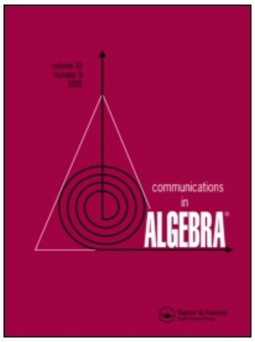

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Cleft extensions of a family of braided Hopf algebras

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ABSTRACT
We introduce a family of braided Hopf algebras that (in characteristic zero) generalizes the rank 1 Hopf algebras introduced by Krop and Radford and we study its cleft extensions.

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1. Introduction

Let k be an arbitrary field. In [4], the authors associated a Hopf algebra $H_{\mathcal{D}}$ over k, with each data $\mathcal{D} := (G, \chi, z, \lambda)$, consisting of:
– a finite group G,
– a central element z of G,
– a character χ of G with values in k such that $\chi(z) \neq 1$,
– an element $\lambda \in k$,
such that $\chi^n = 1$ or $\lambda(z^n - 1_G) = 0$, where n is the order of $\chi(z)$. As an algebra $H_{\mathcal{D}}$ is generated by G and x, subject to the group relations of G,

$$xg = \chi(g)gx \qquad \text{for all } g \in G \qquad \text{and} \qquad x^n = \lambda(z^n - 1_G).$$

The coalgebra structure of $H_{\mathcal{D}}$ is determined by

$$\Delta(g) := g \otimes g \quad \text{for } g \in G, \qquad \Delta(x) := 1 \otimes x + x \otimes z,$$
$$\epsilon(g) := 1 \quad \text{for } g \in G, \qquad \epsilon(x) := 0,$$

and its antipode is given by

$$S(g) = g^{-1} \qquad \text{and} \qquad S(x) = -xz^{-1}.$$

As was point out in [4], as a vector space $H_{\mathcal{D}}$ has basis $\{gx^m | g \in G, 0 \leq m < n\}$. Consequently $\dim H_{\mathcal{D}} = n|G|$. The simplest examples are the Taft algebras H_{n^2}, which are the rank 1 Hopf algebra obtained taking $\mathcal{D} := (C_n, \chi, g, 1)$, where $C_n = \langle g \rangle$ is the cyclic group of order $n > 1$ and $\chi(g)$ is a primitive n-th root of 1.

Also in [4] the authors introduced the concept of rank 1 Hopf algebras (and more generally of rank n Hopf algebras for $n \in \mathbb{N}$) and they prove that when k is an algebraically closed characteristic zero field,

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rank 1 Hopf algebras are the algebras $H_{\mathcal{D}}$. In positive characteristic the Hopf algebras $H_{\mathcal{D}}$ also have rank 1 but they are not all (for the classification see [6])

In [5] Masuoka studied the cleft extensions of the Taft algebras, giving a very elegant description of its crossed product systems. In [2] the Masuoka description was reproduced with simplified proofs, and studying this description were derived several interesting consequences. Motivated by these works, in this paper we introduce a family of braided Hopf algebras that generalize the algebras $H_{\mathcal{D}}$, and we study their cleft extensions. Although the results we get are valid in this broader context we are particularly interested in the case of the algebras $H_{\mathcal{D}}$.

The paper is organized in the following way:

In Section 2 we recall the well known notions of Gaussian binomial coefficients and braided Hopf algebra, and we make a quick review of some results obtained in [3]. The unique new result in this section is the formulas obtained in Theorem 2.36(5). In Section 3 we associate a braided Hopf algebra with each data $\mathcal{D} = (G, \chi, z, \lambda, q)$ (where G is a finite group, χ is a character of G, z is a central element of G and q, λ are scalars) that satisfy suitable conditions. The main result is Corollary 3.8, in which the algebras $H_{\mathcal{D}}$ are introduced. When $q = 1$, we recover the rank 1 Hopf algebras defined by Krop y Radford. In Section 4 we describe the $H_{\mathcal{D}}$-space structures, the $H_{\mathcal{D}}$-comodule structures and the $H_{\mathcal{D}}$-comodule algebra structures (in Proposition 4.5, Corollary 4.7 and Theorem 4.12, respectively). In Section 5 we characterize the $H_{\mathcal{D}}$ cleft extensions, and, finally, in Section 5 we study two particular examples.

2. Preliminaries

In this paper k is an arbitrary field, we work in the category of k-vector spaces, and consequently all the maps are k-linear maps. Moreover we let $U \otimes V$ denote the tensor product $U \otimes_k V$ of each pair of vector spaces U and V. We assume that the algebras are associative unitary and the coalgebras are coassociative counitary. For an algebra A and a coalgebra C, we let $\mu\colon A\otimes A \to A$, $\eta\colon k \to A$, $\Delta\colon C \to C\otimes C$ and $\eta\colon C \to k$ denote the multiplication map, the unit, the comultiplication map and the counit, respectively, specified with a subscript if necessary.

2.1. Gaussian binomial coefficients

Let q be a variable. For any $j \in \mathbb{N}_0$ set

$$(j)_q := \sum_{i=0}^{j-1} q^i = \frac{q^j - 1}{q-1} \qquad \text{and} \qquad (j)!_q := (1)_q(2)_q\cdots(j)_q = \frac{(q-1)(q^2-1)\cdots(q^j-1)}{(q-1)^j}.$$

The Gauss binomials are the rational functions in q defined by

$$\binom{i}{j}_q := \frac{(i)!_q}{(j)!_q(i-j)!_q} \qquad \text{for } 0 \le j \le i. \tag{2.1}$$

A direct computation shows that

$$\binom{r}{0}_q = \binom{i}{i}_q = 1 \qquad \text{and} \qquad \binom{i}{j}_q = q^{i-j}\binom{i-1}{j-1}_q + \binom{i-1}{j}_q \qquad \text{for } 0 < j < i.$$

From these equalities it follows easily that the Gauss binomials are actually polynomials. The Gauss binomials can be evaluated in arbitrary elements of k, but the equality (2.1) only makes sense for $q = 1$ and for $q \neq 1$ such that $q^l \neq 1$ for all $l \le \max(j, i-j)$. We will need the following well known result (q-binomial formula). Let B be a k-algebra and $q \in k$. If $x, y \in B$ satisfy $yx = qxy$, then

$$(x+y)^i = \sum_{j=0}^{i} \binom{i}{j}_q x^j y^{i-j} \qquad \text{for each } i \ge 0. \tag{2.2}$$

Let $i, j \geq 0$ and let $0 \leq l \leq i+j$. Using the equality (2.2) to compute $(x+y)^i(x+y)^j$ in two different ways and comparing coefficients we obtain that

$$\binom{i+j}{l}_q = \sum_{\substack{0\leq s\leq i\\ 0\leq t\leq j\\ s+t=l}} q^{(i-s)t}\binom{i}{s}_q\binom{j}{t}_q. \tag{2.3}$$

2.2. Braided Hopf algebras

Let V, W be vector spaces and let $c\colon V\otimes W\to W\otimes V$ be a map. Recall that:

– If V is an algebra, then c is compatible with the algebra structure of V if

$$c\circ(\eta\otimes W) = W\otimes\eta \qquad\text{and}\qquad c\circ(\mu\otimes W) = (W\otimes\mu)\circ(c\otimes V)\circ(V\otimes c).$$

– If V is a coalgebra, then c is compatible with the coalgebra structure of V if

$$(W\otimes\epsilon)\circ c = \epsilon\otimes W \qquad\text{and}\qquad (W\otimes\Delta)\circ c = (c\otimes V)\circ(V\otimes c)\circ(\Delta\otimes W).$$

More precisely, the first equality in the first item says that c is *compatible with the unit of* V and the second one says that it is *compatible with the multiplication map of* V, while the first equality in the second item says that c is *compatible with the counit of* V and the second one says that it is *compatible with the comultiplication map of* V. Of course, there are similar compatibilities when W is an algebra or a coalgebra.

Definition 2.1. A *braided bialgebra* is a vector space H endowed with an algebra structure, a coalgebra structure and a braiding operator $c\in \mathrm{Aut}_k(H\otimes H)$, called the *braid* of H, such that c is compatible with the algebra and coalgebra structures of H, η is a coalgebra morphism, ϵ is an algebra morphism and

$$\Delta\circ\mu = (\mu\otimes\mu)\circ(H\otimes c\otimes H)\circ(\Delta\otimes\Delta).$$

Furthermore, if there exists a map $S\colon H\to H$, which is the inverse of the identity map for the convolution product, then we say that H is a *braided Hopf algebra* and we call S the *antipode* of H.

Usually H denotes a braided bialgebra, understanding the structure maps, and c denotes its braid. If necessary, we will write c_H instead of c.

Let A and B be algebras. It is well known that if a map $c\colon B\otimes A\longrightarrow A\otimes B$ is compatible with the algebra structures of A and B, then $A\otimes B$ endowed with the multiplication map

$$\mu := (\mu_A\otimes\mu_B)\circ(A\otimes c\otimes B),$$

is an associative algebra with unit $1\otimes 1$, which is called *the twisted tensor product of* A *with* B *associated with* c and denoted $A\otimes_c B$. Similarly, if C and D are coalgebras and $c\colon C\otimes D\longrightarrow D\otimes C$ is a map compatible with the coalgebra structures of C and D, then $C\otimes D$ endowed with the comultiplication map

$$\Delta := (C\otimes c\otimes D)\circ(\Delta_C\otimes\Delta_D),$$

is a coassociative coalgebra with counit $\epsilon\otimes\epsilon$, which is called *the twisted tensor product of* C *with* D *associated with* c and denoted $C\otimes^c D$.

Remark 2.2. Let H be a vector space which is both an algebra and a coalgebra, and let

$$c\colon H\otimes H\longrightarrow H\otimes H$$

be a braiding operator. Assume that c is compatible with the algebra and the coalgebra structures of H. Then H is a braided bialgebra iff its comultiplication map $\Delta\colon H \to H \otimes_c H$ and its counit $\epsilon\colon H \to k$ are algebra maps.

2.3. Left H-spaces, left H-algebras and left H-coalgebras

Definition 2.3. Let H be a braided bialgebra. A *left H-space* (V, s) is a vector space V, endowed with a bijective map $s\colon H \otimes V \longrightarrow V \otimes H$, called the *left transposition of H on V*, which is compatible with the bialgebra structure of H and satisfies

$$(s \otimes H) \circ (H \otimes s) \circ (c \otimes V) = (V \otimes c) \circ (s \otimes H) \circ (H \otimes s)$$

(compatibility of s with the braid). Sometimes, when it is evident, the map s is not explicitly specified. In these cases we will say that V is a left H-braided space in order to point out that there is a left transposition involved in the definition. We adopt a similar convention for all the definitions below. Let (V', s') be another left H-space. A k-linear map $f\colon V \to V'$ is said to be a *morphism of left H-spaces*, from (V, s) to (V', s'), if $(f \otimes H) \circ s = s' \circ (H \otimes f)$.

Remark 2.4. Let $s\colon H \otimes V \longrightarrow V \otimes H$ be a map compatible with the unit, the multiplication map and the braid of H and let $X \subseteq H$ be a set that generates H as an algebra. In order to show that s is a left transposition it suffices to check the compatibility of s with the counit and the comultiplication map of H on simple tensors $h \otimes v$ with $h \in X$ and $v \in V$.

We let $\mathcal{LHB}$ denote the category of all left H-braided spaces. It is easy to check that this is a monoidal category with

- unit (k, τ), where $\tau\colon H \otimes k \to k \otimes H$ is the flip,
- tensor product

  $$(U, s_U) \otimes (V, s_V) := (U \otimes V, s_{U \otimes V}),$$

  where $s_{U\otimes V}$ is the map $s_{U\otimes V} := (U \otimes s_V) \circ (s_U \otimes V)$,
- the usual associativity and unit constraints.

Definition 2.5. A *left H-algebra* (A, s) is an algebra in $\mathcal{LHB}$.

Definition 2.6. A *left transposition of H on an algebra A* is a bijective map $s\colon H \otimes A \longrightarrow A \otimes H$, satisfying
(1) (A, s) is a left H-space,
(2) s is compatible with the algebra structure of A.

Remark 2.7. A left H-algebra is nothing but a pair (A, s) consisting of an algebra A and a left transposition $s\colon H \otimes A \longrightarrow A \otimes H$. Let (A', s') be another left H-algebra. A map $f\colon A \to A'$ is a morphism of left H-algebras, from (A, s) to (A', s'), iff it is a morphism of standard algebras and $(f \otimes H) \circ s = s' \circ (H \otimes f)$.

Definition 2.8. A *left H-coalgebra* (C, s) is a coalgebra in $\mathcal{LHB}$.

Definition 2.9. A *left transposition of H on a coalgebra C* is a bijective map $s\colon H \otimes C \longrightarrow C \otimes H$, satisfying
(1) (C, s) is a left H-space,
(2) s is compatible with the coalgebra structure of C.

Remark 2.10. A left H-coalgebra is nothing but a pair (C, s) consisting of a coalgebra C and a left transposition $s\colon H \otimes C \longrightarrow C \otimes H$. Let (C', s') be another left H-coalgebra. A map $f\colon C \to C'$ is a morphism of left H-coalgebras, from (C, s) to (C', s'), iff it is a morphism of standard coalgebras and $(f \otimes H) \circ s = s' \circ (H \otimes f)$.

Since (H, c) is an algebra and a coalgebra in $\mathcal{LHB}$, it makes sense to consider (H, c)-modules and (H, c)-comodules in this monoidal category.

2.4. Left H-modules and left H-module algebras

Definition 2.11. We will say that (V, s) is a *left H-module* to mean that it is a left (H, c)-module in $\mathcal{LHB}$. Notice that the classical left H-modules can be identified with the left H-modules (V, s) in which s is the flip.

Remark 2.12. A left H-space (V, s) is a left H-module iff V is a standard left H-module and

$$s \circ (H \otimes \rho) = (\rho \otimes H) \circ (H \otimes s) \circ (c \otimes V),$$

where ρ denotes the action of H on V. Let (V', s') be another left H-module. A map $f: V \to V'$ is a *morphism of left H-modules*, from (V, s) to (V', s'), iff it is an H-linear map and the equality $(f \otimes H) \circ s = s' \circ (H \otimes f)$ holds. We let ${}_H\mathcal{LHB}$ denote the category of left H-modules in $\mathcal{LHB}$.

Proposition 2.13 ([3, Proposition 5.6]). *The category ${}_H\mathcal{LHB}$ is monoidal. Its unit is (k, τ) endowed with the trivial left H-module structure, and the tensor product of the left H-modules (U, s_U) and (V, s_V), with actions ρ_U and ρ_V respectively, is the left H-space $(U, s_U) \otimes (V, s_V)$, endowed with the left H-module action given by*

$$\rho_{U \otimes V} := (\rho_U \otimes \rho_V) \circ (H \otimes s_U \otimes V) \circ (\Delta_H \otimes U \otimes V).$$

The associativity and unit constraints are the usual ones.

Definition 2.14. We say that (A, s) is a *left H-module algebra* if it is an algebra in ${}_H\mathcal{LHB}$.

Remark 2.15. (A, s) is a left H-module algebra iff the following facts hold:
(1) A is an algebra and a standard left H-module,
(2) s is a left transposition of H on A,
(3) $s \circ (H \otimes \rho) = (\rho \otimes H) \circ (H \otimes s) \circ (c \otimes A)$,
(4) $\rho \circ (H \otimes \mu_A) = \mu_A \circ (\rho \otimes \rho) \circ (H \otimes s \otimes A) \circ (\Delta_H \otimes A \otimes A)$,
(5) $\rho(h \otimes 1_A) = \epsilon(h) 1_A$ for all $h \in H$,
where ρ denotes the action of H on A.

Let (A', s') be another left H-module algebra. A map $f: A \to A'$ is a *morphism of left H-module algebras*, from (A, s) to (A', s'), iff it is an H-linear morphism of standard algebras that satisfies $(f \otimes H) \circ s = s' \circ (H \otimes f)$.

2.5. Right H-comodules and right H-comodule algebras

Definition 2.16. We will say that (V, s) is a *right H-comodule* if it is a right (H, c)-comodule in $\mathcal{LHB}$.

Remark 2.17. A left H-space (V, s) is a right H-comodule iff V is a standard right H-comodule and

$$(\nu \otimes H) \circ s = (V \otimes c) \circ (s \otimes H) \circ (H \otimes \nu), \tag{2.4}$$

where ν denotes the coaction of H on V. Let (V', s') be another right H-comodule. A map $f: V \to V'$ is a *morphism of right H-comodules*, from (V, s) to (V', s'), iff it is an H-colinear map and $(f \otimes H) \circ s = s' \circ (H \otimes f)$. We let $\mathcal{LHB}^H$ denote the category of right H-comodules in $\mathcal{LHB}$.

Definition 2.18. Let (V, s) be a right H-comodule. An element $v \in V$ is said to be *coinvariant* if $\nu(v) = v \otimes 1_H$.

Remark 2.19. For each right H-comodule (V, s), the set V^{coH}, of coinvariant elements of V, is a vector subspace of V. Furthermore, $s(H \otimes V^{\mathrm{coH}}) = V^{\mathrm{coH}} \otimes H$, and the pair $(V^{\mathrm{coH}}, s_{V^{\mathrm{coH}}})$, where $s_{V^{\mathrm{coH}}}: H \otimes V^{\mathrm{coH}} \to V^{\mathrm{coH}} \otimes H$ is the restriction of s, is a left H-space.

Proposition 2.20 ([3, Proposition 5.2]). *The category $\mathcal{LHB}^H$ is monoidal. Its unit is (k, τ), endowed with the trivial right H-comodule structure, and the tensor product of the right H-comodules (U, s_U) and (V, s_V), with coactions ν_U and ν_V respectively, is $(U, s_U) \otimes (V, s_V)$, endowed with the right H-comodule coaction*

$$\nu_{U\otimes V} := (U \otimes V \otimes \mu_H) \circ (U \otimes s_V \otimes H) \circ (\nu_U \otimes \nu_V).$$

The associativity and unit constraints are the usual ones.

Definition 2.21. We say that (A, s) is a *right H-comodule algebra* if it is an algebra in $\mathcal{LHB}^H$.

Remark 2.22. (A, s) is a right H-comodule algebra iff the following facts hold:
(1) A is an algebra and a standard right H-comodule,
(2) s is a left transposition of H on A,
(3) $(\nu \otimes H) \circ s = (A \otimes c) \circ (s \otimes H) \circ (H \otimes \nu)$,
(4) $\nu \circ \mu_A = (\mu_A \otimes \mu_H) \circ (A \otimes s \otimes H) \circ (\nu \otimes \nu)$,
(5) $\nu(1_A) = 1_A \otimes 1_H$,
where ν denotes the coaction of H on A.

Let (A', s') be another right H-comodule algebra. A map $f: A \to A'$ is a *morphism of right H-comodule algebras*, from (A, s) to (A', s'), iff it is an H-colinear morphism of standard algebras that satisfies $(f \otimes H) \circ s = s' \circ (H \otimes f)$.

Recall that $A \otimes_s H$ denotes the algebra with underlying vector space $A \otimes H$, multiplication map

$$\mu_{A\otimes_s H} := (\mu_A \otimes \mu_H) \circ (A \otimes s \otimes H)$$

and unit $1_A \otimes 1_H$. Conditions (4) and (5) of Remark 2.22 say that $\nu: A \to A \otimes_s H$ is a morphism of algebras.

2.6. Hopf crossed products and H-extensions

Definition 2.23. A left H-space (V, s), endowed with a map $\rho: H \otimes V \to V$, is said to be a *weak left H-module* if
(1) $\rho(1_H \otimes v) = v$, for all $v \in V$,
(2) $s \circ (H \otimes \rho) = (\rho \otimes H) \circ (H \otimes s) \circ (c \otimes V)$.

The category ${}_{wH}\mathcal{LHB}$, of weak left H-modules in $\mathcal{LHB}$, becomes a monoidal category in the same way that ${}_H\mathcal{LHB}$ does. A *weak left H-module algebra* (A, s) is, by definition, an algebra in ${}_{wH}\mathcal{LHB}$.

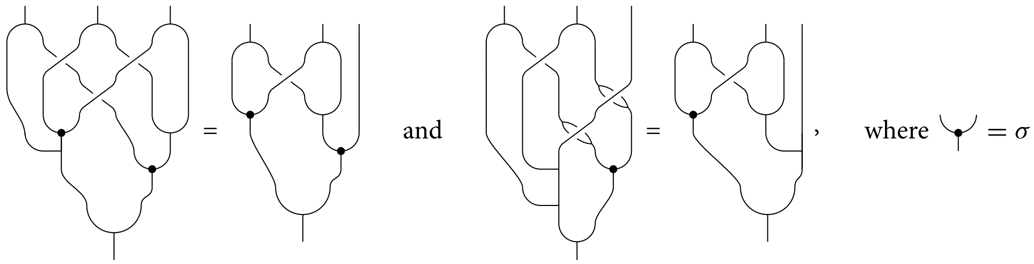
Remark 2.24. (A, s) is a left weak H-module algebra iff A is an usual algebra, s is a left transposition of H on A and the structure map ρ satisfies the following conditions:
(1) $\rho(1_H \otimes a) = a$, for all $a \in A$,
(2) $s \circ (H \otimes \rho) = (\rho \otimes H) \circ (H \otimes s) \circ (c \otimes A)$,
(3) $\rho \circ (H \otimes \mu_A) = \mu_A \circ (\rho \otimes \rho) \circ (H \otimes s \otimes A) \circ (\Delta_H \otimes A \otimes A)$,
(4) $\rho(h \otimes 1_A) = \epsilon(h)1_A$, for all $h \in H$.

Let A be an algebra and $s\colon H\otimes A \longrightarrow A\otimes H$ a left transposition. A map $\rho\colon H\otimes A\to A$ is said to be a *weak action* of H on (A,s) or a *weak s-action* of H on A, if it satisfies the conditions of the above remark.

Definition 2.25. Let A be an algebra, $s\colon H\otimes A\longrightarrow A\otimes H$ a left transposition and $\rho\colon H\otimes A\to A$ a weak action of H on (A,s). Let $\sigma: H\otimes H\to A$ be a map. We say that σ is *normal* if

$$\sigma(1_H\otimes h)=\sigma(h\otimes 1_H)=\epsilon(h)\qquad \text{for all } h\in H,$$

and that σ is a *cocycle that satisfies the twisted module condition* if



More precisely, the first equality is the cocycle condition and the second one is the twisted module condition. Finally we say that σ is *compatible with s* if it is a map in $\mathcal{LHB}$. In other words, if

$$(\sigma\otimes H)\circ(H\otimes c)\circ(c\otimes H)=s\circ(H\otimes\sigma).$$

Let $s\colon H\otimes A\to A\otimes H$ be a left transposition, $\rho\colon H\otimes A\to A$ a weak s-action and $\sigma: H\otimes H\to A$ a normal cocycle compatible with s, that satisfies the twisted module condition. Consider the maps $\chi: H\otimes A\longrightarrow A\otimes H$ and $\mathcal{F}\colon H\otimes H\longrightarrow A\otimes H$ defined by

$$\chi := (\rho\otimes H)\circ(H\otimes s)\circ(\Delta\otimes A)\qquad\text{and}\qquad \mathcal{F}:=(\sigma\otimes\mu_H)\circ(H\otimes c\otimes H)\circ(\Delta\otimes\Delta).$$

Definition 2.26. The *crossed product associated with* (s,ρ,σ) is the k-algebra $A\#^s_{\rho,\sigma}H$, whose underlying k-vector space is $A\otimes H$ and whose multiplication map is

$$\mu := (\mu_A\otimes H)\circ(\mu_A\otimes\mathcal{F})\circ(A\otimes\chi\otimes H).$$

From now on, a simple tensor $a\otimes h$ of $A\#^s_{\rho,\sigma}H$ will usually be written $a\#h$.

Theorem 2.27 ([3, Theorems 2.3, 6.3 and 9.3])**.** *The algebra $A\#^s_{\rho,\sigma}H$ is associative and has unity $1_A\#1_H$.*

Theorem 2.28 ([3, Propositions 10.3 and 10.4])**.** *The map*

$$\widehat{s}: H\otimes A\#^s_{\rho,\sigma}H\longrightarrow A\#^s_{\rho,\sigma}H\otimes H,$$

defined by $\widehat{s}:=(A\otimes c)\circ(s\otimes H)$ is a left transposition of H on $A\#^s_{\rho,\sigma}H$ and the pair $(A\#^s_{\rho,\sigma}H,\widehat{s})$, endowed with the coaction $\nu_{A\#^s_{\rho,\sigma}H}:=A\otimes\Delta$, is a right H-comodule algebra.

Definition 2.29. Let (B,s) be a right H-comodule algebra and let $i\colon A\hookrightarrow B$ be an algebra inclusion. We say that $(i\colon A\hookrightarrow B,s)$ is an *H-extension* of A if $i(A)=B^{\mathrm{coH}}$. Let $(i'\colon A\hookrightarrow B',s')$ be another H-extension of A. We say that $(i\colon A\hookrightarrow B,s)$ and $(i'\colon A\hookrightarrow B',s')$ are *equivalent* if there is a right H-comodule algebra isomorphism $f\colon (B,s)\to(B',s')$, which is also a left A-module homomorphism.

Remark 2.30. For each H-extension $(i\colon A\hookrightarrow B,s)$ of A, the map $s_A\colon H\otimes A\longrightarrow A\otimes H$, induced by s, is a left transposition (in other words, (A,s_A) is a left H-algebra).

Example 2.31. $(i\colon A \hookrightarrow A\#^{s}_{\rho,\sigma}H, \widehat{s})$, where $i(a) := a\#1_H$, is an H-extension of A.

Definition 2.32. Let $(i\colon A \hookrightarrow B, s)$ be an H-extension. We say that:
(1) (i,s) is *cleft* if there is a convolution invertible right H-comodule map $\gamma\colon (H,c) \to (B,s)$,
(2) (i,s) is *H-Galois* if the map $\beta_B\colon B\otimes_A B \longrightarrow B\otimes H$, defined by $\beta_B(b\otimes b') = (b\otimes 1_H)\nu(b')$, where ν denotes the coaction of B, is bijective,
(3) (i,s) has the *normal basis property* if there exists a left A-linear and right H-colinear isomorphism $\phi\colon (A\otimes H, \widehat{s_A}) \longrightarrow (B,s)$, where the coaction of $A\otimes H$ is $A\otimes\Delta$ and $\widehat{s_A} = (A\otimes c)\circ(s_A\otimes H)$.

Definition 2.33. Let $(i\colon A \hookrightarrow B, s)$ be an H-extension of A. If (i,s) is cleft, then each one of the maps γ satisfying the conditions required in item (1) of Definition 2.32 is called a *cleft map* of (i,s), and if (i,s) has the normal basis property, then each one of the left A-linear right and H-colinear isomorphism $\phi\colon (A\otimes H, \widehat{s_A}) \longrightarrow (B,s)$ is called a *normal basis* of B.

Remark 2.34 ([3, Section 10]). If γ is a *cleft map* of $(i\colon A \hookrightarrow B, s)$, then $\gamma(1_H)\in B^{\times}$ and the map $\gamma' := \gamma(1_H)^{-1}\gamma$ is a cleft map that satisfies $\gamma'(1_H) = 1_B$.

Lemma 2.35. *Let H be a braided Hopf algebra and let $(i\colon A \hookrightarrow B, s)$ be a cleft H-extension, with a cleft map γ. The map $f\colon H\otimes A \to B$, defined by*

$$f := \mu_B\circ(\mu_B\otimes B)\circ(\gamma\otimes i\otimes\gamma^{-1})\circ(H\otimes s_A)\circ(\Delta\otimes A)$$

takes its values in $i(A)$.

Proof. Let $\lambda_r^X\colon X\to X\otimes k$ be the canonical map. We must prove that

$$\nu\circ f = (f\otimes\eta)\circ(H\otimes\lambda_r^A).$$

A direct computation shows that

$$\begin{aligned}
\nu\circ f &= \nu\circ\mu_B\circ(\mu_B\otimes B)\circ(\gamma\otimes i\otimes\gamma^{-1})\circ(H\otimes s_A)\circ(\Delta\otimes A)\\
&= (\mu_B\otimes\mu_H)\circ(B\otimes s\otimes H)\circ(\nu\otimes\nu)\circ(\mu_B\otimes B)\circ(\gamma\otimes i\otimes\gamma^{-1})\circ(H\otimes s_A)\circ(\Delta\otimes A)\\
&= (\mu_B\otimes\mu_H)\circ(\mu_B\otimes s\otimes H)\circ(B\otimes s\otimes\nu)\circ(\nu\otimes i\otimes\gamma^{-1})\circ(\gamma\otimes s_A)\circ(\Delta\otimes A)\\
&= (\mu_B\otimes\mu_H)\circ(\mu_B\otimes s\otimes H)\circ(B\otimes i\otimes H\otimes\nu)\circ(\gamma\otimes s_A\otimes\gamma^{-1})\circ(\Delta\otimes s_A)\circ(\Delta\otimes A)\\
&= (\mu_B\otimes\mu_H)\circ(B\otimes s\otimes H)\circ(\mu_B\otimes H\otimes\nu\circ\gamma^{-1})\circ(B\otimes i\otimes\Delta)\circ(\gamma\otimes s_A)\circ(\Delta\otimes A)\\
&= (\mu_B\otimes H)\circ(\mu_B\otimes L)\circ(B\otimes i\otimes H)\circ(\gamma\otimes s_A)\circ(\Delta\otimes A),
\end{aligned}$$

where

$$L := (B\otimes\mu_H)\circ(s\otimes H)\circ(H\otimes\nu\circ\gamma^{-1})\circ\Delta.$$

Since, by [3, Lemma 10.7],

$$\begin{aligned}
L &= (B\otimes\mu_H)\circ(s\otimes H)\circ(H\otimes\gamma^{-1}\otimes S)\circ(H\otimes c\circ\Delta)\circ\Delta\\
&= (\gamma^{-1}\otimes\mu_H)\circ(c\otimes S)\circ(H\otimes c)\circ(H\otimes\Delta)\circ\Delta\\
&= (\gamma^{-1}\otimes\mu_H)\circ(c\otimes S)\circ(H\otimes c)\circ(\Delta\otimes H)\circ\Delta\\
&= (\gamma^{-1}\otimes H)\circ c\circ(\mu_H\otimes H)\circ(H\otimes S\otimes H)\circ(\Delta\otimes H)\circ\Delta\\
&= (\gamma^{-1}\otimes H)\circ c\circ(\eta\circ\epsilon\otimes H)\circ\Delta\\
&= \gamma^{-1}\otimes\eta,
\end{aligned}$$

we have

$$\nu\circ f = (\mu_B\otimes H)\circ(B\otimes\gamma^{-1}\otimes\eta)\circ(\mu_B\otimes\lambda_r^H)\circ(B\otimes i\otimes H)\circ(\gamma\otimes s_A)\circ(\Delta\otimes A) = (f\otimes\eta)\circ(H\otimes\lambda_r^A),$$

as desired. □

Theorem 2.36. *Let H be a braided Hopf algebra and let $(i\colon A \hookrightarrow B, s)$ be an H-extension. The following assertions are equivalent:*

(1) (i,s) is cleft.

(2) (i,s) is H-Galois with a normal basis.

(3) There is a crossed product $A\#^{s_A}_{\rho,\sigma}H$, with convolution invertible cocycle $\sigma : H \otimes^c H \to A$, and a right H-comodule algebra isomorphism

$$(B,s) \longrightarrow (A\#^{s_A}_{\rho,\sigma}H, \widehat{s_A}),$$

which is also left A-linear.

Furthermore, if γ is a cleft map of (i,s) with $\gamma(1_H) = 1_B$, then

(4) The map $\phi\colon (A\otimes H, \widehat{s_A}) \longrightarrow (B,s)$, defined by $\phi(a\otimes h) := i(a)\gamma(h)$, is a normal basis of B.

(5) The weak action ρ and the cocycle σ are given by

$$i\circ\rho = \mu_B\circ(\mu_B\otimes B)\circ(\gamma\otimes i\otimes\gamma^{-1})\circ(H\otimes s_A)\circ(\Delta\otimes A) \tag{2.5}$$

and

$$i\circ\sigma = \mu_B\circ(\mu_B\otimes\gamma^{-1})\circ(\gamma\otimes\gamma\otimes\mu_H)\circ\Delta_{H\otimes^c H}. \tag{2.6}$$

Proof. The equivalence between the first three items is [3, Theorem 10.6], and the fourth item was proved in the proof of that Theorem. It remains to check the last one. By item (4), the discussion below [3, Definition 10.5] and the proof of Theorem 10.6 of [3], we know that ϕ is bijective, that

$$(i\otimes H)\circ\phi^{-1}(b) = b_{(0)}\gamma^{-1}(b_{(1)})\otimes b_{(2)},$$

and that the maps $\rho : H\otimes A\to A$ and $\sigma : H\otimes H\to A$ are given by

$$\rho(h\otimes a) := (A\otimes\epsilon)\circ\phi^{-1}(\gamma(h)i(a)) \qquad \text{and} \qquad \sigma(h\otimes l) := (A\otimes\epsilon)\circ\phi^{-1}(\gamma(h)\gamma(l)).$$

We must check that ρ and σ satisfy (2.5) and (2.6), respectively. Let f be as in Lemma 2.35 and let i^{-1} be the compositional inverse of $i\colon A\to i(A)$. Since

$$\begin{aligned}
\mu_B\circ(\gamma\otimes i) &= \mu_B\circ(\mu_B\otimes\eta_B\circ\epsilon)\circ(B\otimes i\otimes H)\circ(\gamma\otimes s_A)\circ(\Delta\otimes A)\\
&= \mu_B\circ(B\otimes\mu_B)\circ(B\otimes\gamma^{-1}\otimes\gamma)\circ(\mu_B\otimes\Delta)\circ(B\otimes i\otimes H)\circ(\gamma\otimes s_A)\circ(\Delta\otimes A)\\
&= \mu_B\circ(\mu_B\otimes\mu_B)\circ(\gamma\otimes i\otimes\gamma^{-1}\otimes\gamma)\circ(H\otimes s_A\otimes H)\circ(\Delta\otimes s_A)\circ(\Delta\otimes A)\\
&= \mu_B\circ(f\otimes\gamma)\circ(H\otimes s_A)\circ(\Delta\otimes A),
\end{aligned}$$

and, by Lemma 2.35,

$$\mu_B\circ(f\otimes\gamma) = \phi\circ(i^{-1}\circ f\otimes H),$$

we have

$$\begin{aligned}
i\circ\rho &= (i\otimes\epsilon)\circ\phi^{-1}\circ\mu_B\circ(\gamma\otimes i)\\
&= (i\otimes\epsilon)\circ\phi^{-1}\circ\mu_B\circ(f\otimes\gamma)\circ(H\otimes s_A)\circ(\Delta\otimes A)\\
&= (i\otimes\epsilon)\circ(i^{-1}\circ f\otimes H)\circ(H\otimes s_A)\circ(\Delta\otimes A)\\
&= \mu_B\circ(\mu_B\otimes B)\circ(\gamma\otimes i\otimes\gamma^{-1})\circ(H\otimes s_A)\circ(\Delta\otimes A).
\end{aligned}$$

Finally,

$$\begin{aligned}
i\circ\sigma &= (i\otimes\epsilon)\circ\phi^{-1}\circ\mu_B\circ(\gamma\otimes\gamma)\\
&= \mu_B\circ(B\otimes\gamma^{-1})\circ\nu\circ\mu_B\circ(\gamma\otimes\gamma)\\
&= \mu_B\circ(B\otimes\gamma^{-1})\circ(\mu_B\otimes\mu_H)\circ(B\otimes s\otimes B)\circ(\nu\otimes\nu)\circ(\gamma\otimes\gamma)\\
&= \mu_B\circ(\mu_B\otimes\gamma^{-1})\circ(\gamma\otimes\gamma\otimes\mu_H)\circ\Delta_{H\otimes^c H},
\end{aligned}$$

as desired. □

Remark 2.37. In the proof of Theorem 10.6 of [3], it was also shown that $\phi\colon A\#^{s_A}_{\rho,\sigma}H \to B$ is an algebra isomorphism.

3. A family of braided Hopf algebras

Let G be a finite group, $\chi : G \to k^\times$ a character, $n > 1$ in $\mathbb{N}$, and $T \in kG$, where kG denotes the group algebra of G with coefficients in k. Set $\mathcal{E} := (G, \chi, T, n)$ and write $T := \sum_{g\in G} \lambda_g g$.

Proposition 3.1. *There exists an associative algebra $B_{\mathcal{E}}$ such that*
- *$B_{\mathcal{E}}$ is generated by G and an element $x \in B_{\mathcal{E}} \backslash kG$,*
- *$\mathcal{B} := \{gx^i : g \in G$ and $0 \le i < n\}$ is a basis of $B_{\mathcal{E}}$ as a k-vector space,*
- *the multiplication of elements of $\mathcal{B}$ is given by:*

$$gx^ihx^j := \begin{cases} \chi^i(h)ghx^{i+j} & \text{if } i+j<n,\\ \chi^i(h)ghTx^{i+j-n} & \text{if } i+j\ge n,\end{cases}$$

iff $\lambda_{hgh^{-1}} = \chi^n(h)\lambda_g$ for all $h, g \in G$, and $\chi(g) = 1$ for all $g \in G$ such that $\lambda_g \neq 0$.

Proof. Let $V := kx_0 \oplus \cdots \oplus kx_{n-1}$, where $x_0, \dots, x_{n-1}$ are indeterminate. We will prove the result by showing that there is an associative and unitary algebra $kG\#V$, with underlying vector space $kG \otimes V$, whose multiplication map satisfies

$$(g \otimes x_i)(g' \otimes x_0) = \chi^i(g')gg' \otimes x_i \qquad \text{for all } i \text{ and all } g \in G$$

and

$$(1_G \otimes x_i)(1_G \otimes x_j) = \begin{cases} 1_G \otimes x_{i+j} & \text{if } i+j<n,\\ T \otimes x_{i+j-n} & \text{if } i+j \ge n,\end{cases}$$

iff
(1) $\chi(g) = 1$ for all $g \in G$ such that $\lambda_g \neq 0$,
(2) $\lambda_{hgh^{-1}} = \chi^n(h)\lambda_g$ for all $h, g \in G$.
By the theory of general crossed products developed in [1], for this it suffices to check that the maps

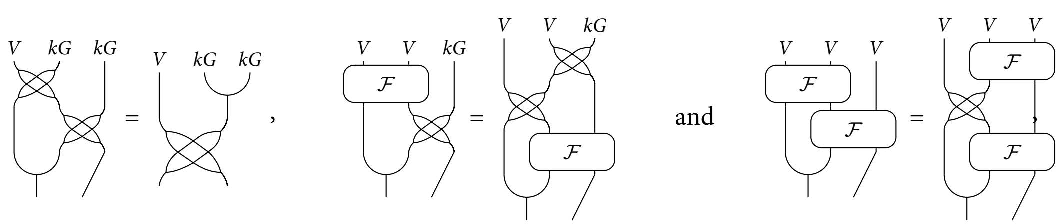
$$\Phi\colon V \otimes kG \longrightarrow kG \otimes V \qquad \text{and} \qquad \mathcal{F}\colon V \otimes V \longrightarrow kG \otimes V,$$

given by

$$\Phi(x_i \otimes g) := \chi^i(g)g \otimes x_i \qquad \text{and} \qquad \mathcal{F}(x_i \otimes x_j) := \begin{cases} 1_G \otimes x_{i+j} & \text{if } i+j<n,\\ T \otimes x_{i+j-n} & \text{if } i+j \ge n,\end{cases}$$

satisfy

$$\Phi(x_i \otimes 1_G) = 1_G \otimes x_i, \qquad \Phi(x_0 \otimes g) = g \otimes x_0, \qquad \mathcal{F}(x_0 \otimes x_i) = \mathcal{F}(x_i \otimes x_0) = 1_G \otimes x_i,$$



where ⌧ stands for Φ, iff conditions (1) and (2) are fulfilled.

By the very definitions of Φ and $\mathcal{F}$, the first four conditions always hold. Assume that the other ones hold. Evaluating the fifth one in $x_1 \otimes x_{n-1} \otimes h$ we see that

$$\sum_{g\in G} \lambda_g gh \otimes x_0 = \sum_{g\in G} \chi^n(h)\lambda_g hg \otimes x_0 \quad \text{for all } g, h \in G,$$

or equivalently,

$$\lambda_{hgh^{-1}} = \chi^n(h)\lambda_g \quad \text{for all } g, h \in G,$$

and evaluating the sixth one in $x_1 \otimes x_{n-1} \otimes x_1$ we see that

$$\chi(g) = 1 \quad \text{for all } g \in G \text{ with } \lambda_g \neq 0.$$

Conversely, a direct computation proves that if these facts are true, then the equalities in the last two diagrams hold. □

Corollary 3.2. *If there is an algebra $B_{\mathcal{E}}$ satisfying the conditions required in Proposition 3.1, and there exists g in the center $\mathrm{Z}G$ of G with $\lambda_g \neq 0$, then $\chi^n = 1$.*

Remark 3.3. It is clear that if there exists, then $B_{\mathcal{E}}$ is a k-algebra unitary with unit $1_G x^0$, that kG is a subalgebra of $B_{\mathcal{E}}$ and that $B_{\mathcal{E}}$ is unique up to isomorphism.

Remark 3.4. Using that $B_{\mathcal{E}}$ has dimension $n|G|$ it is easy to see that it is canonically isomorphic to the algebra generated by the group G and the element x subject to the relations $x^n = T$ and $xg = \chi(g)gx$ for all $g \in G$.

Given $q \in k^{\times}$, let

$$c_q\colon B_{\mathcal{E}} \otimes B_{\mathcal{E}} \longrightarrow B_{\mathcal{E}} \otimes B_{\mathcal{E}}$$

be the k-linear map defined by $c_q(gx^i \otimes hx^j) := q^{ij}\, hx^j \otimes gx^i$. It is easy to check that c_q is a braiding operator that is compatible with the unit of $B_{\mathcal{E}}$. Furthermore,

- a direct computation shows that c_q is compatible with the multiplication map of $B_{\mathcal{E}}$ iff $T = 0$ or $q^n = 1$,
- by Remark 3.4 there exists an algebra map $\epsilon\colon B_{\mathcal{E}} \to k$ such that $\epsilon(x) = 0$ and $\epsilon(g) = 1$ for all $g \in G$ iff $\sum_{g\in G} \lambda_g = 0$. Moreover, in this case, c_q is compatible with ϵ.

Proposition 3.5. *Let $\mathcal{E}$ be as at the beginning of this section, $z \in G$ and $q \in k^{\times}$. Assume that $B_{\mathcal{E}}$ exists. Then, the algebra $B_{\mathcal{E}}$ is a braided bialgebra with braid c_q and comultiplication map Δ defined by*

$$\Delta(x) := 1 \otimes x + x \otimes z \qquad \textit{and} \qquad \Delta(g) := g \otimes g \quad \textit{for } g \in G \tag{3.7}$$

iff
(1) $\binom{n}{j}_{q\chi(z)} = 0$ for all $0 < j < n$,
(2) $z \in \mathrm{Z}G$ and $T = \lambda(z^n - 1_G)$ for some $\lambda \in k$, where $\lambda = 0$ if $z^n \neq 1_G$ and $q^n \neq 1$.

Proof. Since $B_{\mathcal{E}}$ is generated by the group G and the element x subject to the relations $x^n = T$ and $xg = \chi(g)gx$ for all $g \in G$, there exists an algebra map $\Delta\colon B_{\mathcal{E}} \longrightarrow B_{\mathcal{E}} \otimes_{c_q} B_{\mathcal{E}}$ such that (3.7) is satisfied iff the equalities

$$(h \otimes h)(g \otimes g) = hg \otimes hg,$$
$$(1 \otimes x + x \otimes z)(g \otimes g) = \chi(g)(g \otimes g)(1 \otimes x + x \otimes z)$$

and

$$(1 \otimes x + x \otimes z)^n = \sum_{l\in G} \lambda_l\, l \otimes l \tag{3.8}$$

hold in $B_{\mathcal{E}} \otimes_{c_q} B_{\mathcal{E}}$ for all $h, g \in G$. The first equality is trivial, while the second one is equivalent to

$$\chi(g)(g \otimes gx + gx \otimes zg) = \chi(g)(g \otimes gx + gx \otimes gz) \qquad \text{for all } g \in G,$$

and so it is fulfilled iff z is in the center of G. In order to deal with the last one, we note that, in $B_{\mathcal{E}} \otimes_{c_q} B_{\mathcal{E}}$,

$$(1 \otimes x)(x \otimes z) = q\chi(z)\, x \otimes zx = q\chi(z)(x \otimes z)(1 \otimes x),$$

and so, by formula (2.2),

$$(1 \otimes x + x \otimes z)^n = \sum_{j=0}^{n} \binom{n}{j}_{q\chi(z)} (x \otimes z)^j (1 \otimes x)^{n-j} = \sum_{j=0}^{n} \binom{n}{j}_{q\chi(z)} x^j \otimes z^j x^{n-j}.$$

Hence, equality (3.8) holds iff

$$\sum_{j=0}^{n} \binom{n}{j}_{q\chi(z)} x^j \otimes z^j x^{n-j} = \sum_{l \in G} \lambda_l\, l \otimes l,$$

which is clearly equivalent to

$$\binom{n}{j}_{q\chi(z)} = 0 \qquad \text{for all } 0 < j < n,$$

and

$$\sum_{l \in G} \lambda_l\, l \otimes l = 1 \otimes x^n + x^n \otimes z^n = 1 \otimes T + T \otimes z^n = \sum_{l \in G} 1 \otimes \lambda_l l + \sum_{l \in G} \lambda_l l \otimes z^n.$$

If $z^n = 1_G$ this happens iff $\lambda_l = 0$ for all $l \in G$, while if $z^n \neq 1_G$, this happens iff $\lambda_l = 0$ for all $l \neq 1_G, z^n$ and if $\lambda_{z^n} = -\lambda_{1_G}$. By the way, this computation shows that if Δ exists, then the augmentation ϵ introduced above, is well defined. Moreover, by formula (2.2),

$$\Delta(gx^i) := \sum_{j=0}^{i} \binom{i}{j}_{q\chi(z)} (g \otimes g)(x \otimes z)^j (1 \otimes x)^{i-j} = \sum_{j=0}^{i} \binom{i}{j}_{q\chi(z)} gx^j \otimes gz^j x^{i-j} \tag{3.9}$$

for all $g \in G$ and $i \geq 0$. Using this it is easy to see that c_q is compatible with Δ. Since we already know that c_q is compatible with $1_{B_{\mathcal{E}}}$, the multiplication map of $B_{\mathcal{E}}$ and ϵ, in order to finish the proof we only must check that Δ is coassociative and that ϵ is its counit. But, since c_q is compatible with Δ and Δ is an algebra map, it suffices to verify these facts on x and $g \in G$, which is trivial. □

Remark 3.6. Let $\mathcal{E}$ be as at the beginning of this section. If $T = \lambda(z^n - 1_G)$ with $z \in ZG$, $z^n \neq 1_G$ and $\lambda \in k^\times$, then the hypothesis of Proposition 3.1 are equivalent to $\chi^n = 1$, while if $T = 0$, then the hypothesis of Proposition 3.1 are automatically satisfied.

Remark 3.7. It is easy to see that $\binom{n}{1}_{q\chi(z)} = 0$ implies $(q\chi(z))^n = 1$ and that if $q\chi(z)$ is an n-th primitive root of unit, then $\binom{n}{j}_{q\chi(z)} = 0$ for all $0 < j < n$.

Corollary 3.8. *Each data $\mathcal{D} = (G, \chi, z, \lambda, q)$ consisting of:*
– *a finite group G,*
– *a character χ of G with values in k,*
– *a central element z of G,*
– *elements $q \in k^\times$ and $\lambda \in k$,*
such that
– *$q\chi(z)$ is a root of 1 of order n greater than 1,*
– *if $\lambda(z^n - 1_G) \neq 0$, then $\chi^n = 1$,*

has associated a braided Hopf algebra $H_{\mathcal{D}}$. As an algebra, $H_{\mathcal{D}}$ is generated by the group G and the element x subject to the relations $x^n = \lambda(z^n - 1_G)$ and $xg = \chi(g)gx$ for all $g \in G$, the coalgebra structure of $H_{\mathcal{D}}$ is determined by

$$\Delta(g) := g \otimes g \quad \text{for } g \in G, \quad \Delta(x) := 1 \otimes x + x \otimes z,$$
$$\epsilon(g) := 1 \quad \text{for } g \in G, \quad \epsilon(x) := 0,$$

the braid c_q of $H_{\mathcal{D}}$ is defined by

$$c_q(gx^i \otimes hx^j) := q^{ij}\, hx^j \otimes gx^i, \tag{3.10}$$

and its antipode is given by

$$S(gx^i) := (-1)^i (q\chi(z))^{\frac{i(i-1)}{2}} x^i z^{-i} g^{-1}. \tag{3.11}$$

Furthermore, as a vector space $H_{\mathcal{D}}$ has basis

$$\{gx^i : g \in G \text{ and } 0 \le i < n\},$$

and consequently, $\dim\big(H_{\mathcal{D}}\big) = n|G|$.

Proof. Let $\mathcal{E} := \big(G, \chi, \lambda(z^n - 1_G), n\big)$ and let $B_{\mathcal{E}}$ be the algebra obtained applying Proposition 3.1. Now note that if $\lambda(z^n - 1_G)$, then $\chi^n = 1$ and so $q^n = q^n\chi(z)^n = 1$. Hence, we can apply Proposition 3.5, which implies that $B_{\mathcal{E}}$ has a braided bialgebra structure with comultiplication map, counit and braid as in its statement. Let $H_{\mathcal{D}}$ denote this bialgebra. It remains to check that the map S given by (3.11) is the antipode of $H_{\mathcal{D}}$. Since

$$S \circ \mu(gx^i \otimes hx^j) = \mu \circ (S \otimes S) \circ c_q(gx^i \otimes hx^j),$$

for this it suffices to verify that

$$S(x) + xS(z) = S(1)x + S(x)z = 0 \qquad \text{and} \qquad S(g)g = gS(g) = 1 \quad \text{for all } g \in G,$$

which is evident. □

Remark 3.9. If $\lambda(z^n - 1_G) = 0$, then we can assume without loss of generality (and we do it), that $\lambda = 0$.

Remark 3.10. Assume that $n > 1$. The previous corollary also holds if the hypothesis that $q\chi(z)$ is a root of 1 of order n is replaced by $\binom{n}{j}_{q\chi(z)} = 0$ for all $0 < j < n$. However, from now on we will consider that $q\chi(z)$ is a root of 1 of order n.

4. Right $H_{\mathcal{D}}$-comodule algebras

Let G be a group, V be a k-vector space and $s\colon k[G] \otimes V \to V \otimes k[G]$ a k-linear map. Evidently, there is a unique family of maps $(\alpha_x^y\colon V \to V)_{x,y \in G}$, such that

$$s(x \otimes v) = \sum_{y \in G} \alpha_x^y(v) \otimes y.$$

Proposition 4.1. *The pair (V, s) is a left $k[G]$-space iff s is a bijective map and the following conditions hold:*

(1) $(\alpha_x^y)_{y \in G}$ is a complete family of orthogonal idempotents, for all $x \in G$,
(2) $\alpha_1^1 = \mathrm{id}$,
(3) $\alpha_{xy}^z = \sum_{uw=z} \alpha_x^u \circ \alpha_y^w$, for all $x, y, z \in G$.

Proof. Mimic the proof of [3, Proposition 4.10]. □

For $x, y \in G$, let $V_x^y := \{v \in V : s(x \otimes v) = v \otimes y\}$.

Proposition 4.2. *The pair (V, s) is a left $k[G]$-space iff:*
(1) $\bigoplus_{z\in G} V_x^z = V = \bigoplus_{z\in G} V_z^x$, for all $x \in G$,
(2) $V_1^1 = V$,
(3) $V_{xy}^z = \bigoplus_{uw=z} V_x^u \cap V_y^w$, for all $x, y, z \in G$.

Proof. Mimic the proof of [3, Propositions 4.11 and 4.13]. □

Theorem 4.3. *If G is a finitely generated group, then each left $k[G]$-space (V, s) determines an* Aut(G)-*gradation*

$$V = \bigoplus_{\zeta\in \mathrm{Aut}(G)} V_\zeta$$

on V, by

$$V_\zeta := \bigcap_{x\in G} V_x^{\zeta(x)} = \{v \in V : s(x \otimes v) = v \otimes \zeta(x) \text{ for all } x \in G\}.$$

Moreover, the correspondence that each left $k[G]$-space (V, s), with underlying vector space V, assigns the Aut(G)-*gradation of V obtained as above, is bijective.*

Proof. Mimic the proof of [3, Theorem 4.14]. □

In the sequel $\mathcal{D} := (G, \chi, z, \lambda, q)$ and $H_{\mathcal{D}}$ are as in Corollary 3.8 and we will freely use the notations and properties established there. Furthermore, to abbreviate expressions we set $p := \chi(z)$. We now begin with the study of the right $H_{\mathcal{D}}$-braided comodule algebras. We let $\mathrm{Aut}_{\chi,z}(G)$ denote the subgroup of Aut(G) consisting of all the automorphism ϕ such that $\phi(z) = z$ and $\chi \circ \phi = \chi$.

Proposition 4.4. *If $(p, q) \neq (1, -1)$, then for all left $H_{\mathcal{D}}$-space (V, s) it is true that*

$$s(kG \otimes V) = V \otimes kG,$$

$$s(z \otimes v) = v \otimes z \quad \text{for all } v \in V,$$

and there exists $\varphi \in \mathrm{Aut}(V)$ such that

$$s(x \otimes v) = \varphi(v) \otimes x \quad \text{for all } v \in V. \tag{4.12}$$

Proof. Write

$$s(gx^i \otimes v) = \sum_{\substack{h\in G\\ 0\le j<n}} \beta_{h,j}^{g,i}(v) \otimes hx^j.$$

Since $S^2(gx^i) = q^{i(i-1)}p^{-i}gx^i$, we have

$$\begin{aligned} q^{i(i-1)}p^{-i} \sum_{\substack{h\in G\\ 0\le j<n}} \beta_{h,j}^{g,i}(v) \otimes hx^j &= q^{i(i-1)}p^{-i}s(gx^i \otimes v)\\ &= s \circ (S^2 \otimes V)(gx^i \otimes v)\\ &= (V \otimes S^2) \circ s(gx^i \otimes v)\end{aligned}$$

$$= \sum_{\substack{h\in G\\ 0\le j<n}} \beta^{g,i}_{h,j}(v)\otimes S^2(hx^j)$$

$$= \sum_{\substack{h\in G\\ 0\le j<n}} q^{j(j-1)}p^{-j}\beta^{g,i}_{h,j}(v)\otimes hx^j,$$

and consequently,

$$\beta^{g,i}_{h,j}\neq 0 \Longrightarrow q^{j(j-1)-i(i-1)} = p^{j-i}. \tag{4.13}$$

Using now that s is compatible with Δ, we obtain that

$$\begin{aligned}
&\sum_{\substack{h\in G\\ 0\le i<n}}\sum_{j=0}^{i}\binom{i}{j}_{qp}\beta^{g,0}_{h,i}(v)\otimes hx^j\otimes hz^jx^{i-j}\\
&= (V\otimes\Delta)\circ s(g\otimes v)\\
&= (s\otimes H_{\mathcal{D}})\circ(H_{\mathcal{D}}\otimes s)\circ(\Delta\otimes V)(g\otimes v)\\
&= \sum_{\substack{h\in G\\ 0\le i<n}}\sum_{\substack{h'\in G\\ 0\le i'<n}}\beta^{g,0}_{h,i}\circ\beta^{g,0}_{h',i'}(v)\otimes hx^i\otimes h'x^{i'}.
\end{aligned} \tag{4.14}$$

Hence,

$$\beta^{g,0}_{h,i}\circ\beta^{g,0}_{h',i'} = \begin{cases}\binom{i+i'}{i}_{qp}\beta^{g,0}_{h,i+i'} & \text{if } h' = hz^i \text{ and } i+i' < n,\\ 0 & \text{otherwise.}\end{cases} \tag{4.15}$$

Combining this with (4.13) we obtain that

$$\beta^{g,0}_{h,i}\neq 0 \Longrightarrow \beta^{g,0}_{h,j}\neq 0 \text{ for all } j\le i \Longrightarrow q^{j(j-1)} = p^j \text{ for } j\le i.$$

Consequently, if $\beta^{g,0}_{h,i}\neq 0$ for some $g\in G$ and $i\ge 1$, then $p = q^0 = 1$. Hence if $p\neq 1$, then $\beta^{g,0}_{h,i} = 0$ for all $g\in G$ and $i\ge 1$. Assume that $p = 1$. If $\beta^{g,0}_{h,i}\neq 0$ for some $g\in G$ and $i\ge 2$, then $q^2 = p^2 = 1$. But this is impossible, since it implies that $n := \mathrm{ord}(qp) = \mathrm{ord}(q)\le 2$, which contradicts that $i < n$. Therefore

$$s(g\otimes v) = \begin{cases}\sum_{h\in G}\beta^{g,0}_{h,0}(v)\otimes h & \text{if } p\neq 1,\\ \sum_{h\in G}\beta^{g,0}_{h,0}(v)\otimes h + \sum_{h\in G}\beta^{g,0}_{h,1}(v)\otimes hx & \text{if } p = 1.\end{cases} \tag{4.16}$$

On the other hand, due to s is compatible with the counit of $H_{\mathcal{D}}$, we get

$$\sum_{h\in G}\beta^{g,0}_{h,0} = \mathrm{id} \quad \text{for all } g\in G,$$

which, combined with the particular case of (4.15) obtained by taken $i = i' = 0$, shows that

$$\left(\beta^{g,0}_{h,0}\right)_{h\in G} \text{ is a complete family of orthogonal idempotents for all } g\in G. \tag{4.17}$$

Equality (4.16) shows that if $p \neq 1$, then $s(kG \otimes V) \subseteq V \otimes kG$. Assume now that $p = 1$ and $q \neq -1$ (which implies $n > 2$). Using that s is compatible with the multiplication map of $H_{\mathcal{D}}$ we get that

$$
\begin{aligned}
v \otimes 1 &= s(g^{-1}g \otimes v) \\
&= (v \otimes \mu) \circ (s \otimes H_{\mathcal{D}}) \circ (H_{\mathcal{D}} \otimes s)(g^{-1} \otimes g \otimes v) \\
&= \sum_{h\in G}\sum_{l\in G} \beta_{h,0}^{g^{-1},0} \circ \beta_{1,0}^{g,0}(v) \otimes hl + \sum_{h\in G}\sum_{l\in G} \chi(l)\beta_{h,1}^{g^{-1},0} \circ \beta_{l,0}^{g,0}(v) \otimes hlx \\
&\quad + \sum_{h\in G}\sum_{l\in G} \beta_{h,0}^{g^{-1},0} \circ \beta_{l,1}^{g,0}(v) \otimes hlx + \sum_{h\in G}\sum_{l\in G} \chi(l)\beta_{h,1}^{g^{-1},0} \circ \beta_{l,1}^{g,0}(v) \otimes hlx^2. \qquad (4.18)
\end{aligned}
$$

Consequently,

$$
\sum_{h\in G} \beta_{h^{-1},0}^{g^{-1},0} \circ \beta_{g,0}^{h,0} = \mathrm{id}_V,
$$

which by (4.17) implies that

$$
\beta_{g^{-1},0}^{h^{-1},0}(v) = v \quad \text{for all } v \in \mathrm{Im}\big(\beta_{h,0}^{g,0}\big) \text{ and } h, g \in G.
$$

Since $\big(\beta_{g,0}^{h,0}\big)_{h\in G}$ and $\big(\beta_{h,0}^{g^{-1},0}\big)_{h\in G}$ are complete families of orthogonal idempotents, from this it follows that

$$
\beta^{g^{-1},0}{}_{h^{-1},0} = \beta_{h,0}^{g,0} \quad \text{for all } h, g \in G.
$$

Combining this with (4.18), we conclude that

$$
\begin{aligned}
0 &= \sum_{h\in G}\sum_{l\in G} \chi(l)\beta_{h,1}^{g^{-1},0} \circ \beta_{l,0}^{g,0}(v) \otimes hlx + \sum_{h\in G}\sum_{l\in G} \beta_{h,0}^{g^{-1},0} \circ \beta_{l,1}^{g,0}(v) \otimes hlx \\
&= \sum_{h\in G}\sum_{l\in G} \chi(l)\beta_{h,1}^{g^{-1},0} \circ \beta_{l^{-1},0}^{g^{-1},0}(v) \otimes hlx + \sum_{h\in G}\sum_{l\in G} \beta_{h^{-1},0}^{g,0} \circ \beta_{l,1}^{g,0}(v) \otimes hlx \\
&= \sum_{h\in G} \chi(z^{-1}h^{-1})\beta_{h,1}^{g^{-1},0}(v) \otimes z^{-1}x + \sum_{h\in G} \beta_{h,1}^{g,0}(v) \otimes x, \qquad (4.19)
\end{aligned}
$$

where the last equality follows from the fact that by (4.15)

$$
\beta_{h,1}^{g,0} \circ \beta_{h',0}^{g,0} = \begin{cases} \beta_{h,1}^{g,0} & \text{if } h' = hz, \\ 0 & \text{otherwise,} \end{cases} \quad \text{and} \quad \beta_{h,0}^{g,0} \circ \beta_{h',1}^{g,0} = \begin{cases} \beta_{h,1}^{g,0} & \text{if } h' = h, \\ 0 & \text{otherwise.} \end{cases} \qquad (4.20)
$$

Note that by (4.17) and the second equality in (4.20), the images of the maps $\beta_{h,1}^{g,0}$ are in direct sum, for each $g \in G$. Hence, from (4.19) it follows that if $z \neq 1$, then $\beta_{h,1}^{g,0} = 0$ for all $g, h \in G$, which by equality (4.16) implies that $s(kG \otimes V) \subseteq V \otimes kG$. We now assume additionally that $z = 1$. Then $\mathrm{Prim}(H_{\mathcal{D}}) = kx$ and so, by [3, Proposition 4.4] there exists an automorphism φ of V such that $s(x \otimes v) = \varphi(v) \otimes x$ for all $v \in V$. Furthermore, by the compatibility of s with c_q,

$$
\begin{aligned}
&\sum_{h\in G} \beta_{h,0}^{g,0} \circ \varphi(v) \otimes x \otimes h + q\sum_{h\in G} \beta_{h,1}^{g,0} \circ \varphi(v) \otimes x \otimes hx \\
&\quad = (V \otimes c_q) \circ (s \otimes H_{\mathcal{D}}) \circ (H_{\mathcal{D}} \otimes s)(g \otimes x \otimes v) \\
&\quad = (s \otimes H_{\mathcal{D}}) \circ (H_{\mathcal{D}} \otimes s) \circ (c_q \otimes V)(g \otimes x \otimes v) \\
&\quad = \sum_{h\in G} \varphi \circ \beta_{h,0}^{g,0}(v) \otimes x \otimes h + \sum_{h\in G} \varphi \circ \beta_{h,1}^{g,0}(v) \otimes x \otimes hx
\end{aligned}
$$

for all $g \in G$, and therefore

$$\varphi \circ \beta^{g,0}_{h,0} = \beta^{g,0}_{h,0} \circ \varphi \quad \text{and} \quad \varphi \circ \beta^{g,0}_{h,1} = q\beta^{g,0}_{h,1} \circ \varphi \quad \text{for all } h, g \in G. \tag{4.21}$$

Using now that s is compatible with the multiplication map of $H_{\mathcal{D}}$, we obtain that

$$\begin{aligned}
&\chi(g)\sum_{h\in G}\beta^{g,0}_{h,0}\circ\varphi(v)\otimes hx + \chi(g)\sum_{h\in G}\beta^{g,0}_{h,1}\circ\varphi(v)\otimes hx^2\\
&= (V\otimes\mu)\circ(s\otimes H_{\mathcal{D}})\circ(H_{\mathcal{D}}\otimes s)(\chi(g)g\otimes x\otimes v)\\
&= (s\otimes H_{\mathcal{D}})\circ(H_{\mathcal{D}}\otimes s)\circ(\mu\otimes V)(\chi(g)g\otimes x\otimes v)\\
&= (s\otimes H_{\mathcal{D}})\circ(H_{\mathcal{D}}\otimes s)\circ(\mu\otimes V)(x\otimes g\otimes v)\\
&= (V\otimes\mu)\circ(s\otimes H_{\mathcal{D}})\circ(H_{\mathcal{D}}\otimes s)(x\otimes g\otimes v)\\
&= \sum_{h\in G}\chi(h)\,\varphi\circ\beta^{g,0}_{h,0}(v)\otimes hx + \sum_{h\in G}\chi(h)\,\varphi\circ\beta^{g,0}_{h,1}(v)\otimes hx^2
\end{aligned}$$

for all $g \in G$, which combined with (4.21) gives

$$\chi(g)\,\beta^{g,0}_{h,0}\circ\varphi = \chi(h)\,\varphi\circ\beta^{g,0}_{h,0} = \chi(h)\,\beta^{g,0}_{h,0}\circ\varphi \tag{4.22}$$

and

$$\chi(g)\,\beta^{g,0}_{h,1}\circ\varphi = \chi(h)\,\varphi\circ\beta^{g,0}_{h,1} = \chi(h)q\,\beta^{g,0}_{h,1}\circ\varphi \tag{4.23}$$

for all $g, h \in G$. Since φ is bijective, from (4.22) it follows that if $\beta^{g,0}_{h,0} \neq 0$, then $\chi(g) = \chi(h)$. Combining this with (4.20), we see that $\beta^{g,0}_{h,1} \neq 0 \Rightarrow \beta^{g,0}_{h,0} \neq 0 \Rightarrow \chi(g) = \chi(h)$. Therefore, from (4.23) it follows that if there exist $g, h \in G$ such that $\beta^{g,0}_{h,1} \neq 0$, then $q = 1$, which is false. So, $\beta^{g,0}_{h,1} = 0$ for all $g, h \in G$. This concludes the proof that $s(kG \otimes V) \subseteq V \otimes kG$. But then a similar computation with s replaced by s^{-1} shows that $s(kG \otimes V) \supseteq V \otimes kG$, and so the equality holds.

We now return to the general case and we claim that

(1) $\beta^{1,1}_{h,j} = 0$ for all $h \in G$ and $j \geq 2$,

(2) $\beta^{1,1}_{h,1} = 0$ for $h \neq 1_G$,

(3) $\beta^{1,1}_{1,1}$ is bijective,

(4) $\beta^{z,0}_{z,0} = \mathrm{id}$ and $\beta^{z,0}_{h,0} = 0$ for $h \neq z$,

(5) $\beta^{1,1}_{h,0} = 0$ for $h \notin \{1_G, z\}$,

(6) If $z \neq 1$, then $\beta^{1,1}_{z,0} = -\beta^{1,1}_{1,0}$ while if $z = 1_G$, then $\beta^{1,1}_{1,0} = 0$,

In fact, $s(kG \otimes V) \subseteq V \otimes kG$ means that $\beta^{g,0}_{h,j} = 0$ for all $g, h \in G$ and $j > 0$. Hence, by the compatibility of s with Δ,

$$\begin{aligned}
\sum_{\substack{h\in G\\0\le i<n}}\sum_{j=0}^{i}\binom{i}{j}_{qp}\beta^{1,1}_{h,i}(v)\otimes hx^j\otimes hz^jx^{i-j} &= (V\otimes\Delta)\circ s(x\otimes v)\\
&= (s\otimes H_{\mathcal{D}})\circ(H_{\mathcal{D}}\otimes s)\circ(\Delta\otimes V)(x\otimes v)\\
&= \sum_{\substack{h\in G\\0\le i<n}}\beta^{1,1}_{h,i}(v)\otimes 1\otimes hx^i\\
&\quad+\sum_{\substack{h,l\in G\\0\le i<n}}\beta^{1,1}_{l,i}\circ\beta^{z,0}_{h,0}(v)\otimes lx^i\otimes h.
\end{aligned} \tag{4.24}$$

This implies that items (1) and (2) are true and that

(8) $\beta^{1,1}_{1,1} = \beta^{1,1}_{1,1} \circ \beta^{z,0}_{z,0}$ and $\beta^{1,1}_{1,1} \circ \beta^{z,0}_{h,0} = 0$ for all $h \in G\backslash\{z\}$,

(9) $\beta^{1,1}_{h,0} = \beta^{1,1}_{h,0} \circ \beta^{z,0}_{h,0}$ and $\beta^{1,1}_{h,0} = -\beta^{1,1}_{1,0} \circ \beta^{z,0}_{h,0}$ for all $h \in G \backslash \{1_G\}$,
(10) $\beta^{1,1}_{h,0} \circ \beta^{z,0}_{1,0} = 0$ for all $h \in G$.
By items (1) and (2) and condition (4.13),

$$s(x \otimes v) = \begin{cases} \beta^{1,1}_{1,1}(v) \otimes x & \text{if } p \neq 1, \\ \beta^{1,1}_{1,1}(v) \otimes x + \sum_{h \in G} \beta^{1,1}_{h,0}(v) \otimes h & \text{if } p = 1. \end{cases} \tag{4.25}$$

This immediately implies that $\beta^{1,1}_{1,1}$ is injective. In fact, if $\beta^{1,1}_{1,1}(v) = 0$, then $s(x \otimes v) \in s(kG \otimes V)$, and so $v = 0$ since $s\colon H_{\mathcal{D}} \otimes V \to V \otimes H_{\mathcal{D}}$ is injective. Item (4) follows from item (8) and the injectivity of $\beta^{1,1}_{1,1}$. Hence, by item (9), we have $\beta^{1,1}_{h,0} = \beta^{1,1}_{h,0} \circ \beta^{z,0}_{h,0} = 0$ for all $h \in G \backslash \{1_G, z\}$, proving item (5). Note also that by items (4), (9) and (10),

$$\beta^{1,1}_{z,0} = \begin{cases} -\beta^{1,1}_{1,0} \circ \beta^{z,0}_{z,0} = -\beta^{1,1}_{1,0} & \text{if } z \neq 1_G, \\ \beta^{1,1}_{1,0} \circ \beta^{1,0}_{1,0} = 0 & \text{if } z = 1_G, \end{cases}$$

which proves item (6). Combining item (4) with the fact that $\beta^{z,0}_{h,j} = 0$ for all $h \in G$ and $j > 0$, we deduce that

$$s(z \otimes v) = v \otimes z \qquad \text{for all } v \in V.$$

Furthermore, by items (5) and (6), equality (4.25) becomes

$$s(x \otimes v) = \begin{cases} \beta^{1,1}_{1,1}(v) \otimes x & \text{if } p \neq 1 \text{ or } z = 1_G, \\ \beta^{1,1}_{1,1}(v) \otimes x + \beta^{1,1}_{1,0}(v) \otimes 1_{H_{\mathcal{D}}} - \beta^{1,1}_{1,0}(v) \otimes z & \text{otherwise.} \end{cases} \tag{4.26}$$

Next we prove that if $p = 1$ and $q \neq -1$, then $\beta^{1,1}_{1,0} = 0$. If $z = 1_G$ this was checked above. So we can assume that $z \neq 1_G$. To abbreviate expressions we set $\varphi := \beta^{1,1}_{1,1}$ and $\beta := \beta^{1,1}_{1,0}$. Evaluating

$$(s \otimes H_{\mathcal{D}}) \circ (H_{\mathcal{D}} \otimes s) \circ (c_q \otimes V) \qquad \text{and} \qquad (V \otimes c_q) \circ (s \otimes H_{\mathcal{D}}) \circ (H_{\mathcal{D}} \otimes s)$$

in $x \otimes x \otimes v$ for all $v \in V$, and using (4.26) and that these maps coincide, we see that

$$q\beta \circ \varphi = \varphi \circ \beta \qquad \text{and} \qquad q\varphi \circ \beta = \beta \circ \varphi.$$

Then $q^2 \varphi \circ \beta = \varphi \circ \beta$, and so $\beta = 0$, since $q^2 \neq 1$ and φ is injective. Hence (4.26) becomes

$$s(x \otimes v) = \varphi(v) \otimes v \qquad \text{for all } v \in V.$$

Consequently $s(x \otimes V) \subseteq V \otimes x$ and a similar computation with s replaced by s^{-1} shows that $s(x \otimes V) \supseteq V \otimes x$, which immediately proves that φ is a surjective map. □

In the rest of the paper we assume that $(p, q) \neq (1, -1)$.

Proposition 4.5. *Let V be a k-vector space endowed with an $\mathrm{Aut}_{\chi,z}(G)$-gradation*

$$V = \bigoplus_{\zeta \in \mathrm{Aut}_{\chi,z}(G)} V_\zeta$$

and an automorphism $\varphi\colon V \to V$ fulfilling
– *$\varphi(V_\zeta) = V_\zeta$ for all $\zeta \in \mathrm{Aut}_{\chi,z}(G)$,*
– *$\varphi^n = \mathrm{id}$ if $\lambda(z^n - 1_G) \neq 0$.*

Then the pair (V, s), where $s\colon H_{\mathcal{D}} \otimes V \longrightarrow V \otimes H_{\mathcal{D}}$ is the map defined by

$$s(gx^i \otimes v) := \varphi^i(v) \otimes \zeta(g)x^i \qquad \text{for all } v \in V_\zeta, \tag{4.27}$$

is a left $H_{\mathcal{D}}$-space. Furthermore, all the left $H_{\mathcal{D}}$-spaces with underlying k-vector space V have this form.

Proof. It is easy to check that the map s defined by (4.27) is compatible with the unit, the counit, the multiplication map and the braid of $H_{\mathcal{D}}$. So, by Remark 2.4, in order to check that s is a left transposition it suffices to verify that

$$(s \otimes H_{\mathcal{D}}) \circ (H_{\mathcal{D}} \otimes s) \circ (\Delta \otimes V)(x \otimes v) = (V \otimes \Delta) \circ s(x \otimes v)$$

and

$$(s \otimes H_{\mathcal{D}}) \circ (H_{\mathcal{D}} \otimes s) \circ (\Delta \otimes V)(g \otimes v) = (V \otimes \Delta) \circ s(g \otimes v) \quad \text{for } g \in G,$$

which is clear.

Conversely, assume that (V, s) is a left $H_{\mathcal{D}}$-space. By Proposition 4.4 and Theorem 4.3, we know that there exists an automorphism φ of V and a gradation

$$V = \bigoplus_{\zeta \in \mathrm{Aut}(G)} V_\zeta$$

of V, such that $s(g \otimes v) = v \otimes \zeta(g)$ and $s(x \otimes v) = \varphi(v) \otimes x$ for all $g \in G$ and $v \in V_\zeta$. Again by Proposition 4.4, we also know that $s(z \otimes v) = v \otimes z$ for all $v \in V$. Therefore, if $V_\zeta \neq 0$, then $\zeta(z) = z$. Now, let $g \in G$ and $v \in V_\zeta \setminus \{0\}$. A direct computation shows that

$$\begin{aligned}
\varphi(v) \otimes x\zeta(g) &= s(xg \otimes v)\\
&= s\big(\chi(g)gx \otimes v\big)\\
&= \sum_{\phi \in \mathrm{Aut}(G)} \varphi(v)_\phi \otimes \chi(g)\phi(g)x\\
&= \sum_{\phi \in \mathrm{Aut}(G)} \varphi(v)_\phi \otimes \chi(g)\chi(\phi(g))^{-1}x\phi(g).
\end{aligned}$$

Since g is arbitrary, from this it follows that $\varphi(v)_\phi = 0$ for $\phi \neq \zeta$ and that $\chi(\zeta(g)) = \chi(g)$. So

$$\varphi(V_\zeta) = V_\zeta \qquad \text{and} \qquad \chi \circ \zeta = \chi.$$

Lastly, suppose that $\lambda(z^n - 1_G) \neq 0$. Then

$$v \otimes \lambda(z^n - 1_G) = s\big(\lambda(z^n - 1_G) \otimes v\big) = s(x^n \otimes v) = \varphi^n(v) \otimes x^n = \varphi^n(v) \otimes \lambda(z^n - 1_G),$$

for each $v \in V$. This shows that $\varphi^n = \mathrm{id}$ and finishes the proof. □

Our next aim is to characterize the right $H_{\mathcal{D}}$-braided comodule structures. Let (V, s) be a left $H_{\mathcal{D}}$-space and let

$$V = \bigoplus_{\zeta \in \mathrm{Aut}_{\chi,z}(G)} V_\zeta \qquad \text{and} \qquad \varphi\colon V \longrightarrow V$$

be the decomposition and the automorphism associated with the left transposition s. Each map

$$\nu\colon V \longrightarrow V \otimes H_{\mathcal{D}}$$

determines and it is determined by a family of maps

$$\big(U_i^g\colon V \longrightarrow V\big)_{g \in G,\, 0 \le i < n} \tag{4.28}$$

via

$$\nu(v) := \sum_{\substack{g\in G\\ 0\le i<n}} U_i^g(v)\otimes gx^i. \tag{4.29}$$

Proposition 4.6. *The pair (V, s) is a right $H_{\mathcal{D}}$-comodule via ν iff*
(1) $U_i^g(V_\zeta) \subseteq V_\zeta$ *for all* $g\in G$, $\zeta \in \mathrm{Aut}_{\chi,z}(G)$ *and* $i\in\{0,1\}$,
(2) $(U_0^g)_{g\in G}$ *is a complete family of orthogonal idempotents,*
(3) $U_1^g = U_0^g\circ U_1^g = U_1^g\circ U_0^{gz}$ *for all* $g\in G$,
(4) $U_i^g = \frac{1}{(i)!_{qp}} U_1^g\circ U_1^{gz}\circ\cdots\circ U_1^{gz^{i-1}}$ *for all* $g\in G$ *and* $1\le i<n$,
(5) $U_1^g\circ U_1^{gz}\circ\cdots\circ U_1^{gz^{n-1}} = 0$ *for all* $g\in G$,
(6) $\varphi\circ U_0^g = U_0^g\circ\varphi$ *and* $q\varphi\circ U_1^g = U_1^g\circ\varphi$ *for all* $g\in G$.

Proof. For each $v\in V_\zeta$, $h\in G$ and $0\le j<n$, write

$$U_j^h(v) = \sum_{\phi\in\mathrm{Aut}_{\chi,z}(G)} U_j^h(v)_\phi \qquad \text{with } U_j^h(v)_\phi \in V_\phi.$$

Since

$$(\nu\otimes H_{\mathcal{D}})\circ s(gx^i\otimes v) = \sum_{\substack{h\in G\\ 0\le j<n}} U_j^h\big(\varphi^i(v)\big)\otimes hx^j\otimes\zeta(g)x^i$$

and

$$(V\otimes c_q)\circ(s\otimes H_{\mathcal{D}})\circ(H_{\mathcal{D}}\otimes\nu)(gx^i\otimes v) = \sum_{\substack{h\in G\\ 0\le j<n}}\sum_{\phi\in\mathrm{Aut}_{\chi,z}(G)} q^{ij}\varphi^i\big(U_j^h(v)_\phi\big)\otimes hx^j\otimes\phi(g)x^i$$

the map ν satisfies condition (2.4) in Remark 2.17 iff

$$\sum_{\substack{h\in G\\ 0\le j<n}} U_j^h\big(\varphi^i(v)\big)\otimes hx^j\otimes\zeta(g)x^i = \sum_{\substack{h\in G\\ 0\le j<n}}\sum_{\phi\in\mathrm{Aut}_{\chi,z}(G)} q^{ij}\varphi^i\big(U_j^h(v)_\phi\big)\otimes hx^j\otimes\phi(g)x^i,$$

for all $\zeta\in\mathrm{Aut}_{\chi,z}(G)$, $v\in V_\zeta$, $g\in G$ and $0\le i<n$. Since ζ, v and g are arbitrary, $\varphi(V_\phi) = V_\phi$ for all $\phi\in\mathrm{Aut}_{\chi,z}(G)$, and φ is bijective, this happens iff

$$U_j^h(V_\zeta)\subseteq V_\zeta \qquad\text{and}\qquad q^j\varphi\circ U_j^h = U_j^h\circ\varphi, \tag{4.30}$$

for all h, j, and ζ. On the other hand, since $\epsilon(gx^i) = \delta_{0i}$, the map ν is counitary iff

$$\sum_{g\in G} U_0^g = \mathrm{id}, \tag{4.31}$$

and since

$$(V\otimes\Delta)\circ\nu(v) = \sum_{\substack{g\in G\\ 0\le i<n}} U_i^g(v)\otimes\Delta(gx^i) = \sum_{\substack{g\in G\\ 0\le i<n}}\sum_{j=0}^{i}\binom{i}{j}_{qp} U_i^g(v)\otimes gx^j\otimes gz^jx^{i-j}$$

and

$$(\nu\otimes H_{\mathcal{D}})\circ\nu(v) = \sum_{\substack{h\in G\\ 0\le l<n}} \nu\big(U_l^h(v)\big)\otimes hx^l = \sum_{\substack{h\in G\\ 0\le l<n}}\sum_{\substack{g\in G\\ 0\le j<n}} U_j^g\big(U_l^h(v)\big)\otimes gx^j\otimes hx^l,$$

it is coassociative iff

$$U_j^g \circ U_l^h = \begin{cases} \binom{j+l}{j}_{qp} U_{j+l}^g & \text{if } h = gz^j \text{ and } j+l < n, \\ 0 & \text{otherwise.} \end{cases} \tag{4.32}$$

Thus, in order to prove this proposition we must show items (1)–(6) are equivalent to conditions (4.30), (4.31) and (4.32). It is evident that (4.30) implies items (1) and (6), while items (2) and (3) follow from (4.31) and (4.32). Finally, using (4.32) again it is easy to prove by induction on j that items (4) and (5) are also satisfied. Conversely, assume that the maps U_i^g satisfy items (1)–(6). It is clear that item (2) implies condition (4.31), and equality (4.30) follows from items (1), (4) and (6). It remains to prove equality (4.32). We claim that

$$U_i^f \circ U_j^{fz^i} = \begin{cases} \binom{i+j}{i}_{qp} U_{i+j}^f & \text{if } i+j < n, \\ 0 & \text{if } i+j \geq n. \end{cases} \tag{4.33}$$

By item (2) this is true if $j = i = 0$. In order to check it when $j > 0$ and $i = 0$ or $j = 0$ and $i > 0$, it suffices to note that by items (3) and (4),

$$U_0^f \circ U_i^f = \frac{1}{(i)!_{qp}} U_0^f \circ U_1^f \circ \cdots \circ U_1^{fz^{i-1}} = \frac{1}{(i)!_{qp}} U_1^f \circ \cdots \circ U_1^{fz^{i-1}} = U_i^f$$

and

$$U_i^f \circ U_0^{fz^i} = \frac{1}{(i)!_{qp}} U_1^f \circ \cdots \circ U_1^{fz^{i-1}} \circ U_0^{fz^i} = \frac{1}{(i)!_{qp}} U_1^f \circ \cdots \circ U_1^{fz^{i-1}} = U_i^f,$$

respectively. Assume now that $j > 0$ and $i > 0$. Then, by item (4),

$$U_i^f \circ U_j^{fz^i} = \frac{1}{(i)!_{qp}} \frac{1}{(j)!_{qp}} U_1^f \circ \cdots \circ U_1^{fz^{i-1}} \circ U_1^{fz^i} \circ \cdots \circ U_1^{fz^{i+j-1}},$$

and the claim follows immediately from items (4) and (5). Note now that (4.33) implies that

$$U_i^f \circ U_j^h = U_i^f \circ U_0^{fz^i} \circ U_0^h \circ U_j^h$$

which, combined with item (2), shows that

$$U_i^f \circ U_j^h = 0 \quad \text{if } h \neq fz^i,$$

finishing the proof of (4.32). □

Corollary 4.7. *Let V be a k-vector space. Each data consisting of*
- *a $G \times \mathrm{Aut}_{\chi,z}(G)$-gradation $V = \bigoplus_{(g,\zeta)\in G\times \mathrm{Aut}_{\chi,z}(G)} V_{g,\zeta}$ of V,*
- *an automorphism $\varphi\colon V \to V$ of V such that*

$$\varphi^n = \mathrm{id} \text{ if } \lambda(z^n - 1_G) \neq 0 \qquad \text{and} \qquad \varphi(V_{g,\zeta}) = V_{g,\zeta} \text{ for all } (g,\zeta) \in G \times \mathrm{Aut}_{\chi,z}(G),$$

- *a map $U\colon V \to V$, such that*

$$U \circ \varphi = q\varphi \circ U, \quad U^n = 0 \qquad \text{and} \qquad U(V_{g,\zeta}) \subseteq V_{gz^{-1},\zeta} \text{ for all } (g,\zeta) \in G \times \mathrm{Aut}_{\chi,z}(G),$$

determines univocally a right $H_{\mathcal{D}}$-comodule (V,s), in which

– *$s\colon H_{\mathcal{D}} \otimes V \longrightarrow V \otimes H_{\mathcal{D}}$ is the left transposition of $H_{\mathcal{D}}$ on V associated as in (4.27) with the map φ and the $\mathrm{Aut}_{\chi,z}(G)$-gradation of V*

$$V = \bigoplus_{\zeta\in \mathrm{Aut}_{\chi,z}(G)} V_\zeta, \qquad \text{where } V_\zeta := \bigoplus_{g\in G} V_{g,\zeta},$$

– *the coaction $\nu\colon V \to V \otimes H_{\mathcal{D}}$ of (V,s) is defined by*

$$\nu(v) := \sum_{j=0}^{n-1} \frac{1}{(j)!_{qp}} U^j(v) \otimes z^{-j} gx^j \qquad \text{for all } v \in V_g,$$

where, for all $g \in G$,

$$V_g := \bigoplus_{\zeta\in \mathrm{Aut}_{\chi,z}(G)} V_{g,\zeta}.$$

Furthermore, all the right $H_{\mathcal{D}}$-braided comodules with underlying k-vector space V have this form.

Proof. Assume we have a data as in the statement. Then we define a family of maps as in (4.28), by
– $U_0^g(v) := \pi_g(v)$, where $\pi_g\colon V \to V_g$ is the projection onto V_g along $\bigoplus_{h\in G\setminus\{g\}} V_h$,
– $U_1^g := U_0^g \circ U \circ U_0^{gz}$,
– $U_j^g = \frac{1}{(j)!_{qp}} U_1^g \circ U_1^{gz} \circ \cdots \circ U_1^{gz^{j-1}}$ for all $1 < j < n$.
We must check that these maps satisfy the conditions required in Proposition 4.6. Item (1) is fulfilled since $U(V_\zeta) \subseteq V_\zeta$ and $U_0^g(V_\zeta) \subseteq V_\zeta$ for all $g \in G$, items (2)–(4) hold by the very definition of the maps U_i^g, and item (6) is fulfilled since $\varphi(V_g) = V_g$ for all $g \in G$ and $U \circ \varphi = q\,\varphi \circ U$. We next prove item (5). Since $U^n = 0$, this trivially follows if we prove that, for all $j \geq 1$,

$$U_1^g \circ U_1^{gz} \circ \cdots \circ U_1^{gz^{j-1}}(v) = \begin{cases} U^j(v) & \text{if } v \in V_{gz^j}, \\ 0 & \text{if } v \in V_h \text{ with } h \neq gz^j. \end{cases}$$

Clearly if $v \in V_h$ with $h \neq gz^j$, then $U_0^{gz^j}(v) = 0$, and so

$$U_1^g \circ U_1^{gz} \circ \cdots \circ U_1^{gz^{j-1}}(v) = U_1^g \circ \cdots \circ U_1^{gz^{j-1}} \circ U_0^{gz^j}(v) = 0.$$

It remains to consider the case $v \in V_{gz^j,\zeta}$. We proceed by induction on j. If $j = 1$, then

$$U_1^g(v) = U_0^g \circ U \circ U_0^{gz}(v) = U_0^g \circ U(v) = U(v),$$

because $U(v) \in V_g$. Assume now $j > 1$ and the result is valid for $j - 1$. Then

$$U_1^g \circ U_1^{gz} \circ \cdots \circ U_1^{gz^{j-1}}(v) = U_1^g \circ \cdots \circ U_1^{gz^{j-2}} \circ U_1^{gz^{j-1}}(v) = U_1^g \circ \cdots \circ U_1^{gz^{j-2}} \circ U(v) = U^j(v),$$

where the last equality follows from the inductive hypothesis and the fact that $U(v) \in V_{gz^{j-1}}$.

Conversely assume that (V,s) is a right $H_{\mathcal{D}}$-comodule via a coaction $\nu\colon V \to V \otimes H_{\mathcal{D}}$. Let

$$V = \bigoplus_{\zeta\in \mathrm{Aut}_{\chi,z}(G)} V_\zeta \qquad \text{and} \qquad \varphi\colon V \longrightarrow V$$

be the decomposition and the automorphism associated with s (see Proposition 4.5). By items (1) and (2) of Proposition 4.6, we know that, for each $\zeta \in \mathrm{Aut}_{\chi,z}(G)$, the maps U_0^g's determine by restriction a complete family $(U_0^g\colon V_\zeta \to V_\zeta)_{g\in G}$ of orthogonal idempotents. Let

$$V_\zeta = \bigoplus_{g\in G} V_{g,\zeta}$$

be the decomposition associated with this family. Clearly

$$V = \bigoplus_{(g,\zeta)\in G\times \mathrm{Aut}_{\chi,z}(G)} V_{g,\zeta}.$$

By item (6) of Proposition 4.6, we have $\varphi \circ U_0^g = U_0^g \circ \varphi$ for all $g \in G$. Since, by Proposition 4.5 we know that $\varphi(V_\zeta) = V_\zeta$ for all $\zeta \in \mathrm{Aut}(G)$, this implies that

$$\varphi(V_{g,\zeta}) = V_{g,\zeta} \qquad \text{for all } (g,\zeta) \in G \times \mathrm{Aut}_{\chi,z}(G).$$

We now define a map $U\colon V \to V$ by

$$U(v) = U_1^g(v) \qquad \text{for all } v \in V_{gz,\zeta}.$$

From items (3) and (5) of Proposition 4.6, it follows that $U^n = 0$, and using the second equality in item (6) of the same proposition, we obtain that $\varphi \circ U = q\, U \circ \varphi$. Finally, by items (1) and (3) of Proposition 4.6, we have $U(V_{g,\zeta}) \subseteq V_{z^{-1}g,\zeta}$ for all $(g,\zeta) \in G \times \mathrm{Aut}_{\chi,z}(G)$.

We leave the reader the task to prove that the construction given in the two parts of this proof are reciprocal one of each other. □

Remark 4.8. Assume that $q = 1$, or equivalently, that $H_{\mathcal{D}}$ is a Krop-Radford Hopf algebra. In this case (V, s) is a standard $H_{\mathcal{D}}$-comodule (that is, s is the flip) iff $V_{g,\zeta} = 0$ for $\zeta \neq \mathrm{id}$ and φ is the identity map. Hence, in order to obtain the standard $H_{\mathcal{D}}$-comodule structures, the conditions that we need verify (given in Corollary 4.7) are considerably simplified.

Corollary 4.9. *With the notations of the previous corollary,* $V^{\mathrm{coH}} = V_{1_G} \cap \ker(U)$.

Proof. This is an immediate consequence of Corollary 4.7. □

Proposition 4.10. *Let B be an algebra. If*

$$B = \bigoplus_{\zeta\in \mathrm{Aut}_{\chi,z}(G)^{\mathrm{op}}} B_\zeta \tag{4.34}$$

is an $\mathrm{Aut}_{\chi,z}(G)^{\mathrm{op}}$-gradation of B as an algebra and $\varphi\colon B \to B$ an automorphism of algebras that satisfies

– $\varphi(B_\zeta) = B_\zeta$ *for all* $\zeta \in \mathrm{Aut}_{\chi,z}(G)$,

– $\varphi^n = \mathrm{id}$ *if* $\lambda(z^n - 1_G) \neq 0$,

then, the map $s\colon H_{\mathcal{D}} \otimes B \longrightarrow B \otimes H_{\mathcal{D}}$, given by

$$s(gx^i \otimes b) = \varphi^i(b) \otimes \zeta(g)x^i \quad \text{for all } b \in B_\zeta, \tag{4.35}$$

is a left transposition of $H_{\mathcal{D}}$ on the algebra B. Furthermore, all the left transpositions of $H_{\mathcal{D}}$ on B have this form.

Proof. By Proposition 4.5 in order to prove this it suffices to check that the formula (4.35) defines a map compatible with the unit and the multiplication map of B iff (4.40) is a gradation of B as an algebra and φ is an automorphism of algebras. We left this task to the reader. □

The group $\mathrm{Aut}_{\chi,z}(G)$ acts on G^{op} via $\zeta \cdot g := \zeta(g)$. So, it makes sense to consider the semidirect product $G(\chi, z) := G^{\mathrm{op}} \rtimes \mathrm{Aut}_{\chi,z}(G)$.

Definition 4.11. Let $\mathcal{D} = (G, \chi, z, \lambda, q)$ be as in Corollary 3.8 and let B be an algebra endowed with an algebra automorphism $\varphi\colon B \to B$, a map $U\colon B \to B$ and a $G(\chi, z)^{\mathrm{op}}$-gradation

$$B = \bigoplus_{(g,\zeta)\in G(\chi,z)^{\mathrm{op}}} B_{g,\zeta}, \tag{4.36}$$

of B as a vector space. We will say that the decomposition (4.36) of B is *compatible with the pair* $(U,\mathcal{D})$ if one of the following conditions is fulfilled:

(1) $\lambda(z^n - 1_G) = 0$ and (4.36) is a gradation of B as an algebra.

(2) $\lambda(z^n - 1_G) \neq 0$, $1_B \in B_{1_G,\mathrm{id}}$,

$$B_{g,\zeta}B_{h,\phi} \subseteq B_{\phi(g)h,\phi\circ\zeta} \oplus B_{z^{-n}\phi(g)h,\phi\circ\zeta} \qquad \text{for all } (g,\zeta),(h,\phi) \in G(\chi,z)^{\mathrm{op}}, \tag{4.37}$$

and, for each $b \in B_{g,\zeta}$ and $c \in B_{h,\phi}$, the homogeneous component $(bc)_{z^{-n}\phi(g)h,\phi\circ\zeta}$ of bc of degree $(z^{-n}\phi(g)h,\phi\circ\zeta)$ is given by

$$(bc)_{z^{-n}\phi(g)h,\phi\circ\zeta} := -\lambda \sum_{j=1}^{n-1} \frac{p^{j^2}\chi(h)^j}{(j)!_{qp}(n-j)!_{qp}} U^j(b)\varphi^j\big(U^{n-j}(c)\big). \tag{4.38}$$

Theorem 4.12. *Let B be an algebra. Each data consisting of*

– *a $G(\chi,z)^{\mathrm{op}}$-gradation*

$$B = \bigoplus_{(g,\zeta)\in G(\chi,z)^{\mathrm{op}}} B_{g,\zeta}, \tag{4.39}$$

of B as a vector space,

– *an algebra automorphism $\varphi\colon B \to B$ of B such that*

$$\varphi^n = \mathrm{id} \text{ if } \lambda(z^n - 1_G) \neq 0 \qquad \text{and} \qquad \varphi(B_{g,\zeta}) = B_{g,\zeta} \text{ for all } (g,\zeta) \in G(\chi,z)^{\mathrm{op}},$$

– *a map $U\colon B \to B$ such that*

$$\begin{aligned}
&\text{the decomposition (3.39) is compatible with the pair } (U,\mathcal{D}),\\
&U\circ\varphi = q\varphi\circ U,\\
&U^n = 0,\\
&U(B_{g,\zeta}) \subseteq B_{z^{-1}g,\zeta} \qquad \text{for all } (g,\zeta) \in G(\chi,z)^{\mathrm{op}}
\end{aligned}$$

and

$$U(bc) = bU(c) + \chi(h)U(b)\varphi(c) \quad \text{for all } b \in B \text{ and } c \in B_h, \tag{4.40}$$

where

$$B_h := \bigoplus_{\zeta\in\mathrm{Aut}_{\chi,z}(G)} B_{h,\zeta} \qquad \text{for all } h \in G,$$

determines a right $H_{\mathcal{D}}$-comodule algebra (B,s), in which $s\colon H_{\mathcal{D}}\otimes B \longrightarrow B\otimes H_{\mathcal{D}}$ is the left transposition of $H_{\mathcal{D}}$ on B associated with the map φ and the $\mathrm{Aut}_{\chi,z}(G)^{\mathrm{op}}$-gradation of B

$$B = \bigoplus_{\zeta\in\mathrm{Aut}_{\chi,z}(G)^{\mathrm{op}}} B_\zeta, \tag{4.41}$$

where $B_\zeta := \bigoplus_{g\in G} B_{g,\zeta}$. The coaction $\nu\colon B \longrightarrow B\otimes H_{\mathcal{D}}$ of (B,s) is given by

$$\nu(b) := \sum_{i=0}^{n-1} \frac{1}{(i)!_{qp}} U^i(b)\otimes z^{-i}gx^i \qquad \text{for all } b \in B_g. \tag{4.42}$$

Furthermore, all the right $H_{\mathcal{D}}$-braided comodule algebra structures with underlying algebra B are obtained in this way.

Proof. Let (B,s) be a right $H_{\mathcal{D}}$-comodule, with s a left transposition of $H_{\mathcal{D}}$ on the algebra B. Consider the subspaces $B_{g,\zeta}$ of B and the maps φ and U associated with (B,s) as in Corollary 4.7. By that corollary, Proposition 4.10, and Remarks 2.17 and 2.22, in order to finish the proof it suffices to show that the coaction ν of (B,s) satisfy

$$\nu(1_B)=1_B\otimes 1_{H_\mathcal{D}} \qquad \text{and} \qquad \nu\circ\mu_B=(\mu_B\otimes\mu_{H_\mathcal{D}})\circ(B\otimes s\otimes H_\mathcal{D})\circ(\nu\otimes\nu) \tag{4.43}$$

iff the decomposition

$$B=\bigoplus_{(g,\zeta)\in G(\chi,z)^{\mathrm{op}}} B_{g,\zeta}, \tag{4.44}$$

of B is compatible with $(U,\mathcal{D})$ and U satisfies condition (4.40). First we make some remarks. Let $b\in B_{g,\zeta}$ and $c\in B_{h,\phi}$. By the definition of ν,

$$\nu(bc)=\sum_{i=0}^{n-1}\sum_{f\in G}\frac{1}{(i)!_{qp}}U^i\big((bc)_f\big)\otimes z^{-i}fx^i. \tag{4.45}$$

On the other hand, a direct computation shows that

$$\begin{aligned} F(b,c)&=\sum_{j=0}^{n-1}\sum_{i=0}^{n-1}\frac{p^{-ij}\chi(h)^j}{(j)!_{qp}(i)!_{qp}}U^j(b)\varphi^j\big(U^i(c)\big)\otimes z^{-i-j}\phi(g)hx^{i+j}\\ &=\sum_{u=0}^{2n-2}\sum_{\substack{j=0\\0\le u-j<n}}^{n-1}\frac{p^{-(u-j)j}\chi(h)^j}{(j)!_{qp}(u-j)!_{qp}}U^j(b)\varphi^j\big(U^{u-j}(c)\big)\otimes z^{-u}\phi(g)hx^u, \end{aligned} \tag{4.46}$$

where to abbreviate expressions we write

$$F(b,c):=(\mu_B\otimes\mu_{H_\mathcal{D}})\circ(B\otimes s\otimes H_\mathcal{D})(\nu(b)\otimes\nu(c)).$$

Set

$$A_u^j(b,c):=\frac{p^{(j-u)j}\chi(h)^j}{(j)!_{qp}(u-j)!_{qp}}U^j(b)\varphi^j\big(U^{u-j}(c)\big).$$

Since $x^n=\lambda(z^n-1_G)$, equality (4.46) becomes

$$\begin{aligned} F(b,c)&=\sum_{i=0}^{n-1}\sum_{j=0}^{i}A_i^j(b,c)\otimes z^{-i}\phi(g)hx^i+\lambda\sum_{i=0}^{n-1}\sum_{j=i+1}^{n-i}A_{i+n}^j(b,c)\otimes z^{-n-i}\phi(g)h(z^n-1_G)x^i\\ &=\sum_{i=0}^{n-1}\left(\sum_{j=0}^{i}A_i^j(b,c)+\lambda\sum_{j=i+1}^{n-1}A_{i+n}^j(b,c)\right)\otimes z^{-i}\phi(g)hx^i\\ &\quad-\sum_{i=0}^{n-1}\sum_{j=i+1}^{n-1}\lambda A_{i+n}^j(b,c)\otimes z^{-n-i}\phi(g)hx^i. \end{aligned} \tag{4.47}$$

Next we prove the part $\Rightarrow$). We assume that $\lambda(z^n-1_G)\neq 0$ and leave the case $\lambda(z^n-1_G)=0$, which is easier, to the reader. To begin with note that by the first equality in (4.43),

$$1_B\in B_{1_G}=\bigoplus_{\zeta\in\mathrm{Aut}_{\chi,z}(G)}B_{1_G,\zeta}.$$

Since, on the other hand, (4.41) is a gradation of B as an algebra, necessarily

$$1_B\in B_{\mathrm{id}}=\bigoplus_{g\in G}B_{g,\mathrm{id}},$$

and so $1_B \in B_{1_G,\mathrm{id}}$. Recall that $b \in B_{g,\zeta}$ and $c \in B_{h,\phi}$. Since by the second equality in (4.43), equations (4.45) and (4.47) coincide, we have

$$(bc)_f = \begin{cases} A_0^0(b,c) + \lambda \sum_{j=1}^{n-1} A_n^j(b,c) & \text{if } f = \phi(g)h, \\ -\lambda \sum_{j=1}^{n-1} A_n^j(b,c) & \text{if } f = z^{-n}\phi(g)h, \\ 0 & \text{otherwise,} \end{cases} \tag{4.48}$$

and

$$U\big((bc)_f\big) = \begin{cases} \sum_{j=0}^{1} A_1^j(b,c) + \lambda \sum_{j=2}^{n-1} A_{1+n}^j(b,c) & \text{if } f = \phi(g)h, \\ -\lambda \sum_{j=2}^{n-1} A_{1+n}^j(b,c) & \text{if } f = z^{-n}\phi(g)h, \\ 0 & \text{otherwise.} \end{cases} \tag{4.49}$$

Since, by Proposition 4.10, we know that $bc \in B_{\phi\circ\zeta}$, from equality (4.48) it follows easily that the decomposition (4.39) is compatible with $(U, \mathcal{D})$ (recall that if $\lambda(z^n - 1_G) \neq 0$, then $p^n = 1$). Finally, by (4.49)

$$\begin{aligned} U(bc) &= \sum_f U\big((bc)_f\big) \\ &= \sum_{j=0}^{1} A_1^j(b,c) + \lambda \sum_{j=2}^{n-1} A_{1+n}^j(b,c) - \lambda \sum_{j=2}^{n-1} A_{1+n}^j(b,c) \\ &= A_1^0(b,c) + A_1^1(b,c), \end{aligned}$$

and so, (4.40) is true.

We now prove the part $\Leftarrow$). So we assume that the decomposition (4.44) is compatible with $(U, \mathcal{D})$ and that U satisfies (4.40). Again we consider the case $\lambda(z^n - 1_G) \neq 0$ and leave the case $\lambda(z^n - 1_G) = 0$, which is easier, to the reader. To begin with note that $\nu(1_B) = 1_B \otimes 1_{H_{\mathcal{D}}}$, because

$$1_B \in B_{1_G} \qquad \text{and} \qquad U(1_B) = 1_B U(1_B) + U(1_B)1_B \Rightarrow U(1_B) = 0.$$

So, we are reduced to prove that the second condition in (4.43) is fulfilled. By equalities (4.45) and (4.47), this is equivalent to prove that for all $0 \leq i < n$ and $f \in G$,

$$\frac{1}{(i)!_{qp}} U^i\big((bc)_f\big) = \begin{cases} \sum_{j=0}^{i} A_i^j(b,c) + \lambda \sum_{j=i+1}^{n-1} A_{i+n}^j(b,c) & \text{if } f = \phi(g)h, \\ -\lambda \sum_{j=i+1}^{n-1} A_{i+n}^j(b,c) & \text{if } f = z^{-n}\phi(g)h, \\ 0 & \text{otherwise.} \end{cases} \tag{4.50}$$

For $i = 0$ this follows easily from the fact that equality (4.38) holds and

$$A_0^0(b,c) = bc = (bc)_{\phi(g)h,\phi\circ\zeta} + (bc)_{z^{-n}\phi(g)h,\phi\circ\zeta},$$

since the decomposition (4.44) is compatible with $(U,\mathcal{D})$. Assume by inductive hypothesis that equality (4.38) is true for i and that $i < n-1$. This implies that

$$\frac{1}{(i)!_{qp}}U^{i+1}\big((bc)_f\big) = \begin{cases} \sum_{j=0}^{i} U\big(A_i^j(b,c)\big) + \lambda \sum_{j=i+1}^{n-1} U\big(A_{i+n}^j(b,c)\big) & \text{if } f = \phi(g)h,\\ -\lambda \sum_{j=i+1}^{n-1} U\big(A_{i+n}^j(b,c)\big) & \text{if } f = z^{-n}\phi(g)h,\\ 0 & \text{otherwise.}\end{cases}$$

So, we must prove that

$$\frac{1}{(i+1)_{qp}}\sum_{j=i+1}^{n-1} U\big(A_{i+n}^j(b,c)\big) = \sum_{j=i+2}^{n-1} A_{i+1+n}^j(b,c) \tag{4.51}$$

and

$$\frac{1}{(i+1)_{qp}}\sum_{j=0}^{i} U\big(A_i^j(b,c)\big) = \sum_{j=0}^{i+1} A_{i+1}^j(b,c). \tag{4.52}$$

Recall again that $b \in B_{g,\zeta}$ and $c \in B_{h,\phi}$. Using the equality (4.40) and the facts that $U \circ \varphi = q\varphi \circ U$, $U^u(c) \in B_{z^{-u}h,\phi}$ for all $u \in \mathbb{N}$, and $p^n = 1$, we obtain

$$\begin{aligned} U\big(A_i^j(b,c)\big) &= \frac{p^{(j-i)j}\chi(h)^j}{(j)!_{qp}(i-j)!_{qp}} U\big(U^j(b)\varphi^j\big(U^{i-j}(c)\big)\big)\\ &= \frac{p^{(j-i)j}\chi(h)^j}{(j)!_{qp}(i-j)!_{qp}} \Big(q^jU^j(b)\varphi^j\big(U^{i+1-j}(c)\big) + p^{j-i}\chi(h)U^{j+1}(b)\varphi^{j+1}\big(U^{i-j}(c)\big)\Big) \end{aligned}$$

and

$$\begin{aligned} U\big(A_{i+n}^j(b,c)\big) &= \frac{p^{(j-i)j}\chi(h)^j}{(j)!_{qp}(i+n-j)!_{qp}} U\big(U^j(b)\varphi^j\big(U^{i+n-j}(c)\big)\big)\\ &= \frac{p^{(j-i)j}\chi(h)^j}{(j)!_{qp}(i+n-j)!_{qp}} \Big(q^jU^j(b)\varphi^j\big(U^{j+1+n-j}(c)\big) + p^{j-i}\chi(h)U^{j+1}(b)\varphi^{j+1}\big(U^{i+n-j}(c)\big)\Big). \end{aligned}$$

Since $U^n = 0$, this implies that

$$\begin{aligned} \sum_{j=0}^{i} U\big(A_i^j(b,c)\big) &= \sum_{j=0}^{i} \frac{p^{(j-i)j}\chi(h)^jq^j}{(j)!_{qp}(i-j)!_{qp}} U^j(b)\varphi^j\big(U^{i+1-j}(c)\big)\\ &\quad + \sum_{j=0}^{i} \frac{p^{(j-i)(j+1)}\chi(h)^{j+1}}{(j)!_{qp}(i-j)!_{qp}} U^{j+1}(b)\varphi^{j+1}\big(U^{i-j}(c)\big)\\ &= \sum_{j=0}^{i} \frac{p^{(j-i)j}\chi(h)^jq^j}{(j)!_{qp}(i-j)!_{qp}} U^j(b)\varphi^j\big(U^{i+1-j}(c)\big)\\ &\quad + \sum_{j=1}^{i+1} \frac{p^{(j-i-1)j}\chi(h)^j}{(j-1)!_{qp}(i+1-j)!_{qp}} U^j(b)\varphi^j\big(U^{i+1-j}(c)\big) \end{aligned}$$

and

$$\sum_{j=i+1}^{n-1} U\big(A^j_{i+n}(b,c)\big) = \sum_{j=i+1}^{n-1} \frac{p^{(j-i)j}\chi(h)^j q^j}{(j)!_{qp}(i+n-j)!_{qp}} U^j(b)\varphi^j\big(U^{i+1+n-j}(c)\big)$$
$$+ \sum_{j=i+1}^{n-1} \frac{p^{(j-i)(j+1)}\chi(h)^{j+1}}{(j)!_{qp}(i+n-j)!_{qp}} U^{j+1}(b)\varphi^{j+1}\big(U^{i+n-j}(c)\big)$$
$$= \sum_{j=i+2}^{n-1} \frac{p^{(j-i)j}\chi(h)^j q^j}{(j)!_{qp}(i+n-j)!_{qp}} U^j(b)\varphi^j\big(U^{i+1+n-j}(c)\big)$$
$$+ \sum_{j=i+2}^{n-1} \frac{p^{(j-i-1)j}\chi(h)^j}{(j-1)!_{qp}(i+1+n-j)!_{qp}} U^j(b)\varphi^j\big(U^{i+1+n-j}(c)\big).$$

Consequently in order to finish the proof of equalities (4.51) and (4.52) it suffices to see that

$$\frac{(i+1)_{qp}}{(i+1)!_{qp}} = \frac{1}{(i)!_{qp}},$$
$$\frac{(i+1)_{qp}\chi(h)^{i+1}}{(i+1)!_{qp}} = \frac{\chi(h)^{i+1}}{(i)!_{qp}},$$
$$\frac{(i+1)_{qp}p^{(j-i-1)j}\chi(h)^j}{(j)!_{qp}(i+1-j)!_{qp}} = \frac{p^{(j-i)j}\chi(h)^j q^j}{(j)!_{qp}(i-j)!_{qp}} + \frac{p^{(j-i-1)j}\chi(h)^j}{(j-1)!_{qp}(i+1-j)!_{qp}} \qquad \text{for } 1\le j\le i$$

and

$$\frac{(i+1)_{qp}p^{(j-i-1)j}\chi(h)^j}{(j)!_{qp}(i+1+n-j)!_{qp}} = \frac{p^{(j-i)j}\chi(h)^j q^j}{(j)!_{qp}(i+n-j)!_{qp}} + \frac{p^{(j-i-1)j}\chi(h)^j}{(j-1)!_{qp}(i+1+n-j)!_{qp}} \qquad \text{for } i+2\le j< n.$$

But the first two equalities are trivial and the last ones are equivalent to

$$(i+1)_{qp} = p^j q^j (i+1-j)_{qp} + (j)_{qp} \qquad \text{for } 1\le j\le i$$

and

$$(i+1)_{qp} = p^j q^j (i+1+n-j)_{qp} + (j)_{qp} \qquad \text{for } i+2\le j< n,$$

which can be easily checked. □

Remark 4.13. Using that $\chi\circ\phi=\chi$ for all $\phi\in \mathrm{Aut}_{\chi,z}(G)$ and that if $\lambda(z^n-1_G)\ne 0$, then $\chi(z)^n=1$, it is easy to see that B is a B-bimodule via

$$b_1\cdot b_2\cdot b_3 := \chi(h)b_1b_2\varphi(b_3) \qquad \text{for all } b_1,b_2\in B \text{ and } b_2\in B_h.$$

Let $B\ltimes B$ be the cartesian product $B\times B$ endowed with the multiplication

$$(b_1,c_1)(b_2,c_2) = (b_1b_2, b_1\cdot c_2 + c_1\cdot b_2).$$

It is well known that $B\ltimes B$ is an unitary associative algebra and that a map $U\colon B\to B$ satisfies equality (4.40) if and only if the map $\theta\colon B\to B\ltimes B$, defined by $\theta(b):=(b,U(b))$, is a morphism of algebras.

Remark 4.14. Assume that $q=1$. Then (B,s) is a standard $H_{\mathcal{D}}$-comodule algebra (that is, s is the flip) iff $B_{g,\zeta}=0$ for $\zeta\ne \mathrm{id}$ and φ is the identity map. Hence, in order to obtain the standard $H_{\mathcal{D}}$-comodule algebra structures, the conditions that we need to verify (given in Theorem 4.12) are considerably simplified.

Example 4.15. Fix $n > 1$ in $\mathbb{N}$. Assume that k contains a root of unity ξ of order n. Let

$$H_{n^2} := k\langle g, x | g^n = 1,\ x^n = 0 \text{ and } gx + \xi xg = 0\rangle$$

be the Taft algebra. Clearly $H_{n^2} = H_{\mathcal{D}}$, where $\mathcal{D} = (G, \chi, z, \lambda, q)$ with

$$G := \{1, g, \dots, g^{n-1}\}, \quad z = g, \quad \lambda = 1, \quad q = 1 \quad \text{and} \quad \chi(g) = \xi.$$

Note that $\mathrm{Aut}_{\chi,g}(G) = \mathrm{id}$, and hence $G(\chi, g) = G$. Consider the k-algebra

$$B = \left(\frac{\alpha, \beta, \delta}{k}\right) := k\langle u, v | u^n = \alpha,\ v^n = \beta \text{ and } vu - \xi uv = \delta u^2\rangle,$$

where $\alpha, \beta, \delta \in k$. It is easy to see that B is a G-graded algebra via

$$B = B_1 \oplus B_g \oplus \cdots \oplus B_{g^{n-1}},$$

where

$$B_{g^i} := \bigoplus_{j=0}^{i} k \cdot u^j v^{i-j} \oplus \bigoplus_{j=1}^{n-i-1} k \cdot u^{i+j} v^{n-j}.$$

Let $U\colon B \to B$ be the k-linear map defined by

$$U(u^i v^j) := (1 + \xi + \cdots + \xi^{j-1}) u^i v^{j-1} \qquad \text{for all } i, j \geq 0.$$

Clearly

$$U^n = 0, \qquad U(B_{g^0}) \subseteq B_{g^{n-1}} \qquad \text{and} \qquad U(B_{g^i}) \subseteq B_{g^{i-1}} \quad \text{for } 1 \leq i < n.$$

Moreover, it follows easily from Remark 4.13 that U satisfies equality (4.40) with $\varphi := \mathrm{id}$. Thus, by Theorem 4.12, we know that B is an H_{n^2}-comodule algebra with coaction $\nu\colon B \to B \otimes H_{n^2}$, given by

$$\nu(b) := \sum_{l=0}^{n-1} \frac{1}{(l)!_\xi} U^l(b) \otimes g^{i-l} x^l \qquad \text{for } b \in B_{g^i}.$$

Consequently,

$$\nu(u^j v^{i-j}) = \sum_{l=0}^{i-j} \binom{i-j}{l}_\xi u^j v^{i-j-l} \otimes g^{i-l} x^l \qquad \text{for } 0 \leq j \leq i$$

and

$$\nu(u^{i+j} v^{n-j}) = \sum_{l=0}^{n-j} \binom{n-j}{l}_\xi u^{i+j} v^{n-j-l} \otimes g^{i-l} x^l \qquad \text{for } 1 \leq j < n - i.$$

Finally, by Corollary 4.9, we have $B^{\mathrm{co}H_{n^2}} = B_{g^0} \cap \ker(U) = k$.

Remark 4.16. Let (B, s) be a right $H_{\mathcal{D}}$-comodule algebra and let $B_G := \{b \in B : \nu(b) \in B \otimes kG\}$. Note that

$$B_G = \ker(U) = \bigoplus_{(g,\zeta) \in G(\chi,z)^{\mathrm{op}}} B_{g,\zeta} \cap \ker(U).$$

Moreover,

$$\nu(B_G) \subseteq B_G \otimes kG,$$

because $(\nu \otimes H_{\mathcal{D}}) \circ \nu = B \otimes \Delta) \circ \nu$. Consequently, since

$$(B \otimes c_q) \circ (s \otimes H_{\mathcal{D}}) \circ (H_{\mathcal{D}} \otimes \nu) = (\nu \otimes H_{\mathcal{D}}) \circ s,$$

we have $s(H_{\mathcal{D}} \otimes B_G) \subseteq B_G \otimes H_{\mathcal{D}}$. Similarly $s^{-1}(B_G \otimes H_{\mathcal{D}}) \subseteq H_{\mathcal{D}} \otimes B_G$, and so,

$$s(H_{\mathcal{D}} \otimes B_G) = B_G \otimes H_{\mathcal{D}}.$$

Furthermore B_G is a subalgebra of B because

$$\nu \circ \mu = (\mu_B \otimes \mu_{H_{\mathcal{D}}}) \circ (B \otimes s \otimes H_{\mathcal{D}}) \circ (\nu \otimes \nu).$$

Clearly s induces by restriction a left transposition $\tilde{s}$ of kG on B_G. From the previous discussion it follows that $(B_G, \tilde{s})$ is a right kG-comodule algebra.

5. $H_{\mathcal{D}}$ cleft extensions

Throughout this section we use freely the notations introduced in Section 4 and the characterization of right $H_{\mathcal{D}}$-comodule algebras obtained in Theorem 4.12. Let (B, s) be a right $H_{\mathcal{D}}$-comodule algebra and let $C := B^{\mathrm{co}H_{\mathcal{D}}}$. Recall that, by Corollary 4.9,

$$C = B_{1_G} \cap \ker(U) = \bigoplus_{\zeta \in \mathrm{Aut}_{\chi,z}(G)} B_{1_G,\zeta} \cap \ker(U).$$

Theorem 5.1. *The extension $(C \hookrightarrow B, s)$ is cleft iff there exists $b_x \in B$ and a family $(b_g)_{g\in G}$ of elements of $B^{\times}$, such that*

(a) $b_g \in B_{g,\mathrm{id}} \cap \ker(U)$ *for all* $g \in G$,
(b) $b_x \in B_{z,\mathrm{id}} \cap U^{-1}(1_B)$,
(c) $\varphi(b_x) = qb_x$,
(d) $\varphi(b_g) = b_g$ *for all* $g \in G$.

If this is the case, then the map $\gamma\colon H_{\mathcal{D}} \to B$, defined by $\gamma(gx^i) := b_g b_x^i$, is a cleft map, and its convolution inverse is given by

$$\gamma^{-1}(gx^i) = (-1)^i (qp)^{\frac{i(i-1)}{2}} b_x^i b_{gz^i}^{-1}.$$

Proof. Assume that $(C \hookrightarrow B, s)$ is a cleft extension and fix a cleft map $\gamma\colon H_{\mathcal{D}} \to B$ such that $\gamma(1) = 1$. For every $g \in G$ and $0 \le i < n$, set $b_{gx^i} := \gamma(gx^i)$. Since γ is a right comodule map,

$$\nu(b_g) = b_g \otimes g \qquad \text{and} \qquad \nu(b_x) = 1_B \otimes x + b_x \otimes z,$$

which, by formula (4.42), is equivalent to

$$b_g \in B_g \cap \ker(U) \qquad \text{and} \qquad b_x \in B_z \cap U^{-1}(1).$$

Moreover b_g is invertible for each $g \in G$, because γ is convolution invertible. On the other hand evaluating the equality $(\gamma \otimes H_{\mathcal{D}}) \circ c_q = s \circ (H_{\mathcal{D}} \otimes \gamma)$ in $h \otimes x$, $x \otimes x$, $h \otimes g$ and $x \otimes g$, where $h \in G$ is arbitrary, we obtain that

$$b_x \in B_{\mathrm{id}}, \qquad \varphi(b_x) = qb_x, \qquad b_g \in B_{\mathrm{id}} \qquad \text{and} \qquad \varphi(b_g) = b_g,$$

for all $g \in G$. Thus, items (a)–(d) hold. Conversely, assume that there exist $b_x \in B$ and a family $(b_g)_{g\in G}$ of elements of $B^{\times}$ satisfying statements (a)–(d). We are going to prove that $(C \hookrightarrow B, s)$ is cleft and the map $\gamma\colon H_{\mathcal{D}} \to B$, defined by $\gamma(gx^i) := b_g b_x^i$, is a cleft map. First note that

$$(\gamma \otimes H_{\mathcal{D}}) \circ c_q(gx^i \otimes hx^j) = q^{ij} b_h b_x^j \otimes gx^i = s(gx^i \otimes b_h b_x^j) = s \circ (H_{\mathcal{D}} \otimes \gamma)(gx^i \otimes hx^j),$$

for all $h, g \in G$ and $0 \le i, j < n$. So we must only check that γ is convolution invertible and

$$\nu \circ \gamma(gx^i) = (\gamma \otimes H_{\mathcal{D}}) \circ \Delta(gx^i) \quad \text{for all } g \in G \text{ and } 0 \le i < n. \tag{5.53}$$

For $g = 1$ and $i = 0$ it is evident that this is true. Assume it is true for $g = 1$ and $i = i_0$, and that $i_0 < n-1$. Then

$$\begin{aligned}
\nu \circ \gamma(x^{i_0+1}) &= \nu\big(\gamma(x^{i_0})b_x\big) \\
&= (\mu_B \otimes \mu_{H_{\mathcal{D}}}) \circ (B \otimes s \otimes H_{\mathcal{D}})\big(\nu(\gamma(x^{i_0})) \otimes \nu(b_x)\big) \\
&= \sum_{j=0}^{i_0} \binom{i_0}{j}_{qp} (\mu_B \otimes \mu_{H_{\mathcal{D}}}) \circ (B \otimes s \otimes H_{\mathcal{D}})(b_x^j \otimes z^j x^{i_0-j} \otimes 1_B \otimes x) \\
&\quad + \sum_{j=0}^{i_0} \binom{i_0}{j}_{qp} (\mu_B \otimes \mu_{H_{\mathcal{D}}}) \circ (B \otimes s \otimes H_{\mathcal{D}})(b_x^j \otimes z^j x^{i_0-j} \otimes b_x \otimes z) \\
&= \sum_{j=0}^{i_0} \binom{i_0}{j}_{qp} b_x^j \otimes z^j x^{i_0+1-j} + \sum_{j=0}^{i_0} \binom{i_0}{j}_{qp} b_x^j \varphi^{i_0-j}(b_x) \otimes z^j x^{i_0-j} z \\
&= \sum_{j=0}^{i_0} \binom{i_0}{j}_{qp} b_x^j \otimes z^j x^{i_0+1-j} + \sum_{j=1}^{i_0+1} \binom{i_0}{j-1}_{qp} q^{i_0+1-j} p^{i_0+1-j} b_x^j \otimes z^j x^{i_0+1-j} \\
&= \sum_{j=0}^{i_0} \binom{i_0+1}{j}_{qp} b_x^j \otimes z^j x^{i_0+1-j}.
\end{aligned}$$

Thus, equality (5.53) holds when $g = 1_G$. But then

$$\begin{aligned}
\nu \circ \gamma(gx^i) &= \nu\big(b_g \gamma(x^i)\big) \\
&= \sum_{j=0}^{i} \binom{i}{j}_{qp} (\mu_B \otimes \mu_{H_{\mathcal{D}}}) \circ (B \otimes s \otimes H_{\mathcal{D}})(b_g \otimes g \otimes b_x^j \otimes z^j x^{i-j}) \\
&= \sum_{j=0}^{i} \binom{i}{j}_{qp} b_g b_x^j \otimes g z^j x^{i-j}.
\end{aligned}$$

It remains to check that γ is convolution invertible. As was noted in [2, Section 3],

$$\sum_{j=0}^{i} (-1)^j (qp)^{\frac{j(j-1)}{2}} \binom{i}{j}_{qp} = \begin{cases} 1 & \text{if } i = 0, \\ 0 & \text{if } 0 < i < n. \end{cases} \tag{5.54}$$

Using this it is easy to prove that γ is invertible with

$$\gamma^{-1}(gx^i) = (-1)^i (qp)^{\frac{i(i-1)}{2}} b_x^i b_{gz^i}^{-1},$$

which finishes the proof. □

In the previous theorem we can assume without loss of generality that $b_1 = 1_B$.

Example 5.2. With the notations and assumptions of Example 4.15, the extension $(k \hookrightarrow B, s)$, where s is the flip is cleft if and only if there exist

$$b_x \in B_g \cap U^{-1}(1) = \{\lambda_u \cdot u + v : \lambda_u \in k\} \qquad \text{and} \qquad b_{g^i} \in B_{g^i} \cap \ker(U) = \{\lambda_i \cdot u^i : \lambda_i \in k\},$$

such that the b_{g^i}'s are invertible. Note that since $u^n = \alpha$, this is equivalent to say that $\alpha \in k^\times$.

Theorem 5.3. *Assume that $(C \hookrightarrow B, s)$ is cleft. Take $b_x \in B$ and a family $(b_g)_{g\in G}$ of elements of $B^\times$ with $b_1 = 1_B$, in such a way that conditions (a)–(d) of Theorem 5.1 are fulfilled. Then*

(1) *B is a free left C-module with basis $\{b_g b_x^i : g \in G \text{ and } 0 \le i < n\}$.*

(2) *Set $\mathfrak{b} := b_x^n b_z^{|z|-n}$ and for all $g \in G$ set $\mathfrak{a}_g := b_g^{|g|}$ and $\mathfrak{c}_g := (b_x b_g - \chi(g) b_g b_x) b_g^{-1} b_z^{-1}$. Then $\mathfrak{a}_g \in C^\times$, $\mathfrak{c}_g \in C$, and if $x^n = 0$, then $\mathfrak{b} \in C$.*

(3) *The weak action of $H_{\mathcal{D}}$ on C associated with γ according to item (5) of Theorem 2.36 is given by*

$$gx^i \rightharpoonup c = \sum_{j=0}^{i} (-1)^j (qp)^{\frac{j(j-1)}{2}} \binom{i}{j}_{qp} b_g b_x^{i-j} \varphi^j(c) b_x^j b^{-1}_{\zeta(g) z^i} \quad \text{for } c \in B_{1_G,\zeta} \cap \ker(U).$$

(4) *The two cocycle $\sigma : H_{\mathcal{D}} \otimes H_{\mathcal{D}} \to C$ associated with γ according to item (5) of Theorem 2.36, is given by*

$$\begin{aligned}
\sigma(gx^s \otimes hx^r) = & \sum_{\substack{0\le i\le s\\ 0\le j\le r\\ \xi_{ij}<n}} (-1)^{\xi_{ij}} \binom{s}{i}_{qp} \binom{r}{j}_{qp} (qp)^{\frac{\xi_{ij}(\xi_{ij}-1)}{2}+sj-ij} \chi(h)^{s-i} b_g b_x^i b_h b_x^{s+r-i} b^{-1}_{ghz^{s+r}} \\
& + \lambda \sum_{\substack{0\le i\le s\\ 0\le j\le r\\ \xi_{ij}\ge n}} (-1)^{\xi'_{ij}} \binom{s}{i}_{qp} \binom{r}{j}_{qp} (qp)^{\frac{\xi'_{ij}(\xi'_{ij}-1)}{2}+sj-ij} \chi(h)^{s-i} b_g b_x^i b_h b_x^{\xi_{in}} b^{-1}_{ghz^{s+r}} \\
& - \lambda \sum_{\substack{0\le i\le s\\ 0\le j\le r\\ \xi_{ij}\ge n}} (-1)^{\xi'_{ij}} \binom{s}{i}_{qp} \binom{r}{j}_{qp} (qp)^{\frac{\xi'_{ij}(\xi'_{ij}-1)}{2}+sj-ij} \chi(h)^{s-i} b_g b_x^i b_h b_x^{\xi_{in}} b^{-1}_{ghz^{s+r-n}},
\end{aligned}$$

where $\xi_{ij} := s + r - i - j$ and $\xi'_{ij} := \xi_{ij} - n$.

Proof.

(1) By item (4) of Theorem 2.36, the map $\phi : C \otimes H_{\mathcal{D}} \to B$, given by $\phi(c \otimes y) := c\gamma(y)$, is a normal basis. Item (1) is an immediate consequence of this fact.

(2) Using item (4) of Remark 2.22 it is easy to check by induction on i, that

$$\nu(b_g^i) = b_g^i \otimes g^i \quad \text{and} \quad \nu(b_x^i) = \sum_{j=0}^{i} \binom{i}{j}_{qp} b_x^j \otimes z^j x^{i-j} \quad \text{for all } g \in G \text{ and } i \ge 0. \tag{5.55}$$

By the first equality

$$\nu(\mathfrak{a}_g) = \mathfrak{a}_g \otimes 1_{H_{\mathcal{D}}} \quad \text{for all } g \in G,$$

and so $\mathfrak{a}_g \in C$. Since $\nu : B \to B \otimes_s H$ is an algebra map, we have

$$\nu(\mathfrak{a}_g^{-1}) = \mathfrak{a}_g^{-1} \otimes 1_{H_{\mathcal{D}}} \quad \text{and} \quad \nu(b_g^{-1}) = b_g^{-1} \otimes g^{-1} \quad \text{for all } g \in G, \tag{5.56}$$

which implies in particular that $\mathfrak{a}_g \in C^\times$. Note also that, by the second equality in (5.55),

$$\nu(b_x^n) = 1_B \otimes x^n + b_x^n \otimes z^n.$$

because $\binom{n}{j}_{qp} = 0$ for $0 < j < n$. Consequently,

$$\begin{aligned}
\nu(\mathfrak{b}) &= \nu(b_x^n b_z^{|z|-n}) \\
&= (\mu_B \otimes \mu_{H_{\mathcal{D}}}) \circ (B \otimes s \otimes H_{\mathcal{D}})(1_B \otimes x^n \otimes b_z^{|z|-n} \otimes z^{-n} + b_x^n \otimes z^n \otimes b_z^{|z|-n} \otimes z^{-n}) \\
&= (\mu_B \otimes \mu_{H_{\mathcal{D}}})(1_B \otimes b_z^{|z|-n} \otimes x^n \otimes z^{-n} + b_x^n \otimes b_z^{|z|-n} \otimes z^n \otimes z^{-n}) \\
&= b_z^{|z|-n} \otimes x^n z^{-n} + \mathfrak{b} \otimes 1_{H_{\mathcal{D}}},
\end{aligned}$$

which implies that, if $x^n = 0$, then $\mathfrak{b} \in C$. Furthermore, for all $g \in G$,

$$\begin{aligned} \nu(b_x b_g) &= (\mu_B \otimes \mu_{H_{\mathcal{D}}}) \circ (B \otimes s \otimes H_{\mathcal{D}})(1_B \otimes x \otimes b_g \otimes g + b_x \otimes z \otimes b_g \otimes g) \\ &= (\mu_B \otimes \mu_{H_{\mathcal{D}}})(1_B \otimes b_g \otimes x \otimes g + b_x \otimes b_g \otimes z \otimes g) \\ &= \chi(g)\, b_g \otimes gx + b_x b_g \otimes zg \end{aligned}$$

and

$$\begin{aligned} \nu(\chi(g)\, b_g b_x) &= (\mu_B \otimes \mu_{H_{\mathcal{D}}}) \circ (B \otimes s \otimes H_{\mathcal{D}})(\chi(g)\, b_g \otimes g \otimes 1_B \otimes x + \chi(g)\, b_g \otimes g \otimes b_x \otimes z) \\ &= (\mu_B \otimes \mu_{H_{\mathcal{D}}})(\chi(g)\, b_g \otimes 1_B \otimes g \otimes x + \chi(g)\, b_g \otimes b_x \otimes g \otimes z) \\ &= \chi(g)\, b_g \otimes gx + \chi(g)\, b_g b_x \otimes zg. \end{aligned}$$

Combining this with the second equality in (5.56), we obtain

$$\begin{aligned} \nu(\mathfrak{c}_g) &= \nu\big((b_x b_g - \chi(g)\, b_g b_x) b_g^{-1} b_z^{-1}\big) \\ &= (\mu_B \otimes \mu_{H_{\mathcal{D}}}) \circ (B \otimes s \otimes H_{\mathcal{D}})\big((b_x b_g - \chi(g)\, b_g b_x) \otimes zg \otimes b_g^{-1} b_z^{-1} \otimes g^{-1} z^{-1}\big) \\ &= (\mu_B \otimes \mu_{H_{\mathcal{D}}})\big((b_x b_g - \chi(g)\, b_g b_x) \otimes b_g^{-1} b_z^{-1} \otimes zg \otimes g^{-1} z^{-1}\big) \\ &= (b_x b_g - \chi(g)\, b_g b_x) b_g^{-1} b_z^{-1} \otimes 1_{H_{\mathcal{D}}}, \end{aligned}$$

and so $\mathfrak{c}_g \in C$, as desired.

(3) This follows by a direct computation from item (5) of Theorem 2.36, using equalities (3.9) and (4.27), and the formulas for γ and γ^{-1} that appears in Theorem 5.1.

(4) This follows by a direct computation from item (5) of Theorem 2.36, using equalities (3.9) and (3.10), and the formulas for γ and γ^{-1} that appears in Theorem 5.1. □

The following proposition and its corollary is useful to simplify the computation of the the right side in the equality in Theorem 5.3(4).

Proposition 5.4. *Let $r, s, i \geq 0$ with $0 \leq i \leq s$. With the notations of the previous result, we have*

$$\sum_{j=0}^{r} (-1)^{\xi_{ij}} \binom{r}{j}_{qp} (qp)^{\frac{\xi_{ij}(\xi_{ij}-1)}{2} + sj - ij} = \begin{cases} (-1)^{s-i} (qp)^{\frac{(s-i)(s-i-1)}{2}} & \text{if } r = 0, \\ 0 & \text{if } 0 < r < n. \end{cases}$$

Proof. Let $a := s - i$ and $b := r - j$. Since $\xi_{ij} = a + b$, we have

$$\begin{aligned} \sum_{j=0}^{r} (-1)^{\xi_{ij}} \binom{r}{j}_{qp} (qp)^{\frac{\xi_{ij}(\xi_{ij}-1)}{2} + sj - ij} &= \sum_{b=0}^{r} (-1)^{a+b} \binom{r}{b}_{qp} (qp)^{\frac{(a+b)(a+b-1)}{2} + ar - ab} \\ &= (-1)^a (qp)^{ar + \frac{a(a-1)}{2}} \sum_{b=0}^{r} (-1)^b \binom{r}{b}_{qp} (qp)^{\frac{b(b-1)}{2}}, \end{aligned}$$

which combined with (5.54) gives the desired result. □

Corollary 5.5. *Let $r, s, i \geq 0$ with $0 \leq i \leq s$. If $0 < r < n - s + i$, then*

$$\sum_{\substack{0 \leq j \leq r \\ \xi_{ij} < n}} (-1)^{\xi_{ij}} \binom{r}{j}_{qp} (qp)^{\frac{\xi_{ij}(\xi_{ij}-1)}{2} + sj - ij} = 0.$$

Proof. By Proposition 5.4. □

6. Examples

In this section we consider two examples of the braided Hopf algebras $H_{\mathcal{D}}$ defined in Corollary 3.8 and we apply the results obtained in the previous section in order to determine their cleft extensions.

6.1. First example

Consider the datum $\mathcal{D} = (C_2 \times C_2 \times C_2, \chi, z, \lambda, q)$, where:
- $C_2 = \{1, g\}$ is the multiplicative group of order 2,
- $\chi\colon C_2 \times C_2 \times C_2 \longrightarrow \mathbb{C}$ is the character given by $\chi(g^{i_1}, g^{i_2}, g^{i_3}) := (-1)^{i_1+i_2+i_3}$,
- $z := (g, g, g)$,
- $q = 1$ and $\lambda = 1$.

In this case $p := \chi(z) = -1$, $n = 2$ and the Hopf braided $\mathbb{C}$-algebra $H_{\mathcal{D}}$ of Corollary 3.8 is the $\mathbb{C}$-algebra generated by the group $G := C_2 \times C_2 \times C_2$ and an element x subject to the relations

$$x^2 = z^2 - 1_G = 0 \qquad \text{and} \qquad x(g^{i_1}, g^{i_2}, g^{i_3}) = (-1)^{i_1+i_2+i_3}(g^{i_1}, g^{i_2}, g^{i_3})x,$$

endowed with the standard Hopf algebra structure with comultiplication map Δ, counit ϵ and antipode S, given by

$$\begin{aligned} \Delta(\mathbf{g}) &:= \mathbf{g} \otimes \mathbf{g}, \qquad \Delta(x) := 1 \otimes x + x \otimes z,\\ \epsilon(\mathbf{g}) &:= 1, \qquad \epsilon(x) := 0\\ S(\mathbf{g}) &:= \mathbf{g}^{-1}, \qquad S(\mathbf{g}x) := -xz^{-1}\mathbf{g}^{-1}, \end{aligned}$$

where $\mathbf{g}$ denotes an arbitrary element of G. Let S_3 be the symmetric group in $\{1, 2, 3\}$. It is easy to check that the map

$$\theta\colon S_3^{\mathrm{op}} \to \mathrm{Aut}_{\chi,z}(G),$$

defined by $\theta(\sigma)(g^{i_1}, g^{i_2}, g^{i_3}) := (g^{i_{\sigma(1)}}, g^{i_{\sigma(2)}}, g^{i_{\sigma(3)}})$, is an isomorphism.

6.1.1. $H_{\mathcal{D}}$-spaces

Let V be a $\mathbb{C}$-vector space. By Proposition 4.5 to have a left $H_{\mathcal{D}}$-space structure with underlying vector space V is "the same" that to have a gradation

$$V = \bigoplus_{\sigma \in S_3} V_\sigma$$

and an automorphism $\varphi\colon V \to V$ such that $\varphi(V_\sigma) = V_\sigma$ for all $\sigma \in S_3$. The structure map

$$s\colon H_{\mathcal{D}} \otimes V \longrightarrow V \otimes H_{\mathcal{D}},$$

constructed from these data, is given by

$$s\big((g^{i_1}, g^{i_2}, g^{i_3}) \otimes v\big) := v \otimes (g^{i_{\sigma(1)}}, g^{i_{\sigma(2)}}, g^{i_{\sigma(3)}})$$

and

$$s\big((g^{i_1}, g^{i_2}, g^{i_3})x \otimes v\big) := \varphi(v) \otimes (g^{i_{\sigma(1)}}, g^{i_{\sigma(2)}}, g^{i_{\sigma(3)}})x,$$

for each $v \in V_\sigma$.

6.1.2. $H_{\mathcal{D}}$-comodules

Let V be a $\mathbb{C}$-vector space. By Corollary 4.7 each right $H_{\mathcal{D}}$-comodule structure (V, s) with underlying vector space V is univocally determined by the following data:

(a) A decomposition

$$V = \bigoplus_{(\mathbf{g},\sigma)\in G\times S_3^{\mathrm{op}}} V_{\mathbf{g},\sigma},$$

(b) An automorphism $\varphi\colon V \to V$ that satisfies $\varphi\big(V_{\mathbf{g},\sigma}\big) = V_{\mathbf{g},\sigma}$ for all $(\mathbf{g},\sigma) \in G \times S_3^{\mathrm{op}}$,
(c) A map $U\colon V \to V$ such that

$$U \circ \varphi = \varphi \circ U, \qquad U^2 = 0 \qquad \text{and} \qquad U\big(V_{\mathbf{g},\sigma}\big) \subseteq V_{\mathbf{g}z,\sigma} \quad \text{for all } (\mathbf{g},\sigma) \in G \times S_3^{\mathrm{op}}.$$

The formula for the transposition s of $H_{\mathcal{D}}$ on V is the one obtained in Section 6.1.1 (where we take $V_\sigma := \bigoplus_{\mathbf{g}\in G} V_{\mathbf{g},\sigma}$ for each $\sigma \in S_3$), while the $H_{\mathcal{D}}$-coaction ν is given by

$$\nu(v) = v \otimes (g^{i_1}, g^{i_2}, g^{i_3}) + U(v) \otimes (g^{i_1+1}, g^{i_2+1}, g^{i_3+1})x,$$

for $v \in \bigoplus_{\sigma\in S_3} V_{\mathbf{g},\sigma}$ with $\mathbf{g} = (g^{i_1}, g^{i_2}, g^{i_3})$.

Next given a decomposition as in item (a), we give a proceeding to construct an automorphism $\varphi\colon V \to V$ and a map $U\colon V \to V$ satisfying the conditions required in items (b) and (c): First we decompose each space $V_{\mathbf{g},\sigma}$ as a direct sum

$$V_{\mathbf{g},\sigma} = V^0_{\mathbf{g},\sigma} \oplus V^1_{\mathbf{g},\sigma},$$

in such a way that $\dim_{\mathbb{C}}(V^1_{\mathbf{g},\sigma}) \le \dim_{\mathbb{C}}(V^0_{\mathbf{g}z,\sigma})$, and we fix an injective morphisms

$$U_{\mathbf{g},\sigma}\colon V^1_{\mathbf{g},\sigma} \longrightarrow V^0_{\mathbf{g}z,\sigma},$$

for each $(\mathbf{g},\sigma) \in G \times S_3^{\mathrm{op}}$. Then we define U on $V_{\mathbf{g},\sigma}$, by

$$U(v) = \begin{cases} U_{\mathbf{g},\sigma}(v) & \text{if } v \in V^1_{\mathbf{g},\sigma}, \\ 0 & \text{if } v \in V^0_{\mathbf{g},\sigma}. \end{cases}$$

It remains to construct φ. Let $(\mathbf{g},\sigma) \in G \times S_3^{\mathrm{op}}$ arbitrary. Since $U \circ \varphi = \varphi \circ U$, $\varphi(V_{\mathbf{g},\sigma}) \subseteq V_{\mathbf{g},\sigma}$ and $U_{\mathbf{g},\sigma}$ is injective, there exist morphisms

$$\varphi^0_{\mathbf{g},\sigma}\colon V^0_{\mathbf{g},\sigma} \longrightarrow V^0_{\mathbf{g},\sigma}, \qquad \varphi^1_{\mathbf{g},\sigma}\colon V^1_{\mathbf{g},\sigma} \longrightarrow V^1_{\mathbf{g},\sigma} \qquad \text{and} \qquad \varphi^{10}_{\mathbf{g},\sigma}\colon V^1_{\mathbf{g},\sigma} \longrightarrow V^0_{\mathbf{g},\sigma},$$

such that

$$\varphi(v_0, v_1) = \big(\varphi^0_{\mathbf{g},\sigma}(v_0) + \varphi^{10}_{\mathbf{g},\sigma}(v_1), \varphi^1_{\mathbf{g},\sigma}(v_1)\big) \qquad \text{for all } (v_0, v_1) \in V^0_{\mathbf{g},\sigma} \oplus V^1_{\mathbf{g},\sigma}.$$

Moreover, since φ is an automorphism, the maps $\varphi^0_{\mathbf{g},\sigma}$ and $\varphi^1_{\mathbf{g},\sigma}$ are also automorphisms. All these maps can be constructed as follows: For each $(\mathbf{g},\sigma) \in G \times S_3^{\mathrm{op}}$ we take an arbitrary automorphism $\varphi^1_{\mathbf{g},\sigma}$ of $V^1_{\mathbf{g},\sigma}$. Then, for each $(\mathbf{g},\sigma) \in G \times S_3^{\mathrm{op}}$, we choose $\varphi^0_{\mathbf{g},\sigma}$ as an automorphism of $V^0_{\mathbf{g},\sigma}$ such that

$$\varphi^0_{\mathbf{g},\sigma}\big(U_{\mathbf{g}z,\sigma}(v)\big) = U_{\mathbf{g},\sigma}\big(\varphi^1_{\mathbf{g}z,\sigma}(v)\big) \qquad \text{for all } v \in V^1_{\mathbf{g}z,\sigma}$$

(which is forced by the condition $U \circ \varphi = \varphi \circ U$). Finally, we take $\varphi^{10}_{\mathbf{g},\sigma}$ as an arbitrary automorphism.

Remark 6.1. By Corollary 4.9 we know that $V^{\mathrm{co}H_{\mathcal{D}}} = V_{1_G} \cap \ker(U)$, where $V_{1_G} = \bigoplus_{\sigma\in S_3} V_{1_G,\sigma}$.

Remark 6.2. We are in the classical case (i.e. s is the flip) iff $V_{\mathbf{g},\sigma} = 0$ for $\sigma \neq \mathrm{id}$ and φ is the identity map. So, in this case the decomposition in item a) above has at most eight nonzero summands, item b) becomes trivial and the first condition in item c) also becomes trivial.

6.1.3. Transpositions of $H_\mathcal{D}$ on an algebra

By Proposition 4.10, for each $\mathbb{C}$-algebra B, to have a transposition $s\colon H_\mathcal{D} \otimes B \longrightarrow B \otimes H_\mathcal{D}$ is equivalent to have an algebra gradation

$$B = \bigoplus_{\sigma \in S_3^{\mathrm{op}}} B_\sigma$$

and an automorphism of algebras $\varphi\colon B \to B$ such that $\varphi(B_\sigma) = B_\sigma$ for all $\sigma \in S_3$. The structure map $s\colon H_\mathcal{D} \otimes B \longrightarrow B \otimes H_\mathcal{D}$, constructed from these data, is the same as in Section 6.1.1.

6.1.4. Right $H_\mathcal{D}$-comodule algebras

By the discussion above Definition 4.11 we know that the group S_3^{op} acts on G via

$$\sigma \cdot (g^{i_1}, g^{i_2}, g^{i_3}) := (g^{i_{\sigma(1)}}, g^{i_{\sigma(2)}}, g^{i_{\sigma(3)}}).$$

Consider the semidirect product $G(\chi, z) := G \rtimes S_3^{\mathrm{op}}$. We are going to work with $G(\chi, z)^{\mathrm{op}}$. Its underlying set is $C_2 \times C_2 \times C_2 \times S_3$ and its product is given by

$$(g^{i_1}, g^{i_2}, g^{i_3}, \sigma)(g^{j_1}, g^{j_2}, g^{j_3}, \tau) = (g^{j_1+i_{\tau(1)}}, g^{j_2+i_{\tau(2)}}, g^{j_3+i_{\tau(3)}}, \sigma \circ \tau).$$

Let B be a $\mathbb{C}$-algebra. By Theorem 4.12 to have a right $H_\mathcal{D}$-comodule algebra (B, s) is equivalent to have

(a) a $G(\chi, z)^{\mathrm{op}}$-gradation

$$B = \bigoplus_{(\mathbf{g},\sigma) \in G(\chi,z)^{\mathrm{op}}} B_{\mathbf{g},\sigma}$$

of B as an algebra,

(b) an automorphism of algebras $\varphi\colon B \to B$ such that

$$\varphi\big(B_{\mathbf{g},\sigma}\big) \subseteq B_{\mathbf{g},\sigma} \quad \text{for all } (\mathbf{g}, \sigma) \in G(\chi, z)^{\mathrm{op}},$$

(c) a map $U\colon B \to B$ such that
- $U \circ \varphi = \varphi \circ U$,
- $U^2 = 0$,
- $U\big(B_{\mathbf{g},\sigma}\big) \subseteq B_{\mathbf{g}z,\sigma}$ for all $(\mathbf{g}, \sigma) \in G(\chi, z)^{\mathrm{op}}$,
- the equality

$$U(bc) = bU(c) + (-1)^{i_1+i_2+i_3} U(b)\varphi(c)$$

holds for all $b \in B$ and $c \in B_{(g^{i_1}, g^{i_2}, g^{i_3})} := \bigoplus_{\sigma \in S_3} B_{(g^{i_1}, g^{i_2}, g^{i_3}),\sigma}$.

Remark 6.3. We are in the classical case (i.e. s is the flip) iff $B_{\mathbf{g},\sigma} = 0$ for $\sigma \neq \mathrm{id}$ and φ is the identity map. So, in this case the gradation in item (a) is a G-gradation, item (b) is trivial, and item (c) is considerably simplified.

6.1.5. Right $H_\mathcal{D}$-cleft extensions

Let $C := B^{\mathrm{coH}_\mathcal{D}}$. By Corollary 4.9 we know that

$$C = B_{1_G} \cap \ker(U) = \bigoplus_{\sigma \in S_3} B_{1_G,\sigma} \cap \ker(U).$$

By Theorem 5.1 and the comment below that result, the extension $(C \hookrightarrow B, s)$ is cleft iff there exist $b_x \in B$ and a family $(b_\mathbf{g})_{\mathbf{g} \in G}$ of elements of $B^\times$, such that

(a) $b_{1_G} = 1$,
(b) $b_\mathbf{g} \in B_{\mathbf{g},\mathrm{id}} \cap \ker(U)$ for all $\mathbf{g} \in G$,
(c) $b_x \in B_{(g,g,g),\mathrm{id}} \cap U^{-1}(1)$,
(d) $\varphi(b_x) = b_x$,

(e) $\varphi(b_{\mathbf{g}}) = b_{\mathbf{g}}$ for all $\mathbf{g} \in G$.

By Theorem 5.3 we know that

(1) B is a free left C-module with basis $\{b_g b_x^i : g \in G \text{ and } 0 \le i \le 1\}$.

(2) By Theorem 5.3(3), the weak action of $H_{\mathcal{D}}$ on C associated with γ according to item (5) of Theorem 2.36 is given by

$$(g^{i_1}, g^{i_2}, g^{i_3}) \rightharpoonup c = b_{(g^{i_1}, g^{i_2}, g^{i_3})} c b^{-1}_{(g^{i_{\sigma(1)}}, g^{i_{\sigma(2)}} g^{i_{\sigma(3)}})}$$

and

$$(g^{i_1}, g^{i_2}, g^{i_3})x \rightharpoonup c = b_{(g^{i_1}, g^{i_2}, g^{i_3})} \big(b_x c - \varphi(c) b_x\big) b^{-1}_{(g^{i_{\sigma(1)}+1}, g^{i_{\sigma(2)}+1}, g^{i_{\sigma(3)}+1})}$$

for $c \in B_{1_G,\sigma} \cap \ker(U)$.

(3) By Theorem 5.3(3), the two cocycle $\sigma : H_{\mathcal{D}} \otimes H_{\mathcal{D}} \to C$, associated with γ according to item (5) of Theorem 2.36 is given by

$$\begin{aligned} \sigma(\mathbf{g} \otimes \mathbf{h}) &= b_{\mathbf{g}} b_{\mathbf{h}} b^{-1}_{\mathbf{gh}}, \\ \sigma(\mathbf{g}x \otimes \mathbf{h}) &= -\chi(\mathbf{h}) b_{\mathbf{g}} b_{\mathbf{h}} b_x b^{-1}_{\mathbf{gh}(g,g,g)} + b_{\mathbf{g}} b_x b_{\mathbf{h}} b^{-1}_{\mathbf{gh}(g,g,g)}, \\ \sigma(\mathbf{g} \otimes \mathbf{h}x) &= 0 \end{aligned}$$

and

$$\sigma(\mathbf{g}x \otimes \mathbf{h}x) = \chi(h) b_{\mathbf{g}} b_{\mathbf{h}} b_x^2 b^{-1}_{\mathbf{gh}},$$

for $\mathbf{g}, \mathbf{h} \in G$.

Remark 6.4. It is clear that once chosen $b_g^{(1)} := b_{(g,1,1)}$, $b_g^{(2)} := b_{(1,g,1)}$ and $b_g^{(3)} := b_{(1,1,g)}$, one can take $b_{(g,g,1)} := b_g^{(1)} b_g^{(2)}$, $b_{(g,1,g)} := b_g^{(1)} b_g^{(3)}$, $b_{(1,g,g)} := b_g^{(2)} b_g^{(3)}$ and $b_{(g,g,g)} := b_g^{(1)} b_g^{(2)} b_g^{(3)}$.

6.2. *Second example*

Consider the datum $\mathcal{D} = (C_6, \chi, z, \lambda, q)$, where:

- $C_6 = \{1, g, g^2, g^3, g^4, g^5\}$ is the multiplicative cyclic group of order 6,
- $\chi : C_6 \longrightarrow \mathbb{C}$ is the character given by $\chi(g^i) := \xi^i$, where ξ is a root of order 3 of 1,
- $z := g$,
- $q = \xi$ and $\lambda = 1$.

In this case $p := \chi(z) = \xi$, $n = 3$ and the Hopf braided $\mathbb{C}$-algebra $H_{\mathcal{D}}$ of Corollary 3.8 is the $\mathbb{C}$-algebra generated by the group C_6 and an element x subject to the relations

$$x^3 = g^3 - 1 = -2 \qquad \text{and} \qquad xg = \xi g x,$$

endowed with the braided Hopf algebra structure with comultiplication map Δ, counit ϵ, antipode S and braid c_ξ, given by

$$\begin{aligned} \Delta(g^i) &:= g^i \otimes g^i, \qquad \Delta(x) := 1 \otimes x + x \otimes g, \\ \epsilon(g^i) &:= 1, \qquad \epsilon(x) := 0 \\ S(g^i x^j) &:= (-1)^j \xi^{j(j-1)} x^j g^{-j-i}, \\ c_\xi(g^i x^j \otimes g^k x^l) &= \xi^{jl} g^k x^l \otimes g^i x^j. \end{aligned}$$

It is clear that $\mathrm{Aut}_{\chi,z}(C_6) = \{\mathrm{id}\}$.

6.2.1. *$H_{\mathcal{D}}$-spaces*

Let V be a $\mathbb{C}$-vector space. By Proposition 4.5 we know that to have an $H_{\mathcal{D}}$-space structure with underlying vector space V is equivalent to have an automorphism $\varphi : V \to V$ such that $\varphi^3 = \mathrm{id}$. The structure map $s\colon H_{\mathcal{D}} \otimes V \longrightarrow V \otimes H_{\mathcal{D}}$ construct from these data is given by

$$s\big(g^i x^j \otimes v\big) := \varphi^j(v) \otimes g^i x^j.$$

6.2.2. $H_{\mathcal{D}}$-comodules

Let V be a $\mathbb{C}$-vector space. By Corollary 4.7 the right $H_{\mathcal{D}}$-comodule structures (V, s) with underlying vector space V are univocally determined by the following data:
(a) a decomposition

$$V = \bigoplus_{g^i \in C_6} V_{g^i} = V_1 \oplus V_g \oplus V_{g^2} \oplus V_{g^3} \oplus V_{g^4} \oplus V_{g^5},$$

(b) an automorphism $\varphi\colon V \to V$ that satisfies $\varphi^3 = \mathrm{id}$ and $\varphi\big(V_{g^i}\big) = V_{g^i}$ for all i,
(c) a map $U\colon V \to V$ such that

$$U \circ \varphi = \xi \varphi \circ U, \qquad U^3 = 0 \qquad \text{and} \qquad U\big(V_{g^i}\big) \subseteq V_{g^{i-1}} \quad \text{for all } i.$$

The formula for the transposition s of $H_{\mathcal{D}}$ on V is the one obtained in Section 6.2.1, while the $H_{\mathcal{D}}$-coaction ν is given by

$$\nu(v) = v \otimes g^i + U(v) \otimes g^{i-1}x - \xi U^2(v) \otimes g^{i-2}x^2 \qquad \text{for all } v \in V_{g^i}.$$

6.2.3. Transpositions of $H_{\mathcal{D}}$ on an algebra

By Proposition 4.10, for each $\mathbb{C}$-algebra B, to have a transposition $s\colon H_{\mathcal{D}} \otimes B \longrightarrow B \otimes H_{\mathcal{D}}$ is equivalent to have an automorphism of algebras $\varphi\colon B \to B$ such that $\varphi^3 = \mathrm{id}$. The structure map $s\colon H_{\mathcal{D}} \otimes B \longrightarrow B \otimes H_{\mathcal{D}}$, constructed from these data, is the same as in Section 6.2.1.

6.2.4. Right $H_{\mathcal{D}}$-comodule algebras

Let B be a $\mathbb{C}$-algebra. By Theorem 4.12 to have a right $H_{\mathcal{D}}$-comodule algebra (B, s) is equivalent to have
(a) a C_6-gradation

$$B = B_1 \oplus B_g \oplus B_{g^2} \oplus B_{g^3} \oplus B_{g^4} \oplus B_{g^5},$$

of B as a vector space such that $1_B \in B_1$ and $B_{g^i}B_{g^j} \subseteq B_{g^{i+j}} \oplus B_{g^{i+j-3}}$ for all i, j.
(b) an automorphism of algebras $\varphi\colon B \to B$ such that

$$\varphi^3 = \mathrm{id} \qquad \text{and} \qquad \varphi\big(B_{g^i}\big) \subseteq B_{g^i} \qquad \text{for all } i,$$

(c) a map $U\colon B \to B$ such that
- $U \circ \varphi = \xi \varphi \circ U$,
- $U^3 = 0$,
- $U\big(B_{g^i}\big) \subseteq B_{g^{i-1}}$ for all i,
- the equality

$$U(bc) = bU(c) + \xi^i U(b)\varphi(c)$$

  holds for all $b \in B$ and $c \in B_{g^i}$,
- For $b \in B_{g^i}$ and $c \in B_{g^j}$, the component $(bc)_{g^{i+j-3}} \in B_{g^{i+j-3}}$ of bc is given by

$$(bc)_{g^{i+j-3}} = \xi^j U(b)\varphi(U^2(c)) + \xi^{2j} U^2(b)\varphi^2(U(c)).$$

6.2.5. Right $H_{\mathcal{D}}$-cleft extensions

Let $C := B^{\mathrm{co}H_{\mathcal{D}}}$. By Corollary 4.9 we know that

$$C = B_1 \cap \ker(U).$$

By Theorem 5.1 and the comment below that result, the extension $(C \hookrightarrow B, s)$ is cleft iff there exist $b_x \in B$ and a family $(b_{g^i})_{g^i \in C_6}$ of elements of $B^\times$, such that
(a) $b_1 = 1$,
(b) $b_{g^i} \in B_{g^i} \cap \ker(U)$ for all $g^i \in C_6$,
(c) $b_x \in B_g \cap U^{-1}(1)$,
(d) $\varphi(b_x) = \xi b_x$,
(e) $\varphi(b_{g^i}) = b_{g^i}$ for all $g^i \in C_6$.
By Theorem 5.3 we know that
(1) B is a free left C-module with basis $\{b_{g^i} b_x^j : g^i \in C_6 \text{ and } 0 \le j \le 2\}$.
(2) The weak action of $H_{\mathcal{D}}$ on C associated with γ according to item (5) of Theorem 2.36, is given by

$$g^i \rightharpoonup c = b_{g^i} c b_{g^i}^{-1},$$
$$g^i x \rightharpoonup c = b_{g^i}\big(b_x c - \varphi(c) b_x\big) b_{g^{i+1}}^{-1}$$

and

$$g^i x^2 \rightharpoonup c = b_{g^i}\big(b_x^2 c + \xi b_x \varphi(c) b_x + \varphi^2(c) b_x^2\big) b_{g^{i+2}}^{-1},$$

for $c \in C$.
(3) The two cocycle $\sigma\colon H_{\mathcal{D}} \otimes H_{\mathcal{D}} \to C$, associated with γ according to item (5) of Theorem 2.36, is given by

$$\begin{aligned}
\sigma(g^i \otimes g^j) &= b_{g^i} b_{g^j} b_{g^{i+j}}^{-1},\\
\sigma(g^i x \otimes g^j) &= -\xi^j b_{g^i} b_{g^j} b_x b_{g^{i+j+1}}^{-1} + b_{g^i} b_x b_{g^j} b_{g^{i+j+1}}^{-1},\\
\sigma(g^i x^2 \otimes g^j) &= \xi^{2j+2} b_{g^i} b_{g^j} b_x^2 b_{g^{i+j+2}}^{-1} + \xi^{j+1} b_{g^i} b_x b_{g^j} b_x b_{g^{i+j+2}}^{-1} + b_{g^i} b_x^2 b_{g^j} b_{g^{i+j+2}}^{-1},\\
\sigma(g^i \otimes g^j x) &= 0,\\
\sigma(g^i x \otimes g^j x) &= 0,\\
\sigma(g^i x^2 \otimes g^j x) &= -\xi^{2j} b_{g^i} b_{g^j} b_{g^{i+j+3}}^{-1} - \xi^{2j} b_{g^i} b_{g^j} b_{g^{i+j}}^{-1} + \xi^{2j} b_{g^i} b_{g^j} b_x^3 b_{g^{i+j+3}}^{-1},\\
\sigma(g^i \otimes g^j x^2) &= 0,\\
\sigma(g^i x \otimes g^j x^2) &= -\xi^j b_{g^i} b_{g^j} b_{g^{i+j+3}}^{-1} - \xi^j b_{g^i} b_{g^j} b_{g^{i+j}}^{-1} + \xi^j b_{g^i} b_{g^j} b_x^3 b_{g^{i+j+3}}^{-1}
\end{aligned}$$

and

$$\begin{aligned}
\sigma(g^i x^2 \otimes g^j x^2) = &-\xi^{2j+1} b_{g^i} b_{g^j} b_x b_{g^{i+j+4}}^{-1} - \xi^{2j+1} b_{g^i} b_{g^j} b_x b_{g^{i+j+1}}^{-1} + \xi^{j+1} b_{g^i} b_x b_{g^j} b_{g^{i+j+4}}^{-1}\\
&+ \xi^{j+1} b_{g^i} b_x b_{g^j} b_{g^{i+j+1}}^{-1} + \xi^{2j+1} b_{g^i} b_{g^j} b_x^4 b_{g^{i+j+4}}^{-1} - \xi^{j+1} b_{g^i} b_x b_{g^j} b_x^3 b_{g^{i+j+4}}^{-1}.
\end{aligned}$$

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