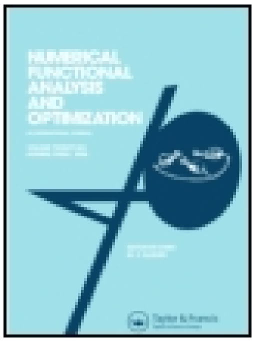

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Abstract

We shall prove some simultaneous localization or concentration inequalities for the Continuous Wavelet Transform. We will also show that simultaneous localization in the scale-time(space) is impossible, in the sense that the scale sections of the support of the wavelet transform of a non null L^p-function can not have finite Lebesgue measure. Finally, some properties of the support of the continuous wavelet transform of band-limited functions are studied.

KEYWORDS: Continuous wavelet transform; L^p spaces; uncertainty principles

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1. INTRODUCTION

The Lebesgue L^p spaces have proved to be useful in modelling signals which are not necessarily restricted to the usual finite energy case of $L^2(\mathbb{R})$, and on the other hand, Wavelet transforms are nowadays a standard signal analysis-synthesis tool [15]. Let $f \in L^p(\mathbb{R}^n)$, $p \in (1, +\infty)$, then its continuous wavelet transform (CWT) is defined as (here we use the L^1 normalization) [10, 16]:

$$\mathcal{W}f(a,b) = \frac{1}{a^n} \int_{\mathbb{R}^n} \psi\left(\frac{x-b}{a}\right) f(x) dx\,, \tag{1}$$

$$(a,b) \in \mathbb{R}_{>0} \times \mathbb{R}^n,\ \mathbb{R}_{>0} = (0, +\infty)\,,$$

for an admissible wavelet ψ [15]. The variable a represents, in some sense, the scales of the signal f "*acting*" in an interval of time centered in the location parameter b. In view of that this integral transform gives an alternative description to the ordinary windowed Fourier transform time-frequency decomposition of f, it is of interest to describe its simultaneous time-frequency or time-scale localization properties. We shall see that if we want to study the time-scale localization in terms of the Lebesgue measure of the support of $\mathcal{W}f$ we have a similar restriction to that given by Benedicks for the Fourier transform, which says:

Theorem 1.1 ([2]). *Let $f \in L^1(\mathbb{R}^n)$, such that both supports of f and $\hat{f}$, have finite Lebesgue measure, then $f = 0$ a.e.*

An analogue result is given in [11] for Wigner distributions. Some generalizations of Benedicks' result are also studied in [19]. There, the results are presented as simultaneous restrictions on the measure of the supports of f and its Fourier transform $\hat{f}$ for certain locally compact groups, and in particular the affine group . It is of also of practical interest to study similar localization principles for wavelets [9][21]. On the other hand, localization properties can also be stated as uncertainty inequalities like Heisenberg's classical principle for the Fourier transform [7, 9]:

Theorem 1.2. *Given $f \in L^2(\mathbb{R})$, then :*

$$\frac{\|f\|_{L^2}^2}{4\pi} \leq \|xf(x)\|_{L^2} \left\|\lambda\hat{f}(\lambda)\right\|_{L^2} .$$

The inequality becomes an equality if and only if $f(x) = Ce^{-kx^2}$, with $C \in \mathbb{C}$ and $k > 0$.

This type of inequalities are studied for example in [4, 17, 18, 20, 21], for the Cohen class transforms in [12], for the Linear Canonical Transform in [5, 13, 24] and in [8] for the Gabor-STFT transforms. Other generalizations can be found in [14].

1.1. Paper outline

We shall first study in section 3 some localization properties of the CWT of an L^p function-signal in terms of some norm inequalities relating its CWT and its Fourier transform. Afterwords, in a similar way to [2, 19], we shall prove in section 4 that the -scale- sections of the support of the wavelet transform of a non null L^p-function cannot have finite Lebesgue measure. In particular, that it cannot have finite Lebesgue measure in the time-scale space (corollary 1). Finally, we shall study briefly some simple properties of the support of the continuous wavelet transform of band limited functions, this intuitive result characterizes in terms of the wavelet transform the bandwidth of a band limited function.

2. SOME PRELIMINARY RESULTS

2.1. Fourier transform

If $\mathcal{S}(\mathbb{R}^n)$ denotes the Schwartz class of functions [7], for $f \in \mathcal{S}(\mathbb{R}^n)$ we shall define its Fourier transform as [7]:

$$\mathcal{F}(f)(\lambda) = \hat{f}(\lambda) = \int_{\mathbb{R}^n} f(x) e^{-i\,2\pi\, x.\lambda} dx\,.$$

This linear operator extends to the Lebesgue spaces $L^p(\mathbb{R}^n)$ and to the dual of $\mathcal{S}(\mathbb{R}^n)$: $\mathcal{S}'(\mathbb{R}^n)$. We assume the reader to be familiar with the usual properties of the Fourier transform [7, 22], the inverse Fourier transform $\mathcal{F}^{-1}f = \overset{\vee}{f}$ and Plancherel's identity, of which an immediate generalization is the *Hausdorff-Young Inequality*: if $p \in (1,2]$, then $\left\|\hat{f}\right\|_{L^q} \leq \|f\|_{L^p}$, with $\frac{1}{p}+\frac{1}{q} = 1$. A sharper inequality is the *Babenko-Beckner inequality* [1]: if $p \in (1,2]$, then

$$\left\|\hat{f}\right\|_{L^q} \leq B(p,q)\, \|f\|_{L^p}\ , \tag{2}$$

with $B(p,q) = \left(\frac{p^{1/p}}{q^{1/q}}\right)^{\frac{n}{2}}$, $\frac{1}{p}+\frac{1}{q} = 1$.

Finally, as it is well known that the Fourier transform is not an involution in $L^p(\mathbb{R}^n)$, we recall an auxiliary a known result which will be useful in the sequel.

Proposition 1. *Let* $f,\ g \in L^p(\mathbb{R}^n)$, $p \in (1,2]$, *then*

$$\int_{\mathbb{R}^n} f(x)\hat{g}(x)dx = \int_{\mathbb{R}^n} \hat{f}(x) g(x) dx\,.$$

The same holds for $\overset{\vee}{f}$. Given a several variables function $f(x_1, \ldots, x_n)$, we denote $\mathcal{F}_{x_j}f$ to the operation of taking Fourier transform only with respect to the variable x_j.

2.2. Wavelet transform

We recall that, for suitable wavelet function ψ, eq. 1 defines a bounded linear operator and indeed the following characterization of the L^p spaces in terms of the continuous Wavelet transform holds [10, 16, 22]:

Theorem 2.1. *Let $f \in L^p(\mathbb{R})$, $p \in (1, +\infty)$, then there exists positive constants $c_\psi(p), C_\psi(p)$, only depending on ψ and p, such that:*

$$A_\psi(p)\,\|f\|_{L^p} \leq \left(\int_{\mathbb{R}_{>0}} \left(\int_{\mathbb{R}} |\mathcal{W}f(a,b)|^2 \frac{da}{a}\right)^{\frac{p}{2}} db\right)^{\frac{1}{p}} \leq B_\psi(p)\,\|f\|_{L^p}\ .$$

Some properties of the continuous wavelet transform in L^p spaces are also studied in e.g. [16] for more references. We will assume that the wavelet function $\psi \in L^1(\mathbb{R}^n)\bigcap L^2(\mathbb{R}^n)$ verifies the admissibility condition on ψ [7, 15]: $C_\psi = \int_{[0,\infty)} \frac{|\hat{\psi}(t\lambda)|^2}{t} dt < \infty$ for every $\lambda \in \mathbb{R}^n \setminus \{0\}$. In our discussion, we will use extensively the following:

2.2.1. Fact

If ψ is a wavelet function then $\psi \in L^p(\mathbb{R}^n)$, for all $p \in [1,2]$. Indeed, this is a direct consequence of Hölder's inequality with appropriate indexes: if $p \in (1,2)$ then

$$\|\psi\|_{L^p}^p = \int_{\mathbb{R}^n} |\psi(x)|^{2p-2} |\psi(x)|^{2-p} dx \leq \|\psi\|_{L^2}^{2(p-1)} \|\psi\|_{L^1}^{2-p} < \infty .$$

3. A HEISENBERG TYPE UNCERTAINTY PRINCIPLE FOR THE WAVELET TRANSFORM

First, we have the following claim on the Fourier Transform. The following result, for the case $p = q = 2$ reduces to the classic uncertainty principle.

Lemma 1. *Let $f \in L^q(\mathbb{R})$ and $p \in [2,\infty)$, then if $\frac{1}{p} + \frac{1}{q} = 1$:*

$$\frac{1}{2\pi p B(p,q)} \left\|\hat{f}\right\|_{L^p}^p \leq \left(\int_{\mathbb{R}} |\lambda|^q |\hat{f}(\lambda)|^p d\lambda\right)^{\frac{1}{q}} \left(\int_{\mathbb{R}} |x|^q |f(x)|^q dx\right)^{\frac{1}{q}} ,$$

where $B(p,q) = \left(\frac{p^{1/p}}{q^{1/q}}\right)^{\frac{1}{2}}$ (the constant of the Babenko-Beckner inequality). If $p \neq 2$, equality is attained if and only if $f = 0$ a.e.

Proof. In case of $\int_{\mathbb{R}} |x|^q |f(x)|^q dx = \infty$ the result is trivial. Otherwise if $f(x)$ and $xf(x) \in L^q(\mathbb{R})$ then the distributional derivative of $\hat{f}$, $(\hat{f})'$ is a function in $L^p(\mathbb{R})$ and moreover $\hat{f}$ coincides a.e. with an absolutely continuous function with derivative $(\hat{f})'$. If $\hat{f}$ is absolutely continuous then, over any finite interval $[-N,N]$, the same holds for $|\hat{f}|^p$, and $(|\hat{f}|^p)' = \frac{p}{2}(|\hat{f}|^2)^{\frac{p}{2}-1} 2\mathbb{R}e(\bar{\hat{f}}(\hat{f})')$. Therefore the integration by parts formula for absolutely continuous functions [23] yields

$$\int_{\mathbb{R}} |\hat{f}(\lambda)|^p d\lambda = \lim_{N\longrightarrow\infty} N|\hat{f}(N)|^p + N|\hat{f}(-N)|^p - \int_{-N}^{N} p\lambda\, (|\hat{f}|^2)^{\frac{p}{2}-1}\, \mathbb{R}e(\bar{\hat{f}}(\hat{f})')\, d\lambda\,. \tag{3}$$

If $\liminf_{N\longrightarrow\infty} N|\hat{f}(N)|^p > 0$ then $|\hat{f}(\lambda)|^p \sim \frac{C}{\lambda}$, which is a contradiction since $\hat{f} \in L^p(\mathbb{R})$. So, by Hölder's inequality eq. 3 becomes

$$\begin{aligned} \int_{\mathbb{R}} |\hat{f}(\lambda)|^p d\lambda &\leq \left| \int_{\mathbb{R}} p\lambda\, (|\hat{f}|^2)^{\frac{p}{2}-1}\, \bar{\hat{f}}(\hat{f})'\, d\lambda \right| \\ &\leq p \left(\int_{\mathbb{R}} |\lambda|^q |\hat{f}(\lambda)|^{q(p-1)} d\lambda \right)^{\frac{1}{q}} \left(\int_{\mathbb{R}} |(\hat{f})'(\lambda)|^p d\lambda \right)^{\frac{1}{p}}, \end{aligned} \tag{4}$$

with $\frac{1}{p} + \frac{1}{q} = 1$ by Hölder's inequality. But $q(p-1) = p$, thus:

$$\int_{\mathbb{R}} |\hat{f}(\lambda)|^p d\lambda \leq \left(\int_{\mathbb{R}} |\lambda|^q |\hat{f}(\lambda)|^p d\lambda \right)^{\frac{1}{q}} \left(\int_{\mathbb{R}} |(\hat{f})'(\lambda)|^p d\lambda \right)^{\frac{1}{p}}$$

Recalling the formula for the derivative of the Fourier transform [7]:

$$|\mathcal{F}^{-1}((\hat{f})')(x)| = 2\pi\, |x||f(x)|\,,$$

and the Babenko-Beckner inequality (eq.2 section 2):

$$\int_{\mathbb{R}} |\hat{f}(\lambda)|^p d\lambda \leq \left(\int_{\mathbb{R}} |\lambda|^q |\hat{f}(\lambda)|^p d\lambda \right)^{\frac{1}{q}} 2\pi B(p,q) \left(\int_{\mathbb{R}} |x|^q |f(x)|^q d\lambda \right)^{\frac{1}{q}}\,. \tag{5}$$

Finally, if equality holds then the same holds for eq.4. But this is true if $\hat{f}$ verifies

$$(\hat{f}(\lambda))' = \alpha\lambda|\lambda|^{p-2}\hat{f}|\hat{f}|^{p(p-2)}\,,$$

but the non null solutions of this differential equation belongs to $L^p(\mathbb{R})$ if and only if $p = 2$. □

3.1. Application to the wavelet transform

We shall analyse the concentration of $\mathcal{W}f$. We will compare the localization of f (resp. $\hat{f}$) with the localization of its wavelet transform proving the following Heisenberg type uncertainty principle for the L^p Wavelet transform (in the variable b):

Theorem 3.1. *Let $f \in L^p(\mathbb{R})$, $p \in [2, +\infty)$ and $\hat{f} \in L^q(\mathbb{R})$ with $\frac{1}{p} + \frac{1}{q} = 1$, then:*

$$\frac{A_\psi^p(p)\, \|f\|_{L^p}^p}{(B(p,q)\, C_\psi 2\pi p)^{\frac{1}{q}}} \leq \left\|\lambda \hat{f}\right\|_{L^q} \left(\int_{\mathbb{R}_{>0}} \left(\int_{\mathbb{R}} |b|^q |\mathcal{W}f(a,b)|^p db\right)^{\frac{2}{p}} \frac{da}{a}\right)^{\frac{p-1}{2}}.$$

Remark

The constant $A_\psi(p)$ is the same of theorem 2.1. However, theorem 2.1 shows that the L^p-norm is essentially equivalent to the inner integral of this double sided inequality. This result, if $p = 2$, reduces to one obtained in [6].

Proof. Recalling the before mentioned fact 2.2.1, is immediate that $\psi \in L^q(\mathbb{R})$ with $q \in (1, 2]$, and as f is a real function $\check{f} = \bar{\hat{f}} \in L^q(\mathbb{R})$, thus by proposition 1, and since $f = \hat{\check{f}}$, eq. 1 can be written as:

$$\begin{aligned} \mathcal{W}f(a,b) &= \frac{1}{a}\int_{\mathbb{R}} \psi\left(\frac{x-b}{a}\right) f(x)dx = \frac{1}{a}\int_{\mathbb{R}} \psi\left(\frac{x-b}{a}\right) \hat{\check{f}}(x)dx \\ &= \int_{\mathbb{R}} \hat{\psi}\,(a\lambda)\bar{\hat{f}}(\lambda) e^{-i2\pi\lambda b} d\lambda\,. \end{aligned} \tag{6}$$

thus, if $\frac{1}{p}+\frac{1}{q}=1$, by lemma 1 :

$$\int_{\mathbb{R}} |\mathcal{W}f(a,b)|^p db$$
$$\leq 2\pi\, p\, B(p,q) \left(\int_{\mathbb{R}} |b|^q |\mathcal{W}f(a,b)|^p db\right)^{\frac{1}{q}} \left(\int_{\mathbb{R}} |\lambda|^q |\hat{\psi}\,(a\lambda)\bar{\hat{f}}(\lambda)|^q d\lambda\right)^{\frac{1}{q}},$$

therefore:

$$\begin{aligned}
I_1 &= \int_{\mathbb{R}_{>0}} \left(\int_{\mathbb{R}} |\mathcal{W}f(a,b)|^p db\right)^{\frac{2}{p}} \frac{da}{a} \\
&\leq \int_{\mathbb{R}_{>0}} \left(2\pi\, pB(p,q) \left(\int_{\mathbb{R}} |b|^q |\mathcal{W}f(a,b)|^p db\right)^{\frac{1}{q}}\right)^{\frac{2}{p}} \\
&\quad \times \left(\left(\int_{\mathbb{R}} |\lambda|^q |\hat{\psi}\,(a\lambda)\bar{\hat{f}}(\lambda)|^q d\lambda\right)^{\frac{1}{q}}\right)^{\frac{2}{p}} \frac{da}{a},
\end{aligned}$$

then by Hölder's inequality:

$$\begin{aligned}
\frac{I_1}{(2\pi\, p\, B(p,q))^{\frac{2}{p}}} &\leq \left(\int_{\mathbb{R}_{>0}} \left(\int_{\mathbb{R}} |b|^q |\mathcal{W}f(a,b)|^p db\right)^{\frac{2}{p}} \frac{da}{a}\right)^{\frac{1}{q}} \\
&\quad \times \left(\int_{\mathbb{R}_{>0}} \left(\int_{\mathbb{R}} |\lambda|^q |\hat{\psi}\,(a\lambda)\bar{\hat{f}}(\lambda)|^q d\lambda\right)^{\frac{2}{q}} \frac{da}{a}\right)^{\frac{1}{p}} \\
&= I_2 I_3,
\end{aligned}$$

But, since $q \leq 2$ then $\frac{2}{q} \geq 1$ and therefore by Minkowski's integral inequality we get that:

$$\begin{aligned}
I_3 &\leq \left(\int_{\mathbb{R}} \left(\int_{\mathbb{R}_{>0}} \left(|\lambda|^q |\hat{\psi}\,(a\lambda)\bar{\hat{f}}(\lambda)|^q\right)^{\frac{2}{q}} \frac{da}{a}\right)^{\frac{q}{2}} d\lambda\right)^{\frac{2}{qp}} \\
&= \left(\int_{\mathbb{R}} |\lambda|^q |\hat{f}(\lambda)|^q \left(\int_{\mathbb{R}_{>0}} |\hat{\psi}\,(a\lambda)\,|^2 \frac{da}{a}\right)^{\frac{q}{2}} d\lambda\right)^{\frac{2}{qp}} \leq C_\psi^{\frac{1}{p}} \left(\int_{\mathbb{R}} |\lambda|^q |\hat{f}(\lambda)|^q d\lambda\right)^{\frac{2}{qp}}.
\end{aligned}$$

Finally, as $p \geq 2$, again by Minkowski's inequality:

$$I_1 \geq \left(\int_{\mathbb{R}} \left(\int_{\mathbb{R}_{>0}} |\mathcal{W}f(a,b)|^2 \frac{da}{a}\right)^{\frac{p}{2}} db\right)^{\frac{2}{p}}$$

Combining these bounds on I_1, I_3, and theorem 2.1 we get the desired result. □

4. ON THE SIZE OF SUBSETS OF THE TIME-SCALE PLANE

Denote the Lebesgue measure of a measurable subset A: $|A|$, and for a measurable function f define $\mathbf{C}(f) := \{f \neq 0\}$. Note that this subset differs with the usual definition of the support of a function, and as we are dealing with $f \in L^p(\mathbb{R}^n)$ which consists of equivalence classes of functions this definition may lead to apparent confusions. However as we shall actually only consider the measure of these sets any contradiction is avoided. In practice it is important to compare the time concentration versus the bandwidth of a signal [3]. For example, this can be done by comparison of the size of the supports of f and $\hat{f}$ or other time-frequency representation. Next we give an uncertainty principle for the continuous wavelet transform. We shall prove that the scale sections of the support of the wavelet transform of a non null L^p-function cannot have finite Lebesgue measure, and in particular that it cannot have finite Lebesgue measure in the time-scale space (corollary 1). For this result we will assume that the measure of the support of $\hat{\psi}$ is finite (but not necessarily band limited). In [6] a similar result is proved for $p = 2$. However that proof which does not need $\hat{\psi}$ to be supported on a set of finite measure, relies heavily on Hilbert space methods, so it cannot be directly modified for the case $p \neq 2$.

Theorem 4.1. *Let $f \in L^p(\mathbb{R}^n)$, $p \in (1, 2]$, and ψ an admissible wavelet such that $|\mathbf{C}(\hat{\psi})| < \infty$. If for almost all $a \in \mathbb{R}_{>0}$:*

$$|S_a|_{\mathbb{R}^n} = \left|\left\{b \in \mathbb{R}^n \ : \ |\mathcal{W}f(a, b)| > 0\right\}\right|_{\mathbb{R}^n} < \infty\,,$$

then $f = 0$ *a.e.*

We remark that the result also holds with the L^2 normalization: $\mathcal{W}f(a,b) = \frac{1}{\sqrt{a^n}} \int_{\mathbb{R}^n} \psi\left(\frac{x-b}{a}\right) f(x)dx$.

Proof. (Of theorem 4.1) Recalling proposition 1 we can take in account again equation 6 (now in $\mathbb{R}^n$), then there exists $A \subset \mathbb{R}_{>0}$ such that $|A^c \cap \mathbb{R}_{>0}| = 0$ such that for all $a \in A$, $|S_a|_{\mathbb{R}^n} < \infty$, and on the other hand:

$$\mathcal{F}_b^{-1}(\mathcal{W}f(a,\,.))(\lambda) = \hat{\psi}(a\lambda)\bar{\hat{f}}(\lambda)\,, \forall a \in A.$$

Then for each $a \in A$:

$$\begin{aligned} &\left|\left\{\lambda \in \mathbb{R} \,:\, |\hat{\psi}(a\lambda)\hat{f}(\lambda)| > 0\right\}\right| \leq \left|\left\{\lambda \in \mathbb{R} \,:\, |\hat{\psi}(a\lambda)| > 0\right\}\right| \\ &= \frac{1}{a^n}\left|\left\{\lambda \in \mathbb{R} \,:\, |\hat{\psi}(\lambda)| > 0\right\}\right| < \infty\,. \end{aligned}$$

Now, let us verify that $\hat{\psi}(a\,.)\bar{\hat{f}}(\,.\,) \in L^1(\mathbb{R}^n)$. Recalling that $\psi \in L^2(\mathbb{R}^n)$, thus by Hölder's inequality:

$$\int_{\mathbb{R}^n} |\hat{\psi}(\lambda)|^p d\lambda \leq \left(\int_{\mathbf{C}(\hat{\psi})} |\hat{\psi}(\lambda)|^2 d\lambda\right)^{\frac{p}{2}} |\mathbf{C}(\hat{\psi})|^{1-\frac{p}{2}}\,.$$

Thus, again by Hölder's inequality and the Hausdorff-Young theorem:

$$\int_{\mathbb{R}^n} |\hat{\psi}(a\lambda)\bar{\hat{f}}(\lambda)| d\lambda \leq \frac{1}{a^{\frac{n}{p}}} \left\|\hat{\psi}\right\|_{L^p} \left\|\hat{f}\right\|_{L^q} \leq \frac{1}{a^{\frac{n}{p}}} \left\|\hat{\psi}\right\|_{L^p} \|f\|_{L^p} < \infty\,,$$

with $\frac{1}{q} = 1 - \frac{1}{p}$. Then by theorem 1.1, we have that if $a \in A$:

$$\mathcal{W}f(a,b) = 0\,,$$

for almost all $b \in \mathbb{R}^n$. Now, by Fubini's theorem:

$$\begin{aligned} &\left|\left\{(a,b) \in \mathbb{R}_{>0} \times \mathbb{R}^n \ : \ |\mathcal{W}f(a,b)| > 0\right\}\right|_{\mathbb{R}\times\mathbb{R}^n} \\ &\quad = \int_A \left|\left\{b \in \mathbb{R}^n \ : \ |\mathcal{W}f(a,b)| > 0\right\}\right|_{\mathbb{R}^n} da = 0\,. \end{aligned}$$

Thus $\mathcal{W}f = 0$ a.e. and then $f = 0$ a.e. □

From this result one obtains immediately:

Corollary 1. *Let $f \in L^p(\mathbb{R}^n)$, $p \in (1,2]$, and ψ an admissible wavelet such that $|\mathbf{C}(\hat{\psi})| < \infty$. If $|\{(a,b) \in \mathbb{R}_{>0} \times \mathbb{R}^n \ : \ |\mathcal{W}f(a,b)| > 0\}|_{\mathbb{R}\times\mathbb{R}^n} < \infty$, then $f = 0$ a.e.*

In theorem 4.1 we gave a result on the size of the support of the wavelet transform, about the impossibility of an instantaneous time-scale description of a non-null function. On the other hand, in a similar way to STFT, we can see briefly that a function cannot be concentrated on small sets in the time-scale plane. We shall find a lower bound for the size, in terms of the Lebesgue measure, of a non null L^2 function:

Theorem 4.2. *Let $\|f\|_{L^2} = 1$, $1 > \epsilon \geq 0$ and $U \subset \mathbb{R}_{>0} \times \mathbb{R}^n$ a measurable set such that:*

$$\frac{1}{\|\psi\|^2_{L^2}} \int\int_U |\mathcal{W}f(a,b)|^2 a^n da db \geq 1 - \epsilon\,, \tag{7}$$

then $|U| \geq 1 - \epsilon$.

Remark

a similar statement holds for the L^2-normalization with eq. 7 replaced by $\frac{1}{\|\psi\|_{L^2}^2}\int\int_U |\mathcal{W}f(a,b)|^2 dadb$.

Proof. From eq. 6 and the Cauchy-Schwartz inequality one immediately obtains:

$$|\mathcal{W}f(a,b)| \leq \left\|\hat{f}\right\|_{L^2} \left\|\hat{\psi}(a\,.\,)\right\|_{L^2} = \frac{\left\|\hat{\psi}\right\|_{L^2}}{\sqrt{a^n}}\,.$$

Then:

$$1-\epsilon \leq \frac{1}{\|\psi\|_{L^2}^2}\int\int_U |\mathcal{W}f(a,b)|^2 a^n dadb \leq \frac{1}{\|\psi\|_{L^2}^2}\int\int_U \left|\frac{\left\|\hat{\psi}\right\|_{L^2}}{\sqrt{a^n}}\right|^2 a^n dadb = |U|\,.$$

□

4.1. On the continuous wavelet transform of Band-Limited functions

Now, in contrast to time localization, we describe briefly some properties of functions whose Fourier transforms are supported on a finite interval.

The main result of this section is:

Theorem 4.3. *Let ψ be a bandpass wavelet, $f \in L^p(\mathbb{R})$, $p \in (1,2]$, $f \neq 0$ and $\mathcal{W}f \in C(\mathbb{R}_{>0}\times\mathbb{R})$. If*

$$\alpha = \sup\left\{x > 0 \,:\, \hat{\psi}(\lambda) = 0\,,\, \forall\,\lambda \in [-x\,,x]\right\},$$

then the following assertions are equivalent:

(1) $\hat{f} \in L^q(\mathbb{R})$ *has compact support,* $\frac{1}{p} + \frac{1}{q} = 1$.

(2) $\exists\, R > 0$ *such that* $\mathcal{W}f(a,b) = 0$ *for every* $a \leq R$ *and all* $b \in \mathbb{R}$. *Moreover if such* R *exists then*

$$\frac{\alpha}{\sigma} = \sup\{R > 0 \ : \ |\mathcal{W}f(a,b)| = 0, \ \ \forall\, (a,b) \in (0,R] \times \mathbb{R}\ \},$$

where $\sigma = \inf\left\{x \in \mathbb{R} \ : \ \hat{f} = 0 \text{ a.e. in } [-x,x]^c\right\}$.

The continuity condition of $\mathcal{W}f$ may look rather restrictive, however this can achieved imposing relatively mild decay conditions on ψ. The following proposition gives sufficient conditions for the continuity of $\mathcal{W}f$.

Proposition 2. *Let* ψ *be an admissible wavelet and a function* η*, such that* $|\hat{\psi}| \leq \eta \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$*, which is an even function that verifies* $\eta(x_1) \leq \eta(x_2)$*, if* $0 \leq x_2 \leq x_1$*. If* $f \in L^p(\mathbb{R})$*,* $p \in (1,2]$*, then* $\mathcal{W}f$*, given by eq. 1, defines a continuous function of the variables* $(a,b) \in \mathbb{R}_{>0} \times \mathbb{R}$.

Proof. To prove the continuity on the variables (a,b): Let $|h| < 1$ and $|k| < 1$, recalling the definition of $\mathcal{W}f(a,b)$ (eq. 1), and by proposition 1, since from the hypothesis $f \in L^p(\mathbb{R})$ and

$\hat{\psi} \in L^p(\mathbb{R})$, we can consider again eq. 6. Indeed: $\left\|\hat{\psi}\right\|_{L^p} \leq \left\|\hat{\psi}\right\|_{L^\infty}^{1-\frac{1}{p}} \left\|\hat{\psi}\right\|_{L^1}^{\frac{1}{p}} < \infty$, and thus:

$$
\begin{aligned}
&|\mathcal{W}f(a+h,b+k) - \mathcal{W}f(a,b)| \\
&= \left| \int_{\mathbb{R}} (\hat{\psi}(a\lambda) - \hat{\psi}((a+h)\lambda) e^{-i2\pi k\lambda}) \bar{\hat{f}}(\lambda)\, d\lambda \right| \\
&\leq \int_{\mathbb{R}} \left| \hat{\psi}(a\lambda) - \hat{\psi}((a+h)\lambda) e^{-i2\pi k\lambda} \right| \left| \hat{f}(\lambda) \right| d\lambda \\
&\leq \|f\|_{L^p} \left(\int_{\mathbb{R}} \left| \hat{\psi}(a\lambda) - \hat{\psi}((a+h)\lambda) e^{-i2\pi k\lambda} \right|^p d\lambda \right)^{\frac{1}{p}}
\end{aligned} \tag{8}
$$

the last inequality is a consequence of Hölder's inequality and the Hausdorff-Young theorem [22] combined on f. Now, the result follows from Lebesgue's dominated convergence theorem and the continuity of $\hat{\psi}$, since $\psi \in L^1(\mathbb{R})$:

$$
\begin{aligned}
\left| \hat{\psi}(a\lambda) - \hat{\psi}((a+h)\lambda) e^{-i2\pi k\lambda} \right|^p &\leq 2^p \left| \hat{\psi}(a\lambda) \right|^p + 2^p \left| \hat{\psi}((a+h)\lambda) \right|^p \\
&\leq 2^{p+1} |\eta(a\lambda)|^p
\end{aligned}
$$

and from this eq. 8 tends to zero . □

Now, one can prove the main result:

Proof. (Of Theorem 4.3) (1) $\Longrightarrow$(2)

Again, recall eq. 6 and write for all $b \in \mathbb{R}$:

$$
\mathcal{W}f(a,b) = \int_{\mathbb{R}} \frac{1}{a} f(x) \psi\left(\frac{x-b}{a}\right) dx = \int_{\mathbb{R}} \hat{\psi}(a\lambda) \bar{\hat{f}}(\lambda)\, e^{-i2\pi b\lambda} d\lambda\,. \tag{9}
$$

Again, these equalities follow from proposition 1, (with the inverse Fourier transform $\mathcal{F}^{-1}$) since $f, \hat{\psi} \in L^p(\mathbb{R})$ and $\psi = (\check{\hat{\psi}})$. Fix $R = \frac{\alpha}{\sigma}$, from eq. 9 it suffices to prove that $|\mathbf{C}(\hat{f}) \cap \mathbf{C}(\psi(\hat{a.}))| = 0$, this follows from the fact, $\hat{\psi}(a\lambda) = 0$ for all $\lambda \leq \sigma$ if $a \leq R$, and that $|\mathbf{C}(\hat{f}) \triangle [-\sigma, \sigma]| = 0$.

(2) $\Longrightarrow$) (1)

Now, from equation 9 we get that there exists $R > 0$ such that for each $a \in (0, R]$, for all $b \in \mathbb{R}$:

$$\mathcal{W}f(a,b) = \int_{\mathbb{R}} \hat{\psi}(a\,\lambda)\bar{\hat{f}}(\lambda)\, e^{-i2\pi\, b\,\lambda} d\lambda = 0\,. \tag{10}$$

From the Hausdorff-Young theorem we have that $\hat{f} \in L^q(\mathbb{R})$, then as $\hat{\psi} \in L^p(\mathbb{R})$ we have $\hat{\psi}(a.)\bar{\hat{f}} \in L^1(\mathbb{R})$, but if the Fourier transform of this L^1 function vanishes everywhere, then for $a \in (0, R]$:

$$\hat{\psi}(a\,\lambda)\bar{\hat{f}}(\lambda) = 0 \textit{ for almost every } \lambda \in \mathbb{R}\,. \tag{11}$$

Now define $E_a = \{\lambda : \ \hat{\psi}(a\,\lambda)\bar{\hat{f}}(\lambda) \neq 0\}$, and taking in account that $\hat{\psi} \in C(\mathbb{R})$, since $\psi \in L^1(\mathbb{R})$, then there exists $\delta > 0$ such that $\hat{\psi}(\lambda) \neq 0$ for all $\lambda \in (\alpha, \alpha + \delta)$, and thus $\hat{\psi}(a\,\lambda) \neq 0$ for all $\lambda \in \left(\frac{\alpha}{a}, \frac{\alpha+\delta}{a}\right)$. From all these facts, we can write:

$$\left\{\lambda : \hat{f}(\lambda) \neq 0,\ |\lambda| > \frac{\alpha}{R}\right\} = \bigcup_{a \in \mathbb{Q}\cap(0,R]} \left\{\lambda : \hat{f}(\lambda) \neq 0,\ |\lambda| \in \left(\frac{\alpha}{a}, \frac{\alpha+\delta}{a}\right)\right\} \subseteq \bigcup_{a \in \mathbb{Q}\cap(0,R]} E_a\,,$$

and since $|E_a| = 0$, then $\left|\left\{\lambda : \hat{f}(\lambda) \neq 0,\ |\lambda| > \frac{\alpha}{R}\right\}\right| = 0$.

Let us prove that,

$$\frac{\alpha}{\sigma} = \sup\{x \in \mathbb{R}_{>0} \,:\, |\mathcal{W}f(a,b)| = 0 \quad \forall\,(a,b) \in (0,x] \times \mathbb{R}\,\}.$$

Note that $\mathbf{C}(\mathcal{W}f) = \{(a,b) \in \mathbb{R}_{>0} \times \mathbb{R} \,:\, |\mathcal{W}f(a,b)| > 0\}$ is an open set in $\mathbb{R}_{>0} \times \mathbb{R}$ since $\mathcal{W}f(a,b) : \mathbb{R}_{>0} \times \mathbb{R} \to \mathbb{C}$ is a continuous function (proposition 2). Defining $I_n = \left(\frac{\alpha}{\sigma}, \frac{\alpha}{\sigma} + \frac{1}{n}\right)$ it suffices to show that for all $n \in \mathbb{N}$ there exists $a \in I_n$, $b \in \mathbb{R}$, $U \subset \mathbb{R}_{>0} \times \mathbb{R}$ open sets such that $(a,b) \in U \subset \mathbf{C}(\mathcal{W}f)$. On the other hand I_n is open and then $I_n \times \mathbb{R}$ is open, and since $\mathbf{C}(\mathcal{W}f)$ is open, thus we need to prove that $I_n \times \mathbb{R} \cap \mathbf{C}(\mathcal{W}f) \neq \oslash$. In order to prove this fact, suppose the contrary. This means that for some $n_0 \in \mathbb{N}$: $\mathcal{W}f(a,b) = 0$ for all $(a,b) \in I_{n_0} \times \mathbb{R}$. Using a similar argument to that of eq. 11, given $a \in I_{n_0}$:

$$\hat{\psi}(a\lambda)\hat{f}(\lambda) = 0 \textit{ for almost every } \lambda \in \mathbb{R} \tag{12}$$

If $\alpha = \sup\left\{x > 0 \,:\, \hat{\psi}(\lambda) = 0,\ \forall\,\lambda \in [-x,x]\right\}$, and since $\psi \in L^1(\mathbb{R})$ is a real function then $\hat{\psi}$ is a continuous even function, $\mathbf{C}\hat{\psi}$ is an open set and $\alpha = \inf(\mathbf{C}(\hat{\psi}) \cap \mathbb{R}_{>0})$. Then for every $n \in \mathbb{N}$ there exists α_n such that $\hat{\psi}(\alpha_n) \neq 0$ and

$$\alpha_n - \frac{\sigma}{n} < \alpha \leq \alpha_n\,. \tag{13}$$

Now taking $n = n_0$ from equation (11) we have $\hat{f}(\lambda) = 0$ a.e. in $\left(\frac{\alpha_n \sigma}{\alpha + \frac{\sigma}{n}}, \frac{\alpha_n \sigma}{\alpha}\right)$. Since $\alpha_n \geq \alpha \Rightarrow \frac{\alpha_n \sigma}{\alpha} \geq \sigma$, and from equation (13) we get $\alpha_n < \alpha + \frac{\sigma}{n}$ and from this $\frac{\alpha_n \sigma}{\alpha + \frac{\sigma}{n}} < \sigma$, on the other hand this implies $\inf\left\{x \in \mathbb{R} \,:\, \hat{f} = 0 \textit{ a.e. in } [-x,x]^c\right\} < \sigma$ which is a contradiction. On the other hand is straightforward to see that $\hat{\psi}(y\lambda)\hat{f}(\lambda) = 0$ for almost every λ, with $y \leq \frac{\alpha}{\sigma}$ and from (10) we get $\mathcal{W}f(y,b) = 0$ for all $b \in \mathbb{R}$ and $y \leq \frac{\alpha}{\sigma}$. □

5. CONCLUSIONS

We characterized some scale-time localization principles for the continuous wavelet transform. These are described as norm inequalities or as qualitative principles related to the size or measure of sets in the time-scale plane.

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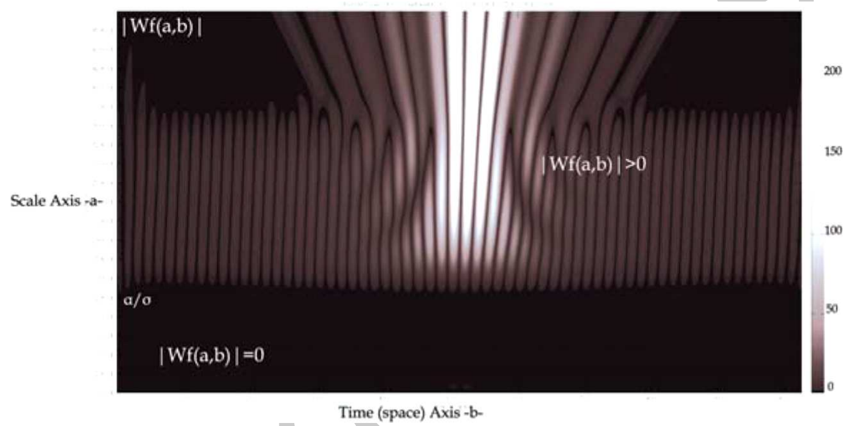

Figure 1: Absolute value of the (Meyer) wavelet transform of the band-limited function $f(x) = 100\frac{sin(10x)}{x}$.