bl\sigma),$$

from which the assertion follows.

The following proposition about multiplicities can be traced back to [6, Corollary 2.6(2)]. The algebraic parallel is not so clear, but the geometric meaning, which will be proved in Proposition 5.18, is that one can cut the support of $\ell_{\rho,\sigma}(P)$ above the diagonal.

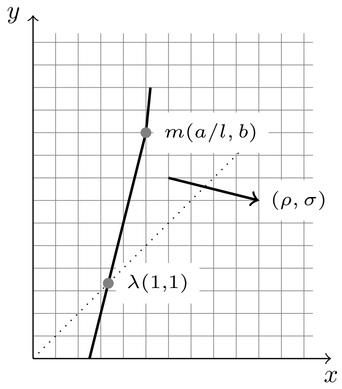

Fig. 2. Remark 5.15 with $\lambda = \frac{m}{l}\left(\frac{a\rho+bl\sigma}{\rho+\sigma}\right)$.

Proposition 5.16 *(*Case II.b)*). Let* (P, Q) *and* $((a/l, b), (\rho, \sigma))$ *be as in* Proposition 5.14*. Assume that* $[\ell_{\rho,\sigma}(P), \ell_{\rho,\sigma}(Q)] = 0$ *and write* $\ell_{\rho,\sigma}(P) = x^{k/l}\mathfrak{p}(z)$ *where* $z := x^{-\sigma/\rho}y$ *and* $\mathfrak{p}(z) \in K[z]$*. If* $\#\,\mathrm{factors}(\mathfrak{p}(z)) > 1$ *and* $v_{1,-1}(\mathrm{st}_{\rho,\sigma}(P)) > 0$*, then there exists* $\lambda \in K^\times$ *such that* $z - \lambda$ *has multiplicity*

$$m_\lambda \geq \frac{m}{l}\left(\frac{a\rho + bl\sigma}{\rho + \sigma}\right)$$

in $\mathfrak{p}(z)$*.*

Proof. Let F be as in Theorem 2.6 and write $F = x^{1+\sigma/\rho}\mathfrak{f}(z)$. In the proof of Proposition 5.14 it was shown that each irreducible factor of $\mathfrak{p}$ divides $\mathfrak{f}$. Hence, there is a linear factor of $\mathfrak{p}$ with multiplicity greater than or equal to $\deg(\mathfrak{p})/\deg(\mathfrak{f})$. Since $\mathrm{en}_{\rho,\sigma}(P) = (ma/l, mb)$, it follows from Remark 2.8, that $\deg(\mathfrak{p}) = mb$. Similarly, if we write $\mathrm{en}_{\rho,\sigma}(F) = (M_0, M)$, then $M = \deg(\mathfrak{f})$, an so $\deg(\mathfrak{p})/\deg(\mathfrak{f}) = mb/M$. Consequently, in order to finish the proof it suffices to check that

$$\frac{mb}{M} = \frac{m}{l}\left(\frac{a\rho + bl\sigma}{\rho + \sigma}\right). \tag{5.8}$$

Since

$$\rho + \sigma = v_{\rho,\sigma}(F) = \rho M_0 + \sigma M,$$

we have

$$M_0 = \frac{1}{\rho}(\rho + \sigma - \sigma M).$$

Hence, by Proposition 5.14(1),

$$\frac{a}{bl} = \frac{M_0}{M} = \frac{\rho + \sigma - \sigma M}{\rho M},$$

which implies

$$M = \frac{bl(\rho + \sigma)}{a\rho + bl\sigma}.$$

Therefore equality (5.8) is true. □

Proposition 5.17 *(*Case III*). Let (P, Q) be an (m, n)-pair in $L^{(l)}$ and let $((a/l, b), (\rho, \sigma))$ be a regular corner of (P, Q). Assume that $[\ell_{\rho,\sigma}(P), \ell_{\rho,\sigma}(Q)] = 0$ and write $\ell_{\rho,\sigma}(P) = x^{k/l}\mathfrak{p}(z)$ where $z := x^{-\sigma/\rho}y$ and $\mathfrak{p}(z) \in K[z]$. If there exist $\mu, \lambda \in K^\times$ and $r \in \mathbb{N}$, such that $\mathfrak{p}(z) = \mu(z - \lambda)^r$, then $\rho \mid l$. Moreover, the automorphism φ of $L^{(l)}$, defined by $\varphi(x^{1/l}) := x^{1/l}$ and $\varphi(y) := y + \lambda x^{\sigma/\rho}$, satisfies*

(1) $\mathrm{en}_{\rho,\sigma}(\varphi(P)) = \mathrm{en}_{\rho,\sigma}(P)$ *and for all $(\rho, \sigma) < (\rho'', \sigma'') < (-\rho, -\sigma)$ the equalities*

$$\ell_{\rho'',\sigma''}(\varphi(P)) = \ell_{\rho'',\sigma''}(P) \quad \text{and} \quad \ell_{\rho'',\sigma''}(\varphi(Q)) = \ell_{\rho'',\sigma''}(Q),$$

hold.

(2) $(\varphi(P), \varphi(Q))$ *is an (m, n)-pair in $L^{(l)}$.*

(3) $((a/l, b), (\rho', \sigma'))$ *is a regular corner of $(\varphi(P), \varphi(Q))$, where $(\rho', \sigma') := \mathrm{Pred}_{\varphi(P)}(\rho, \sigma)$.*

(4) $(a/l, b) = \frac{1}{m}\,\mathrm{st}_{\rho,\sigma}(\varphi(P))$.

Proof. Clearly the conditions imply that

$$\ell_{\rho,\sigma}(P) = x^{k/l}\mu\left(\lambda^r - \binom{r}{1}\lambda^{r-1}z + \cdots\right).$$

Hence $(k/l - \sigma/\rho, 1) \in \mathrm{Supp}(\ell_{\rho,\sigma}(P)) \subseteq \frac{1}{l}\mathbb{Z} \times \mathbb{N}_0$. So $\sigma/\rho \in \frac{1}{l}\mathbb{Z}$, which evidently implies $\rho | l$, because $\gcd(\rho, \sigma) = 1$. From Proposition 3.9 we obtain statement (1). Statement (2) follows easily from Proposition 3.10, statement (1) and the fact that by Proposition 3.9 we know that $\mathrm{en}_{1,0}(\varphi(P)) = \mathrm{en}_{1,0}(P)$, even in the case where $(\rho, \sigma) = (1, 0)$. Finally, statements (3) and (4) follow from Propositions 3.7, 3.9 and 5.4. □

By Proposition 5.16 the hypotheses of the next proposition are always fulfilled in Case II.b). Sometimes they are fulfilled in Case II.a).

Proposition 5.18 *(*Case II*). Let (P, Q) and $((a/l, b), (\rho, \sigma))$ be as in Proposition 5.14 and let $l' := \mathrm{lcm}(\rho, l)$. Assume that $[\ell_{\rho,\sigma}(P), \ell_{\rho,\sigma}(Q)] = 0$ and write $\ell_{\rho,\sigma}(P) = x^{k/l}\mathfrak{p}(z)$ where $z := x^{-\sigma/\rho}y$ and $\mathfrak{p}(z) \in K[z]$. Assume also that $\#\,\mathrm{factors}(\mathfrak{p}(z)) > 1$ and that there exists $\lambda \in K^\times$ such that the multiplicity m_λ of $z - \lambda$ in $\mathfrak{p}(z)$, satisfies*

$$m_\lambda \geq \frac{m}{l}\left(\frac{a\rho + bl\sigma}{\rho + \sigma}\right). \tag{5.9}$$

Define $\varphi \in \operatorname{Aut}(L^{(l')})$ *by* $\varphi(x^{1/l'}) := x^{1/l'}$ *and* $\varphi(y) := y + \lambda x^{\sigma/\rho}$, *and set*

$$A^{(1)} := \frac{1}{m}\operatorname{st}_{\rho,\sigma}(\varphi(P)) \qquad \text{and} \qquad (\rho',\sigma') := \operatorname{Pred}_{\varphi(P)}(\rho,\sigma).$$

Then

(1) *We have* $\operatorname{en}_{\rho,\sigma}(\varphi(P)) = \operatorname{en}_{\rho,\sigma}(P)$ *and for all* $(\rho,\sigma) < (\rho'',\sigma'') < (-\rho,-\sigma)$ *the equalities*

$$\ell_{\rho'',\sigma''}(\varphi(P)) = \ell_{\rho'',\sigma''}(P) \quad \text{and} \quad \ell_{\rho'',\sigma''}(\varphi(Q)) = \ell_{\rho'',\sigma''}(Q)$$

hold.

(2) $(\varphi(P),\varphi(Q))$ *is an* (m,n)*-pair in* $L^{(l')}$.

(3) $(\rho,\sigma) \in \operatorname{Dir}(\varphi(P))$, $\operatorname{st}_{\rho,\sigma}(\varphi(P)) = \left(\frac{k}{l},0\right) + m_\lambda\left(-\frac{\sigma}{\rho},1\right)$ *and* $m \mid m_\lambda$.

(4) $(A^{(1)},(\rho',\sigma'))$ *and* $((a/l,b),(\rho,\sigma))$ *are regular corners of* $(\varphi(P),\varphi(Q))$. *The second one is of type IIa).*

Proof. Statements (1) and (2) follows as in the proof of Proposition 5.17. Now we are going to prove item (3). For this we write $\mathfrak{p}(z) = (z-\lambda)^{m_\lambda}\overline{p}(z)$ with $\overline{p}(\lambda) \neq 0$. Since

$$\varphi(z) = \varphi(x^{-\sigma/\rho})\varphi(y) = x^{-\sigma/\rho}(y + \lambda x^{\sigma/\rho}) = z + \lambda,$$

by Proposition 3.9, we have

$$\ell_{\rho,\sigma}(\varphi(P)) = \varphi(\ell_{\rho,\sigma}(P)) = \varphi(x^{k/l}\mathfrak{p}(z)) = x^{k/l}\varphi((z-\lambda)^{m_\lambda}\overline{p}(z)) = x^{k/l}z^{m_\lambda}\overline{p}(z+\lambda),$$

which implies that $(\rho,\sigma) \in \operatorname{Dir}(\varphi(P))$, because

$$\#\operatorname{factors}(z^{m_\lambda}\overline{p}(z+\lambda)) = \#\operatorname{factors}(\mathfrak{p}(z)) > 1.$$

Moreover, since $\overline{p}(\lambda) \neq 0$, from the first equality in (2.8) it follows that

$$\operatorname{st}_{\rho,\sigma}(\varphi(P)) = (k/l,0) + m_\lambda(-\sigma/\rho,1),$$

and so statement (3) holds. By statement (2) and Remarks 5.10 and 5.11, in order to prove statement (4) it suffices to verify that $(\rho,\sigma) \in A(\varphi(P))$. Since $(\rho,\sigma) \in \operatorname{Dir}(\varphi(P)) \cap I$ we only must check that

$$v_{1,-1}(\operatorname{st}_{\rho,\sigma}(\varphi(P))) < 0 \qquad \text{and} \qquad v_{0,-1}(\operatorname{st}_{\rho,\sigma}(\varphi(P))) < -1. \tag{5.10}$$

Since

$$\frac{k\rho}{l} = v_{\rho,\sigma}\left(\frac{k}{l},0\right) = v_{\rho,\sigma}(P) = v_{\rho,\sigma}\left(\frac{ma}{l},mb\right) = \frac{m}{l}(a\rho + bl\sigma),$$

by inequality (5.9), we have

$$v_{1,-1}(\mathrm{st}_{\rho,\sigma}(\varphi(P))) = \frac{k}{l} - m_\lambda\left(\frac{\sigma}{\rho}+1\right) \le \frac{m}{\rho l}(a\rho + bl\sigma) - \frac{m}{l}\left(\frac{a\rho+bl\sigma}{\rho+\sigma}\right)\left(\frac{\rho+\sigma}{\rho}\right) = 0.$$

But $v_{1,-1}(\mathrm{st}_{\rho,\sigma}(\varphi(P))) = 0$ is impossible by Theorem 2.6(4), and hence the first inequality in (5.10) holds. We next deal with the second one. By Proposition 3.10,

$$[\ell_{\rho,\sigma}(\varphi(P)), \ell_{\rho,\sigma}(\varphi(Q))] = 0,$$

while by Proposition 3.9 and Corollary 5.7(1),

$$v_{\rho,\sigma}(\varphi(P)) = v_{\rho,\sigma}(P) > 0 \qquad \text{and} \qquad v_{\rho,\sigma}(\varphi(Q)) = v_{\rho,\sigma}(Q) > 0.$$

Hence, by Remark 3.1(2), we have $\frac{1}{m}\,\mathrm{st}_{\rho,\sigma}(\varphi(P)) \in \frac{1}{l'}\mathbb{Z}\times\mathbb{N}_0$, and so $m \mid m_\lambda$ and

$$v_{0,-1}(\mathrm{st}_{\rho,\sigma})(\varphi(P)) \le -m < -1,$$

since $v_{0,1}(\mathrm{st}_{\rho,\sigma}(\varphi(P))) = m_\lambda \ge 1$, by statement (3). □

Proposition 5.19 *(First criterion for regular corners). If $(a/l, b)$ is the first entry of a regular corner of an (m,n)-pair in $L^{(l)}$, then it is the first entry of a regular corner of an (possibly different) (m,n)-pair in $L^{(l)}$ of type I or type II. Moreover, in the first case $l - a/b > 1$, while in the second one $\gcd(a,b) > 1$. If $l = 1$ then necessarily case II holds.*

Proof. Assume that we are in case III. By Proposition 5.17(3), there exists $\varphi \in \mathrm{Aut}(L^{(l)})$ such that $((a/l,b),(\rho_1,\sigma_1))$ is a regular corner of $(\varphi(P),\varphi(Q))$, where $(\rho_1,\sigma_1) := \mathrm{Pred}_{\varphi(P)}(\rho,\sigma)$. If Case III holds for this corner, then we can find $(\rho_2,\sigma_2) < (\rho_1,\sigma_1)$ such that $((a/l,b),(\rho_2,\sigma_2))$ is a regular corner. As long as Case III occurs, we can find $(\rho_{k+1},\sigma_{k+1}) < (\rho_k,\sigma_k)$ such that $((a/l,b),(\rho_{k+1},\sigma_{k+1}))$ is a regular corner. But there are only finitely many ρ_k's with $\rho_k|l$. Moreover, $0 < -\sigma_k < \rho_k$, since $(1,-1) < (\rho_k,\sigma_k) < (1,0)$, and so there are only finitely many (ρ_k,σ_k) possible, which proves that eventually cases I or II must occur. In case I Proposition 5.13 gives $l - a/b > 1$ and in case II, by Proposition 5.14(4), we have $\gcd(a,b) > 1$. The last statement is clear, since $1 - a/b < 1$, because $a, b > 0$. □

Proposition 5.20. *For each (m,n)-pair (P,Q) in $L^{(1)}$, there exists an automorphism φ of $L^{(1)}$ such that $(\varphi(P),\varphi(Q))$ is a standard (m,n)-pair with*

$$v_{1,1}(\varphi(P)) = v_{1,1}(P), \quad v_{1,1}(\varphi(Q)) = v_{1,1}(Q) \quad \text{and} \quad \mathrm{en}_{1,0}(\varphi(P)) = \mathrm{en}_{1,0}(P).$$

Moreover, if $(-1,1) < \mathrm{Succ}_P(1,0), \mathrm{Succ}_Q(1,0) < (-1,0)$, then

$$(-1,1) < \mathrm{Succ}_{\varphi(P)}(1,0), \mathrm{Succ}_{\varphi(Q)}(1,0) < (-1,0).$$

Furthermore, if $P,Q \in L$, then we can take $\varphi \in \mathrm{Aut}(L)$.

Proof. If $v_{1,-1}(\mathrm{st}_{1,0}(P)) < 0$, then we can take $\varphi := \mathrm{id}$. Otherwise $\big(\frac{1}{m}\,\mathrm{en}_{1,0}(P),(1,0)\big)$ is a regular corner. Write $(a,b) := \frac{1}{m}\,\mathrm{en}_{1,0}(P)$. Case I is impossible because $a,b>0$ and Proposition 5.13 gives $1-a/b>1$, and the Case II.a) is discarded, because $v_{1,-1}(\mathrm{st}_{1,0}(P)) \ge 0$. By Propositions 5.16 and 5.18 in Case II.b) and by Proposition 5.17 in Case III, we can find a $\varphi \in \mathrm{Aut}(L^{(1)})$ such that

- $\big(\frac{1}{m}\,\mathrm{st}_{1,0}(\varphi(P)),(\rho',\sigma')\big)$ is a regular corner, for some (ρ',σ'),
- $\ell_{1,1}(\varphi(P)) = \ell_{1,1}(P)$, $\ell_{1,1}(\varphi(Q)) = \ell_{1,1}(Q)$ and $\mathrm{en}_{1,0}(\varphi(P)) = \mathrm{en}_{1,0}(P)$,
- If $\mathrm{Succ}_P(1,0), \mathrm{Succ}_Q(1,0) < (-1,0)$, then

$$\mathrm{Succ}_{\varphi(P)}(1,0) = \mathrm{Succ}_P(1,0) \qquad \text{and} \qquad \mathrm{Succ}_{\varphi(Q)}(1,0) = \mathrm{Succ}_Q(1,0).$$

- $\varphi(x) = x$ and $\varphi(y) = y+\lambda$, for some $\lambda \in K^\times$.

The assertions in the statement follow immediately from these facts. □

Corollary 5.21. *If $B<\infty$ (i.e., if the Jacobian conjecture is false), then there exists a Jacobian pair (P,Q) and $m,n \in \mathbb{N}$ coprime with $m,n>1$, such that*

(1) *(P,Q) is a standard (m,n)-pair in L,*
(2) *(P,Q) is a minimal pair (i.e., $\gcd\big(v_{1,1}(P),v_{1,1}(Q)\big) = B$),*
(3) *$\mathrm{st}_{1,1}(P) = \mathrm{en}_{1,0}(P)$,*
(4) *$(-1,1) < \mathrm{Succ}_P(1,0), \mathrm{Succ}_Q(1,0) < (-1,0)$.*

Proof. By Propositions 3.7, 4.7 and 5.20. □

Proposition 5.22. *Each (m,n)-pair (P,Q) in $L^{(l)}$ has a unique regular corner $((a/l,b),(\rho,\sigma))$ with $(\rho,\sigma) \notin A(P)$.*

Proof. If $A(P) \neq \emptyset$, then the existence follows from Remarks 5.10 and 5.11, since

$$\mathrm{Pred}_P(\min(A(P))) \notin A(P).$$

Otherwise, by Proposition 5.4, we know that $(\rho,\sigma) := \mathrm{Pred}_P(\rho_1,\sigma_1) \in I$, where

$$(\rho_1,\sigma_1) := \min\big(\mathrm{Succ}_P(1,0), \mathrm{Succ}_Q(1,0)\big).$$

Clearly item (2) of Definition 5.5 is fulfilled for $\big(\frac{1}{m}\,\mathrm{en}_{\rho,\sigma}(P),(\rho,\sigma)\big)$. Moreover, by Proposition 3.7 and Proposition 4.6(3),

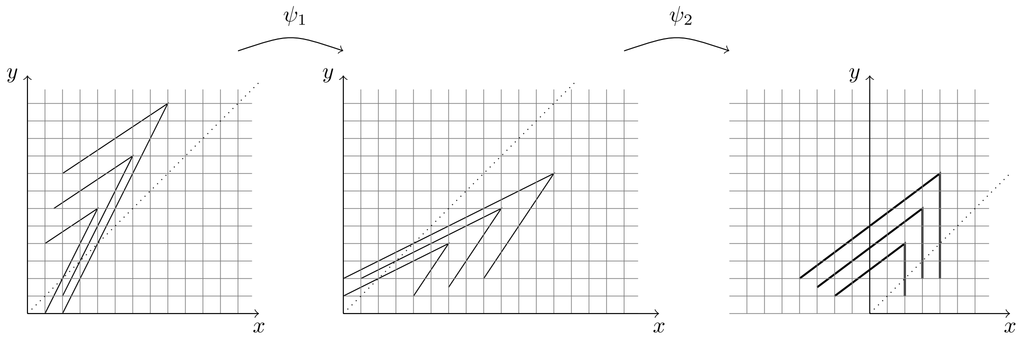

Fig. 3. Applying ψ_1 and ψ_2 to elements P with $v_{2,-1}(P) \leq 4$.

$$\frac{1}{m}\,\mathrm{en}_{\rho,\sigma}(P) = \frac{1}{m}\,\mathrm{en}_{1,0}(P) \in \frac{1}{l}\mathbb{Z} \times \mathbb{N},$$

and so item (3) is also satisfied. In order to prove item (1) we write $(a/l, b) := \frac{1}{m}\,\mathrm{en}_{\rho,\sigma}(P)$. By Definition 4.3,

$$a/l - b = v_{1,-1}\left(\frac{1}{m}\,\mathrm{en}_{\rho,\sigma}(P)\right) = v_{1,-1}\left(\frac{1}{m}\,\mathrm{en}_{1,0}(P)\right) < 0,$$

while by Proposition 4.6(4), we have $b = -v_{0,-1}(\mathrm{en}_{1,0}(P)) > 1$. This ends the proof of the existence. The uniqueness follows from Proposition 5.2 and the fact that, by Proposition 3.7 and Definition 5.5, if $(A, (\rho, \sigma))$ is a regular corner of (P, Q), then $\mathrm{Succ}_P(\rho, \sigma) \in I$ implies that $\mathrm{Succ}_P(\rho, \sigma) \in A(P)$. □

6. Lower bounds

By Corollary 5.21, if $B < \infty$ (i.e., if the Jacobian conjecture is false), then there exists a standard (m, n)-pair (P, Q) in L, which is also a minimal pair (i.e., $\gcd(v_{1,1}(P), v_{1,1}(Q)) = B$). In this section we will first prove that $B \geq 16$. The argument is nearly the same as in [4], but we will also need lower bounds for (m, n)-pairs in $L^{(1)}$, and not only in L. The reason is the following: One technical result, Proposition 6.9, says something about (m, n)-pairs in L with $\frac{1}{m}v_{2,-1}(P) \leq 4$.
Via the flip ψ_1 this is the same as saying something about Jacobian pairs in L with $\frac{1}{m}v_{-1,2}(P) \leq 4$. Applying the automorphism ψ_2 defined by $\psi_2(x) := x$ and $\psi_2(y) := x^2y$, this amounts to proving facts about (m, n)-pairs in $L^{(1)}$ with $\frac{1}{m}v_{1,0}(P) \leq 4$ (Fig. 3), which we will do in the sequel.

Proposition 6.1. *Let (P, Q) be a standard (m, n)-pair. There exists exactly one regular corner $((a, b), (\rho, \sigma))$ of (P, Q) of type II.b). Moreover,*

(1) $\sigma < 0$,
(2) $v_{\rho,\sigma}(P) > 0$ *and* $v_{\rho,\sigma}(Q) > 0$,

(3) $\frac{v_{\rho,\sigma}(P)}{v_{\rho,\sigma}(Q)} = \frac{m}{n}$,
(4) $[\ell_{\rho,\sigma}(P), \ell_{\rho,\sigma}(Q)] = 0$,
(5) $(\rho, \sigma) \in \mathrm{Dir}(P) \cap \mathrm{Dir}(Q)$,
(6) *There exists* $\mu \in \mathbb{Q}$ *greater than* 0 *such that*

$$\mathrm{en}_{\rho,\sigma}(F) = \frac{\mu}{m} \mathrm{en}_{\rho,\sigma}(P),$$

where $F \in L^{(1)}$ *is the* (ρ, σ)*-homogeneous element obtained in Theorem 2.6,*
(7) $v_{1,-1}\big(\mathrm{en}_{\rho,\sigma}(P)\big) < 0$ *and* $v_{1,-1}\big(\mathrm{en}_{\rho,\sigma}(Q)\big) < 0$,
(8) $\frac{v_{\rho',\sigma'}(P)}{v_{\rho',\sigma'}(Q)} = \frac{m}{n}$ *for all* $(\rho, \sigma) < (\rho', \sigma') < (1, 0)$,
(9) $v_{1,1}(\mathrm{en}_{\rho,\sigma}(P)) \le v_{1,1}(\mathrm{en}_{1,0}(P))$.

Proof. The uniqueness follows immediately from the definition of $A(P)$ and Proposition 5.22. The same proposition yields a regular corner $((a, b), (\rho, \sigma))$ such that

$$(\rho, \sigma) \notin A(P). \tag{6.1}$$

Statements (2), (3) and (5) follow now from Corollary 5.7. By Remark 5.8 we have $1 - a/b < 1$. Hence, by Proposition 5.13 we are in case II or in case III, and so statement (4) holds. Furthermore, by Remark 3.1 we have

$$\frac{1}{m} \mathrm{st}_{\rho,\sigma}(P) = \frac{1}{n} \mathrm{st}_{\rho,\sigma}(Q),$$

which implies $\frac{1}{m} \mathrm{st}_{\rho,\sigma}(P) \in \mathbb{Z} \times \mathbb{N}_0$. We will prove that

$$v_{1,-1}(\mathrm{st}_{\rho,\sigma}(P)) > 0. \tag{6.2}$$

Assume by contradiction that $v_{1,-1}(\mathrm{st}_{\rho,\sigma}(P)) \le 0$, which implies $v_{1,-1}(\mathrm{st}_{\rho,\sigma}(P)) < 0$, by Theorem 2.6(4). If $v_{0,-1}(\frac{1}{m} \mathrm{st}_{\rho,\sigma}(P)) \le -1$ then $(\rho, \sigma) \in A(P)$, which contradicts (6.1). Hence $v_{0,-1}(\frac{1}{m} \mathrm{st}_{\rho,\sigma}(P)) = 0$, and so $\mathrm{st}_{\rho,\sigma}(P) = (k, 0)$, for some $k < 0$. But then

$$0 < v_{\rho,\sigma}(P) = \rho k < 0,$$

a contradiction which proves (6.2). This implies statement (1) since, by the definition of standard (m, n)-pair, $v_{1,-1}(\mathrm{st}_{1,0}(P)) < 0$.

Assume that $((a, b), (\rho, \sigma))$ is of type III. Then, by Proposition 5.17, we know that $\rho | l = 1$, and so $(\rho, \sigma) = (1, 0)$, which contradicts statement (1). Hence, by inequality (6.2), we are in case II.b).

Statement (6) follows from items (1) and (4) of Proposition 5.14. Statement (7) for P follows from Definition 5.5, and then it follows for Q, by Corollary 5.7(2).

By Proposition 3.7 and Definition 5.5, if $\mathrm{Succ}_P(\rho, \sigma) \in I$, then $\mathrm{Succ}_P(\rho, \sigma) \in A(P)$. Consequently, by Proposition 5.2,

$$\mathrm{Dir}(P) \cap \,](\rho,\sigma),(1,0)] \subseteq A(P).$$

Statement (8) now follows easily from Proposition 3.7, Remark 3.1 and the fact that, by Proposition 5.4, statement (3) holds for all $(\rho_j,\sigma_j) \in A(P)$. Finally, by Proposition 3.7 and Remark 1.8,

$$v_{1,1}(\mathrm{en}_{\rho',\sigma'}(P)) = v_{1,1}(\mathrm{st}_{\rho'',\sigma''}(P)) < v_{1,1}(\mathrm{en}_{\rho'',\sigma''}(P))$$

for consecutive directions $(\rho',\sigma') < (\rho'',\sigma'')$ in $\mathrm{Dir}(P) \cap I$, from which statement (9) follows. □

Definition 6.2. The *starting triple* of a standard (m,n)-pair (P,Q) is $(A_0, A_0', (\rho,\sigma))$, where $(A_0,(\rho,\sigma))$ is the unique regular corner of (P,Q) with $(\rho,\sigma) \notin A(P)$, and $A_0' = \frac{1}{m}\,\mathrm{st}_{\rho,\sigma}(P)$. The point A_0 is called the *primitive corner* of (P,Q).

Remark 6.3. By Propositions 5.22 and 6.1 and Remark 5.10, in the previous definition $(A_0,(\rho,\sigma))$ is the unique regular corner of type II.b). Consequently $v_{1,-1}(\mathrm{st}_{\rho,\sigma}(P)) > 0$.

Let (P,Q) be a standard (m,n)-pair and $(A_0, A_0', (\rho,\sigma))$ its starting triple. Let λ and m_λ be as in Proposition 5.16, let $\varphi \in \mathrm{Aut}(L^{(\rho)})$ and $A^{(1)}$ be as in Proposition 5.18 and let F be as in Proposition 5.14. Note that $F \in L$. In fact, for $(i,j) \in \mathrm{Supp}(F)$, we have

$$\rho i + \sigma j = v_{\rho,\sigma}(i,j) = v_{\rho,\sigma}(F) = \rho + \sigma > 0,$$

which implies that $i \geq 0$, since $\rho > 0$, $\sigma < 0$ and $j \geq 0$. Write

$$(f_1,f_2) := \mathrm{en}_{\rho,\sigma}(F), \quad (u,v) := A_0, \quad (r',s') := A_0' \quad \text{and} \quad \gamma := \frac{m_\lambda}{m}.$$

Proposition 6.4. *It is true that* $A_0, A_0' \in \mathbb{N}_0 \times \mathbb{N}_0$ *and* $v_{\rho,\sigma}(A_0) = v_{\rho,\sigma}(A_0')$ *(Fig. 4). Moreover,*

(1) $s' < r' < u < v$,
(2) $2 \leq f_1 < u$,
(3) $\gcd(u,v) > 1$,
(4) $\mathrm{en}_{\rho,\sigma}(F) = \mu A_0$ *for some* $0 < \mu < 1$,
(5) $u f_2 = v f_1$ *and* $\rho \leq u$,
(6) $(\rho,\sigma) = \mathrm{dir}(f_1 - 1, f_2 - 1) = \left(\frac{f_2-1}{d}, \frac{1-f_1}{d}\right)$, *where* $d := \gcd(f_1 - 1, f_2 - 1)$,
(7) $A^{(1)} = A_0' + (\gamma - s')\left(-\frac{\sigma}{\rho}, 1\right)$,
(8) *If* $A^{(1)} = (a'/\rho, b')$, *then* $\rho - a'/b' > 1$ *or* $\gcd(a',b') > 1$,
(9) $\gamma \leq (v - s')/\rho$. *Moreover, if* $d = \gcd(f_1 - 1, f_2 - 1) = 1$, *then* $\gamma = (v - s')/\rho$.

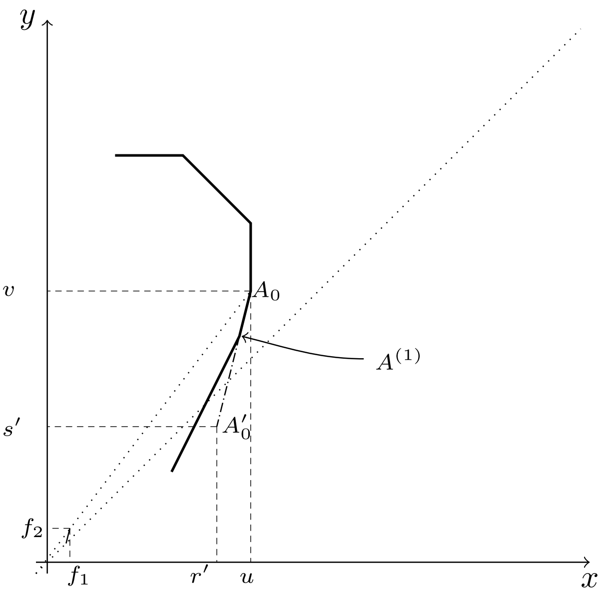

Fig. 4. Illustration of Proposition 6.4.

Proof. By statements (2), (3) and (4) of Proposition 6.1 and statement (2b) of Proposition 2.1, there exist $\lambda_P, \lambda_Q \in K^\times$ and a (ρ,σ)-homogeneous element $R \in L^{(1)}$ such that

$$\ell_{\rho,\sigma}(P) = \lambda_P R^m \quad \text{and} \quad \ell_{\rho,\sigma}(Q) = \lambda_P R^n. \tag{6.3}$$

This implies that

$$A_0 = \mathrm{en}_{\rho,\sigma}(R) \qquad \text{and} \qquad A'_0 = \mathrm{st}_{\rho,\sigma}(R).$$

Hence, $v_{\rho,\sigma}(A_0) = v_{\rho,\sigma}(A'_0)$. Moreover, the same argument given above for F shows that $R \in L$, and so $A_0, A'_0 \in \mathbb{N}_0 \times \mathbb{N}_0$

Statement (1) follows from the fact that, by inequality (6.2) and Proposition 6.1(7)

$$v_{1,-1}(\mathrm{en}_{\rho,\sigma}(P)) < 0 \qquad \text{and} \qquad v_{1,-1}(\mathrm{st}_{\rho,\sigma}(P)) > 0,$$

and, by Remark 1.8,

$$v_{1,0}(\mathrm{st}_{\rho,\sigma}(P)) < v_{1,0}(\mathrm{en}_{\rho,\sigma}(P)), \tag{6.4}$$

since $(1,-1) < (\rho,\sigma) < (1,0)$. Proposition 6.1(6) gives statement (4) except the inequality $\mu < 1$. But this is true because $\mu \geq 1$ implies

$$v_{\rho,\sigma}(A'_0) = v_{\rho,\sigma}(A_0) = \frac{1}{\mu} v_{\rho,\sigma}(F) \leq v_{\rho,\sigma}(F) = \rho + \sigma,$$

which is impossible since, by statement (1) and Proposition 6.1(1),

$$v_{\rho,\sigma}(A_0') = r'\rho + s'\sigma = (r'-s')\rho + s'(\rho+\sigma) \geq (r'-s')\rho \geq \rho > \rho+\sigma.$$

We claim that $v_{1,0}(\mathrm{st}_{\rho,\sigma}(F)) \geq 1$. In fact, otherwise $\mathrm{st}_{\rho,\sigma}(F) = (0,h)$ for some $h \in \mathbb{N}_0$, which implies $v_{\rho,\sigma}(F) = \sigma h \leq 0$. But this is impossible since $v_{\rho,\sigma}(F) = \rho+\sigma > 0$. Hence, by Remark 1.8,

$$f_1 = v_{1,0}(\mathrm{en}_{\rho,\sigma}(F)) > v_{1,0}(\mathrm{st}_{\rho,\sigma}(F)) \geq 1,$$

which combined with $f_1 = \mu u$ and $0 < \mu < 1$ proves statement (2). Moreover, if $\gcd(u,v) = 1$, then there is no $\mu \in \,]0,1[$ such that $\mu(u,v) \in \mathbb{N}_0 \times \mathbb{N}_0$, and so statement (3) is true. Next we prove statement (5). From statement (4) it follows that $uf_2 = vf_1$. Equivalently $u(f_1,f_2) = f_1(u,v)$, and so

$$v_{\rho,\sigma}(f_1 A_0') = f_1 v_{\rho,\sigma}(A_0') = f_1 v_{\rho,\sigma}(A_0) = u v_{\rho,\sigma}(F) = u v_{\rho,\sigma}(1,1) = v_{\rho,\sigma}(u,u).$$

Hence there exists $t \in \mathbb{Z}$ such that $f_1 A_0' = (u,u) - t(-\sigma,\rho)$. Thus

$$u - t\rho = v_{0,1}(f_1 A_0') = f_1 v_{0,1}(A_0') \geq 0,$$

and so $u \geq t\rho$. Therefore, in order to finish the proof of statement (5) we only must note that $t \leq 0$ is impossible, because it implies $f_1 v_{1,-1}(A_0') \leq 0$, contradicting Remark 6.3.

The first equality in statement (6) follows from the fact that $v_{\rho,\sigma}(f_1,f_2) = v_{\rho,\sigma}(1,1)$ and Remark 3.2. So, by (3.2)

$$(\rho,\sigma) = \pm\left(\frac{f_2-1}{d}, \frac{1-f_1}{d}\right),$$

where $d := \gcd(f_1-1, f_2-1)$. Since $\rho+\sigma > 0$ and, by statements (1) and (4), we have $f_2 - f_1 > 0$, necessarily

$$(\rho,\sigma) = \left(\frac{f_2-1}{d}, \frac{1-f_1}{d}\right),$$

which ends the proof of statement (6). Next we prove statement (7). The point $A^{(1)}$ is completely determined by $v_{0,1}(A^{(1)})$ and $v_{\rho,\sigma}(A^{(1)})$. Let φ be as in Proposition 5.18. Since

$$v_{0,1}\left(A_0' + (\gamma - s')\left(-\frac{\sigma}{\rho}, 1\right)\right) = \gamma = v_{0,1}(A^{(1)})$$

by Proposition 5.18(3), and

$$v_{\rho,\sigma}\left(A_0' + (\gamma - s')\left(-\frac{\sigma}{\rho}, 1\right)\right) = v_{\rho,\sigma}(A_0') = v_{\rho,\sigma}(A^{(1)})$$

because φ is (ρ,σ)-homogeneous, statement (7) is true. Statement (8) follows directly from Propositions 5.18(4) and 5.19. It remains to prove statement (9). Write $\ell_{\rho,\sigma}(P) = x^r y^s p(z)$, where $z = x^{-\sigma/\rho}y$ and $p(0) \neq 0$. Since $R \in L$ and $\ell_{\rho,\sigma}(P) = \lambda_P R^m$, we have $\ell_{\rho,\sigma}(P) \in L$, which implies $p(z) \in K[x,y]$. Hence

$$\ell_{\rho,\sigma}(P) = x^r y^s \overline{p}(z^\rho), \quad \text{where } \overline{p}(z^\rho) \in K[z^\rho].$$

By Proposition 5.16 we know that $\lambda \neq 0$, so m_λ is the multiplicity of a root of $p(z)$. Since the multiplicities of the roots of $p(z)$ are the same as the multiplicities of the roots of $\overline{p}(z)$, we have $m_\lambda \le \deg(\overline{p})$. Combining this with Remark 2.8, we obtain

$$m_\lambda \le \deg(\overline{p}) = \frac{v_{0,1}(\mathrm{en}_{\rho,\sigma}(P)) - v_{0,1}(\mathrm{st}_{\rho,\sigma}(P))}{\rho} = m\left(\frac{v-s'}{\rho}\right),$$

which proves the first part of statement (9). We claim that $\mathrm{st}_{\rho,\sigma}(F) = (1,1)$. In fact, otherwise

$$(\alpha,\beta) := \mathrm{st}_{\rho,\sigma}(F) = (1-\sigma i, 1+\rho i), \qquad \text{with } i > 0.$$

Note that $\alpha < \beta$, since $\rho > -\sigma$. But this is impossible because $r' > s'$, and, by Theorem 2.6(2), we have

$$\mathrm{st}_{\rho,\sigma}(F) \sim \mathrm{st}_{\rho,\sigma}(P) = m(r',s').$$

Consequently, if $d = 1$, then by statement (6)

$$v_{0,1}(\mathrm{en}_{\rho,\sigma}(F)) - v_{0,1}(\mathrm{st}_{\rho,\sigma}(F)) = f_2 - 1 = \rho,$$

and so, by Proposition 2.11(4),

$$m_\lambda = \frac{1}{\rho}\left(v_{0,1}(\mathrm{en}_{\rho,\sigma}(P)) - v_{0,1}(\mathrm{st}_{\rho,\sigma}(P))\right) = \frac{m(v-s')}{\rho},$$

as desired. □

Proposition 6.5. *If A_0 is as before Proposition 6.4, then $v_{1,1}(A_0) \ge 16$.*

Proof. By Proposition 6.4 it suffices to prove that there is no pair $A_0 = (u,v)$ with $u+v \le 15$, for which there exist (f_1,f_2), $A_0' = (r',s')$, γ and $A^{(1)}$, such that all the conditions of that proposition are satisfied. In Table 1 we first list all possible pairs (u,v) with $v > u > 2$, $\gcd(u,v) > 1$ and $u+v \le 15$. We also list all the possible $(f_1,f_2) = \mu(u,v)$ with $f_1 \ge 2$ and $0 < \mu < 1$. Then we compute the corresponding (ρ,σ) using Proposition 6.4(6) and we verify if there is an $A_0' := (r',s')$ with $s' < r' < u$ and $v_{\rho,\sigma}(u,v) = v_{\rho,\sigma}(r',s')$. This happens in five cases. In all these cases $d := \gcd(f_1 -$

Table 1
Possible pairs (u,v) with $v>u>2$, $\gcd(u,v)>1$ and $u+v\le 15$.

A_0	(f_1,f_2)	(ρ,σ)	A_0'	d	γ	$A^{(1)}$
(3,6)	(2,4)	(3,−1)	(1,0)	1	2	$(\frac{5}{3},2)$
(3,9)	(2,6)	(5,−1)	×			
(3,12)	(2,8)	(7,−1)	×			
(4,6)	(2,3)	(2,−1)	(1,0)	1	3	$(\frac{5}{2},3)$
(4,8)	(2,4)	(3,−1)	×			
(4,8)	(3,6)	(5,−2)	×			
(4,10)	(2,5)	(4,−1)	×			
(5,10)	(2,4)	(3,−1)	(2,1)	1	3	$(\frac{8}{3},3)$
(5,10)	(3,6)	(5,−2)	(1,0)	1	2	$(\frac{9}{5},2)$
(5,10)	(4,8)	(7,−3)	×			
(6,8)	(3,4)	(3,−2)	×			
(6,9)	(2,3)	(2,−1)	(2,1)	1	4	$(\frac{7}{2},4)$
(6,9)	(4,6)	(5,−3)	×			

$1,f_2-1)=1$. Then, by Proposition 6.4(9), we have $\gamma=(v-s')/\rho$. Using these values, we compute $A^{(1)}$ in each of the five cases using statement (7) of the same proposition. Finally we verify that in none of this cases condition (8) of Proposition 6.4 is satisfied, concluding the proof. □

Corollary 6.6. *We have $B\ge 16$.*

Proof. Suppose $B<\infty$ and take (P,Q) and (m,n) as in Corollary 5.21. Assume that (ρ,σ) and A_0 are as above Proposition 6.4. By Proposition 6.5,

$$B=\gcd\big(v_{1,1}(P),v_{1,1}(Q)\big)=\frac{1}{m}v_{1,1}(P)\ge\frac{1}{m}v_{1,1}(\mathrm{en}_{\rho,\sigma}(P))=v_{1,1}(A_0)\ge 16,$$

as desired. □

Proposition 6.7. *Let (P,Q) be a standard (m,n)-pair and let $A_0=(u,v)$ be as before Proposition 6.4. Then $v\le u(u-1)$ and $u\ge 4$.*

Proof. Let F, $(f_1,f_2)=\mathrm{en}_{\rho,\sigma}(F)$ and $d=\gcd(f_1-1,f_2-1)$ be as before Proposition 6.4. By statements (5) and (6) of Proposition 6.4,

$$\frac{f_1v-u}{du}=\frac{f_2-1}{d}=\rho\le u.$$

Hence

$$v\le\frac{du^2+u}{f_1}=u\frac{du+1}{f_1}\le u\frac{(f_1-1)u+1}{f_1}=u\frac{f_1u-(u-1)}{f_1}=u\left(u-\frac{u-1}{f_1}\right)$$
$$\le u(u-1),$$

where the last inequality follows from Proposition 6.4(2). Again by Proposition 6.4(2), we know that $u\ge 3$, so we must only check that the case $u=3$ is impossible. But if

$A_0 = (3, v)$, then by the first statement necessarily $v \leq 6$, which contradicts Proposition 6.5. $\square$

Remark 6.8. The inequality $u \geq 4$ is related to [5, Proposition 2.22]. It shows that for a standard (m,n)-pair (P,Q), the greatest common divisor of $\deg_x(P) = v_{1,0}(P)$ and $\deg_x(Q) = v_{1,0}(Q)$ is greater than or equal to 4. Using similar techniques as in the proof of Proposition 4.7, one can prove that this inequality holds for any counterexample.

Let $\psi_1 \in \operatorname{Aut}(L)$ be the map defined by $\psi_1(x) := y$ and $\psi_1(y) := -x$. Since $[\psi_1(x), \psi_1(y)] = 1$, by Proposition 3.10, this map preserves Jacobian pairs. Moreover, the action induced by ψ_1 on the Newton polygon of a polynomial P is the orthogonal reflection at the main diagonal, and so, it maps edges of the convex hull of $\operatorname{Supp}(P)$ into edges of the convex hull of $\operatorname{Supp}(\psi_1(P))$, interchanging st and en.

Similarly the automorphism ψ_2 of $L^{(1)}$, defined by $\psi_2(x) := -x^{-1}$ and $\psi_2(y) := x^2 y$ preserves Jacobian pairs and it induces on the Newton polygon of each $P \in L^{(1)}$ a reflection at the main diagonal, parallel to the X-axis. Hence it also maps edges of the convex hull of $\operatorname{Supp}(P)$ into edges of the convex hull of $\operatorname{Supp}(\psi_2(P))$, interchanging st and en.

Moreover, an elementary computation shows that if we define

$$\overline{\psi}_1(\rho,\sigma) := (\sigma,\rho) \quad \text{and} \quad \overline{\psi}_2(\rho,\sigma) := (-\rho, 2\rho+\sigma), \tag{6.5}$$

and set $(\rho_k,\sigma_k) := \overline{\psi}_k(\rho,\sigma)$ for $k = 1,2$, then

$$v_{\rho_k,\sigma_k}(\psi_k(P)) = v_{\rho,\sigma}(P) \quad \text{and} \quad \ell_{\rho_k,\sigma_k}(\psi_k(P)) = \psi_k(\ell_{\rho,\sigma}(P)), \tag{6.6}$$

for all $(\rho,\sigma) \in \mathfrak{V}$ and $P \in L^{(1)}$ (when $k = 1$ we assume $P \in L$).

Proposition 6.9. *Let (P,Q) be a standard (m,n)-pair in L and let (ρ,σ) and A_0 be as before Proposition 6.4. If $(\rho,\sigma) = (2,-1)$, then it is impossible that $v_{\rho,\sigma}(A_0) \leq 3$.*

Proof. Let $\varphi \colon L \to L^{(1)}$ be the morphism defined by $\varphi := \psi_2 \circ \psi_1$. Write $\mathrm{en}_{2,-1}(P) = (a,b)$, so that $A_0 = \frac{1}{m}(a,b)$. We claim that $\mathrm{en}_{1,0}(\varphi(P)) = (2a-b, a)$. In order to prove the claim, note first that

$$\overline{\psi}_1(-1,1) = (1,-1), \quad \overline{\psi}_1(2,-1) = (-1,2), \quad \overline{\psi}_2(-1,2) = (1,0) \quad \text{and}$$
$$\overline{\psi}_2(1,-1) = (-1,1).$$

Since, by Remark 1.8,

$$\operatorname{Supp}\bigl(\ell_{-1,1}(\ell_{2,-1}(P))\bigr) = \mathrm{en}_{2,-1}(P) = (a,b),$$

and, by the second equality in (6.6),

$$\ell_{-1,1}(\ell_{1,0}(\varphi(P))) = \ell_{-1,1}(\psi_2(\ell_{-1,2}(\psi_1(P)))) = \psi_2(\ell_{1,-1}(\psi_1(\ell_{2,-1}P))) \\ = \varphi(\ell_{-1,1}(\ell_{2,-1}(P))),$$

we have, again by Remark 1.8,

$$\mathrm{en}_{1,0}(\varphi(P)) = \mathrm{Supp}(\varphi(x^a y^b)) = (2a-b, a),$$

which proves the claim. Moreover, $(\varphi(P), \varphi(Q))$ is an (m,n)-pair, because by Proposition 3.10, we have $[\varphi(P), \varphi(Q)] = 1$; it is true that

$$v_{1,-1}(\mathrm{en}_{1,0}(\varphi(P))) = a - b = v_{1,-1}(\mathrm{en}_{2,-1}(P)) < 0;$$

and, by the first equality in (6.6), statements (3) and (8) of Proposition 6.1, and the fact that $\overline{\psi}_2(\overline{\psi}_1(2,-1)) = (1,0)$ and $\overline{\psi}_2(\overline{\psi}_1(3,-1)) = (1,1)$, we have

$$\frac{v_{1,0}(\varphi(P))}{v_{1,0}(\varphi(Q))} = \frac{v_{2,-1}(P)}{v_{2,-1}(Q)} = \frac{m}{n} \quad \text{and} \quad \frac{v_{1,1}(\varphi(P))}{v_{1,1}(\varphi(Q))} = \frac{v_{3,-1}(P)}{v_{3,-1}(Q)} = \frac{m}{n}.$$

Applying Proposition 5.20 we obtain a standard (m,n)-pair $(\widetilde{P}, \widetilde{Q})$ with

$$\frac{1}{m}\,\mathrm{en}_{1,0}(\widetilde{P}) = \frac{1}{m}\,\mathrm{en}_{1,0}(\varphi(P)) = \frac{1}{m}(2a-b, a).$$

Hence,

$$\frac{1}{m} v_{1,0}(\widetilde{P}) = \frac{1}{m} v_{1,0}(\mathrm{en}_{1,0}(\widetilde{P})) = \frac{1}{m}(2a-b) = \frac{1}{m} v_{2,-1}(\mathrm{en}_{2,-1}(P)) = v_{\rho,\sigma}(A_0).$$

Let $\widetilde{A}_0 = (u,v)$ be the primitive corner of $(\widetilde{P}, \widetilde{Q})$. Since $m(u,v) \in \mathrm{Supp}(\widetilde{P})$, we have

$$u \le \frac{1}{m} v_{1,0}(\widetilde{P}) = v_{\rho,\sigma}(A_0).$$

So, if $v_{\rho,\sigma}(A_0) \le 3$, then $u \le 3$, which contradicts Proposition 6.7 and concludes the proof. □

Proposition 6.10. *Let (P,Q) be a standard (m,n)-pair in L, $A_0 = (u,v)$ as before Proposition 6.4 and μ as in Proposition 6.4(4). Then $\mu \neq 1/2$ and $\gcd(u,v) \neq 2$.*

Proof. Let (ρ,σ), $A_0' = (r', s')$ and F be as before Proposition 6.4. By Proposition 6.1(1) we know that $(1,-1) < (\rho,\sigma) < (1,0)$ and by Proposition 6.4(1), we have

$$A_0' \neq (2,2) \quad \text{and} \quad v_{1,-1}(A_0') = r' - s' > 0 = v_{1,-1}(2,2). \tag{6.7}$$

Assume by contradiction that $\mu = 1/2$, which implies

$$v_{\rho,\sigma}(A_0') = v_{\rho,\sigma}(A_0) = 2v_{\rho,\sigma}(F) = 2v_{\rho,\sigma}(1,1) = v_{\rho,\sigma}(2,2).$$

Then, by Remark 3.2

$$\operatorname{dir}(A_0' - (2,2)) = (\rho,\sigma) < (1,0).$$

From this, the second inequality in (6.7) and Lemma 3.5, it follows that

$$0 \le s' < r' = v_{1,0}(A_0') < v_{1,0}(2,2) = 2.$$

Hence necessarily $A_0' = (1,0)$. Therefore $(\rho,\sigma) = (2,-1)$ and Lemma 6.9 yields the desired contradiction, since then $v_{2,-1}(A_0) = v_{2,-1}(A_0') = 2$. Finally, since $\mu \neq 1/2$, necessarily $\gcd(u,v) \neq 2$. □

7. More conditions on *B*

In this section we prove that $B \neq 2p$ for all prime p. Abhyankar allegedly developed a proof of this result according to [5, Page 50], but we could not find any published article of Abhyankar with such proof. Heitmann says that it is possible to adapt the proof of [5, Proposition 2.21] to prove $B \neq 2p$, however we were not able to do this. On the other hand this is also claimed to be proven in [12, Theorem 4.12]. But the proof relies on [12, Lemma 4.10], which has a gap, since it claims without proof that $I_2 \subseteq \frac{1}{m}\Gamma(f_2)$, an assertion which cannot be proven to be true. The main technical results in this section are Propositions 7.1 and 7.3, together with its Corollaries 7.2 and 7.4. They are closely related to [9, Propositions 6.3 and 6.4] and seem to be a generalization of them. These results are interesting on their own, but they also allow to establish a very strong criterion for the possible regular corners (Theorem 7.6) which leads to the proof of $B \neq 2p$.

Proposition 7.1. *Let $m,n \in \mathbb{N}$ be coprime with $m,n > 1$ and let $P,Q \in L^{(l)}$ with*

$$[P,Q] \in K^{\times} \quad \textit{and} \quad \frac{v_{1,1}(P)}{v_{1,1}(Q)} = \frac{v_{1,0}(P)}{v_{1,0}(Q)} = \frac{m}{n}.$$

Take $T_0 \in K[P,Q]$ and set $T_j := [T_{j-1},P]$ for $j \ge 1$. Assume that $(\rho_0,\sigma_0) \in \mathfrak{V}_{\ge 0}$ satisfies

1. *$(\rho_0,\sigma_0) \in \operatorname{Dir}(P)$ and $v_{\rho_0,\sigma_0}(P) > 0$,*
2. *$\operatorname{en}_{\rho_0,\sigma_0}(T_j) \sim \operatorname{en}_{\rho_0,\sigma_0}(P)$ for all j with $T_j \neq 0$,*
3. *$\frac{1}{m}\operatorname{en}_{\rho_0,\sigma_0}(P) = \frac{1}{n}\operatorname{en}_{\rho_0,\sigma_0}(Q) \in \frac{1}{l}\mathbb{Z}\times\mathbb{N}$,*
4. *$b > a/l$, where $(a/l,b) := \frac{1}{m}\operatorname{en}_{\rho_0,\sigma_0}(P)$.*

Let $I_0 := [(\rho_0,\sigma_0),(0,-1)[$ and

$$(\tilde{\rho},\tilde{\sigma}) := \max\{(\rho,\sigma)\in \mathrm{Dir}(P)\cap I_0 : v_{\rho',\sigma'}(P)>0 \textit{ for all } (\rho_0,\sigma_0)\le(\rho',\sigma')\le(\rho,\sigma)\}$$

Then for all $(\rho,\sigma)\in\mathfrak{V}$ *with* $(\rho_0,\sigma_0)<(\rho,\sigma)\le(\tilde{\rho},\tilde{\sigma})$ *and all* $j\ge 0$ *we have*

$$[\ell_{\rho,\sigma}(T_j),\ell_{\rho,\sigma}(P)]=0 \quad \textit{and} \quad \frac{v_{\rho,\sigma}(T_j)}{v_{\rho,\sigma}(P)}=\frac{v_{\rho_0,\sigma_0}(T_j)}{v_{\rho_0,\sigma_0}(P)}. \qquad (7.1)$$

Idea of the proof: We must prove that there is a partial homothety between P and T_j for $(\rho,\sigma)>(\rho_0,\sigma_0)$. The basic idea is that otherwise $\mathrm{en}(T_{j+n})\nsim \mathrm{en}(P)$ for some direction and all $n>0$, and then $T_{j+n}\neq 0$ for all $n>0$, which is impossible.

Proof. Let

$$(\rho_1,\sigma_1)<\cdots<(\rho_k,\sigma_k)=(\tilde{\rho},\tilde{\sigma})$$

be the directions in $\mathrm{Dir}(P)$ between (ρ_0,σ_0) and $(\tilde{\rho},\tilde{\sigma})$. We will use freely that $v_{\rho',\sigma'}(P)>0$ for all $(\rho_0,\sigma_0)\le(\rho',\sigma')\le(\tilde{\rho},\tilde{\sigma})$. By Remark 1.15 and conditions (2), (3) and (4), we have

$$v_{1,-1}(\mathrm{en}_{\rho_0,\sigma_0}(P))<0 \qquad \text{and} \qquad \mathrm{en}_{\rho_0,\sigma_0}(T_j)=\mu_j\,\mathrm{en}_{\rho_0,\sigma_0}(P) \quad \text{with } \mu_j\ge 0, \qquad (7.2)$$

for all j with $T_j\neq 0$. We claim that if there exists $0\le i<k$ such that

$$v_{1,-1}(\mathrm{en}_{\rho_i,\sigma_i}(P))<0 \qquad \text{and} \qquad \mathrm{en}_{\rho_i,\sigma_i}(T_j)=\mu_j\,\mathrm{en}_{\rho_i,\sigma_i}(P) \quad \text{with } \mu_j\ge 0, \qquad (7.3)$$

for all j with $T_j\neq 0$, then

(a) If $T_j\neq 0$, then $\mathrm{en}_{\rho_i,\sigma_i}(T_j)=\mathrm{st}_{\rho_{i+1},\sigma_{i+1}}(T_j)$.
(b) $[\ell_{\rho_{i+1},\sigma_{i+1}}(T_j),\ell_{\rho_{i+1},\sigma_{i+1}}(P)]=0$, for all j.

In order to check this, we write

$$\mathrm{en}_{\rho_i,\sigma_i}(P)=r_i(a_i/l,b_i) \quad \text{with } r_i\ge 0 \text{ and } \gcd(a_i,b_i)=1.$$

We define the auxiliary direction

$$(\overline{\rho},\overline{\sigma}) := \frac{1}{d}(lb_i,-a_i), \quad \text{where } d:=\gcd(lb_i,a_i).$$

By (3.2) and the inequality in (7.3), we have $(\overline{\rho},\overline{\sigma})=\mathrm{dir}(a_i/l,b_i)$. Furthermore $r_i\in\mathbb{N}$, because $\gcd(a_i,b_i)=1$ and $\mathrm{en}_{\rho_i,\sigma_i}(P)\neq(0,0)$. Note that

$$(c,d)\sim(a_i/l,b_i) \quad \text{if and only if} \quad v_{\overline{\rho},\overline{\sigma}}(c,d)=0. \qquad (7.4)$$

Since $v_{\rho_i,\sigma_i}(a_i/l,b_i), v_{\rho_{i+1},\sigma_{i+1}}(a_i/l,b_i)>0$, by Remarks 2.10 and 3.3 we know that

$$(\overline{\rho},\overline{\sigma}) < (\rho_i,\sigma_i) < (\rho_{i+1},\sigma_{i+1}) < (-\overline{\rho},-\overline{\sigma}). \tag{7.5}$$

Next we prove condition (a). For this it suffices to prove that if $T_j \neq 0$, then

$$\operatorname{Dir}(T_j)\cap](\rho_i,\sigma_i),(\rho_{i+1},\sigma_{i+1})[= \emptyset.$$

In order to check this fact, assume by contradiction that it is false and set $(\hat{\rho},\hat{\sigma}) := \operatorname{Succ}_{T_j}(\rho_i,\sigma_i)$. Since $(\hat{\rho},\hat{\sigma}) \in](\rho_i,\sigma_i),(\rho_{i+1},\sigma_{i+1})[$, by (7.5) we have

$$(\overline{\rho},\overline{\sigma}) < (\hat{\rho},\hat{\sigma}) < (-\overline{\rho},-\overline{\sigma}). \tag{7.6}$$

By Remark 1.8 and (7.4), we have

$$v_{\overline{\rho},\overline{\sigma}}(\operatorname{en}_{\hat{\rho},\hat{\sigma}}(T_j)) < v_{\overline{\rho},\overline{\sigma}}(\operatorname{st}_{\hat{\rho},\hat{\sigma}}(T_j)) = 0, \tag{7.7}$$

since $(a_i/l,b_i) \sim \operatorname{en}_{\rho_i,\sigma_i}(T_j) = \operatorname{st}_{\hat{\rho},\hat{\sigma}}(T_j)$, by (7.3). We assert that

$$T_{j+k} \neq 0 \quad \text{and} \quad \operatorname{en}_{\hat{\rho},\hat{\sigma}}(T_{j+k}) = \operatorname{en}_{\hat{\rho},\hat{\sigma}}(T_j) + k\operatorname{en}_{\hat{\rho},\hat{\sigma}}(P) - k(1,1), \tag{7.8}$$

for all $k \in \mathbb{N}_0$. We will prove this by induction on k. For $k = 0$ this is trivial. Assume that (7.8) is true for some k. Then,

$$\begin{aligned} v_{\overline{\rho},\overline{\sigma}}(\operatorname{en}_{\hat{\rho},\hat{\sigma}}(T_{j+k})) &= v_{\overline{\rho},\overline{\sigma}}(\operatorname{en}_{\hat{\rho},\hat{\sigma}}(T_j)) + k v_{\overline{\rho},\overline{\sigma}}(\operatorname{en}_{\hat{\rho},\hat{\sigma}}(P)) - k v_{\overline{\rho},\overline{\sigma}}(1,1) \\ &= v_{\overline{\rho},\overline{\sigma}}(\operatorname{en}_{\hat{\rho},\hat{\sigma}}(T_j)) - k(\overline{\rho}+\overline{\sigma}) \\ &< 0, \end{aligned}$$

since $v_{\overline{\rho},\overline{\sigma}}(\operatorname{en}_{\hat{\rho},\hat{\sigma}}(P)) = v_{\overline{\rho},\overline{\sigma}}(\operatorname{en}_{\rho_i,\sigma_i}(P)) = 0$ by Proposition 3.7 and (7.4), $v_{\overline{\rho},\overline{\sigma}}(\operatorname{en}_{\hat{\rho},\hat{\sigma}}(T_j)) < 0$ by (7.7), and $\overline{\rho}+\overline{\sigma} > 0$. But then, again by (7.4),

$$\operatorname{en}_{\hat{\rho},\hat{\sigma}}(T_{j+k}) \nsim \operatorname{en}_{\hat{\rho},\hat{\sigma}}(P) = r_i(a_i/l,b_i).$$

Hence, by Propositions 1.13 and 2.3

$$\ell_{\hat{\rho},\hat{\sigma}}(T_{j+k+1}) = [\ell_{\hat{\rho},\hat{\sigma}}(T_{j+k}),\ell_{\hat{\rho},\hat{\sigma}}(P)] \neq 0.$$

Consequently, by Proposition 2.4(2) and (7.8) for k,

$$\begin{aligned} \operatorname{en}_{\hat{\rho},\hat{\sigma}}(T_{j+k+1}) &= \operatorname{en}_{\hat{\rho},\hat{\sigma}}(T_{j+k}) + \operatorname{en}_{\hat{\rho},\hat{\sigma}}(P) - (1,1) \\ &= \operatorname{en}_{\hat{\rho},\hat{\sigma}}(T_j) + (k+1)\operatorname{en}_{\hat{\rho},\hat{\sigma}}(P) - (k+1)(1,1), \end{aligned}$$

which ends the proof of the assertion. But $T_{j+k} \neq 0$ for all k is impossible, since from $[P,Q] \in K^{\times}$ and $T_0 \in K[P,Q]$ it follows easily that $T_n = 0$ for n large enough. Therefore statement (a) is true.

Now we are going to prove statement (b). Assume by contradiction that

$$[\ell_{\rho_{i+1},\sigma_{i+1}}(T_j),\ell_{\rho_{i+1},\sigma_{i+1}}(P)]\neq 0,$$

which by Proposition 1.13 implies

$$[\ell_{\rho_{i+1},\sigma_{i+1}}(T_j),\ell_{\rho_{i+1},\sigma_{i+1}}(P)]=\ell_{\rho_{i+1},\sigma_{i+1}}([T_j,P])=\ell_{\rho_{i+1},\sigma_{i+1}}(T_{j+1}). \tag{7.9}$$

By (7.5) we have $(\overline{\rho},\overline{\sigma})<(\rho_{i+1},\sigma_{i+1})<(-\overline{\rho},-\overline{\sigma})$ and so, by Remark 1.8

$$\begin{aligned}\mathrm{st}_{\rho_{i+1},\sigma_{i+1}}(P)&=\mathrm{Supp}(\ell_{\overline{\rho},\overline{\sigma}}(\ell_{\rho_{i+1},\sigma_{i+1}}(P))),\\ \mathrm{st}_{\rho_{i+1},\sigma_{i+1}}(T_j)&=\mathrm{Supp}(\ell_{\overline{\rho},\overline{\sigma}}(\ell_{\rho_{i+1},\sigma_{i+1}}(T_j))),\\ \mathrm{st}_{\rho_{i+1},\sigma_{i+1}}(T_{j+1})&=\mathrm{Supp}(\ell_{\overline{\rho},\overline{\sigma}}(\ell_{\rho_{i+1},\sigma_{i+1}}(T_{j+1}))).\end{aligned}$$

But then, by Proposition 1.13 and equivalence (7.4),

$$\begin{aligned}v_{\overline{\rho},\overline{\sigma}}(\mathrm{st}_{\rho_{i+1},\sigma_{i+1}}(T_{j+1}))&=v_{\overline{\rho},\overline{\sigma}}(\ell_{\rho_{i+1},\sigma_{i+1}}(T_{j+1}))\\ &\le v_{\overline{\rho},\overline{\sigma}}(\ell_{\rho_{i+1},\sigma_{i+1}}(T_j))+v_{\overline{\rho},\overline{\sigma}}(\ell_{\rho_{i+1},\sigma_{i+1}}(P))-(\overline{\rho}+\overline{\sigma})\\ &=v_{\overline{\rho},\overline{\sigma}}(\mathrm{st}_{\rho_{i+1},\sigma_{i+1}}(T_j))+v_{\overline{\rho},\overline{\sigma}}(\mathrm{st}_{\rho_{i+1},\sigma_{i+1}}(P))-(\overline{\rho}+\overline{\sigma})\\ &=-(\overline{\rho}+\overline{\sigma})<0,\end{aligned}$$

since by item (a), Proposition 3.7 and (7.3),

$$\mathrm{st}_{\rho_{i+1},\sigma_{i+1}}(T_j)=\mathrm{en}_{\rho_i,\sigma_i}(T_j)\sim(a_i/l,b_i)\sim\mathrm{en}_{\rho_i,\sigma_i}(P)=\mathrm{st}_{\rho_{i+1},\sigma_{i+1}}(P).$$

Hence, by item (a), Proposition 3.7 and (7.4),

$$\mathrm{en}_{\rho_i,\sigma_i}(T_{j+1})=\mathrm{st}_{\rho_{i+1},\sigma_{i+1}}(T_{j+1})\nsim(a_i/l,b_i)\sim\mathrm{en}_{\rho_i,\sigma_i}(P),$$

which contradicts (7.3), thus proving (b) and finishing the proof of the claim.

In order to prove (7.1), we must check that

$$[\ell_{\rho,\sigma}(T_j),\ell_{\rho,\sigma}(P)]=0\qquad\text{and}\qquad\frac{v_{\rho,\sigma}(T_j)}{v_{\rho,\sigma}(P)}=\frac{v_{\rho_i,\sigma_i}(T_j)}{v_{\rho_i,\sigma_i}(P)} \tag{7.10}$$

hold for all (ρ,σ) with $(\rho_{i+1},\sigma_{i+1})\ge(\rho,\sigma)>(\rho_i,\sigma_i)$ and all i. We proceed by induction, using the claim and (7.2). More precisely, we are going to prove for any i, that (7.3) implies that (7.10) hold for all (ρ,σ) with $(\rho_{i+1},\sigma_{i+1})\ge(\rho,\sigma)>(\rho_i,\sigma_i)$, and that condition (7.3) is true for $i+1$.

In fact, if $(\rho_{i+1},\sigma_{i+1})>(\rho,\sigma)>(\rho_i,\sigma_i)$, then by Proposition 3.7,

$$\mathrm{en}_{\rho_i,\sigma_i}(P)=\mathrm{Supp}(\ell_{\rho,\sigma}(P))=\mathrm{st}_{\rho_{i+1},\sigma_{i+1}}(P), \tag{7.11}$$

while, again by Proposition 3.7 and statement (a), for the same (ρ,σ)

$$\mathrm{en}_{\rho_i,\sigma_i}(T_j) = \mathrm{Supp}(\ell_{\rho,\sigma}(T_j)) = \mathrm{st}_{\rho_{i+1},\sigma_{i+1}}(T_j). \tag{7.12}$$

Consequently, since $\mathrm{en}_{\rho_i,\sigma_i}(T_j) \sim \mathrm{en}_{\rho_i,\sigma_i}(P)$,

$$[\ell_{\rho,\sigma}(T_j), \ell_{\rho,\sigma}(P)] = 0 \qquad \text{for all } (\rho,\sigma) \text{ with } (\rho_{i+1},\sigma_{i+1}) > (\rho,\sigma) > (\rho_i,\sigma_i)$$

and

$$\frac{v_{\rho,\sigma}(T_j)}{v_{\rho,\sigma}(P)} = \frac{v_{\rho_i,\sigma_i}(T_j)}{v_{\rho_i,\sigma_i}(P)} \qquad \text{for all } (\rho,\sigma) \text{ with } (\rho_{i+1},\sigma_{i+1}) \geq (\rho,\sigma) > (\rho_i,\sigma_i).$$

Hence the equalities in (7.10) hold for all required (ρ,σ)'s. Next we prove that condition (7.3) is true for $i+1$. We first prove that

$$v_{1,-1}(\mathrm{en}_{\rho_{i+1},\sigma_{i+1}}(P)) < 0. \tag{7.13}$$

If $\rho_{i+1}+\sigma_{i+1} \geq 0$, then by Proposition 3.7, Remark 1.8 and the inequality in (7.3),

$$v_{1,-1}(\mathrm{en}_{\rho_{i+1},\sigma_{i+1}}(P)) \leq v_{1,-1}(\mathrm{st}_{\rho_{i+1},\sigma_{i+1}}(P)) = v_{1,-1}(\mathrm{en}_{\rho_i,\sigma_i}(P)) < 0,$$

as desired. Assume that $\rho_{i+1}+\sigma_{i+1} < 0$ and set $A := \mathrm{en}_{\rho_{i+1},\sigma_{i+1}}(P)$. First we are going to prove that $v_{1,-1}(A) \neq 0$. Otherwise $A = k(1,1)$ for some $k \in \mathbb{N}_0$, which is impossible, since then

$$v_{\rho_{i+1},\sigma_{i+1}}(P) = v_{\rho_{i+1},\sigma_{i+1}}(A) = k(\rho_{i+1}+\sigma_{i+1}) \leq 0,$$

contradicting the definition of $(\tilde{\rho},\tilde{\sigma})$. Assume that

$$v_{1,-1}(A) > 0 = v_{1,-1}(0,0). \tag{7.14}$$

Since $\rho_{i+1}+\sigma_{i+1} < 0$, $(\rho_{i+1},\sigma_{i+1}) \in I_0$ and $A \in \mathrm{Supp}(P)$, we have

$$(-1,1) < (\rho_{i+1},\sigma_{i+1}) < (0,-1) \quad \text{and} \quad v_{0,1}(A) \geq 0 = v_{0,1}(0,0).$$

Thus, by Corollary 3.6 (which we can apply because $(-\rho_{i+1},-\sigma_{i+1}) < (0,1)$ in $\mathfrak{V}_{>0}$), we have

$$v_{\rho_{i+1},\sigma_{i+1}}(P) = v_{\rho_{i+1},\sigma_{i+1}}(A) = -v_{-\rho_{i+1},-\sigma_{i+1}}(A) < -v_{-\rho_{i+1},-\sigma_{i+1}}(0,0) = 0,$$

which contradicts again the definition of $(\tilde{\rho},\tilde{\sigma})$ and ends the proof of (7.13).

It remains to check that the second assertion in (7.3) holds for $i+1$. By equalities (7.11) and (7.12),

$$\mathrm{st}_{\rho_{i+1},\sigma_{i+1}}(T_j) = \mathrm{en}_{\rho_i,\sigma_i}(T_j) = \mu_j \,\mathrm{en}_{\rho_i,\sigma_i}(P) = \mu_j \,\mathrm{st}_{\rho_{i+1},\sigma_{i+1}}(P),$$

which implies $v_{\rho_{i+1},\sigma_{i+1}}(T_j) = \mu_j v_{\rho_{i+1},\sigma_{i+1}}(P) \geq 0$. Therefore, by (b), we can apply Remark 3.1 in order to obtain that

$$\mathrm{en}_{\rho_{i+1},\sigma_{i+1}}(T_j) = \mu_j \,\mathrm{en}_{\rho_{i+1},\sigma_{i+1}}(P),$$

as desired. This proves (7.1) and concludes the proof. □

Corollary 7.2. *Let $m, n \in \mathbb{N}$ be coprime with $m, n > 1$ and let $P, Q \in L^{(l)}$ with*

$$[P, Q] \in K^{\times} \quad \textit{and} \quad \frac{v_{1,1}(P)}{v_{1,1}(Q)} = \frac{v_{1,0}(P)}{v_{1,0}(Q)} = \frac{m}{n}.$$

Assume that $(\rho_0, \sigma_0) \in \mathfrak{V}_{\geq 0}$ satisfies

(1) $(\rho_0, \sigma_0) \in \mathrm{Dir}(P)$ *and* $v_{\rho_0,\sigma_0}(P) > 0$,
(2) $\frac{1}{m}\mathrm{en}_{\rho_0,\sigma_0}(P) = \frac{1}{n}\mathrm{en}_{\rho_0,\sigma_0}(Q) \in \frac{1}{l}\mathbb{Z} \times \mathbb{N}$,
(3) $b > a/l$, *where* $(a/l, b) := \frac{1}{m}\mathrm{en}_{\rho_0,\sigma_0}(P)$.

Let $(\tilde{\rho}, \tilde{\sigma})$ be as in Proposition 7.1 and let $F \in L^{(l)}$ be the (ρ_0, σ_0)-homogeneous element obtained in Theorem 2.6. If there exist $p, q \in \mathbb{N}$ coprime such that $\mathrm{en}_{\rho_0,\sigma_0}(F) = \frac{p}{q}(a/l, b)$, then for all $(\rho, \sigma) \in \mathfrak{V}$ with $(\rho_0, \sigma_0) < (\rho, \sigma) \leq (\tilde{\rho}, \tilde{\sigma})$ there exists a (ρ, σ)-homogeneous element $R \in L^{(l)}$ such that $\ell_{\rho,\sigma}(P) = R^{qm}$.

Proof. Let G_0 and G_1 be as in Theorem 2.6. Since $G_0, P \neq 0$, by the last equality in (2.7) we have

$$[\ell_{\rho_0,\sigma_0}(G_0), \ell_{\rho_0,\sigma_0}(P)] \neq 0,$$

which, by Proposition 1.13, implies that

$$\ell_{\rho_0,\sigma_0}(G_1) = [\ell_{\rho_0,\sigma_0}(G_0), \ell_{\rho_0,\sigma_0}(P)] \neq 0.$$

By Proposition 2.3 and Theorem 2.6(5) there exists $g_1 \in \mathbb{Q}$ such that

$$\mathrm{en}_{\rho_0,\sigma_0}(G_1) = g_1 \,\mathrm{en}_{\rho_0,\sigma_0}(P). \tag{7.15}$$

Moreover,

$$\mathrm{en}_{\rho_0,\sigma_0}(F) = \frac{p}{q}(a/l, b) = \frac{p}{qm}\,\mathrm{en}_{\rho_0,\sigma_0}(P). \tag{7.16}$$

On the other hand, using again the last equality in (2.7), we obtain

$$\ell_{\rho_0,\sigma_0}(G_1)F = \ell_{\rho_0,\sigma_0}(G_0)\ell_{\rho_0,\sigma_0}(P),$$

and hence

$$\mathrm{en}_{\rho_0,\sigma_0}(G_1) + \mathrm{en}_{\rho_0,\sigma_0}(F) = \mathrm{en}_{\rho_0,\sigma_0}(G_0) + \mathrm{en}_{\rho_0,\sigma_0}(P).$$

Consequently, by (7.15) and (7.16),

$$\mathrm{en}_{\rho_0,\sigma_0}(G_0) = \mathrm{en}_{\rho_0,\sigma_0}(G_1) + \mathrm{en}_{\rho_0,\sigma_0}(F) - \mathrm{en}_{\rho_0,\sigma_0}(P) = \Bigl(g_1 + \frac{p}{qm} - 1\Bigr)\,\mathrm{en}_{\rho_0,\sigma_0}(P).$$

Set $g_0 := g_1 + \frac{p}{qm} - 1$ and take $r \in \mathbb{Z}$ and $s \in \mathbb{N}$ coprime, such that $g_0 = r/s$. Note that by (7.15), (7.16) and the fact that $g_1 = \frac{r}{s} + 1 - \frac{p}{qm}$, we have

$$\frac{1}{v_{\rho_0,\sigma_0}(P)}\bigl(v_{\rho_0,\sigma_0}(G_0), v_{\rho_0,\sigma_0}(G_1), v_{\rho_0,\sigma_0}(P), v_{\rho_0,\sigma_0}(Q)\bigr) = \Bigl(\frac{r}{s}, \frac{r}{s} + 1 - \frac{p}{qm}, 1, \frac{n}{m}\Bigr). \tag{7.17}$$

Let $(\rho,\sigma) > (\rho_0,\sigma_0)$. Applying Proposition 7.1 with $T_0 := G_0$, with $T_0 := G_1$ and with $T_0 := Q$, we obtain that

$$[\ell_{\rho,\sigma}(G_0), \ell_{\rho,\sigma}(P)] = 0, \quad [\ell_{\rho,\sigma}(G_1), \ell_{\rho,\sigma}(P)] = 0 \quad \text{and} \quad [\ell_{\rho,\sigma}(Q), \ell_{\rho,\sigma}(P)] = 0.$$

Hence, by Proposition 2.1(2b), there exist $\gamma_0, \gamma_1, \gamma_2, \gamma_3 \in K^\times$, a (ρ,σ)-homogeneous element $R_0 \in L$ and $u_0, u_1, u_2, u_3 \in \mathbb{N}$, such that

$$\ell_{\rho,\sigma}(G_0) = \gamma_0 R_0^{u_0}, \quad \ell_{\rho,\sigma}(G_1) = \gamma_1 R_0^{u_1}, \quad \ell_{\rho,\sigma}(P) = \gamma_2 R_0^{u_2} \quad \text{and} \quad \ell_{\rho,\sigma}(Q) = \gamma_3 R_0^{u_3},$$

and clearly we can assume that $\gcd(u_0,u_1,u_2,u_3) = 1$. But then, by Proposition 7.1 and equality (7.17),

$$\begin{aligned} v_{\rho,\sigma}(R_0)(u_0,u_1,u_2,u_3) &= \bigl(v_{\rho,\sigma}(G_0), v_{\rho,\sigma}(G_1), v_{\rho,\sigma}(P), v_{\rho,\sigma}(Q)\bigr) \\ &= \frac{v_{\rho,\sigma}(P)}{v_{\rho_0,\sigma_0}(P)}\bigl(v_{\rho_0,\sigma_0}(G_0), v_{\rho_0,\sigma_0}(G_1), v_{\rho_0,\sigma_0}(P), v_{\rho_0,\sigma_0}(Q)\bigr) \\ &= \frac{v_{\rho,\sigma}(P)}{sqm}(rqm, rqm + sqm - ps, sqm, sqn), \end{aligned}$$

and so we have $u_2 = \frac{sqm}{d}$, where $d := \gcd(rqm, rqm + sqm - ps, sqm, sqn)$. Since

$$d = \gcd(rqm, ps, sqm, sqn) = \gcd(qm, s),$$

we obtain that $d|s$. Consequently we can write $\ell_{\rho,\sigma}(P) = \gamma_2 R_0^{sqm/d} = \gamma_2 (R_0^{s/d})^{qm}$. We conclude the proof setting $R := \gamma R_0^{s/d}$, where we choose $\gamma \in K^\times$ such that $\gamma^{qm} = \gamma_2$. $\square$

Proposition 7.3. *Let $m, n \in \mathbb{N}$ be coprime with $m, n > 1$ and let $P, Q \in L^{(l)}$ with*

$$[P, Q] \in K^{\times} \quad \text{and} \quad \frac{v_{1,1}(P)}{v_{1,1}(Q)} = \frac{v_{0,1}(P)}{v_{0,1}(Q)} = \frac{m}{n}.$$

Take $T_0 \in K[P, Q]$ and set $T_j := [T_{j-1}, P]$ for $j \geq 1$. Assume that $(\rho_0, \sigma_0) \in \mathfrak{V}_{\geq 0}$ satisfies

(1) $(\rho_0, \sigma_0) \in \mathrm{Dir}(P)$ *and* $v_{\rho_0,\sigma_0}(P) > 0$,
(2) $\mathrm{st}_{\rho_0,\sigma_0}(T_j) \sim \mathrm{st}_{\rho_0,\sigma_0}(P)$ *for all j with $T_j \neq 0$,*
(3) $\frac{1}{m}\mathrm{st}_{\rho_0,\sigma_0}(P) = \frac{1}{n}\mathrm{st}_{\rho_0,\sigma_0}(Q) \in \frac{1}{l}\mathbb{Z} \times \mathbb{N}$,
(4) $b < a/l$, *where* $(a/l, b) := \frac{1}{m}\mathrm{st}_{\rho_0,\sigma_0}(P)$.

Let $I_1 := [(0, -1), (\rho_0, \sigma_0)]$ and

$$(\tilde{\rho}, \tilde{\sigma}) := \min\{(\rho, \sigma) \in \mathrm{Dir}(P) \cap I_1 : v_{\rho',\sigma'}(P) > 0 \text{ for all } (\rho_0, \sigma_0) \geq (\rho', \sigma') \geq (\rho, \sigma)\}.$$

Then for all $(\rho, \sigma) \in \mathfrak{V}$ with $(\tilde{\rho}, \tilde{\sigma}) \leq (\rho, \sigma) < (\rho_0, \sigma_0)$ and all $j \geq 0$ we have

$$[\ell_{\rho,\sigma}(T_j), \ell_{\rho,\sigma}(P)] = 0 \quad \text{and} \quad \frac{v_{\rho,\sigma}(T_j)}{v_{\rho,\sigma}(P)} = \frac{v_{\rho_0,\sigma_0}(T_j)}{v_{\rho_0,\sigma_0}(P)}.$$

Proof. Mimic the proof of Proposition 7.1. □

Corollary 7.4. *Let $m, n \in \mathbb{N}$ be coprime with $m, n > 1$ and let $P, Q \in L^{(l)}$ with*

$$[P, Q] \in K^{\times} \quad \text{and} \quad \frac{v_{1,1}(P)}{v_{1,1}(Q)} = \frac{v_{0,1}(P)}{v_{0,1}(Q)} = \frac{m}{n}.$$

Assume that $(\rho_0, \sigma_0) \in \mathfrak{V}_{\geq 0}$ satisfies

(1) $(\rho_0, \sigma_0) \in \mathrm{Dir}(P)$ *and* $v_{\rho_0,\sigma_0}(P) > 0$,
(2) $\frac{1}{m}\mathrm{st}_{\rho_0,\sigma_0}(P) = \frac{1}{n}\mathrm{st}_{\rho_0,\sigma_0}(Q) \in \frac{1}{l}\mathbb{Z} \times \mathbb{N}$,
(3) $b < a/l$, *where* $(a/l, b) := \frac{1}{m}\mathrm{st}_{\rho_0,\sigma_0}(P)$.

Let $(\tilde{\rho}, \tilde{\sigma})$ be as in Proposition 7.3 and let $F \in L^{(l)}$ be the (ρ_0, σ_0)-homogeneous element obtained in Theorem 2.6. If there exist $p, q \in \mathbb{N}$ coprime, such that $\mathrm{st}_{\rho_0,\sigma_0}(F) = \frac{p}{q}(a/l, b)$, then for all $(\rho, \sigma) \in \mathfrak{V}$ with $(\tilde{\rho}, \tilde{\sigma}) \leq (\rho, \sigma) < (\rho_0, \sigma_0)$ there exists a (ρ, σ)-homogeneous element $R \in L^{(l)}$ such that $\ell_{\rho,\sigma}(P) = R^{qm}$.

Proof. Mimic the proof of Corollary 7.2. □

Remark 7.5. Let $P \in L^{(l)} \setminus \{0\}$ and let (ρ', σ') and (ρ'', σ'') be consecutive elements in $\mathrm{Dir}(P)$. It follows from Remarks 2.10 and 3.3 that if $v_{\rho',\sigma'}(P), v_{\rho'',\sigma''}(P) > 0$, then $v_{\rho,\sigma}(P) > 0$ for all $(\rho', \sigma') < (\rho, \sigma) < (\rho'', \sigma'')$.

The following theorem is related to [5, Proposition 1.10] and also to [12, Remark 5.12]. In this theorem we consider the order in $I =](1,-1),(1,0)]$.

Theorem 7.6. *Let $(A_0,(\rho_0,\sigma_0)),(A_1,(\rho_1,\sigma_1)),\dots,(A_k,(\rho_k,\sigma_k))$ be the regular corners of an (m,n)-pair (P,Q) in $L^{(l)}$, where $(\rho_i,\sigma_i)<(\rho_{i+1},\sigma_{i+1})$ for all $i<k$. The following facts hold:*

(1) *$A(P)=\{(\rho_1,\sigma_1),\dots,(\rho_k,\sigma_k)\}$. In particular, if (P,Q) is a standard (m,n)-pair, then $(A_0,A_0',(\rho_0,\sigma_0))$ is the starting triple of (P,Q), where $A_0' := \frac{1}{m}\operatorname{st}_{\rho_0,\sigma_0}(P)$.*
(2) *For all $j\ge 1$ there exists $d_j\in\mathbb{N}$ maximum such that $\ell_{\rho_j,\sigma_j}(P)=R_j^{md_j}$ for some (ρ_j,σ_j)-homogeneous $R_j\in L^{(l)}$. If A_0 is of type II, then this holds also for $j=0$.*
(3) *For all $j>0$ the element F_j constructed via Theorem 2.6 satisfies*

$$\operatorname{en}_{\rho_j,\sigma_j}(F_j)=\frac{p_j}{q_j}\frac{1}{m}\operatorname{en}_{\rho_j,\sigma_j}(P),$$

where p_j and q_j are coprime. If A_0 is of type II, then this holds also for $j=0$.
(4) *$q_i\nmid d_i$ for all $i>0$.*
(5) *$q_j\mid d_i$ for all $i>j>0$.*
(6) *$q_i\nmid q_j$ for all $i>j>0$.*

Set $D_j:=\gcd(a_j,b_j,a_{j-1},b_{j-1})$, where $A_j=(a_j/l,b_j)$ and $A_{j-1}=(a_{j-1}/l,b_{j-1})$. Then

(7) *$d_j\mid D_j$ and $\Omega(D_j)\ge\Omega(d_j)\ge j-1$ for all $j>0$, where for $n\in\mathbb{N}$ we let $\Omega(n)$ denote the number of prime factors of n, counted with multiplicity.*
(8) *If A_0 is of type II, then $q_0\nmid d_0$ and for all $i>0$, we have*

$$q_0\mid d_i,\quad q_i\nmid q_0,\quad \textit{and}\quad \Omega(d_i)\ge i.$$

Proof. By Remark 5.10 and Propositions 5.2 and 5.22 statement (1) is true. By Corollary 5.7(1) we know that $v_{\rho_j,\sigma_j}(P)>0$ for all j. If A_0 is of type II, then $[\ell_{\rho_0,\sigma_0}(P),\ell_{\rho_0,\sigma_0}(Q)]=0$. In the general case, when $j\ge 1$, by Remark 5.10, we are in Case II.a), and so $[\ell_{\rho_j,\sigma_j}(P),\ell_{\rho_j,\sigma_j}(Q)]=0$. Hence, by Proposition 2.1(2b), statement (2) holds. Statement (3) follows from Remark 5.10 and Proposition 5.14(1).

In order to prove statement (4), assume by contradiction that $q_j\mid d_j$. Then $\widetilde{R}:=R_j^{p_jd_j/q_j}$ satisfies

$$[\widetilde{R},\ell_{\rho_j,\sigma_j}(P)]=0\quad\text{and}\quad v_{\rho_j,\sigma_j}(\widetilde{R})=v_{\rho_j,\sigma_j}(F_j)=\rho_j+\sigma_j,\tag{7.18}$$

where the second equality follows from the fact that

$$\operatorname{en}_{\rho_j,\sigma_j}(F_j)=\frac{p_j}{q_j}\frac{1}{m}\operatorname{en}_{\rho_j,\sigma_j}(P)=\operatorname{en}_{\rho_j,\sigma_j}(\widetilde{R}).$$

But the existence of $\widetilde{R}$ satisfying (7.18) contradicts Proposition 2.11(5) (The condition $s > 0$ or $\#\,\mathrm{factors}(p) > 1$ required in Proposition 2.11(5) is satisfied if and only if $\#\,\mathrm{factors}(\mathfrak{p}(z)) > 1$, which holds because we are in case II).

By Corollary 5.7(1) we have $v_{\rho_j,\sigma_j}(P) > 0$ for all $j \ge 0$, and hence, by Remark 7.5, we have $v_{\rho,\sigma}(P) > 0$ if (ρ,σ) lies between (ρ_0,σ_0) and (ρ_k,σ_k). Let $(\tilde{\rho},\tilde{\sigma})$ be as in Proposition 7.1. By its very definition $(\tilde{\rho},\tilde{\sigma}) \ge (\rho_i,\sigma_i) > (\rho_j,\sigma_j)$. Thus the hypotheses of Corollary 7.2 are satisfied with $(\rho_0,\sigma_0) = (\rho_j,\sigma_j)$ and $(\rho,\sigma) = (\rho_i,\sigma_i)$, and hence we have

$$R_i^{md_i} = \ell_{\rho_i\sigma_i}(P) = R^{mq_j} \quad \text{for some } R \in L^{(l)},$$

which gives statement (5) by the maximality of d_i.

Statement (6) follows from (4) and (5). In order to prove statement (7), note that $d_j|D_j$ since

$$A_j = d_j\,\mathrm{en}_{\rho_j\sigma_j}(R_j) \quad \text{and} \quad A_{j-1} = d_j\,\mathrm{st}_{\rho_j\sigma_j}(R_j),$$

and a straightforward computation using (4), (5) and (6) proves the last assertion of (7). The proof of statement (8) follows along the lines of the proofs of (4), (5), (6) and (7). □

In the proof the next corollary, nearly all facts were more or less known, except Proposition 6.10, which is the missing piece of the puzzle.

Corollary 7.7. *Let (P,Q) be a standard (m,n)-pair in L. Write $(a,b) := \frac{1}{m}\,\mathrm{en}_{1,0}(P)$. Then $(a,b) \in \mathbb{N}\times\mathbb{N}$ and $\gcd(a,b) > 2$. Furthermore $B \neq p$ and $B \neq 2p$ for any prime p, where B is as at the beginning of Section 4.*

Proof. By Remark 5.12 we know that (a,b) is the first component of a regular corner of (P,Q). Hence, by Remark 5.8 we have $(a,b) \in \mathbb{N}\times\mathbb{N}$ and by Proposition 5.19 we know $\gcd(a,b) > 1$. Next we discard $\gcd(a,b) = 2$. Let k be the number of regular corners in $A(P)$. If $k = 0$, then $\gcd(a,b) = 2$ contradicts Proposition 6.10. Assume $k > 0$. Then $(a,b) = (a_k,b_k)$ and the very definitions of q_k and d_k show that $q_k|\gcd(a,b)$ and $d_k|\gcd(a,b)$. Moreover, by Theorem 7.6(8) we have $q_0|d_k$, $q_0 \nmid d_0$ and $q_k \nmid q_0$. Hence $\gcd(a,b)$ is a composite number and so $\gcd(a,b) > 2$.

Now assume that (P,Q) is as in Corollary 5.21. In particular,

$$B = \gcd(v_{1,1}(P), v_{1,1}(Q)) = \frac{1}{m}v_{1,1}(P) \quad \text{and} \quad (a,b) = \frac{1}{m}\,\mathrm{en}_{1,0}(P) = \frac{1}{m}\,\mathrm{st}_{1,1}(P),$$

and so $a + b = B$, which implies $\gcd(a,b) \mid B$. Now, if $B = p$ or $B = 2p$ for some prime p, then $\gcd(a,b) \in \{1,2,p,2p\}$. Since $\gcd(a,b) > 2$ and $\gcd(a,b) = 2p$ is impossible, we have to discard only the case $\gcd(a,b) = p$. But in that case $a = b = p$, which contradicts $a < b$ and finishes the proof. □

In the following proposition we give a condition under which the Newton polygon of P has no vertical edge at the right hand side.

Corollary 7.8. *Let (P,Q) be a standard (m,n)-pair and let (ρ,σ), A_0 and F be as in the discussion above Proposition 6.4. By Proposition 6.4(4) there exist $p,q \in \mathbb{N}$ coprime, such that $\mathrm{en}_{\rho,\sigma}(F) = \frac{p}{q}A_0$. Assume that $A_0 = \frac{1}{m}\mathrm{st}_{1,0}(P)$. If $A_0 = (q,b)$, then $\mathrm{st}_{1,0}(P) = \mathrm{en}_{1,0}(P)$.*

Proof. Along the proof we use the notations of Theorem 7.6. Assume that $\mathrm{st}_{1,0}(P) \neq \mathrm{en}_{1,0}(P)$. Note that $(\rho_0,\sigma_0) = (\rho,\sigma)$, $q_0 = q$, $k = 1$ and $(A_1,(\rho_1,\sigma_1)) = \big(\frac{1}{m}\mathrm{en}_{1,0}(P), (1,0)\big)$. By Theorem 7.6(3),

$$\mathrm{en}_{1,0}(F_1) = \frac{p_1}{q_1}\frac{1}{m}\mathrm{en}_{1,0}(P),$$

and so $1 = v_{1,0}(F_1) = p_1 v_{1,0}(A_0)/q_1 = p_1 q/q_1$. Consequently, $q_1 = q = q_0$, which contradicts Theorem 7.6(6) and concludes the proof. $\square$

Remark 7.9. As long as we are not able to discard the possibility $B = 16$, there can be expected no real progress in proving or disproving the JC just by describing the admissible A_0's. However we submit without proof a complete list of small values. Let

$$B_0 := \frac{1}{m}\mathrm{st}_{1,0}(P) \qquad \text{and} \qquad B_1 := \frac{1}{m}\mathrm{en}_{1,0}(P).$$

If $B \leq 50$, then necessarily

a) A_0 belongs to the following set:

$$\mathcal{X} := \{(4,12),(5,20),(6,15),(6,30),(7,21),(7,35),(7,42),(8,24),(8,28),(9,21),(9,24),(9,36),(10,25),(10,30),(10,40),(11,33),(12,28),(12,30),(12,33),(12,36),(14,35),(15,35),(18,30)\}.$$

b) $B_0 \in \mathcal{X}$ or $B_0 = (8,40)$ and $A_0 = (4,12)$.

c) $B_1 \in \mathcal{X}$ or $B_1 \in \{(8,32),(8,40),(6,18),(6,24),(6,36),(6,42),(9,27)\}$. Furthermore,
- if $B_1 = (8,32)$, then $B_0 = (8,28)$,
- if $B_1 = (8,40)$ then $B_0 = B_1$ or $B_0 = (8,28)$,
- if $B_1 = (6,18+6k)$, then $B_0 = (6,15)$,
- if $B_1 = (9,27)$, then $B_0 = (9,21)$ or $B_0 = (9,24)$.

The cases listed in a) and b) coincide with the list for B_0 given in [5, Theorem 2.24(1)], where $B_0 = (E_1,D_1)$ is written as (D_1,E_1) and $B_1 = (E,D)$ is written as (D,E). However, our list in c), in addition to the cases considered in [5], contains the pairs

$\{(6,18),(6,24),(6,36),(6,42),(9,27)\}$. His result follows from a computer search and some computations on a parameter α, which should be the same as our q.

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