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The minus order and range additivity

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We study the minus order on the algebra of bounded linear operators on a Hilbert space. By giving a characterization in terms of range additivity, we show that the intrinsic nature of the minus order is algebraic. Applications to generalized inverses of the sum of two operators, to systems of operator equations and to optimization problems are also presented.

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1. Introduction

The minus order was introduced by Hartwig [24] and independently by Nambooripad [30], in both cases on semigroups, with the idea of generalizing some classical partial orders. It was extended to operators in infinite dimensional spaces independently by Antezana, Corach and Stojanoff [2] and by Šemrl [36]. There is now an extensive literature devoted to this order and other related partial orders on matrices, operators and elements of various algebraic structures. See for example, [8,28,29].

The main goal of this work is to obtain a new characterization of the minus order for operators acting on Hilbert spaces in terms of the so called range additivity property. Given two linear bounded operators A and B acting on a Hilbert space $\mathscr{H}$, we say that A and B have the *range additivity property* if $R(A+B) = R(A) + R(B)$, where $R(T)$ stands for the range of an operator T. Operators with this property have been studied in [4] and [5] (see also [10]). Recall that if A and B are two bounded linear Hilbert space operators then $A \overset{-}{\leq} B$ (where the symbol "$\overset{-}{\leq}$" stands for the minus order of operators) if and only if there are bounded (oblique) projections, i.e. idempotents, P and Q such that $A = PB$ and $A^* = QB^*$. In this paper, we prove that this is equivalent to the range of B being the direct sum of ranges of A and $B - A$ and the range of B^* being the direct sum of ranges of A^* and $B^* - A^*$. Thus the minus order is intrinsically algebraic in nature. This plays an equivalent role to a known characterization when A and B are matrices [24,29]; that $A \overset{-}{\leq} B$ if and only if the rank of $B - A$ is the difference of the rank of B and the rank of A.

As a consequence, diverse concepts that have been developed for matrices and operators are in fact manifestations of the minus order. These include, for example, the notions of weakly bicomplementary matrices defined due to Werner [37], and quasidirect addition of operators defined by Lešnjak and Šemrl [26]. Although in these papers the minus order does not appear explicitly, these notions when applied to operators A and B are equivalent to saying that $A \overset{-}{\leq} A + B$. The minus order also lurks in the papers of Baksalary and Trenkler [9], Baksalary, Šemrl and Styan [7], Mitra [27] and Arias, Corach and Maestripieri [5].

The minus order can be weakened to what we call left and right minus orders. As with the minus order, these orders are easily derived from a range additivity condition. It happens that they truly differ from the minus order only in the infinite dimensional setting. When $A \overset{-}{\leq} B$, we give some applications to formulas for generalized inverses of sums $A + B$ in terms of generalized inverses of A and B, and we show that certain optimization problems involving the operator $A + B$ can be decoupled into a system of similar problems for A and B.

The paper is organized as follows. In Section 2 we collect some useful known results about range additivity, while in Section 3, the minus order is defined and the connection with range additivity is made. Motivated by the concepts of the left and the right star orders, we define left and the right minus orders on $L(\mathscr{H})$. For matrices, these are equivalent to the minus order, with differences only emerge in the infinite dimensional context. Proposition 3.13 characterizes the left minus order in terms of densely defined, though not necessarily bounded, projections. Additionally, the left minus, the right minus and the minus orders are characterized in terms of (densely defined) inner generalized inverses, generalizing a matricial result (see [28]).

Finally, Section 4 is devoted to applications. We begin by relating the minus partial order to some formulas for reflexive inner inverses of the sum of two operators. In particular, we give an alternative proof for the Fill–Fishkind formula for the Moore–Penrose inverse of a sum, as found in [19] for matrices and extended to $L(\mathscr{H})$ by Arias et al. [5]. We also apply the new characterization of the minus order to systems of equations and least squares problems. We include a final remark about a possible generalization of the minus order involving densely defined projections with closed range.

2. Preliminaries

Throughout, $(\mathscr{H}, \langle\cdot,\cdot\rangle)$ denotes a complex Hilbert space and $L(\mathscr{H})$ the algebra of linear bounded operators on $\mathscr{H}$, $\mathscr{Q}$ is the subset of $L(\mathscr{H})$ of (oblique) projections or idempotents, i.e., $\mathscr{Q} = \{Q \in L(\mathscr{H}) : Q^2 = Q\}$ and $\mathscr{P}$ the subset of $\mathscr{Q}$ of orthogonal projections, i.e., $\mathscr{P} = \{P \in L(\mathscr{H}) : P^2 = P = P^*\}$.

Given $\mathscr{M}$ and $\mathscr{N}$ two subspaces of $\mathscr{H}$, write $\mathscr{M} \dot{+} \mathscr{N}$ for the direct sum of $\mathscr{M}$ and $\mathscr{N}$, $\mathscr{M} \oplus \mathscr{N}$ for the orthogonal sum; if $\mathscr{M}$ and $\mathscr{N}$ are closed denote $\mathscr{M} \ominus \mathscr{N} = \mathscr{M} \cap (\mathscr{M} \cap \mathscr{N})^{\perp}$.

Given $\mathscr{M}$ and $\mathscr{N}$ two subspaces such that $\mathscr{M} \cap \mathscr{N} = \{0\}$, $P_{\mathscr{M}//\mathscr{N}}$ stands for the projection onto $\mathscr{M}$ with nullspace $\mathscr{N}$, defined as $P_{\mathscr{M}//\mathscr{N}} : \mathscr{M} \dot{+} \mathscr{N} \to \mathscr{H}$, $P_{\mathscr{M}//\mathscr{N}}$ is the identity on $\mathscr{M}$ and zero on $\mathscr{N}$. The domain of $P_{\mathscr{M}//\mathscr{N}}$ is $\mathscr{D}(P_{\mathscr{M}//\mathscr{N}}) = \mathscr{M} \dot{+} \mathscr{N}$; $P_{\mathscr{M}//\mathscr{N}}$ is densely defined if $\overline{\mathscr{M} \dot{+} \mathscr{N}} = \mathscr{H}$ and it is bounded, i.e. $P_{\mathscr{M}//\mathscr{N}} \in \mathscr{Q}$, if and only if $\mathscr{M}$ and $\mathscr{N}$ are closed and $\mathscr{M} \dot{+} \mathscr{N} = \mathscr{H}$. See the paper by Ôta [32] for many results on (unbounded) projections.

If $\mathscr{M}$ is a closed subspace, $P_{\mathscr{M}} = P_{\mathscr{M}//\mathscr{M}^{\perp}}$ is the orthogonal projection onto $\mathscr{M}$.

For $A \in L(\mathscr{H})$, $R(A)$ stands for the range of A, $N(A)$ for its nullspace and P_A for $P_{\overline{R(A)}}$. The Moore–Penrose inverse of A is the (densely defined) operator $A^{\dagger} : R(A) \oplus R(A)^{\perp} \to \mathscr{H}$, defined by $A^{\dagger}|_{R(A)} = (A|_{N(A)^{\perp}})^{-1}$ and $N(A^{\dagger}) = R(A)^{\perp}$. It holds that $A^{\dagger} \in L(\mathscr{H})$ if and only if A has a closed range.

Given $\mathscr{M}$ and $\mathscr{N}$ two closed subspaces of $\mathscr{H}$, the *minimal angle* between $\mathscr{M}$ and $\mathscr{N}$ is $\alpha_0(\mathscr{M}, \mathscr{N}) \in [0, \pi/2]$, the cosine of which is

$$c_0(\mathscr{M}, \mathscr{N}) = \sup \{|\langle\xi, \eta\rangle| : \ \xi \in \mathscr{M}, \ \|\xi\| \le 1, \ \ \eta \in \mathscr{N}, \ \|\eta\| \le 1\} \in [0, 1].$$

When the minimal angle between $\mathscr{M}$ and $\mathscr{N}$ is strictly less that 1, then the sum $\mathscr{M} + \mathscr{N}$ is closed and direct, moreover, we have the following.

Proposition 2.1. *Let $\mathscr{M}$ and $\mathscr{N}$ be two closed subspaces of $\mathscr{H}$. The following statements are equivalent:*

(1) $c_0(\mathscr{M},\mathscr{N}) < 1$*;*
(2) $\mathscr{M} \dot{+} \mathscr{N}$ *is closed;*
(3) $\mathscr{H} = \mathscr{M}^\perp + \mathscr{N}^\perp$*.*

For a proof, see Lemma 2.11 and Theorem 2.12 in [13].

For $A, B \in L(\mathscr{H})$, it always holds that $R(A+B) \subseteq R(A) + R(B)$. We say that A and B have the *range additivity property* if $R(A+B) = R(A) + R(B)$. In this case, $R(A) \subseteq R(A+B)$. Conversely, if $R(A) \subseteq R(A+B)$ then, for $x \in \mathscr{H}$, $Bx = (A+B)x - Ax \in R(A+B)$. We have proved the following.

Lemma 2.2 *([5, Proposition 2.4]). For* $A, B \in L(\mathscr{H})$*,* $R(A+B) = R(A) + R(B)$ *if and only if* $R(A) \subseteq R(A+B)$*.*

Operators having the range additivity property were characterized in [5, Theorem 2.10]. Closely related is the following result for operators $A, B \in L(\mathscr{H})$ satisfying the condition $R(A) \cap R(B) = \{0\}$.

Proposition 2.3 *([5, Theorem 2.10]). Consider* $A, B \in L(\mathscr{H})$ *such that* $R(A) \cap R(B) = \{0\}$ *then* $R(A+B) = R(A) \dot{+} R(B)$ *if and only if* $\mathscr{H} = N(A) + N(B)$*.*

The next result will be useful in characterizing the minus order in Section 3 (see [5, Proposition 2.2]).

Proposition 2.4. *For* $A, B \in L(\mathscr{H})$ *consider the following statements:*

(1) $\overline{R(A^*)} \dot{+} \overline{R(B^*)}$ *is closed;*
(2) there exists $Q \in \mathscr{Q}$ *such that* $A^* = Q(A^* + B^*)$*;*
(3) $N(A) + N(B) = \mathscr{H}$*;*
(4) $R(A+B) = R(A) + R(B)$*.*

Then $(1) \Leftrightarrow (2) \Leftrightarrow (3) \Rightarrow (4)$*. The implication* $(4) \Rightarrow (3)$ *holds if* $R(A) \cap R(B) = \{0\}$*.*

For a proof of $(1) \Leftrightarrow (2)$ see [2, Proposition 4.13]. $(1) \Leftrightarrow (3)$ was stated in Proposition 2.1. The implication $(3) \Rightarrow (4)$ follows from the proof of [3, Proposition 2.8]. $(4) \Rightarrow (3)$ follows from Proposition 2.3 since $R(A) \cap R(B) = \{0\}$.

3. The minus order

Different definitions have been given for the *minus (partial) order*. For operators we offer one which equivalent to those appearing in [2] and [36].

Definition 3.1. For $A, B \in L(\mathscr{H})$, $A \overset{-}{\leq} B$ if there exist $P, Q \in \mathscr{Q}$ such that $A = PB$ and $A^* = QB^*$.

Proofs that $\overset{-}{\leq}$ is a partial order on $L(\mathcal{H})$ can be found in [2, Corollary 4.14] and [36, Corollary 3]. It is easy to see that the ranges of P and Q can be fixed so that $R(P) = \overline{R(A)}$ and $R(Q) = \overline{R(A^*)}$. For details, see [2, Proposition 4.13] and the definition of minus order in [36].

In the next proposition we collect some characterizations of the minus order in terms of angle conditions and sum of closed subspaces.

Proposition 3.2. *Consider $A, B \in L(\mathcal{H})$. The following statements are equivalent:*

(1) $A \overset{-}{\leq} B$;
(2) $c_0(\overline{R(A)}, \overline{R(B-A)}) < 1$ *and* $c_0(\overline{R(A^*)}, \overline{R(B^*-A^*)}) < 1$;
(3) $\overline{R(B)} = \overline{R(A)} \dotplus \overline{R(B-A)}$ *and* $\overline{R(B^*)} = \overline{R(A^*)} \dotplus \overline{R(B^*-A^*)}$;
(4) $N(A) + N(B-A) = N(A^*) + N(B^*-A^*) = \mathcal{H}$;
(5) *there exists* $P \in \mathcal{Q}$ *such that* $A = PB$ *and* $R(A) \subseteq R(B)$.

Proof. The equivalences (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (4) follow applying the definition of the minus order and Proposition 2.1 to the operators A, $B - A$, A^* and $B^* - A^*$, see also [2, Proposition 4.13].

For (2) $\Leftrightarrow$ (3), suppose that $c_0(\overline{R(A)}, \overline{R(B-A)}) < 1$ and $c_0(\overline{R(A^*)}, \overline{R(B^*-A^*)}) < 1$. Then $\overline{R(A)} \dotplus \overline{R(B-A)}$ and $\overline{R(A^*)} \dotplus \overline{R(B^*-A^*)}$ are closed. In this case, $\overline{R(B)} \subseteq \overline{R(A)} \dotplus \overline{R(B-A)}$. On the other hand, applying Proposition 2.4, there exists $Q \in \mathcal{Q}$ such that $A^* = QB^*$. Then $N(B^*) \subseteq N(A^*)$ and $N(B^*) \subseteq N(B^* - A^*)$, or equivalently $\overline{R(A)} \subseteq \overline{R(B)}$ and $\overline{R(B-A)} \subseteq \overline{R(B)}$. Then $\overline{R(B)} = \overline{R(A)} \dotplus \overline{R(B-A)}$. Similarly, $\overline{R(B^*)} = \overline{R(A^*)} \dotplus \overline{R(B^*-A^*)}$. See also [36, Theorem 2]. Conversely, if item 3 holds, then $\overline{R(A)} \dotplus \overline{R(B-A)}$ and $\overline{R(A^*)} \dotplus \overline{R(B^*-A^*)}$ are closed or equivalently, by Proposition 2.1, item 2 holds.

Next consider (1) $\Leftrightarrow$ (5). If $A \overset{-}{\leq} B$ then $A = PB = BQ^*$ with $P, Q \in \mathcal{Q}$, so that $A = PB$ and $R(A) \subseteq R(B)$. Conversely, suppose $R(A) \subseteq R(B)$ and there exists $P \in \mathcal{Q}$ such that $A = PB$. Then by Lemma 2.2 it holds that $R(B) = R(A) + R(B-A)$. Moreover $R(A) \cap R(B-A) = \{0\}$ because $R(A) \subseteq R(P)$ and $R(B-A) \subseteq N(P)$, so that $R(B) = R(A) \dotplus R(B-A)$. In this case, (4) $\Rightarrow$ (2) of Proposition 2.4 can be applied so that there exists $Q \in \mathcal{Q}$ such that $A^* = QB^*$. Therefore $A \overset{-}{\leq} B$. $\square$

The following is a key result that will be useful on many occasions throughout the paper. It gives a new characterization of the minus partial order in terms of the range additivity property, showing that the minus order has an algebraic nature.

Theorem 3.3. *Consider $A, B \in L(\mathcal{H})$. Then the following assertions are equivalent:*

(1) $A \overset{-}{\leq} B$;
(2) $R(B) = R(A) \dotplus R(B-A)$ *and* $R(B^*) = R(A^*) \dotplus R(B^*-A^*)$.

Proof. Suppose that $A \bar{\leq} B$. By Proposition 3.2, it follows that $\overline{R(A)} \dotplus \overline{R(B-A)}$ and $\overline{R(A^*)} \dotplus \overline{R((B-A)^*)}$ are closed. In particular, $R(A) \cap R(B-A) = R(A^*) \cap R(B^*-A^*) = \{0\}$. Also, it follows from Proposition 2.4 that $R(B^*) = R(A^*) + R(B^* - A^*)$ and $R(B) = R(A) + R(B-A)$. Therefore, $R(B) = R(A) \dotplus R(B-A)$ and $R(B^*) = R(A^*) \dotplus R(B^* - A^*)$.

Conversely, suppose that $R(B) = R(A) \dotplus R(B-A)$ and $R(B^*) = R(A^*) \dotplus R(B^* - A^*)$. Applying (4) ⇒ (1) in Proposition 2.4, it follows that $\overline{R(A)} \dotplus \overline{R(B-A)}$ and $\overline{R(A^*)} \dotplus \overline{R(B^*-A^*)}$ are closed. Hence, by Proposition 3.2 and Proposition 2.1, $A \bar{\leq} B$. □

Let $A_i \in L(\mathscr{H})$ for $1 \leq i \leq k$. Lešnjak and Šemrl gave the following definition in [26]: the operator $A = \sum_{i=1}^{k} A_i$ is the *quasidirect sum* of the A_is if the range of A is the direct sum of the ranges of the A_is and the closure of the range of A is the direct sum of the closures of the ranges of the A_is. The next result may be restated as saying that B is the quasidirect sum of A and $B-A$ if and only if $A \bar{\leq} B$.

Corollary 3.4. *If $A, B \in L(\mathscr{H})$, the following conditions are equivalent:*

(1) $A \bar{\leq} B$;
(2) $R(B) = R(A) \dotplus R(B-A)$ *and* $\overline{R(B)} = \overline{R(A)} \dotplus \overline{R(B-A)}$.

Proof. (1) ⇒ (2) follows from Proposition 3.2 and Theorem 3.3.

For the converse, since $\overline{R(A)} \dotplus \overline{R(B-A)}$ is closed, from Proposition 2.4 we have that $R(B^*) = R(A^*) + R(B^* - A^*)$. To see that this sum is direct, applying Proposition 2.4 again and using the fact that $R(B) = R(A) \dotplus R(B-A)$ we get that $R(A^*) \cap R(B^*-A^*) = \{0\}$. Thus $R(B^*) = R(A^*) \dotplus R(B^* - A^*)$, and so by Theorem 3.3, $A \bar{\leq} B$. □

The next result shows the behavior of the minus order when the operators have closed ranges.

Corollary 3.5. *Consider $A, B \in L(\mathscr{H})$ such that $A \bar{\leq} B$. Then $R(B)$ is closed if and only if $R(A)$ and $R(B-A)$ are closed.*

Proof. If $A \bar{\leq} B$, then by Corollary 3.4, $\overline{R(B)} = \overline{R(A)} \dotplus \overline{R(B-A)}$ and $R(B) = R(A) \dotplus R(B-A)$. If $R(B)$ is closed then $\overline{R(A)} \dotplus \overline{R(B-A)} = R(A) \dotplus R(B-A)$. Hence $\overline{R(A)} = R(A)$ and $\overline{R(B-A)} = R(B-A)$. In fact, given $x \in \overline{R(A)}$, then $x \in R(A) \dotplus R(B-A)$, so that there exist $x_1 \in R(A)$ and $x_2 \in R(B-A)$ such that $x = x_1 + x_2$. But $x - x_1 = x_2 \in \overline{R(A)} \cap \overline{R(B-A)} = \{0\}$, and so $x = x_1 \in R(A)$; that is, $\overline{R(A)} = R(A)$. Similarly, $\overline{R(B-A)} = R(B-A)$. The converse follows by Corollary 3.4. □

3.1. The left and right minus orders

In this section we define the *left* and *right minus orders* and show that they are a generalization of the left and right star orders. As we will see, these orders are really only interesting on infinite dimensional spaces. For matrices, they coincide with the minus order.

We begin analyzing the properties of the left and right star orders. Originally, Drazin [18] introduced the star order on semigroups equipped with a proper involution, Baksalary and Mitra [6] defined the left and right star orders for complex matrices, and later, Antezana, Cano, Mosconi and Stojanoff [1] extended the star order to the algebra of bounded operators on a Hilbert space. See also Dolinar and Marovt [16], Deng and Wang [12] and Djikić [14].

Given $A, B \in L(\mathscr{H})$, the *star order*, *left star order* and *right star order* are respectively defined by

- $A \overset{*}{\leq} B$ if and only if $A^*A = A^*B$ and $AA^* = BA^*$,
- $A \mathrel{{*}{\leq}} B$ if and only if $A^*A = A^*B$ and $R(A) \subseteq R(B)$, and
- $A \mathrel{{\leq}{*}} B$ if and only if $AA^* = BA^*$ and $R(A^*) \subseteq R(B^*)$.

If $A, B \in L(\mathscr{H})$, then $A \overset{*}{\leq} B$ if and only if there exist $P, Q \in \mathscr{P}$ such that $A = PB$ and $A^* = QB^*$ (see [1, Proposition 2.3] or [16, Theorem 5]). We can always take $P = P_A$ and $Q = P_{A^*}$.

The next result is a straightforward consequence of [12, Theorem 2.1]. We include a simple proof. See also [17].

Proposition 3.6. *Let $A, B \in L(\mathscr{H})$. If $A \mathrel{{*}{\leq}} B$ then $A \overset{-}{\leq} B$.*

Proof. If $A \mathrel{{*}{\leq}} B$, then $A^*A = A^*B$, or equivalently $A^*(A - B) = 0$. Hence $P_A(A - B) = 0$, or $A = P_AB$. Conversely, if $A = P_AB$, then $A^*A = A^*B$. So $A \mathrel{{*}{\leq}} B$ is equivalent to $A = P_AB$ and $R(A) \subseteq R(B)$. By Proposition 3.2(5), this gives $A \overset{-}{\leq} B$. □

The following results characterize the left and right star orders in terms of an orthogonal range additivity property.

Proposition 3.7. *For $A, B \in L(\mathscr{H})$, $A \mathrel{{*}{\leq}} B$ if and only if $R(B) = R(A) \oplus R(B - A)$.*

Proof. From the proof of Proposition 3.6, $A \mathrel{{*}{\leq}} B$ if and only if $A = P_AB$ and $R(A) \subseteq R(B)$. Thus $R(B) = R(A) \oplus R(B - A)$ since $R(B - A) \subseteq N(P_A) = R(A)^\perp$.

Conversely, if $R(B) = R(A) \oplus R(B - A)$, then $R(A) \subseteq R(B)$ and $R(B - A) \subseteq R(A)^\perp$, so that $A = P_AB$. Hence $A \mathrel{{*}{\leq}} B$. □

Corollary 3.8. *For $A, B \in L(\mathscr{H})$, $A \mathrel{{\leq}{*}} B$ if and only if $R(B^*) = R(A^*) \oplus R(B^* - A^*)$.*

The next characterization of the star order follows from the previous results (or alternatively, from Theorem 3.3).

Corollary 3.9. *Given $A, B \in L(\mathscr{H})$, the following statements are equivalent:*

(1) $A \overset{*}{\leq} B$;
(2) $A \;{}^{*}\!\!\leq B$ *and* $A \leq^{*} B$;
(3) $R(B) = R(A) \oplus R(B - A)$ *and* $R(B^*) = R(A^*) \oplus R(B^* - A^*)$.

Proof. Obviously, $A \;{}^{*}\!\!\leq B$ and $A \leq^{*} B$ implies that $A \overset{*}{\leq} B$. On the other hand, if $A \overset{*}{\leq} B$, then by the proof of Proposition 3.6, $A = P_A B$ and $A^* = P_{A^*} B^*$. Hence $R(A^*) \subseteq R(B^*)$ and $R(A) \subseteq R(B)$. Thus (1) $\Leftrightarrow$ (2). The equivalence of these to (3) follows from Proposition 3.7 and Corollary 3.8. $\square$

As a generalization of the left and right star orders, we now define the *left* and *right minus orders.*

Definition 3.10. For $A, B \in L(\mathscr{H})$,

- $A \;{}^{-}\!\!\leq B$ if and only if $R(B) = R(A) \dotplus R(B - A)$, and
- $A \leq^{-} B$ if and only if $R(B^*) = R(A^*) \dotplus R(B^* - A^*)$.

Proposition 3.11. *The relations* ${}^{-}\!\!\leq$ *and* $\leq^{-}$ *define partial orders.*

Proof. We only give the proof for ${}^{-}\!\!\leq$, since the proof for $\leq^{-}$ is identical.

First of all, ${}^{-}\!\!\leq$ is clearly reflexive. So consider $A, B \in L(\mathscr{H})$ such that $A \;{}^{-}\!\!\leq B$ and $B \;{}^{-}\!\!\leq A$. Then $R(B) = R(A) \dotplus R(B - A)$ and $R(A) = R(B) \dotplus R(B - A)$. From the last equality $R(B - A) \subseteq R(A)$. But $R(B - A) \cap R(A) = \{0\}$, so that $R(B - A) = \{0\}$. Therefore $A = B$ thus ${}^{-}\!\!\leq$ is antisymmetric.

To prove ${}^{-}\!\!\leq$ is transitive, consider $A, B, C \in L(\mathscr{H})$ such that $A \;{}^{-}\!\!\leq B$ and $B \;{}^{-}\!\!\leq C$. Then $R(B) = R(A) \dotplus R(B - A)$ and $R(C) = R(B) \dotplus R(C - B)$. Since $R(A) \subseteq R(B) \subseteq R(C)$, by Lemma 2.2, $R(C) = R(A) + R(C - A)$. It remains to show that $R(A) \cap R(C - A) = \{0\}$. Since $R(A) \cap R(C - A) \subseteq R(A) \cap (R(C - B) + R(B - A))$ we can write $x \in R(A) \cap R(C - A)$ as $x = x_1 + x_2$, $x_1 \in R(C - B)$ and $x_2 \in R(B - A)$. Then $x - x_2 = x_1 \in R(B) \cap R(C - B) = \{0\}$, and so $x = x_2$. Hence $x \in R(A) \cap R(B - A) = \{0\}$; that is $x = 0$ and $R(A) \cap R(C - A) = \{0\}$. This implies that $R(C) = R(A) \dotplus R(C - A)$, or equivalently, $A \;{}^{-}\!\!\leq C$, and so ${}^{-}\!\!\leq$ is transitive. $\square$

The next corollary is a consequence of Theorem 3.3.

Corollary 3.12. *For $A, B \in L(\mathscr{H})$, $A \;{}^{-}\!\!\leq B$ and $A \leq^{-} B$ if and only if $A \overset{-}{\leq} B$.*

It follows from Proposition 2.4 that $A \mathrel{-}\!\leq B$ if and only if $A^* = QB^*$ for $Q \in \mathcal{Q}$ and $R(A) \cap R(B-A) = \{0\}$. There is also a characterization of the left minus order similar to that of the left star order as found in the proof of Proposition 3.6. We leave the obvious version for the right minus order unstated.

Proposition 3.13. *For $A, B \in L(\mathcal{H})$, $A \mathrel{-}\!\leq B$ if and only if there exists a densely defined projection P such that $A = PB$ and $R(A) \subseteq R(B)$.*

Proof. If $A \mathrel{-}\!\leq B$ then $R(B) = R(A) \dotplus R(B-A)$ so that $R(A) \subseteq R(B)$. Define $P = P_{R(A)//R(B-A)\oplus N(B^*)}$. Then P is a densely defined projection and it is easy to check that $A = PB$. In fact, $\mathcal{D}(P) = R(A) \dotplus R(B-A) \dotplus N(B^*) = R(B) + R(B)^\perp$. Hence, $\overline{\mathcal{D}(P)} = \overline{R(B) \oplus R(B)^\perp} = \overline{R(B)} \oplus R(B)^\perp = \mathcal{H}$. To see that $A = PB$, first observe that $R(B) \subseteq \mathcal{D}(P)$ so that we can compute PB: $PBx = P(Ax + (B-A)x) = PAx$ for any $x \in \mathcal{H}$, because $(B-A)x \in N(P)$. Conversely, if $A = PB$ for a densely defined projection and $R(A) \subseteq R(B)$ then $R(B) = R(A) + R(B-A)$ by Lemma 2.2, and the sum is direct since $R(A) \subseteq R(P)$ and $R(B-A) \subseteq N(P)$. □

Remark 3.14. The minus order can be seen as a star order after applying suitable weights to the Hilbert spaces involved. Recall that, if $A, B \in L(\mathcal{H}, \mathcal{K})$ are such that $A \overset{-}{\leq} B$ then there exist projections $P \in L(\mathcal{K})$ and $Q \in L(\mathcal{H})$ such that $A = PB = BQ$. The operators $W_1 = Q^*Q + (I-Q^*)(I-Q) \in L(\mathcal{H})$ and $W_2 = P^*P + (I-P^*)(I-P) \in L(\mathcal{K})$ are positive and invertible. Hence the inner products in $\mathcal{H}$ and $\mathcal{K}$ respectively,

$$\langle x, y\rangle_{W_1} = \langle W_1 x, y\rangle, \text{ for } x, y \in \mathcal{H} \quad \text{and} \quad \langle z, w\rangle_{W_2} = \langle W_2 z, w\rangle, \text{ for } z, w \in \mathcal{K}$$

give rise to equivalent norms, because the weights W_1 and W_2 are positive and invertible operators in $L(\mathcal{H})$ and any inner product $\langle\cdot,\cdot\rangle_W$ with $W \in L(\mathcal{H})$ positive and invertible is equivalent to the original inner product $\langle\cdot,\cdot\rangle$ of $\mathcal{H}$. With these new inner products, the projections P and Q are orthogonal in $\mathcal{K}_{W_2} = (\mathcal{K}, \langle\cdot,\cdot\rangle_{W_2})$ and $\mathcal{H}_{W_1} = (\mathcal{H}, \langle\cdot,\cdot\rangle_{W_1})$, respectively, and so $A \overset{*}{\leq} B$ as operators in $L(\mathcal{H}_{W_1}, \mathcal{K}_{W_2})$.

On the other hand, $A \mathrel{-}\!\leq B$ if and only if there exists a densely defined projection P such that $A = PB$ and $R(A) \subseteq R(B)$. In this case, it is possible to find a positive and invertible weight W_2 on $\mathcal{K}$ such that P is symmetric with respect to $\langle\cdot,\cdot\rangle_{W_2}$ (or equivalently $A \mathrel{*}\!\leq B$ in $L(\mathcal{H}, \mathcal{K}_{W_2})$) if and only if P admits a bounded extension $\tilde{P} \in \mathcal{Q}$ (or equivalently $A \overset{-}{\leq} B$).

Here is a proof of the last statement: suppose that there exists a weight W_2 on $\mathcal{K}$ positive and invertible such that P is symmetric with respect to $\langle\cdot,\cdot\rangle_{W_2}$. Since P is a (densely defined) idempotent then $\mathcal{D}(P) = R(P) \dotplus N(P)$, where $\mathcal{D}(P)$ is the domain of P. Moreover, given $x \in R(P)$ and $y \in N(P)$ we have $\langle x, y\rangle_{W_2} = \langle Px, y\rangle_{W_2} = \langle x, Py\rangle_{W_2} = 0$ because P is symmetric with respect to $\langle\cdot,\cdot\rangle_{W_2}$ and $y \in N(P)$. Hence $\mathcal{D}(P) = R(P) \oplus_{W_2} N(P)$, and consequently $\mathcal{H} = \overline{R(P)} \oplus_{W_2} \overline{N(P)}$, where the closures are taken with respect to $\langle\cdot,\cdot\rangle_{W_2}$. Then $\overline{P} = P_{\overline{R(P)}//\overline{N(P)}}$ is a bounded extension of P.

Conversely, suppose that there exists $\tilde{P} \in \mathscr{Q}$ such that $\tilde{P}$ is a bounded extension of P, or, following the standard notation for unbounded operators, $P \subseteq \tilde{P}$, see [35]. Let $W_2 = \tilde{P}^*\tilde{P} + (I-\tilde{P})^*(I-\tilde{P})$, which is positive and invertible and satisfies $W_2\tilde{P} = \tilde{P}^*W_2$. Finally, P is symmetric with respect to $\langle \cdot, \cdot \rangle_{W_2}$. In fact, if $x, y \in \mathscr{D}(P)$ then $\langle Px, y \rangle_{W_2} = \langle \tilde{P}x, y \rangle_{W_2} = \langle W_2\tilde{P}x, y \rangle = \langle x, W_2\tilde{P}y \rangle = \langle W_2 x, \tilde{P}y \rangle = \langle x, \tilde{P}y \rangle_{W_2} = \langle x, Py \rangle_{W_2}$.

Corollary 3.15. *Let $A, B \in L(\mathscr{H})$ be such that $A \mathrel{-\!\!\leq} B$. Then $\overline{R(B^*)} = \overline{R(A^*)} \dotplus \overline{R(B^* - A^*)}$.*

Proof. From Proposition 3.13(3), if $A \mathrel{-\!\!\leq} B$, then $A = PB$ and $N(B) \subseteq N(A)$ or $\overline{R(A^*)} \subseteq \overline{R(B^*)}$ and in the same way, $\overline{R(B^* - A^*)} \subseteq \overline{R(B^*)}$. Then by Proposition 2.4(1), $\overline{R(A^*)} \dotplus \overline{R(B^* - A^*)} \subseteq \overline{R(B^*)}$. On the other hand, $R(B^*) \subseteq R(A^*) + R(B^* - A^*) \subseteq \overline{R(A^*)} \dotplus \overline{R(B^* - A^*)}$. Hence

$$R(B^*) \subseteq \overline{R(A^*)} \dotplus \overline{R(B^* - A^*)} \subseteq \overline{R(B^*)}.$$

But $\overline{R(A^*)} \dotplus \overline{R(B^* - A^*)}$ is closed by Proposition 2.4. Therefore, $\overline{R(B^*)} = \overline{R(A^*)} \dotplus \overline{R(B^* - A^*)}$. □

Corollary 3.16. *Let $A, B \in L(\mathscr{H})$ such that $A \mathrel{-\!\!\leq} B$. If $R(B)$ is closed then $R(A)$ and $R(B - A)$ are closed and $A \overset{-}{\leq} B$.*

Proof. Since $A \mathrel{-\!\!\leq} B$, by Corollary 3.15, $\overline{R(B^*)} = \overline{R(A^*)} \dotplus \overline{R(B^* - A^*)}$. If $R(B)$ is closed, then $R(B^*)$ is closed and

$$\overline{R(A^*)} \dotplus \overline{R(B^* - A^*)} = \overline{R(B^*)} = R(B^*) \subseteq R(A^*) \dotplus R(B^* - A^*).$$

Therefore $\overline{R(A^*)} \dotplus \overline{R(B^* - A^*)} = R(A^*) \dotplus R(B^* - A^*)$. This implies that $\overline{R(A^*)} = R(A^*)$ and $\overline{R(B^* - A^*)} = R(B^* - A^*)$ and $A \overset{-}{\leq} B$. □

The above corollary shows that, unlike the left (right) star order, the left (right) minus order coincides with the minus order when applied to matrices. However for operators these orders are not the same.

Example 3.17 *(See also [7]).* Let $A \in L(\mathscr{H})$ be an operator such that $R(A) \neq \overline{R(A)}$ and that there exists $x \in \overline{R(A)} \setminus R(A)$ which is not orthogonal to $N(A)$. For example, consider $\mathscr{H} = l^2(\mathbb{N})$ the space of all square-summable sequences, operator A defined as $A : (x_n)_{n\in\mathbb{N}} \mapsto ((1/n)x_{n+1})_{n\in\mathbb{N}}$, and take x to be $x = (1/n)_{n\in\mathbb{N}}$. Define operator B as $B = A + P_x$, where P_x is the orthogonal projection onto the one-dimensional subspace spanned by $\{x\}$. Since $N(A) \nsubseteq N(P_x)$, and $N(P_x)$ is of co-dimension one, we have $\mathscr{H} = N(A) + N(P_x)$, which according to Proposition 2.4 shows that A and P_x are range-additive; that is, $R(A) + R(B - A) = R(B)$. We also have $R(A) \cap R(B - A) = \{0\}$

showing that $A \mathrel{-\!\!\leq} B$. On the other hand, $\overline{R(A)} \cap \overline{R(B-A)} \neq \{0\}$ so $A \mathrel{\bar{\leq}} B$ does not hold.

Applying Theorem 3.3 it is possible to define the minus order in terms of the inner generalized inverses of the operators involved. By an inner inverse of an operator $A \in L(\mathscr{H}, \mathscr{K})$ we mean a densely defined operator $A^- : \mathscr{D}(A^-) \subseteq \mathscr{K} \to \mathscr{H}$ satisfying $R(A) \subseteq \mathscr{D}(A^-)$ and $AA^-A = A$.

Proposition 3.18. *For $A, B \in L(\mathscr{H})$, the following conditions are equivalent:*

(1) $A \mathrel{-\!\!\leq} B$;
(2) there exists an inner inverse A^- of A such that $A^-A = A^-B$ and $AA^-x = BA^-x$ for every $x \in \mathscr{D}(A^-)$.

Proof. Suppose that $A \mathrel{-\!\!\leq} B$. If $\mathscr{N}$ is a complement of $\overline{R(B)}$, then $\mathscr{H} = \overline{R(A) \dotplus R(B-A)} \dotplus \mathscr{N}$. From Proposition 2.4 we know that $N(A) + N(B-A) = \mathscr{H}$; so if $\mathscr{M} = N(B-A) \ominus N(A)$, then $\mathscr{H} = N(A) \dotplus \mathscr{M}$. Let A_1 be the restriction of A to $\mathscr{M}$, and define A^- as A_1^{-1} on $R(A)$, and as the null operator on $R(B-A) \dotplus \mathscr{N}$. Then A^- is densely defined and the domain of A^- is $\mathscr{D}(A^-) = R(B) \dotplus \mathscr{N}$. In this case, $(A-B)A^-x = 0$ for every $x \in \mathscr{D}(A^-)$, because $R(A^-) \subseteq N(A-B)$. On the other hand, since $R(A-B) \subseteq \mathscr{D}(A^-)$, we find that $A^-(A-B) = 0$ since $R(A-B) \subseteq N(A^-)$.

For the converse, suppose that there exists an inner inverse A^- of A such that $A^-A = A^-B$ and $AA^-x = BA^-x$ for every $x \in \mathscr{D}(A^-)$. In particular, if $z \in \mathscr{H}$ then $Az \in R(A) \subseteq D(A^-)$, so that $Az = AA^-Az = BA^-Az$. Hence $R(A) \subseteq R(B)$, showing that $R(B) = R(A) + R(B-A)$. From $A^-A = A^-B$ we have $R(A-B) \subseteq N(A^-)$, while $N(A^-) \cap R(A) = \{0\}$, and so $R(B) = R(A) \dotplus R(B-A)$. Therefore, $A \mathrel{-\!\!\leq} B$. $\square$

Corollary 3.19. *For $A, B \in L(\mathscr{H})$, the following conditions are equivalent:*

(1) $A \mathrel{\bar{\leq}} B$;
(2) there exist inner inverses A^- of A and $(A^*)^-$ of A^* such that
  (i) $A^-A = A^-B$ *and* $AA^-x = BA^-x$ *for every* $x \in \mathscr{D}(A^-)$,
  (ii) $(A^*)^-A^* = (A^*)^-B^*$ *and* $A^*(A^*)^-x = B^*(A^*)^-x$ *for every* $x \in \mathscr{D}((A^*)^-)$.

4. Applications

4.1. Generalized inverses of $A + B$

In this section we state the formulas for arbitrary reflexive inverses of $A+B$ in terms of the inverses of A and B, when $A \mathrel{-\!\!\leq} A + B$. For the sake of simplicity, we begin by giving the formula for the Moore–Penrose inverse. Theorem 4.7 states the result in the most general form, and from this theorem many existing results in the subject can be recovered.

If $A \overset{-}{\leq} A+B$ then $A = P(A+B)$ for some $P \in \mathscr{Q}$. Using the projection P we can construct a projection $E \in \mathscr{Q}$ onto $\overline{R(A+B)}$ that will be useful in stating the formula for the Moore–Penrose inverse of $A+B$.

Lemma 4.1. *Let $A, B \in L(\mathscr{H})$ be such that $A \overset{-}{\leq} A+B$, and $P \in \mathscr{Q}$ be such that $A = P(A+B)$. Set*

$$E = P_A P + P_B(I-P).$$

Then $E \in \mathscr{Q}$ and $R(E) = \overline{R(A+B)}$. Moreover, E is selfadjoint if and only if $P = P_{\mathscr{M}//\mathscr{N}}$ where $\mathscr{M} = \overline{R(A)} \oplus \mathscr{M}_1$, $\mathscr{N} = \overline{R(B)} \oplus \mathscr{N}_1$ with $\mathscr{M}_1$ and $\mathscr{N}_1$ closed subspaces such that $\mathscr{M}_1, \mathscr{N}_1 \subseteq N(A^*) \cap N(B^*)$.

Proof. If $A = P(A+B)$ then $\overline{R(A)} \subseteq R(P)$ and $\overline{R(B)} \subseteq N(P)$. Therefore $P_A P$ and $P_B(I-P)$ are projections, with $R(P_A P) = \overline{R(A)}$ and $R(P_B(I-P)) = \overline{R(B)}$. Moreover,

$$P_A P P_B(I-P) = P_B(I-P)P_A P = 0.$$

Therefore $E = P_A P + P_B(I-P)$ is a projection. Also, $\overline{R(A)} = R(P_A P) = R(EP) \subseteq R(E)$. Applying Lemma 2.2, $R(E) = \overline{R(A)} + \overline{R(B)} = \overline{R(A+B)}$ because $A \overset{-}{\leq} A+B$.

Finally, if $P = P_{\mathscr{M}//\mathscr{N}}$ then there exist closed subspaces $\mathscr{M}_1, \mathscr{N}_1$ such that $\mathscr{M} = \overline{R(A)} \oplus \mathscr{M}_1$ and $\mathscr{N} = \overline{R(B)} \oplus \mathscr{N}_1$. Hence $P_A P = P_{\overline{R(A)}//\overline{R(B)} \dotplus \mathscr{N}_1 \dotplus \mathscr{M}_1}$, $P_B(I-P) = P_{\overline{R(B)}//\overline{R(A)} \dotplus \mathscr{M}_1 \dotplus \mathscr{N}_1}$ and $E = P_{\overline{R(A+B)}//\mathscr{M}_1 \dotplus \mathscr{N}_1}$. Since $A \overset{-}{\leq} A+B$, it follows that $E^* = E$ if and only if $\mathscr{M}_1 \dotplus \mathscr{N}_1 = (\overline{R(A+B)})^\perp = (\overline{R(A)} \dotplus \overline{R(B)})^\perp = N(A^*) \cap N(B^*)$, or equivalently, $\mathscr{M}_1$ and $\mathscr{N}_1$ are included in $N(A^*) \cap N(B^*)$. □

Definition 4.2. Let $A, B \in L(\mathscr{H})$ such that $A \overset{-}{\leq} A+B$. Consider $P, Q \in \mathscr{Q}$ such that $A = P(A+B) = (A+B)Q$. Then P will be called *optimal* for A and B if $E = P_A P + P_B(I-P)$ is selfadjoint. In a symmetric way, since $A^* = Q^*(A^* + B^*)$, Q will be called *optimal* for A and B if Q^* is optimal for A^* and B^*, i.e., $F = P_{A^*}Q^* + P_{B^*}(I - Q^*)$ is selfadjoint.

Define $\mathscr{O}_L = \{P \in \mathscr{Q} : A = P(A+B) \text{ and } P_A P + P_B(I-P) \in \mathscr{P}\}$. It follows from the above lemma that if $A \overset{-}{\leq} A+B$, then the set $\mathscr{O}_L$ is not empty; moreover

$$\mathscr{O}_L = \left\{ P \in \mathscr{Q} : P = P_{\overline{R(A)} \oplus \mathscr{M}_1 // \overline{R(B)} \oplus \mathscr{N}_1} \text{ with } \mathscr{M}_1 \dotplus \mathscr{N}_1 = N(A^*) \cap N(B^*) \right\}. \quad (4.1)$$

Applying Lemma 4.1 we derive the Fill and Fishkind [19] formula for the Moore–Penrose inverse of the sum of two operators in an easy way. This formula first appeared in their work for square matrices, while Groß [23] extended it to arbitrary rectangular matrices and Arias et al. [5] proved it for operators on a Hilbert space. The version we give here requires simpler hypotheses, and the formula is given in a more general form. We include a short proof.

Corollary 4.3. *Let $A, B \in L(\mathscr{H})$ be such that $R(A+B)$ is closed and $A \mathrel{\bar{\leq}} A+B$. Then*

$$(A+B)^\dagger = QA^\dagger P + (I-Q)B^\dagger(I-P), \tag{4.2}$$

where $A = P(A+B) = (A+B)Q$, with P, Q optimal projections for A and B.

Proof. From Corollary 3.16 we see that A and B are operators with closed range and $A \mathrel{\bar{\leq}} A+B$. Then the operators $A^\dagger$ and $B^\dagger$ are bounded and $T = QA^\dagger P + (I-Q)B^\dagger(I-P)$ is well defined.

Using that $(A+B)Q = A$ and $(A+B)(I-Q) = B$ we have that

$$(A+B)T = AA^\dagger P + BB^\dagger(I-P) = P_A P + P_B(I-P) = P_{R(A+B)},$$

by Lemma 4.1, because P is optimal. Using that $P(A+B) = A$, $(I-P)(A+B) = B$ we see that

$$T(A+B) = QA^\dagger A + (I-Q)B^\dagger B = QP_{A^*} + (I-Q)P_{B^*} = P_{R(A^*+B^*)} = P_{N(A+B)^\perp}.$$

Therefore $T = (A+B)^\dagger$. $\square$

Remark 4.4. In [5, Theorem 5.2] the Fill–Fishkind formula is stated as follows: let $A, B \in L(\mathscr{H})$ be such that $R(A)$, $R(B)$ are closed, $R(A+B) = R(A) \dotplus R(B)$ and $R(A^*+B^*) = R(A^*) \dotplus R(B^*)$, then

$$(A+B)^\dagger = (I-S)A^\dagger(I-T) + SB^\dagger T,$$

where $S = (P_{N(B)^\perp} P_{N(A)})^\dagger$ and $T = (P_{N(A^*)} P_{N(B^*)^\perp})^\dagger$. It holds that $S, T \in \mathscr{Q}$ (see [33, Lemma 2.3] and [21, Theorem 1] for matrices and [11, Theorem 4.1] for operators in Hilbert spaces). If we denote $Q = I - S$ and $P = I - T$, we in fact have (see [5, Theorem 5.1]) $Q = P^*_{R(A^*)\oplus(N(A)\cap N(B))//R(B^*)}$ and $P = P_{R(A)\oplus(N(A^*)\cap N(B^*))//R(B)}$, which are optimal with respect to $A \mathrel{\bar{\leq}} A+B$.

Recall that if $A \in \mathscr{L}(\mathscr{H}, \mathscr{K})$ is a closed range operator, then any operator $X \in \mathscr{L}(\mathscr{K}, \mathscr{H})$ satisfying $AXA = A$ and $XAX = X$ is called a *reflexive inverse* of A. A reflexive inverse of A is also called $(1,2)$-*inverse* of A or *algebraic generalized inverse* of A. The operator X has closed range, $AX = P_{R(A)//N(X)}$ and $XA = P_{R(X)//N(A)}$. If $\mathscr{M}$ and $\mathscr{N}$ are arbitrary closed subspaces of $\mathscr{H}$ and $\mathscr{K}$ satisfying $R(A) \dotplus \mathscr{M} = \mathscr{K}$ and $\mathscr{N} \dotplus N(A) = \mathscr{H}$, then there is only one reflexive inverse of A with the range $\mathscr{N}$ and the null-space $\mathscr{M}$, see [31]. This reflexive inverse is denoted by $A^{(1,2)}_{\mathscr{N},\mathscr{M}}$ and verifies that

$$AA^{(1,2)}_{\mathscr{N},\mathscr{M}} = P_{R(A)//\mathscr{M}}, \quad A^{(1,2)}_{\mathscr{N},\mathscr{M}}A = P_{\mathscr{N}//N(A)}. \tag{4.3}$$

In what follows we generalize Lemma 4.1 in order to prove a formula similar to (4.2) in Corollary 4.3 for an arbitrary reflexive inner inverse of the sum of two operators.

Lemma 4.5. *Let $A, B \in L(\mathscr{H})$ be such that $A \bar{\leq} A+B$. Let $P \in \mathcal{Q}$ such that $A = P(A+B)$ and consider*

$$E = P_{\overline{R(A)}//\mathcal{N}_1} P + P_{\overline{R(B)}//\mathcal{N}_2}(I-P),$$

where $\mathcal{N}_1$ and $\mathcal{N}_2$ are arbitrary. Then $E \in \mathcal{Q}$ and $R(E) = \overline{R(A+B)}$. Moreover, for every closed subspace $\mathcal{M}$ such that $\overline{R(A+B)} \dotplus \mathcal{M} = \mathscr{H}$ there exist $P \in \mathcal{Q}$ and subspaces $\mathcal{N}_1$ and $\mathcal{N}_2$, such that $A = P(A+B)$ and $E = P_{\overline{R(A)}//\mathcal{N}_1} P + P_{\overline{R(B)}//\mathcal{N}_2}(I-P) = P_{\overline{R(A+B)}//\mathcal{M}}$.

Proof. In the same way as in the proof of Lemma 4.1, it can be proved that $E \in \mathcal{Q}$ and $R(E) = \overline{R(A+B)}$.

To prove the last assertion, take

$$\mathcal{N}_1 = \overline{R(B)} \dotplus \mathcal{M}, \quad \mathcal{N}_2 = \overline{R(A)} \dotplus \mathcal{M}, \tag{4.4}$$

and $P = P_{\overline{R(A)}//\overline{R(B)}\dotplus\mathcal{M}}$. Then $\mathcal{N}_1$ and $\mathcal{N}_2$ are closed. To see that $\mathcal{N}_1$ is closed, observe that $\mathcal{N}_1 = \overline{R(B)}\dotplus\mathcal{M} \subseteq \overline{R(A+B)}\dotplus\mathcal{M}$ because $A \bar{\leq} A+B$ implies that $\overline{R(B)} \subseteq \overline{R(A+B)}$ (see Proposition 3.2). Then $c_0(\overline{R(B)}, \mathcal{M}) \leq c_0(\overline{R(A+B)}, \mathcal{M}) < 1$ because by hypothesis $\overline{R(A+B)}\dotplus\mathcal{M}$ is closed. Applying Proposition 2.1, it follows that $\overline{R(B)}\dotplus\mathcal{M}$ is closed, or equivalently, $\mathcal{N}_1$ is closed. In a similar way, it can be proven that $\mathcal{N}_2$ is closed. Moreover, the projections $P_{\overline{R(A)}//\mathcal{N}_1} = P$ and $P_{\overline{R(B)}//\mathcal{N}_2}$ are well defined and $E = P_{\overline{R(A)}//\mathcal{N}_1} P + P_{\overline{R(B)}//\mathcal{N}_2}(I-P) = P + P_{\overline{R(B)}//\mathcal{N}_2}(I-P) = P_{\overline{R(A+B)}//\mathcal{M}}$. □

Definition 4.6. Let $A, B \in L(\mathscr{H})$ be such that $A \bar{\leq} A+B$ and $P \in \mathcal{Q}$ such that $A = P(A+B)$. Given $\mathcal{M}$ an arbitrary closed subspace such that $\overline{R(A+B)} \dotplus \mathcal{M} = \mathscr{H}$, we say that P *agrees* with $\mathcal{M}$ if there exist subspaces $\mathcal{N}_1, \mathcal{N}_2$ so that

$$P_{\overline{R(A)}//\mathcal{N}_1} P + P_{\overline{R(B)}//\mathcal{N}_2}(I-P) = P_{\overline{R(A+B)}//\mathcal{M}}. \tag{4.5}$$

In a symmetric way, if $A = (A+B)Q$, for $Q \in \mathcal{Q}$ and $\mathcal{N} \dotplus N(A+B) = \mathscr{H}$, we say that Q *agrees* with $\mathcal{N}$ if Q^* agrees with $\mathcal{N}^\perp$. In this case $A^* = Q^*(A^*+B^*)$ and $\overline{R(A^*+B^*)} \dotplus \mathcal{N}^\perp = \mathscr{H}$, and there exist closed subspaces $\mathcal{N}_1{}^*, \mathcal{N}_2{}^*$ such that

$$P_{\overline{R(A^*)}//(\mathcal{N}_1{}^*)^\perp} Q^* + P_{\overline{R(B^*)}//(\mathcal{N}_2{}^*)^\perp}(I-Q^*) = P_{\overline{R(A^*+B^*)}//\mathcal{N}^\perp}, \tag{4.6}$$

or

$$Q P_{\mathcal{N}_1{}^*//N(A)} + (I-Q) P_{\mathcal{N}_2{}^*//N(B)} = P_{\mathcal{N}//N(A+B)}. \tag{4.7}$$

For example, $P = P_{\overline{R(A)}//\overline{R(B)}\dot{+}\mathscr{M}}$ agrees with $\mathscr{M}$, as we saw in the proof of Lemma 4.5, and $Q = (P_{\overline{R(A^*)}//\overline{R(B^*)}\dot{+}\mathscr{N}^\perp})^* = P_{N(B)\cap\mathscr{N}//N(A)}$ agrees with $\mathscr{N}$. The projection P is optimal if P agrees with $R(A+B)^\perp$.

Theorem 4.7. *Let $A, B \in L(\mathscr{H})$ be such that $R(A+B)$ is closed and $A \stackrel{-}{\leq} A+B$. Let $\mathscr{M}$ and $\mathscr{N}$ be two closed subspaces such that $R(A+B) \dot{+} \mathscr{M} = \mathscr{H}$ and $\mathscr{N} \dot{+} N(A+B) = \mathscr{H}$. If $P, Q \in \mathscr{Q}$ satisfy $A = P(A+B) = (A+B)Q$ and agree with $\mathscr{M}$ and $\mathscr{N}$ respectively, then*

$$(A+B)^{(1,2)}_{\mathscr{N},\mathscr{M}} = QA^{(1,2)}_{\mathscr{N}_1^*,\mathscr{N}_1}P + (I-Q)B^{(1,2)}_{\mathscr{N}_2^*,\mathscr{N}_2}(I-P), \tag{4.8}$$

where the subspaces $\mathscr{N}_1$, $\mathscr{N}_2$, $\mathscr{N}_1^*$ and $\mathscr{N}_2^*$ are arbitrary closed subspaces satisfying (4.5) and (4.7).

Proof. From Corollary 3.16 we see that A and B are operators with closed range and $A \stackrel{-}{\leq} A+B$. Let $T = QA^{(1,2)}_{\mathscr{N}_1^*,\mathscr{N}_1}P + (I-Q)B^{(1,2)}_{\mathscr{N}_2^*,\mathscr{N}_2}(I-P)$ and choose $E = P_{R(A)//\mathscr{N}_1}P + P_{R(B)//\mathscr{N}_2}(I-P)$ as in Lemma 4.5. Using that $(A+B)Q = A$ and $(A+B)(I-Q) = B$, the definition of reflexive inverse, equation (4.3) and Lemma 4.5,

$$\begin{aligned}(A+B)T &= AA^{(1,2)}_{\mathscr{N}_1^*,\mathscr{N}_1}P + BB^{(1,2)}_{\mathscr{N}_2^*,\mathscr{N}_2}(I-P)\\ &= P_{R(A)//\mathscr{N}_1}P + P_{R(B)//\mathscr{N}_2}(I-P) = E = P_{R(A+B)//\mathscr{M}},\end{aligned}$$

because P agrees with $\mathscr{M}$ (see Definition 4.6 and (4.5)). In a similar way, since $P(A+B) = A$ and $(I-P)(A+B) = B$,

$$T(A+B) = QP_{\mathscr{N}_1^*//N(A)} + (I-Q)P_{\mathscr{N}_2^*//N(B)} = P_{\mathscr{N}//N(A+B)},$$

because Q agrees with $\mathscr{N}$. Moreover, a direct computation shows that $(A+B)T(A+B) = A+B$, and having in mind that $AQ = A, PA = A, QB = 0$ and $PB = 0$, we also directly obtain $T(A+B)T = T$. Hence, $R(T) = \mathscr{N}$ and $N(T) = \mathscr{M}$ (see (4.3)). $\square$

In fact, the terms on the right hand side of (4.8) do not depend on the choices of the subspaces $\mathscr{N}_1, \mathscr{N}_1^*, \mathscr{N}_2$ and $\mathscr{N}_2^*$.

Proposition 4.8. *Under the hypotheses of Theorem 4.7 it holds*

$$\begin{aligned}QA^{(1,2)}_{\mathscr{N}_1^*,\mathscr{N}_1}P &= A^{(1,2)}_{N(B)\cap\mathscr{N},R(B)\dot{+}\mathscr{M}} \quad \textit{and}\\ (I-Q)B^{(1,2)}_{\mathscr{N}_2^*,\mathscr{N}_2}(I-P) &= B^{(1,2)}_{N(A)\cap\mathscr{N},R(A)\dot{+}\mathscr{M}}\end{aligned} \tag{4.9}$$

regardless of the choice of P, Q, $\mathscr{N}_1$, $\mathscr{N}_2$, $\mathscr{N}_1^*$ and $\mathscr{N}_2^*$. Consequently

$$(A+B)^{(1,2)}_{\mathscr{N},\mathscr{M}} = A^{(1,2)}_{N(B)\cap\mathscr{N},R(B)\dot{+}\mathscr{M}} + B^{(1,2)}_{N(A)\cap\mathscr{N},R(A)\dot{+}\mathscr{M}}. \tag{4.10}$$

Proof. Let $T = QA^{(1,2)}_{\mathscr{N}_1^*,\mathscr{N}_1}P$. Since $A = P(A+B) = (A+B)Q$, we get $A = PA = AQ$. Thus $ATA = A$ and $TAT = T$, so T is a reflexive inverse of A, and moreover $N(T) = N(AT)$ and $R(T) = R(TA)$. Let us prove that $N(AT) = R(B)\dotplus\mathscr{M}$ and $R(TA) = N(B)\cap\mathscr{N}$. We have $AT = P_{R(A)//\mathscr{N}_1}P$ and so $N(AT) = N(P)\dotplus(R(P)\cap\mathscr{N}_1)$. Given that $R(A)\subseteq R(P), R(B)\subseteq N(P), R(A)\dotplus\mathscr{N}_1 = \mathscr{H}$ and $R(B)\dotplus\mathscr{N}_2 = \mathscr{H}$, if we denote by $\mathscr{M}_1 = R(P)\cap\mathscr{N}_1$ and $\mathscr{M}_2 = N(P)\cap\mathscr{N}_2$ then $R(P) = R(A)\dotplus\mathscr{M}_1$ and $N(P) = R(B)\dotplus\mathscr{M}_2$. From (4.5) it follows

$$\begin{aligned} P_{R(A+B)//\mathscr{M}} &= P_{R(A)//\mathscr{N}_1}P + P_{R(B)//\mathscr{N}_2}(I-P) \\ &= P_{R(A)//R(B)\dotplus\mathscr{M}_1\dotplus\mathscr{M}_2} + P_{R(B)//R(A)\dotplus\mathscr{M}_1\dotplus\mathscr{M}_2}. \end{aligned}$$

Hence $\mathscr{M}_1\dotplus\mathscr{M}_2 = \mathscr{M}$. Thus $N(AT) = N(P)\dotplus(R(P)\cap\mathscr{N}_1) = R(B)\dotplus\mathscr{M}_2\dotplus\mathscr{M}_1 = R(B)\dotplus\mathscr{M}$. On the other hand, $R(TA)=R(QP_{\mathscr{N}_1^*//N(A)})=N(P_{R(A^*)//(\mathscr{N}_1^*)^\perp}Q^*)^\perp$. By an identical argument with A^* and B^* in place of A and B we have $N(P_{R(A^*)//(\mathscr{N}_1^*)^\perp}Q^*) = R(B^*)\dotplus\mathscr{N}^\perp$. Hence $R(TA) = N(B)\cap\mathscr{N}$. This shows that $T = A^{(1,2)}_{N(B)\cap\mathscr{N},R(B)\dotplus\mathscr{M}}$. The proof for B follows along similar lines. □

Remark 4.9. Formula (4.10) was given for matrices by Werner in [37]. There the author considers pairs of matrices A and B having the property $R(A)\cap R(B) = R(A^*)\cap R(B^*) = \{0\}$, and calls such matrices *weakly bicomplementary*. Recall that in the finite-dimensional setting, conditions $R(A)\dotplus R(B) = R(A+B)$, $R(A^*)\dotplus R(B^*) = R(A^*+B^*)$ and $R(A)\cap R(B) = \{0\} = R(A^*)\cap R(B^*)$ are all equivalent. Also matrices A and B are weakly bicomplementary if and only if $A \overset{-}{\leq} A+B$. Under this assumption, many of the results from [37] are seen to hold in arbitrary Hilbert spaces.

Recall that if for $A\in L(\mathscr{H})$ the relation $\mathscr{H} = R(A)\dotplus N(A)$ holds, then $R(A)$ is closed, see [15, Proposition 3.7] or [20, Theorem 2.3]. In this case, A is called *group-invertible* and $A^\sharp = A^{(1,2)}_{R(A),N(A)}\in L(\mathscr{H})$ is called the *group inverse* of A.

If A is group invertible, the operator $A^{\oplus} = A^{(1,2)}_{R(A),N(A^*)}\in L(\mathscr{H})$ is called the *core inverse* of A and it was introduced by Baksalary and Trenkler [8], see also [25] and [34].

If we denote by $L^1(\mathscr{H})$ the set of all group invertible operators, then the *sharp partial order* and the *core partial order* on $L^1(\mathscr{H})$ are defined as: $A \overset{\sharp}{\leq} B$ if $AA^\sharp = BA^\sharp$ and $A^\sharp A = A^\sharp B$; $A \overset{\oplus}{\leq} B$ if $AA^\oplus = BA^\oplus$ and $A^\oplus A = A^\oplus B$. It is straightforward to see that for $A,B\in L^1(\mathscr{H})$ we have

$$A \overset{\sharp}{\leq} B \quad\Leftrightarrow\quad A^2 = BA = AB, \tag{4.11}$$

and

$$A \leq^{\oplus} B \quad\Leftrightarrow\quad A^*A = A^*B \quad\text{and}\quad A^2 = BA. \tag{4.12}$$

We recover results from Jose and Sivakumar [25] in the following corollary.

Corollary 4.10. *Let $A, B \in L(\mathscr{H})$ such that $R(A+B)$ is closed.*

(1) *If $A \overset{*}{\leq} A+B$, then $(A+B)^\dagger = A^\dagger + B^\dagger$;*
(2) *If $A, A+B \in L^1(\mathscr{H})$ and $A \overset{\sharp}{\leq} A+B$, then $B \in L^1(\mathscr{H})$ and $(A+B)^\sharp = A^\sharp + B^\sharp$;*
(3) *If $A, A+B \in L^1(\mathscr{H})$ and $A \overset{\textcircled{\#}}{\leq} A+B$, then $B \in L^1(\mathscr{H})$. If moreover $A^* \overset{\textcircled{\#}}{\leq} A^* + B^*$, then $(A+B)^{\textcircled{\#}} = A^{\textcircled{\#}} + B^{\textcircled{\#}}$.*

Proof. All three partial orders stated here induce the minus partial order, so $R(A)$ and $R(B)$ are closed, according to Corollary 3.5.

Item (1) follows from Proposition 4.8. In fact, from Corollary 3.9, it holds that $A \overset{*}{\leq} A+B$ if and only if $R(A+B) = R(A) \oplus R(B)$ and $R(A^*+B^*) = R(A^*) \oplus R(B^*)$. Since $R(A+B)$ is closed it follows that $R(A), R(B), R(A^*)$ and $R(B^*)$ are closed. The Moore–Penrose inverse of $A+B$ corresponds to the choice $\mathscr{M} = R(A+B)^\perp = N(A^*) \cap N(B^*)$ and $\mathscr{N} = N(A+B)^\perp = R(A^*+B^*) = R(A^*) \oplus R(B^*)$. Then, we get that $N(B) \cap \mathscr{N} = R(A^*)$ because $R(A^*) \subseteq R(B^*)^\perp = N(B)$. Also, $R(B) \dot{+} \mathscr{M} = R(B) \dot{+} N(A^*) \cap N(B^*) = R(A)^\perp$: in fact, $R(B) \subseteq R(A)^\perp$ and $\mathscr{M} \subset R(A)^\perp$ so that $R(B) \dot{+} \mathscr{M} \subseteq R(A)^\perp$. To see the other inclusion, $R(A)^\perp = R(B) \oplus \mathscr{S}$, where $\mathscr{S} \subseteq R(A)^\perp \cap R(B)^\perp = \mathscr{M}$. Therefore, $A^{(1,2)}_{N(B)\cap \mathscr{N}, R(B)\dot{+}\mathscr{M}} = A^{(1,2)}_{R(A^*), R(A)^\perp} = A^\dagger$. In the same way, $B^{(1,2)}_{N(A)\cap \mathscr{N}, R(A)\dot{+}\mathscr{M}} = B^\dagger$ and applying (4.10) the formula follows.

(2): By [25, Corollary 3.5] we see that $B \in L^1(\mathscr{H})$ and $B \overset{\sharp}{\leq} A+B$. Recall that $N(A+B) = N(A) \cap N(B)$, because $A \overset{-}{\leq} A+B$, which together with $A+B \in L^1(\mathscr{H})$ gives: $\mathscr{H} = R(A) \dot{+} R(B) \dot{+} (N(A) \cap N(B))$. From $A \overset{\sharp}{\leq} A+B$ we obtain $A^\sharp A = A^\sharp(A+B)$, i.e. $A^\sharp B = 0$, implying $R(B) \subseteq N(A^\sharp) = N(A)$. Hence $N(A) = R(B) \dot{+} (N(A) \cap N(B))$, and similarly $N(B) = R(A) \dot{+} (N(A) \cap N(B))$. If we apply Proposition 4.8 with $\mathscr{N} = R(A+B) = R(A) \dot{+} R(B)$ and $\mathscr{M} = N(A+B) = N(A) \cap N(B)$ we obtain directly the desired result.

(3): From [25, Theorem 4.5] we have that $B \in L^1(\mathscr{H})$ and moreover, from (4.11) and (4.12) we see that $A \overset{*}{\leq} A+B$ and $A \overset{\sharp}{\leq} A+B$. If $\mathscr{M} = N(A^*+B^*) = N(A^*) \cap N(B^*)$, then as in (1) we have $R(B) \dot{+} \mathscr{M} = N(A^*)$ and $R(A) \dot{+} \mathscr{M} = N(B^*)$. If $\mathscr{N} = R(A+B) = R(A) \dot{+} R(B) = R(A) \oplus R(B)$, then as in (2) we have $N(B) \cap \mathscr{N} = R(A)$ and $N(A) \cap \mathscr{N} = R(B)$. If we invoke Proposition 4.8 with such $\mathscr{M}$ and $\mathscr{N}$, we obtain $(A+B)^{\textcircled{\#}} = A^{(1,2)}_{R(A), N(A^*)} + B^{(1,2)}_{R(B), N(B^*)} = A^{\textcircled{\#}} + B^{\textcircled{\#}}$. □

4.2. Systems of equations and least squares problems

Consider $A, B \in L(\mathscr{H})$. In what follows we characterize the left minus order in terms of the solutions of the equation $(A+B)x = c$, for $c \in R(A+B)$.

The following theorem appears in [37] in the matrix case. We give the proof for operator equations.

Proposition 4.11. *If $A, B \in L(\mathscr{H})$ the following statements are equivalent:*

(1) $A \mathrel{-\!\!\le} A+B$;
(2) *Given* $a \in R(A)$ *and* $b \in R(B)$, *the equation*

$$(A+B)x = a+b, \tag{4.13}$$

has a solution $x_0 \in \mathscr{H}$. *Moreover,* x_0 *is a solution of the system*

$$\begin{cases} Ax = a \\ Bx = b. \end{cases} \tag{4.14}$$

Proof. Suppose that $A \mathrel{-\!\!\le} A+B$ and $a \in R(A)$ and $b \in R(B)$. Since $A \mathrel{-\!\!\le} A+B$, then $R(A+B) = R(A) \dotplus R(B)$. Therefore, $a+b \in R(A+B)$, so that there exists $x_0 \in \mathscr{H}$ satisfying $(A+B)x_0 = a+b$. Moreover, $Ax_0 - a = b - Bx_0 = 0$ since $R(A) \cap R(B) = \{0\}$. Hence x_0 is a solution of the system (4.14).

For the converse, let $a \in R(A)$. By hypothesis, there exists $x_0 \in \mathscr{H}$ such that $(A+B)x_0 = a$. Hence $R(A) \subseteq R(A+B)$, so that $R(A+B) = R(A) + R(B)$. Now, let $c \in R(A) \cap R(B)$. Consider $x_0 \in \mathscr{H}$ such that $(A+B)x_0 = c$. Since $c \in R(A) \cap R(B)$ then, also by hypothesis,

$$\begin{cases} Ax_0 = c \\ Bx_0 = 0 \end{cases} \quad \text{and} \quad \begin{cases} Ax_0 = 0 \\ Bx_0 = c. \end{cases}$$

Therefore $c = 0$, and so $R(A+B) = R(A) \dotplus R(B)$, or equivalently, $A \mathrel{-\!\!\le} A+B$. □

More generally, in what follows we relate the least squares solutions of the equation $Cx = y$ to a weighted least squares solution of the system $Ax = y$ and $(C-A)x = y$ when $A \mathrel{-\!\!\le} C$. We introduce the seminorm given by a positive weight. If $W \in L(\mathscr{H})$ is a positive (semidefinite), consider the seminorm $\|.\|_W$ on $\mathscr{H}$ defined by

$$\|x\|_W = \langle Wx, x\rangle^{1/2}, \quad x \in \mathscr{H}.$$

Given $C \in L(\mathscr{H})$ and $y \in \mathscr{H}$, an element $x_0 \in \mathscr{H}$ is said a W-*least squares solution* (W-LSS) of the equation $Cx = y$ if

$$\|Cx_0 - y\|_W = \min_{x \in \mathscr{H}} \|Cx - y\|_W.$$

The vector $x_0 \in \mathscr{H}$ is a W-LSS of the equation $Cx = y$ if and only if x_0 is a solution of the associated *normal equation*,

$$C^* W(Cx - y) = 0,$$

see [22].

If $A \,\text{-}\!\leq A+B$ and $R(A+B)$ is closed then, by Corollary 3.16, it holds that $A \,\bar{\leq}\, A+B$. In this case, it follows from the proof of Lemma 4.1 that the set $\mathscr{O}_L = \{P \in \mathscr{Q} : A = P(A+B) \text{ and } P_A P + P_B(I-P) \in \mathscr{P}\}$ is not empty. Moreover,

$$\mathscr{O}_L = \left\{P \in \mathscr{Q} : P = P_{R(A)\oplus\mathscr{M}_1//R(B)\oplus\mathscr{N}_1} \text{ with } \mathscr{M}_1 \dotplus \mathscr{N}_1 = N(A^*) \cap N(B^*)\right\},$$

see (4.1). Using the (oblique) projections in the set $\mathscr{O}_L$, it is possible to "decouple" a least squares problem into a system of weighted least squares problems.

Proposition 4.12. *Let $A, B \in L(\mathscr{H})$ be such that $R(A+B)$ is closed and $A \,\text{-}\!\leq A+B$ and $c \in \mathscr{H}$. For $P \in \mathscr{O}_L$, let $W = P^*P + (I-P^*)(I-P)$. Then the following statements are equivalent:*

(1) x_0 is a solution of

$$\underset{x\in\mathscr{H}}{\operatorname{argmin}} \|(A+B)x - c\|; \tag{4.15}$$

(2) x_0 is a solution of the system of least squares problems,

$$\begin{cases} \underset{x\in\mathscr{H}}{\operatorname{argmin}} \|Ax - c\|_W \\ \underset{x\in\mathscr{H}}{\operatorname{argmin}} \|Bx - c\|_W . \end{cases} \tag{4.16}$$

Proof. Assume (1) holds. Applying Corollary 3.16, if $A \,\text{-}\!\leq A+B$ and $R(A+B)$ is closed then $R(A)$ and $R(B)$ are closed and $A \,\bar{\leq}\, A+B$. Suppose that x_0 is a solution of (4.15). Then x_0 is a solution of the associated normal equation

$$(A+B)^*((A+B)x - c) = 0.$$

Then $(A+B)x_0 - c \in N(A^* + B^*) = N(A^*) \cap N(B^*)$ so that

$$E((A+B)x_0 - c) = 0,$$

where $E = P_A P + P_B(I-P)$ is the orthogonal projection onto $R(A+B)$ because P is optimal. Therefore,

$$Ax_0 - P_A Pc = -(Bx_0 - P_B(I-P)c),$$

and because $R(A) \cap R(B) = \{0\}$, we have $Ax_0 - P_A Pc = Bx_0 - P_B(I-P)c = 0$. Hence, using that $P_A A = A$ and $P_B B = B$, $P_A(Ax_0 - Pc) = 0$ and $P_B(Bx_0 - (I-P)c) = 0$, or equivalently, multiplying on the left by A^* and B^* and using that $A^* P_A = A^*$, $B^* P_B = B^*$,

$$A^*(Ax_0 - Pc) = 0 \;\text{ and }\; B^*(Bx_0 - (I-P)c) = 0.$$

Thus $A^*P(Ax_0 - c) = 0$ and $B^*(I-P)(Bx_0 - c) = 0$.

Finally, observe that $A^*P = A^*P^*P = A^*W$, where $W = P^*P + (I - P^*)(I - P)$. Then x_0 is a solution of the normal equation

$$A^*W(Ax - c) = 0.$$

Equivalently, x_0 is a solution of

$$\operatorname*{argmin}_{x\in\mathscr{H}} \|Ax - c\|_W.$$

In the same way, the equation $B^*(I - P)(Bx - c) = 0$ is equivalent to $B^*W(Bx - c) = 0$ which is the normal equation of the minimizing problem

$$\operatorname*{argmin}_{x\in\mathscr{H}} \|Bx - c\|_W.$$

Next consider the converse. As we noted above, x_0 is a solution of the system (4.16) if and only if x_0 is a solution of

$$\begin{cases} A^*(Ax - Pc) = 0 \\ B^*(Bx - (I - P)c) = 0. \end{cases}$$

We show in this case, that $A^*(Bx_0 - (I - P)c) = 0$. In fact, from the second equation we have that $Bx_0 - (I - P)c \in R(B)^{\perp}$, since applying $P_{R(B)^{\perp}}$ to $Bx_0 - (I - P)c$, we get that $Bx_0 - (I - P)c = -P_{R(B)^{\perp}}(I - P)c$. If P is optimal, then $P = P_{R(A)\oplus\mathscr{M}_1//R(B)\oplus\mathscr{N}_1}$ where $\mathscr{M}_1 \dotplus \mathscr{N}_1 = N(A^*) \cap N(B^*)$. Thus, $Bx_0 - (I - P)c \in P_{R(B)^{\perp}}(N(P)) = \mathscr{N}_1 \subseteq N(A^*)$. Therefore $A^*(Bx_0 - (I - P)c) = 0$. In the same way, $B^*(Ax_0 - Pc) = 0$. The sum of $A^*(Ax_0 - Pc) = 0$ and $A^*(Bx_0 - (I - P)c) = 0$ gives $A^*((A + B)x_0 - c) = 0$. Analogously, $B^*((A + B)x_0 - c) = 0$. Then $(A^* + B^*)((A + B)x_0 - c) = 0$, and so x_0 is a solution of the problem $\operatorname*{argmin}_{x\in\mathscr{H}} \|(A + B)x - c\|$. $\square$

Corollary 4.13. *Let $A, B \in L(\mathscr{H})$ be such that $R(A + B)$ is closed and $A \overset{*}{\leq} A + B$ and $c \in \mathscr{H}$. Then the following statements are equivalent:*

(1) x_0 is a solution of

$$\operatorname*{argmin}_{x\in\mathscr{H}} \|(A + B)x - c\|;$$

(2) x_0 is a solution of the following system of least squares problems:

$$\begin{cases} \operatorname*{argmin}_{x\in\mathscr{H}} \|Ax - c\| \\ \operatorname*{argmin}_{x\in\mathscr{H}} \|Bx - c\|. \end{cases}$$

Proof. This follows from Proposition 4.12 and the fact that we can take $P = P_A$ as an optimal projection such that $A = P(A + B)$ (see Proposition 3.6). In this case, $W = P^*P + (I - P^*)(I - P) = I$. □

Final remark 4.14. It is possible to define a weak version of the minus order in the following way: Consider $A, B \in L(\mathscr{H})$, we write $A \bar{\leq}_w B$ if there exist two densely defined idempotent operators P, Q with closed ranges such that $A = PB$, $R(P) = \overline{R(A)}$, $A^* = QB^*$ and $R(Q) = \overline{R(A^*)}$. The relation $\bar{\leq}_w$ is a partial order on $L(\mathscr{H})$. In fact, it is not difficult to see that the relation is reflexive and antisymmetric.

For transitivity, consider $A, B, C \in L(\mathscr{H})$ such that $A \bar{\leq}_w B$ and $B \bar{\leq}_w C$. By definition there exist P_1, P_2, Q_1, Q_2 densely defined idempotent operators such that $A = P_1B$, $R(P_1) = \overline{R(A)}$, $A^* = Q_1B^*$, $R(Q_1) = \overline{R(A^*)}$, $B = P_2C$ and $R(P_2) = \overline{R(B)}$, $B^* = Q_2A^*$, and $R(Q_2) = \overline{R(B^*)}$. Observe that $A = P_1P_2C$. Without loss of generality, suppose that $P_1 = P_{\overline{R(A)}//R(B-A)\oplus\mathscr{M}_1}$ and $P_2 = P_{\overline{R(B)}//R(C-B)\oplus\mathscr{M}_2}$, with $\mathscr{M}_1 = R(B)^\perp$ and $\mathscr{M}_2 = R(C)^\perp$. Let $\mathscr{D} = \overline{R(A)} \dotplus R(B - A) \dotplus R(C - B) \oplus \mathscr{M}_2$. Note that $\mathscr{D}$ is dense and $R(C) \subseteq \mathscr{D} \subseteq \mathscr{D}(P_1P_2)$, where the second inclusion follows because $P_2x = 0$ for all $x \in R(C - B) \oplus \mathscr{M}_2$ and $P_2x = x$ for all $x \in \overline{R(A)} \dotplus R(B - A) \subseteq \overline{R(B)} \subseteq \mathscr{D}(P_1)$. Consider $P = P_1P_2|_{\mathscr{D}}$, then $PC = P_1P_2|_{\mathscr{D}}C = P_1P_2C = A$. If $x \in \overline{R(A)}$, then $Px = P_1P_2|_{\mathscr{D}}x = P_1x = x$, because $\overline{R(A)} \subseteq \overline{R(B)}$. Therefore, $\overline{R(A)} \subseteq R(P) \subseteq R(P_1) = \overline{R(A)}$ so that $R(P) = \overline{R(A)}$. Since $R(P) = \overline{R(A)} \subseteq \mathscr{D}(P)$ and $P^2 = P_1P_2P_1P_2|_{\mathscr{D}} = P_1P_1P_2|_{\mathscr{D}} = P$, the operator P is idempotent. Therefore, P is a densely defined projection with closed range such that $A = PC$. Similarly, it follows that there exists a densely defined projection Q with closed range such $A^* = QC^*$. Hence $A \bar{\leq}_w C$, and so $\bar{\leq}_w$ is a partial order.

Moreover, if $A, B \in L(\mathscr{H})$ it can be proved that the following statements are equivalent:

(1) $A \bar{\leq}_w B$;
(2) $\overline{R(A)} \cap R(B - A) = \overline{R(A^*)} \cap R(B^* - A^*) = \{0\}$;
(3) $\overline{R(B)} = \overline{\overline{R(A)} \dotplus R(B - A)}$ and $\overline{R(B^*)} = \overline{\overline{R(A^*)} \dotplus R(B^* - A^*)}$.

In fact: (1) ⇒ (2): Suppose that $A \bar{\leq}_w B$, then there exist two densely defined idempotent operators P, Q such that $A = PB$, $R(P) = \overline{R(A)}$, $A^* = QB^*$ and $R(Q) = \overline{R(A^*)}$. Therefore $B - A = (I - P)B$, so that $R(B - A) \subseteq R(I - P) = N(P)$. Since $\overline{R(A)} = R(P)$, then $\overline{R(A)} \cap R(B - A) = \{0\}$. Similarly, $\overline{R(A^*)} \cap R(B^* - A^*) = \{0\}$.

(2) ⇒ (3): Suppose that $\overline{R(A)} \cap R(B - A) = \overline{R(A^*)} \cap R(B^* - A^*) = \{0\}$. Observe that $B = A + (B - A)$ so that $\overline{R(B)} \subseteq \overline{\overline{R(A)} \dotplus R(B - A)}$. To prove the other inclusion, consider $\mathscr{M}_1, \mathscr{M}_2$ two closed subspaces such that $\mathscr{H} = \overline{\overline{R(A)} \dotplus R(B - A)} \oplus \mathscr{M}_1$ and $\mathscr{H} = \overline{\overline{R(A^*)} \dotplus R(B^* - A^*)} \oplus \mathscr{M}_2$. Let $P = P_{\overline{R(A)}//R(B-A)\oplus\mathscr{M}_1}$ and $Q = P_{\overline{R(A^*)}//R(B^*-A^*)\oplus\mathscr{M}_2}$. Observe that P, Q are two densely defined idempotent operators such that $R(P) = \overline{R(A)}$ and $R(Q) = \overline{R(A^*)}$. Moreover, since $R(B) \subseteq R(A) + R(B - A)$, then $R(B) \subseteq \mathscr{D}(P)$

and $PB = P(B-A) + PA = PA = A$. Similarly, $A^* = QB^*$. Therefore $A^* = QB^*$ and $B^* - A^* = (I-Q)B^*$, so that $N(B^*) \subseteq N(A^*)$ and $N(B^*) \subseteq N(B^* - A^*)$. Then $\overline{R(A)} \subseteq \overline{R(B)}$ and $R(B-A) \subseteq \overline{R(B)}$ so that $\overline{\overline{R(A)} \dot{+} R(B-A)} \subseteq \overline{R(B)}$. Hence $\overline{R(B)} = \overline{\overline{R(A)} \dot{+} R(B-A)}$. Similarly, it follows that $\overline{R(B^*)} = \overline{\overline{R(A^*)} \dot{+} R(B^* - A^*)}$.

(3) $\Rightarrow$ (1): Suppose that $\overline{R(B)} = \overline{\overline{R(A)} \dot{+} R(B-A)}$ and $\overline{R(B^*)} = \overline{\overline{R(A^*)} \dot{+} R(B^* - A^*)}$. Then $\overline{R(A)} \cap R(B-A) = \overline{R(A^*)} \cap R(B^* - A^*) = \{0\}$. Therefore there exist two densely defined idempotent operators P, Q such that $A = PB$, $R(P) = \overline{R(A)}$, $A^* = QB^*$ and $R(Q) = \overline{R(A^*)}$, see the proof of (2) $\to$ (3). Hence $A \bar{\leq}_w B$.

Finally, note that $\bar{\leq}_w$ is weaker than the minus order. In fact, consider $P, Q \in \mathscr{P}$ such that $R(P) \cap R(Q) = 0$ and $c_0(R(P), R(Q)) = 1$. Then by the equivalence (1) $\Leftrightarrow$ (2) above, $P \bar{\leq}_w P + Q$. However, by Proposition 3.2, it is not the case that $P \bar{\leq} P + Q$.

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