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Partial isometries and pseudoinverses in semi-Hilbertian spaces

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In this article the concepts of partial isometries, normal partial isometries and generalized projections in the context of operators defined on a semi-Hilbertian space are developed. In particular, we analyze the relationship between these operators and different notions of pseudoinverses in semi-Hilbertian spaces. Finally, we apply the results obtained to describe the set of weighted projections into closed subspaces.

1. Introduction

Let $\mathcal{H}$ be a Hilbert space with inner product $\langle\ ,\ \rangle$ and $L(\mathcal{H})$ the algebra of bounded linear operators on $\mathcal{H}$. If $A \in L(\mathcal{H})$ is positive (semi-definite) and $\langle\ ,\ \rangle_A$ is the semi-inner product on $\mathcal{H}$ defined by $\langle \xi, \eta\rangle_A = \langle A\xi, \eta\rangle$ for all $\xi, \eta \in \mathcal{H}$ then $(\mathcal{H}, \langle\ ,\ \rangle_A)$ is called

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a semi-Hilbertian space. In the literature there are many papers that study operators defined on semi-Hilbertian spaces. The first work on this subject is due to Krein [15]. There, the author deals with operators which are A-selfadjoint, i.e., which are selfadjoint respect to the semi-inner product $\langle\ ,\ \rangle_A$. In [10,11] the existence of A-selfadjoint idempotents with a fixed closed range is studied. Later, in [3] operators which are isometric, unitary and partially isometric under the structure induced by $\langle\ ,\ \rangle_A$ are described. One of the main characteristics of the operators defined in $(\mathcal{H}, \langle\ ,\ \rangle_A)$ is that the existence of an adjoint operator for $\langle\ ,\ \rangle_A$ is not guaranteed. Therefore, the extension of certain properties of operators in $L(\mathcal{H})$ to bounded linear operators defined on a semi-Hilbertian space is not trivial.

In this article we continue with the study of partial isometries in the context of semi-Hilbertian spaces that began in [3] and followed in [2]. Recall that $T \in L(\mathcal{H})$ is a partial isometry if $\|T\xi\| = \|\xi\|$ for all $\xi \in N(T)^\perp$, where $N(T)$ denotes the nullspace of T. The following equivalent conditions are well-known for $T \in L(\mathcal{H})$:

1. T is a partial isometry;
2. T^*T is an idempotent operator;
3. TT^* is an idempotent operator;
4. $TT^*T = T$;
5. $T^*TT^* = T^*$;
6. $T^* = T^\dagger$;
7. $T^*\eta$ is the unique least square solution with minimal norm of the equation $T\xi = \eta$ for all $\xi \in \mathcal{H}$;

where T^* and $T^\dagger$ denote the adjoint operator of T and the Moore–Penrose inverse of T, respectively.

Given a positive operator $A \in L(\mathcal{H})$, an operator $T \in L(\mathcal{H})$ is an A-partial isometry if $\|T\xi\|_A = \|\xi\|_A$ for all $\xi \in N(AT)^{\perp_A}$; where $\|\xi\|_A = \langle \xi, \xi\rangle_A^{1/2}$ is the seminorm on $\mathcal{H}$ induced by A and $N(AT)^{\perp_A}$ denotes the orthogonal complement of $N(AT)$ with respect to $\langle\ ,\ \rangle_A$. However this definition does not allow to ensure that an A-partial isometry admits an A-adjoint operator. Therefore, it is not trivial how to extend the equivalences 1 to 7 above to A-partial isometries.

One of the main goals of this article is to analyze whether the equivalences stated above for partial isometries are still valid in the context of semi-Hilbertian spaces. For this purpose we deal with A-partial isometries that admit A-adjoint and we fix a distinguished one. There is a preliminary study in this direction. In [3] an equivalence like $1 \leftrightarrow 2$ above is shown in the context of semi-Hilbertian spaces under different hypotheses, for example the existence of an A-selfadjoint idempotent with a fixed closed range; or the closedness of the range of A. Later, in [2] the relationship between A-partial isometries and generalized inverses is investigated. In particular, the connection between A-partial isometries and A-generalized inverses is described. Here, the A-generalized inverses will play the roll of the Moore–Penrose inverses in the semi-Hilbertian space $(\mathcal{H}, \langle\ ,\ \rangle_A)$. This study is related to equivalences $1 \leftrightarrow 4 \leftrightarrow 6$ above. In Proposition 3.4 we study equivalences $1 \leftrightarrow 2 \leftrightarrow 3 \leftrightarrow 5$ for A-partial isometries. Equivalence $1 \leftrightarrow 4$ is not valid in this context, but we establish necessary and sufficient conditions for an A-partial isometry to satisfy 4 considering the particular A-adjoint. The results obtained are included in Theorem 4.1. Moreover, in order to analyze equivalences $1 \leftrightarrow 6 \leftrightarrow 7$ we study the relationship between A-partial isometries and a kind of weighted inverses. The generalization of these equivalent conditions is included in Theorem 4.5.

Another goal of this article is to describe the class of A-partial isometries called A-normal A-partial isometries and the class called A-generalized projections, which extend to semi-Hilbertian spaces the concepts of normal partial isometries and generalized projections, respectively. Groß and Trenkler [13] defined the generalized projections, which are a subset of partial isometries, as a generalization of the notion of orthogonal projection where the idempotency condition is not required. Generalized projections were widely studied in [7,5,6,1,17]. Finally, we apply the results obtained along this work to describe a class of A-selfadjoint A-partial isometries called A-projections which are defined in [8].

The article is organized as follows:

Section 2 is devoted to give notation, terminology and preliminary results that will be used along this work.

In Section 3 we continue the study of A-partial isometries that has began in [3]. We give some necessary and sufficient conditions for an operator T to be an A-partial isometry by means of a distinguished A-adjoint of T. The choice of this particular A-adjoint is also justified in this section.

In Section 4 we study the relationship between A-partial isometries, generalized inverses and weighted inverses. Theorem 4.1 and its corollary deal with the connection between A-partial isometries and generalized inverses. Both results contribute to [2, Theorem 2.5] where this link is analyzed. In Theorem 4.5 we relate A-partial isometries to a class of weighted inverses. This result also generalizes the fact that an operator is a partial isometry if and only if its adjoint operator coincides with its Moore–Penrose inverse.

Section 5 is devoted to the study of A-normal A-partial isometries. There we analyze some properties of normal partial isometries in the context of semi-Hilbertian spaces. We generalize the fact that a given partial isometry is normal if and only if the range and the nullspace of its adjoint decomposes H.

In Section 6 we define the A-generalized projections and we present some characterizations for these operators. In addition we remark certain differences and similarities with the generalized projections. For example, every generalized projection is a normal partial isometry and it decomposes $\mathcal{H}$ as the orthogonal sum of its range and its kernel or as the orthogonal sum of the range and the nullspace of its adjoint. We show that these facts are not valid for an A-generalized projection, in general. However, given an A-generalized projection these facts are true for a distinguished A-adjoint of the A-generalized projection.

Finally, in Section 7 we apply some results obtained in the previous sections to describe the set of A-projections into a closed subspace. This concept was introduced by Mitra and

Rao [16] for matrices and then it was extended to bounded linear operators by Corach et al. [8]. An A-projection into a closed subspace $\mathcal{S}$ is an operator which acts as an orthogonal projection on $\mathcal{S}$ when the seminorm induced by $\langle\ ,\ \rangle_A$ is considered in $\mathcal{H}$. In particular, if an operator is an A-projection into the closure of its range then it is called an A-projection. In this section we give a parametrization of the set of A-projections into $\mathcal{S}$ by means of an A-partial isometry with range $\mathcal{S}$ (Proposition 7.4) and also we describe the A-projections by means of generalized projections (Proposition 7.5).

2. Preliminaries

In this paper $\mathcal{H}$ is a complex Hilbert space with inner product $\langle\ ,\ \rangle$. By $L(\mathcal{H})$ we denote the algebra of bounded linear operators from $\mathcal{H}$ to $\mathcal{H}$ and by $L(\mathcal{H})^+$ the cone of positive (semi-definite) operators of $L(\mathcal{H})$. Given $T \in L(\mathcal{H})$, the range and the nullspace of T are respectively denoted by $R(T)$ and $N(T)$. In addition, T^* denotes the adjoint operator of T. By $T^\dagger$ we denote the Moore–Penrose inverse of T, i.e., $T^\dagger$ is the operator $T^\dagger : R(T) \oplus R(T)^\perp \to N(T)^\perp$, defined by $T^\dagger|_{R(T)} = T^{-1}|_{R(T)}$ and $N(T^\dagger) = R(T)^\perp$. It holds that $T^\dagger \in L(\mathcal{H})$ if and only if T has closed range.

Given $\mathcal{S}$ and $\mathcal{W}$ two closed subspaces of $\mathcal{H}$ then $\mathcal{S}\dot{+}\mathcal{W}$ and $\mathcal{S}\oplus\mathcal{W}$ denote the direct sum and the orthogonal sum between $\mathcal{S}$ and $\mathcal{W}$, respectively. Moreover, $\mathcal{S}\ominus\mathcal{W} = \mathcal{S}\cap(\mathcal{S}\cap\mathcal{W})^\perp$. If $\mathcal{S}\dot{+}\mathcal{W} = \mathcal{H}$, then $Q_{\mathcal{S}//\mathcal{W}}$ denotes the idempotent operator in $L(\mathcal{H})$ with range $\mathcal{S}$ and nullspace $\mathcal{W}$. In particular, $P_\mathcal{S} = Q_{\mathcal{S}//\mathcal{S}^\perp}$ is the orthogonal projection onto $\mathcal{S}$. If $T \in L(\mathcal{H})$, the matrix decomposition of T in terms of the decomposition $\mathcal{H} = \mathcal{S} \oplus \mathcal{S}^\perp$ is given by

$$T = \begin{pmatrix} x & y \\ z & w \end{pmatrix},$$

where $x = P_\mathcal{S} T P_\mathcal{S}|_\mathcal{S} \in L(\mathcal{S})$, $y = P_\mathcal{S} T P_{\mathcal{S}^\perp}|_{\mathcal{S}^\perp} \in L(\mathcal{S}^\perp, \mathcal{S})$, $z = P_{\mathcal{S}^\perp} T P_\mathcal{S}|_\mathcal{S} \in L(\mathcal{S}, \mathcal{S}^\perp)$ and $w = P_{\mathcal{S}^\perp} T P_{\mathcal{S}^\perp}|_{\mathcal{S}^\perp} \in L(\mathcal{S}^\perp)$.

Every $A \in L(\mathcal{H})^+$ defines a semi-inner product on $\mathcal{H}$ given by $\langle\ ,\ \rangle_A : \mathcal{H} \times \mathcal{H} \to \mathbb{C}$ where $\langle \xi, \eta \rangle_A = \langle A\xi, \eta \rangle$ for every $\xi, \eta \in \mathcal{H}$. The pair $(\mathcal{H}, \langle , \rangle_A)$ is called a semi-Hilbertian space. Given $T \in L(\mathcal{H})$, an operator $D \in L(\mathcal{H})$ is an A-adjoint of T if $\langle T\xi, \eta \rangle_A = \langle \xi, D\eta \rangle_A$ for every $\xi, \eta \in \mathcal{H}$; or equivalently, if $AD = T^*A$. Moreover, T is A-selfadjoint if $AT = T^*A$. Note that the existence of an A-adjoint of T is equivalent to the existence of a solution of the equation $AX = T^*A$. This kind of equation can be studied applying the following range inclusion theorem due to Douglas [12]:

Theorem 2.1. *Given $B, C \in L(\mathcal{H})$ the following conditions are equivalent:*

1. *the equation $BX = C$ has solution in $L(\mathcal{H})$;*
2. *$R(C) \subseteq R(B)$;*
3. *there exists $\lambda > 0$ such that $CC^* \leq \lambda BB^*$.*

If one of the above conditions holds then there exists a unique $D \in L(\mathcal{H})$ *such that* $BD = C$ *and* $R(D) \subseteq N(B)^{\perp}$*; namely* $D = B^{\dagger}C$*. Moreover,* $\|D\|^2 = inf\{\lambda > 0 : CC^* \leq \lambda BB^*\}$*.*

By Douglas' theorem it holds that the set of all operators of $L(\mathcal{H})$ which admit an A-adjoint operator is the set

$$L_A(\mathcal{H}) = \{T \in L(\mathcal{H}) : R(T^*A) \subseteq R(A)\}.$$

If $T \in L_A(\mathcal{H})$ then it can admit only one A-adjoint operator or infinite A-adjoint operators. In any case, there exists a distinguished A-adjoint of T provided by Douglas' theorem; namely, $T^{\sharp} = A^{\dagger}T^*A$. Therefore, it holds that $R(T^{\sharp}) \subseteq \overline{R(A)}$ and $N(T^{\sharp}) = N(T^*A)$.

Given $A \in L(\mathcal{H})^+$ and a subspace $\mathcal{S}$ of $\mathcal{H}$, $\mathcal{S}^{\perp_A} = (A\mathcal{S})^{\perp}$ denotes the A-orthogonal subspace of $\mathcal{S}$. In addition, for subspaces $\mathcal{S}, \mathcal{W} \subseteq \mathcal{H}$ we denote $\mathcal{S} \oplus_A \mathcal{W}$ the direct and A-orthogonal sum between $\mathcal{S}$ and $\mathcal{W}$. If $\mathcal{S}$ is a closed subspace and there exists a bounded A-selfadjoint idempotent operator Q with $R(Q) = \mathcal{S}$ then the pair $(A, \mathcal{S})$ is called *compatible*. The theory of compatibility was developed by Corach et al. in [10,11]. In the next result we collect some facts about compatibility that will be useful in the sequel. For its proofs see [10] and [11].

Proposition 2.2. *Let* $A \in L(\mathcal{H})^+$ *and* $\mathcal{S} \subseteq \mathcal{H}$ *a closed subspace. Then the following assertions are equivalent:*

1. *the pair* $(A, \mathcal{S})$ *is compatible;*
2. *there exists an idempotent operator* Q *with* $R(Q) = \mathcal{S}$ *and* $N(Q) \subseteq \mathcal{S}^{\perp_A}$*;*
3. $\mathcal{S} + \mathcal{S}^{\perp_A} = \mathcal{H}$*.*

If one of the above conditions holds then the following statements are satisfied:

1. $\mathcal{S} + N(A)$ *is closed;*
2. *there exists an* A*-selfadjoint projection* $P_{A,\mathcal{S}}$ *such that* $R(P_{A,\mathcal{S}}) = \mathcal{S}$ *and* $N(P_{A,\mathcal{S}}) = \mathcal{S}^{\perp_A} \ominus \mathcal{N}$*, where* $\mathcal{N} = \mathcal{S} \cap N(A)$*;*
3. *if* Q *is an* A*-selfadjoint projection onto* $\mathcal{S}$ *then* $\|(I - P_{A,\mathcal{S}})\xi\| \leq \|(I - Q)\xi\|$ *for all* $\xi \in \mathcal{H}$*.*

For a given $T \in L_A(\mathcal{H})$, in the next lemma we add some other properties of the operator $T^{\sharp}$ to those given in [3,4]. From now on, given $A \in L(\mathcal{H})^+$, P_A denotes the orthogonal projection onto $\overline{R(A)}$.

Lemma 2.3. *Let* $T \in L_A(\mathcal{H})$*. Then the following assertions hold:*

1. $TT^{\sharp}$ *and* $T^{\sharp}T$ *are* A*-selfadjoint operators;*
2. $R(T)^{\perp_A} = R(TT^{\sharp})^{\perp_A} = N(T^{\sharp}) = N(TT^{\sharp})$*;*

3. $R(T^\sharp)^{\perp_A} = R(T^\sharp T)^{\perp_A} = N(AT) = N(T^\sharp T)$;
4. *if* $TT^\sharp$ *is an idempotent operator then* $TT^\sharp = Q_{R(TT^\sharp)//N(T^*A)}$;
5. *if* $T^\sharp T$ *is an idempotent operator then* $T^\sharp T = Q_{R(T^\sharp)//N(AT)}$.

Proof. 1. It is straightforward.

2. Note that $R(T)^{\perp_A} = R(AT)^\perp = N(T^*A) = N(T^\sharp)$ and $R(TT^\sharp)^{\perp_A} = R(ATT^\sharp)^\perp = R((T^\sharp)^*AT^\sharp)^\perp = N(AT^\sharp) = N(T^\sharp)$. Finally, observe that $N(T^\sharp) \subseteq N(TT^\sharp) \subseteq R(TT^\sharp)^{\perp_A} = N(T^\sharp)$ where the second inclusion holds because $TT^\sharp$ is A-selfadjoint.

3. It is easy to see that $R(T^\sharp)^{\perp_A} = R(T^\sharp T)^{\perp_A} = N(AT)$. Furthermore, $N(AT) \subseteq N(T^\sharp T) \subseteq R(T^\sharp T)^{\perp_A} = N(AT)$ where the second inclusion holds because $T^\sharp T$ is A-selfadjoint. Then $N(AT) = N(T^\sharp T)$ and the assertion follows.

4. It follows from item 3.

5. Since $T^\sharp T$ is an A-selfadjoint projection then it holds that $\mathcal{H} = R(T^\sharp T) \oplus_A N(T^\sharp T) = R(T^\sharp T) \oplus_A N(AT) \subseteq R(T^\sharp) \oplus_A N(AT)$. Therefore, $R(T^\sharp T) = R(T^\sharp)$ and the assertion follows. □

3. *A*-partial isometries

Partial isometries in semi-Hilbertian spaces were introduced in [3] as follows.

Definition 3.1. Consider $A \in L(\mathcal{H})^+$. An operator $T \in L(\mathcal{H})$ is called an A-partial isometry if $\|T\xi\|_A = \|\xi\|_A$ for all $\xi \in N(AT)^{\perp_A}$.

Remarks 3.2.

1. An A-partial isometry does not necessarily admit an A-adjoint operator. In fact, let $\mathcal{S}$ be a closed subspace of $\mathcal{H}$ and $b \in L(\mathcal{S}^\perp, \mathcal{S})$ with non closed range. Consider $A = \begin{pmatrix} 1 & 0 \\ 0 & b^*b \end{pmatrix}$ and $T = \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix}$ under the decomposition $\mathcal{H} = \mathcal{S} \oplus \mathcal{S}^\perp$. Observe that $AT = T$ and $N(AT)^{\perp_A} = (A(\mathcal{S} \oplus N(b)))^\perp = (A\mathcal{S})^\perp = \mathcal{S}^\perp$. Therefore, consider $\xi \in \mathcal{S}^\perp$ then $\|\xi\|_A = \|(b^*b)^{1/2}\xi\| = \|b\xi\| = \|T\xi\|_A$, so that T is an A-partial isometry. Finally, it holds that $T \notin L_A(\mathcal{H})$. Indeed, $R(b^*) \not\subseteq R(b^*b)$ because $R(b)$ is not closed. Now, since $R(A) = \mathcal{S} \oplus R(b^*b)$ and $R(T^*A) = R(b^*)$, then $R(T^*A) \not\subseteq R(A)$. Hence, $T \notin L_A(\mathcal{H})$.
2. If $R(A)$ is closed and T is an A-partial isometry then T admits an A-adjoint operator. In fact, since T is an A-partial isometry, it is not difficult to see that $N(A) \subseteq N(AT)$. Then $R(T^*A) \subseteq \overline{R(T^*A)} \subseteq R(A)$ because $R(A)$ is closed. Therefore, $T \in L_A(\mathcal{H})$.

Along this work we deal with A-partial isometries in $L_A(\mathcal{H})$.

Proposition 3.3. *An operator* $T \in L_A(\mathcal{H})$ *is an* A*-partial isometry if and only if* $\|T\xi\|_A = \|\xi\|_A$ *for all* $\xi \in \overline{R(T^\sharp T)}$.

Proof. See [3, Proposition 4.4]. □

In [3], the authors gave some characterizations of an A-partial isometry $T \in L_A(\mathcal{H})$ under the hypothesis of compatibility between A and the closed subspace $\overline{R(T^\sharp T)}$ and also under the hypothesis of the closedness of $R(A)$. Here we present some characterizations of an A-partial isometry T assuming only that $T \in L_A(\mathcal{H})$.

Proposition 3.4. *Let $T \in L_A(\mathcal{H})$. Then the following assertions are equivalent:*

1. *T is an A-partial isometry;*
2. *$T^\sharp T$ is an idempotent operator;*
3. *$T^\sharp$ is an A-partial isometry;*
4. *$T^\sharp T T^\sharp = T^\sharp$;*
5. *$ATT^\sharp T = AT$;*
6. *$TT^\sharp$ is an idempotent operator.*

Proof. $1 \to 2$. Since T is an A-partial isometry then, by Proposition 3.3, $\|T\xi\|_A = \|\xi\|_A$ for all $\xi \in \overline{R(T^\sharp T)}$, so that, $T^*AT = A$ in $\overline{R(T^\sharp T)}$. In particular, $A(T^\sharp T)^2 = T^*ATT^\sharp T = AT^\sharp T$, so that $A((T^\sharp T)^2 - T^\sharp T) = 0$. Therefore, $(T^\sharp T)^2 = T^\sharp T$ because $R(T^\sharp T) \subseteq \overline{R(A)}$. Then $T^\sharp T$ is idempotent.

$2 \to 3$. It holds that $R((T^\sharp)^\sharp T^\sharp) = R(P_A TT^\sharp)$ because $(T^\sharp)^\sharp = P_A T$. Therefore, $\|T^\sharp(P_A TT^\sharp\xi)\|_A^2 = \|T^\sharp TT^\sharp\xi\|_A^2 = \|T^\sharp\xi\|_A^2 = \langle AT^\sharp\xi, T^\sharp\xi\rangle = \langle AT^\sharp TT^\sharp\xi, T^\sharp\xi\rangle = \langle ATT^\sharp\xi, TT^\sharp\xi\rangle = \|P_A TT^\sharp\xi\|_A^2$, where the second and the fourth equality follow because $T^\sharp T$ is an idempotent operator on $R(T^\sharp)$ (see Lemma 2.3). Then $T^\sharp$ is an A-partial isometry.

$3 \to 4$. Since $T^\sharp$ is an A-partial isometry, by implication $1 \to 2$, it holds that $(T^\sharp)^\sharp T^\sharp$ is idempotent. Then $(T^\sharp)^\sharp T^\sharp = P_A TT^\sharp$ is an idempotent operator with $R(P_A TT^\sharp) = R(P_A T)$, see Lemma 2.3. Then, $(T^\sharp T)^2 = T^\sharp TT^\sharp T = T^\sharp P_A TT^\sharp P_A T = T^\sharp P_A T = T^\sharp T$. Then, by Lemma 2.3, $R(T^\sharp T) = R(T^\sharp)$ so that $T^\sharp TT^\sharp = T^\sharp$.

$4 \to 5$. Note that $ATT^\sharp T = (TT^\sharp)^*AT = (TT^\sharp)^*(T^\sharp)^*A = (T^\sharp TT^\sharp)^*A = (T^\sharp)^*A = AT$, where the first equality follows because $TT^\sharp$ is A-selfadjoint and the fourth equality follows from the fact that $T^\sharp TT^\sharp = T^\sharp$.

$5 \to 6$. Note that $AT^\sharp TT^\sharp = (T^\sharp T)^*AT^\sharp = (T^\sharp T)^*T^*A = (TT^\sharp T)^*A = T^*A = AT^\sharp$, where the first equality follows because $TT^\sharp$ is A-selfadjoint and the fourth equality follows from the fact that $ATT^\sharp T = AT$. Therefore, by left multiplication by $A^\dagger$ we get that $T^\sharp TT^\sharp = P_A T^\sharp TT^\sharp = P_A T^\sharp = T^\sharp$. In consequence, $(TT^\sharp)^2 = TT^\sharp$.

$6 \to 1$. Observe that $(P_A TT^\sharp)^2 = P_A TT^\sharp P_A TT^\sharp = P_A(TT^\sharp)^2 = P_A TT^\sharp$. Since $P_A TT^\sharp = (T^\sharp)^\sharp T^\sharp$ is idempotent then $R(P_A TT^\sharp) = R(P_A T)$ (see Lemma 2.3). Therefore, it holds that $\|T(T^\sharp T\xi)\|_A^2 = \langle ATT^\sharp T\xi, TT^\sharp T\xi\rangle = \langle AP_A TT^\sharp P_A T\xi, TT^\sharp T\xi\rangle = \langle AT\xi, TT^\sharp T\xi\rangle = \langle AT^\sharp T\xi, T^\sharp T\xi\rangle = \|T^\sharp T\xi\|_A^2$, so that T is an A-partial isometry. □

Remarks 3.5.

1. The above proposition is not valid for any A-adjoint of T. In fact, let $A = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \in L(\mathbb{C}^3)^+$ and $T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{pmatrix} \in L(\mathbb{C}^3)$. Since $T^*A = A$ then $T \in L_A(\mathbb{C}^3)$ and $T^\sharp = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$. Observe that $T^\sharp T$ is an idempotent operator. However, observe that $W = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 2 & 0 & 4 \end{pmatrix}$ is an A-adjoint of T and WT is not an idempotent operator.
2. Every A-selfadjoint projection Q is an A-partial isometry. In fact, note that $Q^\sharp Q = A^\dagger Q^* AQ = A^\dagger AQ^2 = P_A Q$ and $(Q^\sharp Q)^2 = P_A Q P_A Q = P_A Q^2 = P_A Q = Q^\sharp Q$. Then, by Proposition 3.4, Q is an A-partial isometry.
3. The range of an A-partial isometry is not necessarily closed. In fact, consider $A \in L(\mathcal{H})^+$ and $T = P_A + dP_{N(A)} \in L(\mathcal{H})$, where $d \in L(N(A))^+$ has dense non-closed range. Then the range of T is not closed and T is an A-partial isometry because $T^\sharp T = P_A$.
4. If T is an A-partial isometry then $T^\sharp$ has closed range. It holds because $T^\sharp T$ is idempotent and $R(T^\sharp) = R(T^\sharp T)$, see Lemma 2.3 and the above proposition.

4. *A*-partial isometries and pseudoinverses

In this section we deal with the relationship between partial isometries and generalized inverses in semi-Hilbertian spaces. If $T \in L(\mathcal{H})$, the following well-known equivalent conditions relate partial isometries and pseudoinverses:

i. T is a partial isometry,
ii. T^* is a generalized inverse of T,
iii. $T^* = T^\dagger$.

Recall that a linear operator $T' \in L(\mathcal{H})$ is a generalized inverse of $T \in L(\mathcal{H})$ if T' satisfies the equation:

$$TXT = T. \tag{1}$$

Observe that a necessary condition for equation (1) to admit a solution is that the operator T has closed range. In fact, $R(T) = R(TX)$ and TX is a bounded idempotent.

Equivalence i $\leftrightarrow$ ii does not hold in the context of semi-Hilbertian spaces. In [2] it is proved that if $T \in L_A(\mathcal{H})$ and $TT^\sharp T = T$ then T is an A-partial isometry. But the converse is false, in general. Moreover, [2, Theorem 2.5] characterizes when the equivalence holds. In the next theorem we also study the A-partial isometries for which $T^\sharp$ satisfies equation (1). Equivalence $1 \leftrightarrow 3$ of the following result is contained in [2, Theorem 2.5].

Theorem 4.1. *Let $T \in L_A(\mathcal{H})$ be an A-partial isometry. Then the following statements are equivalent:*

1. $TT^\sharp T = T$;
2. $\mathcal{H} = R(T)\dot{+}N(T^\sharp)$;
3. $\mathcal{H} = R(T^\sharp)\dot{+}N(T)$;
4. $R(T) \cap N(A) = \{0\}$ *and* $R(T)$ *is closed.*

In addition, if one of the above conditions holds then $T^\sharp T = Q_{R(T^\sharp)//N(T)}$ *and* $TT^\sharp = Q_{R(T)//N(T^\sharp)}$.

Proof. $1 \to 2$. Since T is an A-partial isometry then $TT^\sharp$ is idempotent. Then, $\mathcal{H} = R(TT^\sharp)\dot{+}N(TT^\sharp) \subseteq R(T) + N(T^\sharp)$, because $N(TT^\sharp) = N(T^\sharp)$ (see Lemma 2.3). Now, consider $\xi \in R(T)\cap N(T^\sharp)$. Then $\xi = T\eta$ for some $\eta \in \mathcal{H}$ and $0 = T^\sharp\xi = T^\sharp T\eta$. Therefore $0 = TT^\sharp T\eta = T\eta$, or equivalently $\eta \in N(T)$, so that $\xi = 0$.

$2 \to 3$. Observe that $R(T) \cap N(A) \subseteq R(T) \cap N(T^\sharp) = \{0\}$, then $N(AT) = N(T)$. Since T is an A-partial isometry then $\mathcal{H} = R(T^\sharp T) \oplus_A R(T^\sharp T)^{\perp_A} = R(T^\sharp) \oplus_A N(AT) = R(T^\sharp) \oplus_A N(T)$, where the first and the second equality hold by Lemma 2.3. Then the assertion follows.

$3 \to 4$. Consider $\xi \in R(T) \cap N(A)$, then $\xi = T\eta$ for some $\eta \in \mathcal{H}$. Since $\mathcal{H} = R(T^\sharp)\dot{+}N(T)$, then $\eta = T^\sharp\eta_1 + \eta_2$, with $\eta_2 \in N(T)$. Therefore, $0 = A\xi = AT\eta = AT(T^\sharp\eta_1 + \eta_2) = ATT^\sharp\eta_1 = (T^\sharp)^*AT^\sharp\eta_1$, so that $\eta_1 \in N((T^\sharp)^*AT^\sharp) = N(AT^\sharp) = N(T^\sharp)$. Hence $\eta = \eta_2 \in N(T)$ and $\xi = T\eta_2 = 0$. It is left to prove that T has closed range. Let $\{\xi_n\} \subseteq \mathcal{H}$ be such that $T\xi_n \xrightarrow[n\to\infty]{} \omega$. Consider $\xi_n = T^\sharp\theta_n + \rho_n$, where $\rho_n \in N(T)$, then $T\xi_n = TT^\sharp\theta_n \xrightarrow[n\to\infty]{} \omega$. Therefore, since $TT^\sharp$ is idempotent, $\omega \in R(TT^\sharp) \subseteq R(T)$, so that $R(T)$ is closed.

$4 \to 1$. Since T is an A-partial isometry such that $R(T) \cap N(A) = \{0\}$ then $\mathcal{H} = R(TT^\sharp)\dot{+}R(TT^\sharp)^{\perp_A}$. By Lemma 2.3, it holds that $R(TT^\sharp)^{\perp_A} = R(T)^{\perp_A}$. Therefore, $\mathcal{H} = R(TT^\sharp)\dot{+}R(T)^{\perp_A} \subseteq R(T)\dot{+}R(T)^{\perp_A}$, where the sum is direct because $R(T)\cap N(A) = \{0\}$. Since $R(T)$ is closed, then $R(TT^\sharp) = R(T)$, so that the assertion follows because $TT^\sharp$ is an idempotent operator with $R(TT^\sharp) = R(T)$.

Moreover, suppose T is an A-partial isometry such that $TT^\sharp T = T$, or equivalently, $R(T) \cap N(A) = \{0\}$ and $R(T)$ is closed. Then, by Lemma 2.3, $T^\sharp T = Q_{R(T^\sharp)//N(AT)}$. Since $R(T) \cap N(A) = \{0\}$ then $N(AT) = N(T)$, so that $T^\sharp T = Q_{R(T^\sharp)//N(T)}$. On the other hand, observe that, by Lemma 2.3, $TT^\sharp = Q_{R(TT^\sharp)//N(T^\sharp)}$. By the proof of $4 \to 1$, it holds that $R(TT^\sharp) = R(T)$. Therefore, $TT^\sharp = Q_{R(T)//N(T^\sharp)}$. $\square$

The following result shows that every A-partial isometry is the sum of two operators, where one of them is an A-partial isometry for which its distinguished A-adjoint satisfies equation (1).

Corollary 4.2. *Let $T \in L_A(\mathcal{H})$ be an A-partial isometry. Then $T = W + P_{N(A)}T$ where W is an A-partial isometry such that $WW^\sharp W = W$.*

Proof. Consider $W = (T^\sharp)^\sharp = P_A T$, then $T = W + P_{N(A)}T$ and, by Proposition 3.4, W is an A-partial isometry. By item 4 of Remarks 3.5 and Theorem 4.1, we get that $WW^\sharp W = W$. $\square$

In order to analyze equivalence i $\leftrightarrow$ iii at the begining of this section it is necessary to consider extensions of the notion of the Moore–Penrose inverse to semi-Hilbertian spaces. It is well known that if $T \in L(\mathcal{H})$ has closed range, $T^\dagger$ is the Moore–Penrose inverse of T if and only if one of the next conditions holds:

(a) $T^\dagger$ satisfies the following equations $TXT = T$, $XTX = X$, $TX = (TX)^*$, $XT = (XT)^*$;
(b) $T^\dagger\eta$ is the unique least squares solution with minimal norm of the equation $T\xi = \eta$ for all $\xi \in \mathcal{H}$.

The reader is referred to [14] for the proof of the above equivalences. Different extensions of the concept of the Moore–Penrose inverse to semi-Hilbertian spaces can be considered. In [9] this notion was extended generalizing condition (a). More precisely, given $A_1, A_2 \in L(\mathcal{H})^+$ and $T, T' \in L(\mathcal{H})$ it is said that T' is an A_1, A_2-generalized inverse of T if T' satisfies the following four equations:

$$TXT = T, \quad XTX = X, \quad A_1(TX) = (TX)^*A_1, \quad A_2(XT) = (XT)^*A_2. \tag{2}$$

Observe that the existence of A_1, A_2-generalized inverses is related to the existence of A_1-selfadjoint and A_2-selfadjoint projections. Not every operator in $L(\mathcal{H})$ admits an A_1, A_2-generalized inverse. Given $T \in L(\mathcal{H})$ with closed range, there exists an A_1, A_2-generalized inverse of T if and only if the pairs $(A_1, R(T))$ and $(A_2, N(T))$ are compatible, see [9, Theorem 3.1]. If $A_1 = A_2 = A$ and T' satisfies equations (2) then T' is called an A-generalized inverse of T. In [2, Proposition 2.2] it is proved that if $T \in L_A(\mathcal{H})$ has closed range, then $TT^\sharp T = T$ if and only if $T^\sharp$ is an A-generalized inverse of T. This fact generalizes the equivalence ii $\leftrightarrow$ iii.

In what follows, we deal with an extension of the concept of the Moore–Penrose inverse to semi-Hilbertian spaces taking into account condition (b) stated above. The next definition was given in [8].

Definition 4.3. Consider $T \in L(\mathcal{H})$ with closed range and $A_1, A_2 \in L(\mathcal{H})^+$.

1. An operator $G \in L(\mathcal{H})$ is an A_1-*inverse* of T if for each $\eta \in \mathcal{H}$, $G\eta$ is an A_1-least square solution (A_1-LSS) of $T\xi = \eta$, for every $\xi \in \mathcal{H}$, i.e.,

$$\|\eta - TG\eta\|_{A_1} \leq \|\eta - T\xi\|_{A_1}, \text{ for every } \xi \in \mathcal{H}.$$

2. An operator $G \in L(\mathcal{H})$ is an A_1, A_2-*inverse* of T if G is an A_1-inverse of T and, for each $\eta \in H$, $G\eta$ has minimum A_2-seminorm among the A_1-LSS of $T\xi = \eta$.

The next result provides a characterization of A_1-inverses and A_1, A_2-inverses. For its proof see [8, Proposition 5.9] and [8, Proposition 5.17].

Proposition 4.4. *Consider $T \in L(\mathcal{H})$ and $A_1, A_2 \in L(\mathcal{H})^+$. Then*

1. *$G \in L(\mathcal{H})$ is an A_1-inverse of T if and only if $T^*A_1TG = T^*A_1$.*
2. *$G \in L(\mathcal{H})$ is an A_1, A_2-inverse of T if and only if $T^*A_1TG = T^*A_1$ and $R(A_2G) \subseteq N(A_1T)^\perp$*

Theorem 4.5. *Consider $T \in L_A(\mathcal{H})$ with closed range. The following statements are equivalent:*

1. *T is an A-partial isometry;*
2. *$T^\sharp$ is an A-inverse of T;*
3. *T is an A-inverse of $T^\sharp$;*
4. *$P_{N(T^\sharp)^\perp}T$ is an A, I-inverse of $T^\sharp$;*
5. *$P_{N(AT)^\perp}T^\sharp$ is an A, I-inverse of T.*

Proof. $1 \to 2$. Since T is an A-partial isometry then, by Proposition 3.4, $T^\sharp TT^\sharp = T^\sharp$. Then, by left multiplication by A, $T^*ATT^\sharp = T^*A$ and the assertion follows by Proposition 4.4.

$2 \to 3$. Since $TT^\sharp$ is A-selfadjoint it holds that $(T^\sharp)^*AT^\sharp T = ATT^\sharp T = (T^*ATT^\sharp)^* = (T^*A)^* = AT = (T^\sharp)^*A$. Then the assertion follows by Proposition 4.4.

$3 \to 4$. Observe that $(T^\sharp)^*AT^\sharp P_{N(T^\sharp)^\perp}T = (T^\sharp)^*AT^\sharp T = (T^\sharp)^*A$, then $P_{N(T^\sharp)^\perp}T$ is an A-inverse of $T^\sharp$. Also, note that $R(P_{N(T^\sharp)^\perp}T) \subseteq N(T^\sharp)^\perp = N(AT^\sharp)^\perp$. Then, again by Proposition 4.4, it follows that $P_{N(T^\sharp)^\perp}T$ is an A, I-inverse of $T^\sharp$.

$4 \to 5$. Since $P_{N(T^\sharp)^\perp}T$ is an A, I-inverse of $T^\sharp$ then, by Proposition 4.4, it holds that $(T^\sharp)^*AT^\sharp P_{N(T^\sharp)^\perp}T = (T^\sharp)^*A$, or equivalently, $(T^\sharp)^*AT^\sharp T = (T^\sharp)^*A$. Therefore, note that $T^*ATP_{N(AT)^\perp}T^\sharp = T^*ATT^\sharp = T^*(T^\sharp)^*AT^\sharp = ((T^\sharp)^*AT^\sharp T)^* = ((T^\sharp)^*A)^* = AT^\sharp = T^*A$. Hence $P_{N(AT)^\perp}T^\sharp$ is an A-inverse of T. Then, by Proposition 4.4, it follows that $P_{N(AT)^\perp}T^\sharp$ is an A, I-inverse of T, because $R(P_{N(AT)^\perp}T^\sharp) \subseteq N(AT)^\perp$.

$5 \to 1$. Since $P_{N(AT)^\perp}T^\sharp$ is an A, I-inverse of T, then $T^*ATP_{N(AT)^\perp}T^\sharp = T^*A$. Observe that $AT^\sharp TT^\sharp = T^*ATT^\sharp = T^*ATP_{N(AT)^\perp}T^\sharp = T^*A = AT^\sharp$. Then, by Proposition 3.4, T is an A-partial isometry. $\square$

Observe that equivalence $1 \leftrightarrow 5$ of the above proposition generalizes equivalence i $\leftrightarrow$ iii given in the introduction of this section.

Consider a closed range A-partial isometry T. The following result gives a parametrization of the set of A-inverses of T and the set of A-inverses of $T^\sharp$.

Proposition 4.6. *Consider $T \in L_A(\mathcal{H})$ an A-partial isometry with closed range. Then*

1. $P_{N(AT)^\perp}T^\sharp + L(\mathcal{H}, N(AT))$ *is the set of A-inverses of T;*
2. $P_{N(T^\sharp)^\perp}T + L(\mathcal{H}, N(T^\sharp))$ *is the set of A-inverses of $T^\sharp$.*

Proof. 1. By [8, Corollary 5.10], the set of A-inverses of T is given by $((T)^*AT)^\dagger T^*A + L(\mathcal{H}, N(AT))$. Since T is an A-partial isometry then it follows that $T^\sharp TT^\sharp = T^\sharp$. Therefore, $((T)^*AT)^\dagger T^*A = (AT^\sharp T)^\dagger AT^\sharp = (AT^\sharp T)^\dagger AT^\sharp TT^\sharp T = P_{N(AT)^\perp}T$. Then the assertion follows.

2. The proof is similar to the proof of item 1. □

Remark 4.7. The equivalence $1 \leftrightarrow 3$ of Theorem 4.5 and item 2 of Proposition 4.6 are still valid even if $R(T)$ is not closed.

5. A-normal A-partial isometries

Definition 5.1. An operator $T \in L_A(\mathcal{H})$ is called A-normal if $T^\sharp T = TT^\sharp$.

Remark 5.2. Consider $T \in L_A(\mathcal{H})$ an A-normal operator, then $R(TT^\sharp) \subseteq \overline{R(A)}$. If T is also an A-partial isometry then $TT^\sharp = T^\sharp T = Q_{R(T^\sharp)//N(T^\sharp)}$, see Lemma 2.3.

Given a partial isometry T, it is well known that T is normal if and only if $R(T^*) \oplus N(T^*) = \mathcal{H}$ or equivalently $R(T) \oplus N(T) = \mathcal{H}$. In what follows we analyze these equivalences for A-partial isometries.

Proposition 5.3. *Let $T \in L_A(\mathcal{H})$ be an A-partial isometry. Then, T is A-normal if and only if $R(T^\sharp) \oplus_A N(T^\sharp) = \mathcal{H}$ and $R(T^\sharp) \subseteq R(TT^\sharp)$.*

Proof. Since T is an A-partial isometry then $T^\sharp T$ and $TT^\sharp$ are idempotent operators. Suppose that T is A-normal. Then $\mathcal{H} = R(T^\sharp T) \oplus_A R(T^\sharp T)^{\perp_A} = R(T^\sharp T) \oplus_A R(TT^\sharp)^{\perp_A}$. By Lemma 2.3, $R(TT^\sharp)^{\perp_A} = N(T^\sharp)$ and $R(T^\sharp) = R(T^\sharp T)$, so that the first implication follows because $T^\sharp T = TT^\sharp$. Conversely, let $\xi = T^\sharp\eta + \theta \in \mathcal{H}$, where $\theta \in N(T^\sharp)$. Then $T^\sharp T\xi = T^\sharp T(T^\sharp\eta + \theta) = T^\sharp\eta$. In fact, by Lemma 2.3 and by the fact that $R(T^\sharp) \subseteq R(TT^\sharp)$, then $N(T^\sharp) = R(TT^\sharp)^{\perp_A} \subseteq R(T^\sharp)^{\perp_A} = N(T^\sharp T)$. Furthermore, since $R(T^\sharp) \subseteq R(TT^\sharp)$ and $TT^\sharp$ is idempotent, then $TT^\sharp\xi = TT^\sharp(T^\sharp\eta + \theta) = T^\sharp\eta$. Therefore T is A-normal. □

Remarks 5.4. The following facts establish certain differences between partial isometries, A-partial isometries and A-normal partial isometries:

1. The Hilbert space decomposition given in the above proposition does not hold for A-partial isometries, in general. In fact, consider $\mathcal{H} = \ell^2$ and $\mathcal{S} = span\{e_{2n-1}$:

$n \in \mathbb{N}\}$. Define $T \in L(\mathcal{H})$ such that $T(e_{2n-1}) = e_{2n}$ and $T(\mathcal{S}^\perp) = \{0\}$. Then T is a partial isometry, $R(T) + N(T) = \mathcal{S}^\perp$ and $R(T^*) + N(T^*) = \mathcal{S}$.

2. Observe that even if T is an A-normal A-partial isometry then the decomposition $R(T) + N(T) = \mathcal{H}$ does not hold, in general. In fact, consider $A \in L(\mathcal{H})^+$ and $T = P_A + dP_{N(A)} \in L(\mathcal{H})$, where $d \in L(N(A))^+$ has dense non-closed range. Then $T^\sharp T = P_A = TT^\sharp$, so that T is an A-normal A-partial isometry and $R(T) + N(T) = R(A) \oplus R(d) \subsetneq \mathcal{H}$.

Observe that if T is an A-normal A-partial isometry $T^\sharp$ does not necessary satisfy condition (1). In fact, consider $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{pmatrix} \in \mathbb{C}^{3\times 3}$ and $T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \in \mathbb{C}^{3\times 3}$. It is easy to check that $T \in L_A(\mathbb{C}^{3\times 3})$ and $TT^\sharp = T^\sharp T$ is an idempotent operator, so that T is an A-normal A-partial isometry. However, $R(T) \cap N(T^\sharp) \neq \{0\}$. Then, by Theorem 4.1, $TT^\sharp T \neq T$. We finish this section by characterizing when $T^\sharp$ is a generalized inverse of an A-normal A-partial isometry T.

Proposition 5.5. *Let $T \in L_A(\mathcal{H})$ be an A-normal A-partial isometry. Then $TT^\sharp T = T$ if and only if $R(T) = R(T^\sharp)$.*

Proof. Since $T \in L_A(\mathcal{H})$ is an A-normal A-partial isometry then $TT^\sharp = T^\sharp T$ is idempotent and $R(T^\sharp) = R(T^\sharp T) = R(TT^\sharp)$, where the first equality follows from Lemma 2.3. Therefore, if $TT^\sharp T = T$ then $R(T) = R(TT^\sharp) = R(T^\sharp)$. Conversely, if $R(T) = R(T^\sharp)$ then $TT^\sharp = T^\sharp T$ is an idempotent operator with $R(TT^\sharp) = R(T)$. Hence, $TT^\sharp T = T$. $\square$

6. *A*-generalized projections

Generalized projections were defined in 1997 by Groß and Trenkler [13] for matrices. This concept was extended for operators in $L(\mathcal{H})$ in [17]. Given $T \in L(\mathcal{H})$ it is said that T is a generalized projection if $T^2 = T^*$. While generalized projections are a weakened version of the concept of orthogonal projections, they preserve many properties that the orthogonal projections have. For example, if T is a generalized projection then T has closed range, $R(T) \oplus N(T) = R(T^*) \oplus N(T^*) = \mathcal{H}$ and T is a normal partial isometry. The reader is referred to [13,17,1] for these and other properties of generalized projections. In this section we extend the concept of generalized projections to operators on semi-Hilbertian spaces and we analyze the differences between the two concepts.

Definition 6.1. Consider $A \in L(\mathcal{H})^+$. An operator $T \in L(\mathcal{H})$ is called an A-generalized projection if $AT^2 = T^*A$.

Remarks 6.2. The next facts follow from the definition of an A-generalized projection:

1. T is an A-generalized projection if and only if T^2 is an A-adjoint of T.
2. If T is an A-generalized projection then $T^\sharp = P_A T^2$.

Proposition 6.3. *Consider $T \in L(\mathcal{H})$ an A-generalized projection. Then the following assertions hold:*

1. *T is an A-partial isometry.*
2. *$N(AT) = N(AT^n)$ for all $n \in \mathbb{N}$.*
3. *$T^\sharp T = Q_{R(T^\sharp)//N(T^\sharp)}$.*

Proof. 1. If T is an A-generalized projection then $\|T\xi\|_A = \|T^n\xi\|_A$ for all $\xi \in \mathcal{H}$. In fact, since $AT^3 = T^*AT \in L(\mathcal{H})^+$ we get that $\|T\xi\|_A^2 = \langle AT\xi, T\xi\rangle = \langle T^*AT\xi, \xi\rangle = \langle AT^3\xi, \xi\rangle = \langle (T^3)^*A\xi, \xi\rangle = \langle T^*A\xi, T^2\xi\rangle = \langle AT^2\xi, T^2\xi\rangle = \|T^2\xi\|_A^2$. If $n \in \mathbb{N}$, $n > 2$, then $\|T^n\xi\|_A = \|T^2(T^{n-2}\xi)\|_A = \|T(T^{n-2})\xi\|_A = \|T^{n-1}\xi\|_A$. Therefore, $\|T^n\xi\|_A = \|T\xi\|_A$ for all $n \in \mathbb{N}$. Now, let $\xi = P_A T^3\eta \in R(T^\sharp T)$. Then $\|T\xi\|_A = \|T(P_A T^3\eta)\|_A = \|T^4\eta\|_A = \|T^3\eta\|_A = \|P_A T^3\eta\|_A = \|\xi\|_A$, so that T is an A-partial isometry.

2. If T is an A-generalized projection then, by the proof of item 1, $\|T\xi\|_A = \|T^n\xi\|_A$ for every $\xi \in \mathcal{H}$. In consequence, $T^*AT = (T^n)^*AT^n$ for all $n \in \mathbb{N}$. Therefore, $N(AT) = N(AT^n)$.

3. Since T is an A-generalized projection then $T^\sharp T$ is an idempotent operator such that $R(T^\sharp T) = R(T^\sharp)$ and $N(T^\sharp T) = R(T^\sharp T)^{\perp_A}$. Now, by item 2. and by the fact that $AT^2 = T^*A$ we get that $R(T^\sharp T)^{\perp_A} = N(AT) = N(AT^2) = N(T^*A) = N(T^\sharp)$. Therefore the assertion follows. $\square$

Remarks 6.4. The next facts provide some differences between generalized projections and A-generalized projections:

1. If T is an A-generalized projection then $R(T)$ is not necessarily closed. In fact, the operator T considered in item 2 of Remarks 3.5 is an A-generalized projection with non-closed range.
2. If T is an A-generalized projection then $T^\sharp$ is an A-generalized projection with closed range.
3. If T is an A-generalized projection then $R(T) + N(T) \subsetneq \mathcal{H}$ in general. In fact, if A and T are the operators given in item 2 of Remarks 3.5 then T is an A-generalized projection such that $R(T) + N(T) \subsetneq \mathcal{H}$.
4. Every A-generalized projection T provides the decomposition $R(T^\sharp) \oplus_A N(T^\sharp) = \mathcal{H}$.

The following result gives necessary and sufficient conditions for an operator in $L_A(\mathcal{H})$ to be an A-generalized projection.

Proposition 6.5. *Consider $T \in L_A(\mathcal{H})$. The following statements are equivalent:*

1. *T is an A-generalized projection;*
2. *$T^\sharp$ is an A-generalized projection;*
3. *any A-adjoint of T is an A-generalized projection;*
4. *$T^\sharp$ is an A-normal A-partial isometry and $(T^\sharp)^4 = T^\sharp$;*
5. *$T^\sharp T$ is an idempotent operator, $AT^\sharp T = AT^3$ and $N(T^\sharp) = N(AT^2)$.*

Proof. $1 \leftrightarrow 2$. Observe that $A(T^\sharp)^2 = (T^*)^2 A$ and $(T^\sharp)^* A = AT$. Therefore, T is an A-generalized projection if and only if $T^\sharp$ is an A-generalized projection.

$2 \leftrightarrow 3$. If W is an A-adjoint of T then $W = T^\sharp + P_{N(A)} Z$, where $Z \in L(\mathcal{H})$. Then $AW^2 = A(T^\sharp + P_{N(A)}Z)W = AT^\sharp W = A(T^\sharp)^2$ because $N(A) \subseteq N(T^\sharp)$. In addition, $W^* A = ((T^\sharp)^* + Z^* P_{N(A)})A = (T^\sharp)^* A$. Therefore, W is an A-generalized projection if and only if $T^\sharp$ is an A-generalized projection.

$3 \to 4$. Suppose that any A-adjoint of T is an A-generalized projection. By $1 \leftrightarrow 3$, it holds that T is an A-generalized projection, i.e., $AT^2 = T^* A$. By Proposition 6.3, T is an A-partial isometry. Moreover, observe that $(T^\sharp)^\sharp T^\sharp = P_A T T^\sharp = P_A T A^\dagger T^* A = P_A T P_A T^2 = P_A T^3 = A^\dagger A T^2 T = A^\dagger T^* A T = T^\sharp T = T^\sharp P_A T = T^\sharp (T^\sharp)^\sharp$. Therefore, $T^\sharp$ is A-normal. On the other hand, $(T^\sharp)^2 = A^\dagger T^* P_A T^* A = A^\dagger (T^*)^2 A = A^\dagger A T = P_A T$, so that $(T^\sharp)^4 = P_A T P_A T = P_A T^2 = T^\sharp$.

$4 \to 5$. Since $T^\sharp$ is an A-normal A-partial isometry then $P_A T T^\sharp = T^\sharp T$ is idempotent. Since $(T^\sharp)^4 = T^\sharp$, then $T^* A = AT^\sharp = A(T^\sharp)^4 = (T^*)^4 A$, or equivalently $AT = AT^4$. Therefore, $AT^\sharp T = AT^\sharp P_A T = AT^\sharp P_A T^4 = AT^\sharp T P_A T^3 = AT^3$, where the last equality follows because $T^\sharp T = P_A T T^\sharp$ is idempotent with $R(T^\sharp T) = R(P_A T T^\sharp) = R((T^\sharp)^\sharp) = R(T^\sharp) = R(P_A T)$ (see Lemma 2.3) and $R(P_A T^3) \subseteq R(P_A T)$. It is left to prove that $N(T^\sharp) = N(AT^2)$. First note that $N(AT) = N(T^\sharp T) = N(P_A T T^\sharp) = N((T^\sharp)^\sharp T^\sharp) = N(AT^\sharp) = N(T^\sharp) = N(T^* A)$, where the second equality holds because $T^\sharp$ is A-normal and the fourth equality follows by Lemma 2.3. Then, it holds that $\xi \in N(AT^2)$ if and only if $T\xi \in N(AT) = N(T^* A)$, equivalently $\xi \in N(T^* A T) = N(AT) = N(T^* A)$. Therefore, $N(T^\sharp) = N(T^* A) = N(AT^2)$.

$5 \to 1$. Since $T^\sharp T$ is idempotent then, by Lemma 2.3, $R(T^\sharp T) = R(T^\sharp)$. Therefore, by Proposition 3.4, $TT^\sharp$ is also idempotent. Then $\mathcal{H} = R(TT^\sharp) + R(TT^\sharp)^{\perp_A} = R(TT^\sharp) + N(T^\sharp)$. Take $\xi = TT^\sharp \eta + \theta$, where $\theta \in N(T^\sharp) = N(AT^2)$. Therefore it follows that $AT^2 \xi = AT^2(TT^\sharp \eta + \theta) = AT^3 T^\sharp \eta = AT^\sharp T T^\sharp \eta = T^* A(TT^\sharp \eta + \theta) = T^* A \xi$. Hence, T is an A-generalized projection. $\square$

Consider $A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \in \mathbb{C}^{2\times 2}$ and $T = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \in \mathbb{C}^{2\times 2}$. It is not difficult to see that T is an A-generalized projection such that $TT^\sharp T \neq T$. Next we study when $T^\sharp$ satisfies equation (1) for an A-generalized projection T.

Theorem 6.6. *Let $T \in L(\mathcal{H})$ be an A-generalized projection. Then the following assertions are equivalent:*

1. $TT^\sharp T = T$;
2. $N(T^\sharp) \subseteq N(T)$;
3. $R(T)\dot{+}N(T) = \mathcal{H}$ *and* $R(T) \cap N(A) = \{0\}$.

In addition, if one of the above conditions holds then $TT^\sharp = Q_{R(T)//N(T)}$.

Proof. Since T is an A-generalized projection then by Proposition 6.3, it follows that $T^\sharp T = Q_{R(T^\sharp)//N(T^\sharp)}$ and $N(T^\sharp) = N(AT)$.

$1 \leftrightarrow 2$. Suppose $TT^\sharp T = T$. Then $N(T^\sharp) = N(T^\sharp T) \subseteq N(TT^\sharp T) = N(T)$ and so, the assertion follows. Conversely, if $N(T^\sharp) \subseteq N(T)$ then $T = TQ_{R(T^\sharp)//N(T^\sharp)} + TQ_{N(T^\sharp)//R(T^\sharp)} = TQ_{R(T^\sharp)//N(T^\sharp)} = TT^\sharp T$.

$1 \leftrightarrow 3$. If $TT^\sharp T$ then, by Theorem 4.1, it holds that $R(T)\dot{+}N(T^\sharp) = \mathcal{H}$ and $R(T) \cap N(A) = \{0\}$. Therefore, as $N(T^\sharp) = N(AT)$ (see Lemma 2.3 and Proposition 6.3) and $R(T) \cap N(A) = \{0\}$ then $N(T^\sharp) = N(T)$ and so $R(T)\dot{+}N(T) = \mathcal{H}$. The converse is similar, noticing that in this case $R(T)$ is closed.

Moreover, suppose T is an A-generalized projection then $TT^\sharp$ is an idempotent operator with $N(TT^\sharp) = N(T^\sharp)$. Since $TT^\sharp T = T$ then $R(TT^\sharp) = R(T)$ and $N(T^\sharp) = N(T)$, so that the assertion follows. □

Corollary 6.7. *Let T be an A-generalized projection. Then $T = W + P_{N(A)}T$, where W is an A-generalized projection such that $WW^\sharp W = W$.*

Proof. The proof is similar to the proof of Corollary 4.2. □

An A-generalized projection is not A-normal, in general. In fact, consider $A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \in \mathbb{C}^{2\times 2}$ and $T = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \in \mathbb{C}^{2\times 2}$. Then it is easy to check that T is an A-generalized projection which is not A-normal. The following result describes the A-generalized projections which are A-normal.

Proposition 6.8. *Let T be an A-generalized projection. Then, T is A-normal if and only if $R(TT^\sharp) \subseteq \overline{R(A)}$.*

Proof. If T is A-normal then, by Remark 5.2, $R(TT^\sharp) \subseteq \overline{R(A)}$. Conversely, if T is an A-generalized projection such that $R(TT^\sharp) \subseteq \overline{R(A)}$ then $TT^\sharp = P_A TT^\sharp = P_A T P_A T^2 = P_A T^3 = T^\sharp T$. Then T is an A-normal operator. □

Proposition 6.9. *Let T be an A-normal A-generalized projection. Then, $TT^\sharp T = T$ if and only if $R(T) \subseteq \overline{R(A)}$.*

Proof. Since T is an A-generalized projection then $T^\sharp = P_A T^2$ and $P_A T^4 = P_A T$. Suppose that $R(T) \subseteq \overline{R(A)}$. Then, since T is A-normal we get that $TT^\sharp T = T^\sharp TT = P_A T^4 = P_A T = T$. Conversely, if $TT^\sharp T = T$ then $R(T) = R(TT^\sharp) = R(T^\sharp T) = R(T^\sharp) \subseteq \overline{R(A)}$. $\square$

We finish this section with a characterization of the A-normal A-partial isometries which are A-generalized projections.

Proposition 6.10. *Let T be an A-normal A-partial isometry. Then the following statements are equivalent:*

1. *T is an A-generalized projection;*
2. *$(T^\sharp)^4 = T^\sharp$;*
3. *$(T^\sharp)^3 = TT^\sharp$.*

Proof. First note that $T^\sharp$ is A-normal because T is A-normal. Then $1 \leftrightarrow 2$ follows by Proposition 6.5.

$2 \leftrightarrow 3$. If $(T^\sharp)^3 = TT^\sharp$, then $(T^\sharp)^4 = T^\sharp TT^\sharp = T^\sharp$. Conversely, since T is an A-normal A-partial isometry, by Remark 5.2, it holds that $TT^\sharp = T^\sharp T = Q_{R(T^\sharp)/N(T^\sharp)}$. Then $(T^\sharp)^3 = Q_{R(T^\sharp)/N(T^\sharp)}(T^\sharp)^3 = TT^\sharp(T^\sharp)^3 = T(T^\sharp)^4 = TT^\sharp$. $\square$

7. Applications to weighted projections

In 1974, Mitra and Rao [16] introduced the notion of A-projections into subspaces of finite dimensional spaces. In [8], this concept was studied for infinite dimensional spaces. In this section we apply the results obtained along this work to relate the A-projections into closed subspaces with the A-partial isometries and A-generalized projections.

An operator $T \in L(\mathcal{H})$ is called an *A-projection into $\mathcal{S}$* if $R(T) \subseteq \mathcal{S}$ and

$$\|y - Ty\|_A \leq \|y - s\|_A, \quad \text{for all } y \in \mathcal{H}, \quad \text{for all } s \in \mathcal{S}.$$

Moreover, T is called an *A-projection* if T is an A-projection into $\overline{R(T)}$. See [8] for the proof of the following characterizations of A-projections.

Proposition 7.1. *Let $T \in L(\mathcal{H})$. Then the following assertions hold:*

1. *T is an A-projection if and only if $AT = T^*A = AT^2$, i.e., T is an A-selfadjoint A-generalized projection.*
2. *T is an A-projection into $\mathcal{S}$ if and only if T is an A-projection and $ATP_\mathcal{S} = AP_\mathcal{S}$.*

Define

$$\Pi(A, \mathcal{S}) = \{T \in L(\mathcal{H}) : T \text{ is an } A\text{-projection into } \mathcal{S}\}.$$

Next we show that the set $\Pi(A, \mathcal{S})$ can be described by means of an A-partial isometry with range $\mathcal{S}$. Before that we study the relationship between the compatibility of a pair $(A, \mathcal{S})$ and the existence of A-partial isometries with range $\mathcal{S}$.

Proposition 7.2. *Consider $A \in L(\mathcal{H})^+$ and $\mathcal{S}$ a closed subspace of $\mathcal{H}$. The pair $(A, \mathcal{S})$ is compatible if and only if there exists an A-partial isometry with range $\mathcal{S}$.*

Proof. First observe that if T is an A-partial isometry then $\mathcal{H} = R(T) + R(T)^{\perp_A}$. In fact, since T is an A-partial isometry then $TT^\sharp$ is idempotent and, by Lemma 2.3, $N(TT^\sharp) = R(T)^{\perp_A}$. Hence, $\mathcal{H} = R(TT^\sharp) \dot{+} N(TT^\sharp) \subseteq R(T) + R(T)^{\perp_A}$, as claimed. Now, suppose T is an A-partial isometry with range $\mathcal{S}$. Then the pair $(A, \mathcal{S})$ is compatible because of the above assertion. Conversely, suppose that the pair $(A, \mathcal{S})$ is compatible and consider Q and A-selfadjoint projection with range $\mathcal{S}$. By item 1 of Remarks 3.5 it holds that Q is an A-partial isometry with range $\mathcal{S}$. $\square$

Corollary 7.3. *Consider $A \in L(\mathcal{H})^+$ and $T \in L(\mathcal{H})$ an A-partial isometry with $R(T) = \mathcal{S}$. Then $TT^\sharp = P_{A,\mathcal{S}} - F$, where $F \in L(\mathcal{H})$ is an idempotent operator with $R(F) = \mathcal{S} \cap N(A)$.*

Proof. Let $\mathcal{N} = \mathcal{S} \cap N(A)$. By Proposition 7.2, the pair $(A, \mathcal{S})$ is compatible. Moreover, it holds that $P_{\mathcal{S} \ominus \mathcal{N}} TT^\sharp = P_{A, \mathcal{S} \ominus \mathcal{N}}$. In fact, it is not difficult to see that $P_{\mathcal{S} \ominus \mathcal{N}} TT^\sharp$ is an A-selfadjoint idempotent operator. In order to prove that $R(P_{\mathcal{S} \ominus \mathcal{N}} TT^\sharp) = \mathcal{S} \ominus \mathcal{N}$, consider $T_0 = P_{\mathcal{S} \ominus \mathcal{N}} T$. Note that $R(T_0) = \mathcal{S} \ominus \mathcal{N}$. Observe that $T_0^\sharp = A^\dagger T^* P_{\mathcal{S} \ominus \mathcal{N}} A = A^\dagger T^* A = T^\sharp$ and $T_0^\sharp T_0 = T^\sharp P_{\mathcal{S} \ominus \mathcal{N}} T = T^\sharp (P_{\mathcal{S} \ominus \mathcal{N}} + P_{\mathcal{N}}) T = T^\sharp T$. Since T is an A-partial isometry then $T_0^\sharp T_0 = T^\sharp T$ is idempotent, so that T_0 is an A-partial isometry. Therefore, by Theorem 4.1 it holds that $T_0 T_0^\sharp T_0 = T_0$. Then, $R(T_0 T_0^\sharp) = R(T_0) = \mathcal{S} \ominus \mathcal{N}$. Since $P_{\mathcal{S} \ominus \mathcal{N}} TT^\sharp = T_0 T_0^\sharp$ then $R(P_{\mathcal{S} \ominus \mathcal{N}} TT^\sharp) = \mathcal{S} \ominus \mathcal{N}$. Hence, $P_{\mathcal{S} \ominus \mathcal{N}} TT^\sharp$ is an A-selfadjoint idempotent operator with range $\mathcal{S} \ominus \mathcal{N}$, so that by Proposition 2.2, $P_{\mathcal{S} \ominus \mathcal{N}} TT^\sharp = P_{A, \mathcal{S} \ominus \mathcal{N}}$, as claimed.

Then, $TT^\sharp = P_{A, \mathcal{S} \ominus \mathcal{N}} + P_{\mathcal{N}} TT^\sharp = P_{A, \mathcal{S} \ominus \mathcal{N}} + P_{\mathcal{N}} + P_{\mathcal{N}} TT^\sharp - P_{\mathcal{N}} = P_{A,\mathcal{S}} + P_{\mathcal{N}}(TT^\sharp - I)$. Observe that $I - TT^\sharp$ is an idempotent operator because T is an A-partial isometry. Moreover, $R(I - TT^\sharp) = N(TT^\sharp) = R(T)^{\perp_A} = (A\mathcal{S})^\perp$, where the second equality holds by Lemma 2.3. Finally, consider $F = P_{\mathcal{N}}(I - TT^\sharp)$. Since $\mathcal{N} \subseteq (A\mathcal{S})^\perp = R(I - TT^\sharp)$, then F is idempotent and $R(F) = \mathcal{N}$. $\square$

Proposition 7.4. *If $(A, \mathcal{S})$ is compatible then*

$$\Pi(A, \mathcal{S}) = TT^\sharp + L(\mathcal{H}, \mathcal{S} \cap N(A)),$$

where T is an A-partial isometry with $R(T) = \mathcal{S}$.

Proof. Let $\mathcal{N} = \mathcal{S} \cap N(A)$. Since the pair $(A, \mathcal{S})$ is compatible then, by [8, Proposition 4.14], $\Pi(A, \mathcal{S})$ is not empty. By Corollary 7.3, there exists an A-partial isometry T with $R(T) = \mathcal{S}$ and $TT^\sharp = P_{A,\mathcal{S}} - F$, where F is an idempotent operator with $R(F) = \mathcal{N}$.

Finally, observe that, by [8, Proposition 4.17], $E \in \Pi(A, \mathcal{S})$ if and only if $E = P_{A,\mathcal{S}} + Z$, where $Z \in L(\mathcal{H}, \mathcal{N})$. Equivalently, $E = P_{A,\mathcal{S}} - F + (Z + F) = TT^\sharp + Z + F$, with $R(Z + F) \subseteq \mathcal{N}$. □

We finish this article by showing that every A-projection can be obtained from an A-generalized projection.

Proposition 7.5. *Consider $E \in L_A(\mathcal{H})$. Then the following statements are equivalent:*

1. *E is an A-projection;*
2. *$AE = AT^\sharp T$, with T an A-generalized projection;*
3. *$E^\sharp = T^\sharp T$, with T an A-generalized projection;*
4. *$E = T^\sharp T + Z$, T an A-generalized projection and $Z \in L(\mathcal{H}, N(A))$.*

Proof. $1 \to 2$. Suppose that E is an A-projection. In particular, E is an A-generalized projection. Therefore $AE^\sharp E = AA^\dagger E^* AE = AA^\dagger AE^2 = AA^\dagger AE = AE$.

$2 \to 3$. Note that $E^\sharp = A^\dagger E^* A = A^\dagger (T^\sharp T)^* A = A^\dagger AT^\sharp T = P_A T^\sharp T = T^\sharp T$.

$3 \to 4$. Observe that, $P_A E = (E^\sharp)^\sharp = A^\dagger (T^\sharp T)^* A = P_A T^\sharp T = T^\sharp T$, where the third equality follows because $T^\sharp T$ is A-selfadjoint. Then $E = P_A E + (I - P_A)E = T^\sharp T + Z$, where T is an A-generalized projection and $Z = I - P_A \in L(\mathcal{H}, N(A))$.

$4 \to 1$. Suppose $E = T^\sharp T + Z$, with T an A-generalized projection and $Z \in L(\mathcal{H}, N(A))$. Then $AE = AT^\sharp T = E^* A$. Also, observe that $AE^2 = E^* AE = E^* AT^\sharp T = A(T^\sharp T)^2 = AT^\sharp T = AE$, where the forth equality holds because $T^\sharp T$ is idempotent (see Proposition 6.5). Therefore, $AE^2 = AE = E^* A$, so that E is an A-projection. □

Observe that the above result generalizes the fact that $E \in L(\mathcal{H})$ is an orthogonal projection if and only if $E = T^* T$ for some generalized projection T.

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References

[1] E. Andruchow, G. Corach, M. Mbekhta, A note on the differentiable structure of generalized idempotents, Cent. Eur. J. Math. (2012) 1–16.
[2] M.L. Arias, M. Mbekhta, A-partial isometries and generalized inverses, Linear Algebra Appl. 439 (2013) 1286–1293.
[3] M.L. Arias, G. Corach, M.C. Gonzalez, Partial isometries in semi-Hilbertian spaces, Linear Algebra Appl. 428 (7) (2008) 1460–1475.
[4] M.L. Arias, G. Corach, M.C. Gonzalez, Metric properties of projections in semi-Hilbertian spaces, Integral Equations Operator Theory 62 (2008) 11–28.
[5] J.K. Baksalary, O.M. Baksalary, X. Liu, Further properties of generalized and hypergeneralized projectors, Linear Algebra Appl. 389 (2004) 295–303.

[6] J.K. Baksalary, O.M. Baksalary, X. Liu, G. Trenkler, Further results on generalized and hypergeneralized projectors, Linear Algebra Appl. 429 (2008) 1038–1050.
[7] J.K. Baksalary, X. Liu, An alternative characterization of generalized projectors, Linear Algebra Appl. 388 (2004) 61–65.
[8] G. Corach, G. Fongi, A. Maestripieri, Weighted projections into closed subspaces, Stud. Math. 216 (2013) 131–148.
[9] G. Corach, A. Maestripieri, Weighted generalized inverses, oblique projections, and least-squares problems, Numer. Funct. Anal. Optim. 26 (2005) 659–673.
[10] G. Corach, A. Maestripieri, D. Stojanoff, Schur complements and oblique projections, Acta Sci. Math. 67 (2001) 337–356.
[11] G. Corach, A. Maestripieri, D. Stojanoff, A Classification of Projectors, vol. 67, Banach Center Publication, Institute of Mathematics, Polish Academy of Sciences, Warszawa, 2004, pp. 145–160.
[12] R.G. Douglas, On majorization, factorization and range inclusion of operators in Hilbert space, Proc. Amer. Math. Soc. 17 (1966) 413–416.
[13] J. Groß, G. Trenkler, Generalized and hypergeneralized projectors, Linear Algebra Appl. 264 (1997) 463–474.
[14] C.W. Groetsch, Generalized Inverses of Linear Operators: Representation and Approximation, Monographs and Textbooks in Pure and Applied Mathematics, vol. 37, Marcel Dekker, New York, Basel, 1977.
[15] M.G. Krein, Compact linear operators on functional spaces with two norms, Integral Equations Operator Theory 30 (1998) 140–162 (translation from the Ukrainian of a paper published in 1937).
[16] S.K. Mitra, C. Rao, Projections under seminorms and generalized Moore Penrose inverses, Linear Algebra Appl. 9 (1974) 155–167.
[17] S. Radosavljević, D. Djordjević, On pairs of generalized and hypergeneralized projections in a Hilbert space, Funct. Anal. Approx. Comput. (2013) 67–75.