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Minimal length curves in unitary orbits of a Hermitian compact operator

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ABSTRACT

We study some examples of minimal length curves in homogeneous spaces of $\mathcal{B}(\mathcal{H})$ under a left action of a unitary group. Recent results relate these curves with the existence of minimal (with respect to a quotient norm) anti-Hermitian operators Z in the tangent space of the starting point. We show minimal curves that are not of this type but nevertheless can be approximated uniformly by those.
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1. Introduction

Let $\mathcal{H}$ be a separable Hilbert space and $\mathcal{K}(\mathcal{H})$ be the algebra of compact operators. In this work we consider the orbit manifold of a self-adjoint compact operator A by a particular unitary group, that is

$$\mathcal{O}_A = \{uAu^* : u \text{ unitary in } \mathcal{B}(\mathcal{H}) \text{ and } u - 1 \in \mathcal{K}(\mathcal{H})\}.$$

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Given two points, $x, y \in \mathcal{O}_A$, the rectifiable distance between them is the infimum of the lengths of all the smooth curves in $\mathcal{O}_A$ that join x and y. Our purpose is to study the existence and properties of some particular minimal length curves in $\mathcal{O}_A$.

The tangent space at any $b \in \mathcal{O}_A$ is

$$(T\mathcal{O}_A)_b = \{zb - bz : z \in \mathcal{K}(\mathcal{H}),\ z^* = -z\}$$

endowed with the Finsler metric given by the usual operator norm $\|\cdot\|$. If $x \in (T\mathcal{O}_A)_b$, the existence of a (not necessarily unique) minimal element z_0 such that

$$\|x\|_b = \|z_0\| = \inf\{\|z\| : z \in \mathcal{K}(\mathcal{H}),\ z^* = -z,\ zb - bz = x\}$$

allows in [1] the description of minimal length curves of the manifold by the parametrization

$$\gamma(t) = e^{tz_0}\ b\ e^{-tz_0},\ t \in \left[-\frac{\pi}{2\,\|z_0\|}, \frac{\pi}{2\,\|z_0\|}\right].$$

These z_0 can be described as $i(C + D)$, with $C \in \mathcal{K}(\mathcal{H})$, $C^* = C$ and D a real diagonal operator in an orthonormal basis of eigenvectors of A.

If we consider $\mathcal{B} \subset \mathcal{A}$ von Neumann algebras and $a \in \mathcal{A}$, $a^* = a$, there always exists an element b_0 in $\mathcal{B}$ such that $\|a + b_0\| \leq \|a + b\|$, for all $b \in \mathcal{B}$ (see [4]). The element $a + b_0$ is called minimal in the class $[a]$ of $\mathcal{A}^h/\mathcal{B}^h$. However, in the case of $\mathcal{A} = \mathcal{K}(\mathcal{H})$, a C^*-algebra which is not a von Neumann algebra, and $\mathcal{B} \subset \mathcal{K}(\mathcal{H})$ a subalgebra there is not always a minimal compact operator in any class in $\mathcal{K}(\mathcal{H}^h)/\mathcal{B}^h$. In [2] we exhibit an example of this fact. In this case, the existence of a best approximant for $C \in \mathcal{K}(\mathcal{H})$, $C^* = C$ is guaranteed when C, for example, has finite rank (see Proposition 5.1 in [1]).

The above motivated us to study the following, among other issues, in the unitary orbit of a Hermitian operator. Let $b \in \mathcal{O}_A$ and $x \in (T\mathcal{O}_A)_b$ and suppose that there exists a uniparametric curve $\psi(t) = e^{tZ}be^{-tZ}$ which is a minimal length curve among all the smooth curves joining b and $\psi(t)$ in $\mathcal{O}_A$ for $t \in \left[-\frac{\pi}{2\|Z\|}, \frac{\pi}{2\|Z\|}\right]$:

- Would Z be a compact minimal lifting of x (i.e. $x = Zb - bZ$ and $\|Z\| = \|x\|_b$)?
- Can ψ be approximated in $\mathcal{O}_A$ by a sequence of minimal length curves of matrices?

The present work continues the analysis made in [1] of these homogeneous spaces and we use minimality characterizations that we developed in [2].

The results in this paper are divided in three parts. In the first we describe and study minimal length curves in the orbit of a particular compact Hermitian operator. In the second part we construct a sequence of minimal length curves of matrices which converges uniformly to the minimal length curves found in the first part. Finally, in the third part we study cases of anti-Hermitian compact operators whose best bounded diagonal approximants are not compact and we also study the properties of the minimal curves they determine.

2. Preliminaries and notation

Let $(\mathcal{H}, \langle, \rangle)$ be a separable Hilbert space. We denote by $\|h\| = \langle h, h\rangle^{1/2}$ the norm for each $h \in \mathcal{H}$. Let $\mathcal{B}(\mathcal{H})$ denote the set of bounded operators (with the identity operator I) and $\mathcal{K}(\mathcal{H})$, the two-sided closed ideal of compact operators on $\mathcal{H}$. Given $\mathcal{A} \subset \mathcal{B}(\mathcal{H})$, we use the superscript ^{ah} (resp. ^{h}) to note the subset of anti-Hermitian (resp. Hermitian) elements of $\mathcal{A}$.

We consider the group of unitary operators in $\mathcal{B}(\mathcal{H})$

$$\mathcal{U}(\mathcal{H}) = \{u \in \mathcal{B}(\mathcal{H}) : \ uu^* = u^*u = I\}$$

and the unitary Fredholm group, defined as

$$\mathcal{U}_c(\mathcal{H}) = \{u \in \mathcal{U}(\mathcal{H}) : u - I \in \mathcal{K}(\mathcal{H})\}.$$

We denote with $\|\cdot\|$ the usual operator norm in $\mathcal{B}(\mathcal{H})$ and with $[\ ,\ ]$ the commutator operator, that is, for any $T, S \in \mathcal{B}(\mathcal{H})$

$$[T, S] = TS - ST.$$

It should be clear from the context the use of the same notation $\|\cdot\|$ to refer to the operator norm or the norm on $\mathcal{H}$.

We define the unitary orbit of a fixed $A = A^* \in \mathcal{K}(\mathcal{H})$ as

$$\mathcal{O}_A = \{uAu^* : u \in \mathcal{U}_c(\mathcal{H})\} \ \subset \ \mathcal{K}(\mathcal{H}). \tag{2.1}$$

If A has spectral multiplicity one then $\mathcal{O}_A$ becomes a smooth homogeneous space if we consider the action $\pi_b : \mathcal{U}_c(\mathcal{H}) \to \mathcal{O}_A$, $\pi_b(u) = ubu^*$. In order to prove this, we will use the following lemma.

Lemma 1. *If $A = A^* \in \mathcal{K}(\mathcal{H})$ has spectral multiplicity one, then there exists a conditional expectation from $\mathcal{K}(\mathcal{H})$ onto $\{A\}' \cap \mathcal{K}(\mathcal{H})$.*

Proof. The hypothesis on A implies that the sequence $\{E_n\}_{n\geq 1}$ of projections onto the nonzero eigenspaces of A consists of rank-one projections and $\mathbf{1} = \sum_{n\geq 1} E_n$ in the strong operator topology. Moreover, one has

$$\{A\}' = \{T \in \mathcal{B}(\mathcal{H}) : \ \forall n \geq 1, \ TE_n = E_nT\}.$$

Then the map

$$\mathcal{E} : \mathcal{K}(\mathcal{H}) \to \{A\}' \cap \mathcal{K}(\mathcal{H}), \qquad \mathcal{E} := \sum_{n\geq 1} E_nTE_n,$$

is well defined by a norm-convergent series and is a conditional expectation as claimed. To see that the above series converges in the norm topology, one uses the Cauchy criterion. More specifically, for any $T \in \mathcal{K}(\mathcal{H})$ one has $\lim_{n\to\infty} \|E_nTE_n\| = 0$, and then, using that $E_{n_1}E_{n_2} = 0$ if $n_1 \neq n_2$, one obtains

$$\left\| \sum_{n\geq m} E_nTE_n \right\| = \sup_{n\geq m} \|E_nTE_n\| \to 0 \text{ as } m \to \infty,$$

which concludes the proof. □

Then it can be proved that for the group action

$$\mathcal{U}_c(\mathcal{H}) \times \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H}), \qquad (u, A) \mapsto uAu^*$$

the isotropy group of any $A = A^* \in \mathcal{K}(\mathcal{H})$ having spectral multiplicity one is a Banach–Lie subgroup of $\mathcal{U}_c(\mathcal{H})$, because it is an algebraic subgroup in the sense of the Harris–Kaup theorem, and its Lie algebra has a complement in the Lie algebra of $\mathcal{U}_c(\mathcal{H})$ by the above lemma. Therefore, the unitary orbit $\mathcal{O}_A$ has a smooth structure.

For each $b \in \mathcal{O}_A$, the isotropy group $\mathcal{I}_b$ is

$$\mathcal{I}_b = \{u \in \mathcal{U}_c(\mathcal{H}) : \ ubu^* = b\}.$$

Since for each $u \in \mathcal{U}_c(\mathcal{H})$ there always exists $X \in \mathcal{K}(\mathcal{H})^{ah}$ such that $u = e^X$ (see Proposition 4), the isotropy can be redefined by

$$\mathcal{I}_b = \{e^X \in \mathcal{U}_c(\mathcal{H}) : \ X \in \mathcal{K}(\mathcal{H})^{ah}, \ [X, b] = 0\}.$$

For each $b \in \mathcal{O}_A$, its tangent space is

$$(T\mathcal{O}_A)_b = \{Yb - bY : Y \in \mathcal{K}(\mathcal{H})^{ah}\} \subset \mathcal{K}(\mathcal{H})^{ah}.$$

Consider a smooth curve (i.e. C^1 and with derivative non-equal to zero) $u : [0,1] \to \mathcal{U}_c(\mathcal{H})$ such that $u(0) = 1$ and $u'(0) = Y$, then the differential of the surjective map π_b at 1 is

$$\begin{aligned}(d\pi_b)_1(Y) &= \frac{d}{dt}\ \pi_b(u(t))|_{t=0} = u'(0)b\ u^*(0) + u(0)b\ u'(0)^* \\ &= Yb1^* + 1bY^* = Yb - bY = [Y, b].\end{aligned}$$

For every $b \in \mathcal{O}_A$ we consider each tangent space as

$$(T\mathcal{O}_A)_b \cong (T\mathcal{U}_c(\mathcal{H}))_1/(T\mathcal{I}_b)_1 \cong \mathcal{K}(\mathcal{H})^{ah}/(\{b\}')^{ah},$$

being $\{b\}'$ the set of elements that commute with b in a C^*-algebra $\mathcal{A}$ (in this particular case $\mathcal{A} = \mathcal{K}(\mathcal{H})$). Let us consider the Finsler metric, defined for each $x \in (T\mathcal{O}_A)_b$ as

$$\|x\|_b = \inf\{\|Y\| : Y \in \mathcal{K}(\mathcal{H})^{ah} \text{ such that } [Y, b] = x\}$$

This metric can be expressed in terms of the projection to the quotient $\mathcal{K}(\mathcal{H})^{ah}/(\{b\}')^{ah}$ as

$$\|Yb - bY\|_b = \|[Y]\| = \inf_{C \in (\{b\}')^{ah}} \|Y + C\|$$

for each class $[Y] = \left\{Y + C : \ C \in (\{b\}')^{ah}\right\}$. This Finsler norm is invariant under the action of $\mathcal{U}_c(\mathcal{H})$.

There always exists $Z \in \mathcal{B}(\mathcal{H})^{ah}$ such that $[Z, b] = x$ and $\|Z\| = \|x\|_b$. Such element Z is called minimal lifting for x, and Z may not be compact and/or unique (see [2]). Consider piecewise smooth curves $\beta : [a, b] \to \mathcal{O}_A$. We define the rectifiable length of β as

$$\mathrm{L}(\beta) = \int_a^b \|\beta'(t)\|_{\beta(t)}\ dt,$$

and the rectifiable distance between two points of $\mathcal{O}_A$, named c_1 and c_2, as

$$\mathrm{dist}(c_1, c_2) = \inf\{\mathrm{L}(\beta) : \beta \text{ is smooth}, \beta(a) = c_1, \beta(b) = c_2\}.$$

If $\mathcal{A}$ is any C^*-algebra of $\mathcal{B}(\mathcal{H})$ and $\{e_k\}_{k=1}^{\infty}$ is a fixed orthonormal basis of $\mathcal{H}$, we denote with $\mathcal{D}(\mathcal{A})$ the set of diagonal operators with respect to this basis, that is

$$\mathcal{D}(\mathcal{A}) = \{T \in \mathcal{A} : \ \langle Te_i, e_j\rangle = 0\ , \quad \text{for all } i \neq j\}\,.$$

Given an operator $Z \in \mathcal{A}$, if there exists an operator $D_1 \in \mathcal{D}(\mathcal{A})$ such that

$$\|Z + D_1\| = \text{dist}\left(Z, \mathcal{D}\left(\mathcal{A}\right)\right),$$

we say that D_1 is a best approximant of Z in $\mathcal{D}(\mathcal{A})$. In other terms, the operator $Z+D_1$ verifies the following inequality

$$\|Z + D_1\| \leq \|Z + D\|$$

for all $D \in \mathcal{D}(\mathcal{A})$. In this sense, we call $Z + D_1$ a minimal operator or similarly we say that D_1 is minimal for Z. If Z is anti-Hermitian it holds that

$$\text{dist}\left(Z, \mathcal{D}\left(\mathcal{A}\right)\right) = \text{dist}\left(Z, \mathcal{D}\left(\mathcal{A}^{ah}\right)\right),$$

since $\|Im(X)\| \leq \|X\|$ for every $X \in \mathcal{A}$.

Let $T \in \mathcal{B}(\mathcal{H})$ and consider the coefficients $T_{ij} = \langle Te_i, e_j\rangle$ for each $i, j \in \mathbb{N}$, that define an infinite matrix $(T_{ij})_{i,j\in\mathbb{N}}$. The jth-column and ith-row of T are the vectors in ℓ^2 given by $c_j(T) = (T_{1j}, T_{2j}, \ldots)$ and $f_i(T) = (T_{i1}, T_{i2}, \ldots)$, respectively.

We use $\sigma(T)$ and $R(T)$ to denote the spectrum and range of $T \in \mathcal{B}(\mathcal{H})$, respectively.

We define $\Phi : \mathcal{B}(\mathcal{H}) \to \mathcal{D}(\mathcal{B}(\mathcal{H}))$, $\Phi(X) = \text{Diag}(X)$, that takes the main diagonal (i.e. the elements of the form $\{\langle Xe_i, e_i\rangle\}_{i\in\mathbb{N}}$) of an operator X and builds a diagonal operator in the chosen fixed basis of $\mathcal{H}$. For a given bounded sequence $\{d_n\}_{n\in\mathbb{N}} \subset \mathbb{C}$ we denote with $\text{Diag}\big(\{d_n\}_{n\in\mathbb{N}}\big)$ the diagonal (infinite) matrix with $\{d_n\}_{n\in\mathbb{N}}$ in its diagonal and 0 elsewhere.

The following theorem is similar to Theorem 1 in [2] but this version only requires that $T \in \mathcal{B}(\mathcal{H})^h$, instead of $T \in \mathcal{K}(\mathcal{H})^h$. The proof follows exactly the same arguments that the one in the case where T is compact.

Theorem 2. *Let $T \in \mathcal{B}(\mathcal{H})^h$ described as an infinite matrix by $(T_{ij})_{i,j\in\mathbb{N}}$. Suppose that T satisfies:*

- *$T_{ij} \in \mathbb{R}$ for each $i, j \in \mathbb{N}$,*
- *there exists $i_0 \in \mathbb{N}$ satisfying $T_{i_0 i_0} = 0$, with $T_{i_0 n} \neq 0$, for all $n \neq i_0$,*
- *if $T^{[i_0]}$ is the operator T with zero in its i_0th-column and i_0th-row then*

$$\|c_{i_0}(T)\| \geq \left\|T^{[i_0]}\right\|$$

  (where $\|c_{i_0}(T)\|$ denotes the Hilbert norm of the i_0th-column of T), and
- *$\langle c_{i_0}(T), c_n(T)\rangle = 0$ for each $n \in \mathbb{N}$, $n \neq i_0$.*

Then,

1. *$\|T\| = \|c_{i_0}(T)\|$.*
2. *T is minimal, that is*

$$\|T\| = \inf_{D\in\mathcal{D}(\mathcal{B}(\mathcal{H})^h)} \|T + D\| = \inf_{D\in\mathcal{D}(\mathcal{K}(\mathcal{H})^h)} \|T + D\|,$$

  and $D = Diag\big(\{T_{nn}\}_{n\in\mathbb{N}}\big)$ is the unique bounded minimal diagonal operator for T.

Proof. Without loss of generality suppose that $T = (T_{ij})_{i,j\in\mathbb{N}} \in \mathcal{B}(\mathcal{H})^h$ has real entries T_{ij} and $i_0 = 1$. The hypothesis in this case are

- $i_0 = 1$ with $T_{1n} \neq 0$, $\forall n \in \mathbb{N} - \{1\}$.
- $T_{11} = 0$.
- $\|c_1(T)\| \geq \left\|T^{[1]}\right\|$.
- $\langle c_1(T), c_n(T)\rangle = 0$ for each $n \in \mathbb{N}$, $n \neq 1$.

There are some remarks to be made:

- The last hypothesis implies that

$$T_{nn} = -\frac{\langle c_1(T), c_n(T)\rangle - T_{nn}T_{1n}}{T_{1n}} = -\frac{1}{T_{1n}}\sum_{i\neq n} T_{i1}T_{in}\ , \quad \text{for every } n \in \mathbb{N}\setminus\{1\}$$

and observe that

$$|T_{nn}| = \left|\left\langle T^{[1]}e_n, e_n\right\rangle\right| \leq \left\|T^{[1]}e_n\right\|\|e_n\| \leq \left\|T^{[1]}\right\| \leq \|c_1(T)\| < \infty.$$

Namely, $\{T_{nn}\}_{n\in\mathbb{N}}$ is a bounded sequence (each T_{nn} is a diagonal element of $T^{[1]}$ in the fixed basis).

- A direct computation proves that $\|c_1(T)\|$ and $-\|c_1(T)\|$ are eigenvalues of T with

$$v_+ = \frac{1}{\sqrt{2}\,\|c_1(T)\|}\left(\|c_1(T)\|\,e_1 + c_1(T)\right) \text{ and}$$
$$v_- = \frac{1}{\sqrt{2}\,\|c_1(T)\|}\left(\|c_1(T)\|\,e_1 - c_1(T)\right),$$

which are eigenvectors of $\|c_1(T)\|$ and $-\|c_1(T)\|$, respectively. Let us consider the space $V = Gen\{v_+, v_-\}$:

a) For every $w \in V$, there exist $\alpha, \beta \in \mathbb{R}$ such that $w = \alpha v_+ + \beta v_-$. Then

$$\begin{aligned}\|Tw\|_2^2 &= \|T(\alpha v_+ + \beta v_-)\|_2^2 = \|\alpha\,\|c_1(T)\|\,v_+ - \beta\,\|c_1(T)\|\,v_-\|_2^2\\ &= |\alpha|^2\|c_1(T)\|^2 + |\beta|^2\|c_1(T)\|^2 = \|c_1(T)\|^2\left(|\alpha|^2 + |\beta|^2\right)\\ &= \|c_1(T)\|^2\|w\|^2.\end{aligned}$$

b) If $y \in V^\perp$, $Ty = \begin{pmatrix} 0 & c_1^t(T)\\ c_1(T) & T^{[1]}\end{pmatrix}\begin{pmatrix}0\\ y\end{pmatrix} = \begin{pmatrix}0\\ T^{[1]}y\end{pmatrix}$. Then,

$$\|Ty\| = \left\|\begin{pmatrix}0\\ T^{[1]}y\end{pmatrix}\right\| = \left\|T^{[1]}y\right\| \leq \left\|T^{[1]}\right\|\|y\|.$$

Now we are in conditions to prove the statements of the theorem:

1. **Norm of $\|T\|$:** For every $x = w + y \in \mathcal{H}$, with $w \in V$ and $y \in V^\perp$:

$$\begin{aligned}\|Tx\|^2 &= \|T(w+y)\|^2 = \|Tw\|^2 + \|Ty\|^2 + 2\ Re\langle Tw, Ty\rangle\\ &= \|Tw\|^2 + \|Ty\|^2\end{aligned}$$

where the last equality follows since V and $V^\perp$ are stable by the Hermitian operator T. Then

$$
\begin{aligned}
\|Tx\|^2 = \|Tw\|^2 + \|Ty\|^2 &\le \|c_1(T)\|^2 \|w\|^2 + \left\|T^{[1]}\right\|^2 \|y\|^2 \\
&\le \|c_1(T)\|^2 \|x\|^2 ,
\end{aligned}
$$

and hence $\|T\| = \|c_1(T)\|$.

2. **Minimality and uniqueness of D:** Let $D' \in \mathcal{D}(\mathcal{B}(\mathcal{H})^h)$ and define $(T + D')e_n = T'(e_n) = c_n(T')$ for each $n \in \mathbb{N}$, then the following properties are satisfied:
   - If $D'_{11} \neq 0$ then

   $$
   \begin{aligned}
   \|T'(e_1)\|^2 &= \|c_1(T')\|^2 = |D'_{11}|^2 + \|c_1(T)\|^2 > \|c_1(T)\|^2 = \|T\|^2 \\
   &\Rightarrow \|T'\| > \|T\| .
   \end{aligned}
   $$

   Therefore, we can assume that if $T + D'$ is minimal then $D'_{11} = 0$.
   - Now suppose that there exists $i \in \mathbb{N}$, $i > 1$, such that T' does not have its ith-column orthogonal to the first one, that is:

   $$
   \langle T'e_1, T'e_i\rangle = \langle c_1(T'), c_i(T')\rangle = a_i \neq 0.
   $$

   Then,

   $$
   \begin{aligned}
   T'\left(\frac{c_1(T)}{\|c_1(T)\|}\right) &= \left(\|c_1(T)\| , \frac{a_2}{\|c_1(T)\|}, \dots , \frac{a_i}{\|c_1(T)\|}, \dots\right) \\
   &\Rightarrow \|T'(c_1(T))\|^2 > \|c_1(T)\|^2 = \|T\| .
   \end{aligned}
   $$

   Hence, $\|T'\| > \|T\|$.

   Therefore, $D = \mathrm{Diag}\big(\{T_{nn}\}_{n\in\mathbb{N}}\big)$ is the unique minimal diagonal for T and it is bounded. □

3. The unitary Fredholm orbit of a Hermitian compact operator

In this section we consider the unitary Fredholm orbit $\mathcal{O}_A$ of a particular case of a Hermitian compact operator, that is: $A \in \mathcal{K}(\mathcal{H})^h$, $A = u\mathrm{Diag}\left(\{\lambda_i\}_{i\in\mathbb{N}}\right) u^*$, with $u \in \mathcal{U}_c(\mathcal{H})$ and $\{\lambda_i\}_{i\in\mathbb{N}} \subset \mathbb{R}$ such that $\lambda_i \neq \lambda_j$ for each $i \neq j$. Consider $\mathcal{O}_A$ as defined in section 2 and $b = \mathrm{Diag}\left(\{\lambda_i\}_{i\in\mathbb{N}}\right) \in \mathcal{O}_A$. The isotropy $\mathcal{I}_b$ is the set $\{e^d : \ d \in \mathcal{D}(\mathcal{K}(\mathcal{H})^{ah})\}$ and $(T\mathcal{O}_A)_b$ can be identified with the quotient space $\mathcal{K}(\mathcal{H})^{ah}/\mathcal{D}(\mathcal{K}(\mathcal{H})^{ah})$.

Proposition 3. *Let $b = \mathrm{Diag}\left(\{\lambda_i\}_{i\in\mathbb{N}}\right) \in \mathcal{O}_A$ with distinct eigenvalues. For each $x \in (T\mathcal{O}_A)_b$, if $Z \in \mathcal{K}(\mathcal{H})^{ah}$ is such that $[Z, b] = x$, then*

$$
\|x\|_b = \inf_{D\in\mathcal{D}(\mathcal{K}(\mathcal{H})^{ah})} \|Z + D\| \tag{3.1}
$$

Proof. If $Y_1, Y_2 \in \{Y \in \mathcal{K}(\mathcal{H})^{ah} : [Y, b] = x\}$ then

$$
Y_1 - Y_2 \in \{D : \ [D, b] = Db - bD = 0\} = \{b\}'
$$

and since b is a diagonal operator with distinct eigenvalues, then every D is diagonal. Thus

$$
Y_1 - Y_2 = D, \text{ with } D \text{ diagonal}
$$

or equivalently: $Y_1 = Y_2 + D$, with $D \in \mathcal{D}(\mathcal{K}(\mathcal{H})^{ah})$. Then,

$$
\|x\|_b = \inf\{\|Y\| : Y \in \mathcal{K}(\mathcal{H})^{ah} \text{ such that } Y = Y_2 + D, \text{ with } D \in \mathcal{D}(\mathcal{K}(\mathcal{H})^{ah})\}. \qquad \square
$$

Fix $x = [Z_r, b] = Z_r b - bZ_r \in \mathcal{B}(\mathcal{H})^{ah}$, where Z_r is an anti-Hermitian operator defined as the infinite matrix defined by

$$(Z_r)_{j,k} = \begin{cases} 0 & \text{if } \ j=k=1, \\ i\ d_k & \text{if } \ j=k \ \ (\text{for } j\neq 1 \text{ and } k\neq 1), \\ i\ \gamma^{\max\{j,k\}-2} & \text{if } \ j\neq k \text{ and (for } j\neq 1 \text{ and } k\neq 1), \\ i\ r\gamma^{|j-k|} & \text{if } \ j=1 \text{ or } \ k=1 \ \ (\text{for } j\neq 1 \text{ or } k\neq 1). \end{cases}$$

Therefore

$$Z_r = i \begin{pmatrix} 0 & r\gamma & r\gamma^2 & r\gamma^3 & r\gamma^4 & \cdots \\ r\gamma & d_2 & \gamma & \gamma^2 & \gamma^3 & \cdots \\ r\gamma^2 & \gamma & d_3 & \gamma^2 & \gamma^3 & \cdots \\ r\gamma^3 & \gamma^2 & \gamma^2 & d_4 & \gamma^3 & \cdots \\ r\gamma^4 & \gamma^3 & \gamma^3 & \gamma^3 & d_5 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \tag{3.2}$$

$$= \underbrace{i \begin{pmatrix} 0 & r\gamma & r\gamma^2 & r\gamma^3 & r\gamma^4 & \cdots \\ r\gamma & 0 & \gamma & \gamma^2 & \gamma^3 & \cdots \\ r\gamma^2 & \gamma & 0 & \gamma^2 & \gamma^3 & \cdots \\ r\gamma^3 & \gamma^2 & \gamma^2 & 0 & \gamma^3 & \cdots \\ r\gamma^4 & \gamma^3 & \gamma^3 & \gamma^3 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}}_{Y_r} + \underbrace{i \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & d_2 & 0 & 0 & 0 & \cdots \\ 0 & 0 & d_3 & 0 & 0 & \cdots \\ 0 & 0 & 0 & d_4 & 0 & \cdots \\ 0 & 0 & 0 & 0 & d_5 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}}_{D_0} = Y_r + D_0. \tag{3.3}$$

The entries of the operator Z_r are such that:

1. $\gamma \in \mathbb{R}$ such that $|\gamma| < 1$.
2. The diagonal entries d_k are defined by imposing that the relation $\langle c_1(Z_r), c_k(Z_r)\rangle = 0$ is satisfied. Then $d_2 = -\sum_{j=1}^{\infty} \gamma^{2j}$, and for each $k \in \mathbb{N}$, $k > 2$:

$$d_k = -\left(\sum_{j=0}^{k-3} \gamma^j\right) - \left(\sum_{j=0}^{\infty} \gamma^{k+2j}\right) = \frac{1-\gamma^{k-2}}{\gamma - 1} - \frac{\gamma^k}{1-\gamma^2}.$$

   Notice that $\lim\limits_{k\to\infty} d_k = \frac{1}{\gamma-1}$.

3. $r \geq \frac{\left\| Y^{[1]} + D_0 \right\|}{\left(\sum_{j=1}^{\infty} \gamma^{2j}\right)^{1/2}}$, where $Y^{[1]} = Y_r - \begin{pmatrix} 0 & r\gamma & r\gamma^2 & r\gamma^3 & \cdots \\ r\gamma & 0 & 0 & 0 & \cdots \\ r\gamma^2 & 0 & 0 & 0 & \cdots \\ r\gamma^3 & 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$.

Observe that the definition of each d_k is independent of the parameter r. Also note that Y_r is a Hilbert–Schmidt operator. Indeed,

- $(Y_r^* Y_r)_{11} = \langle f_1(Y_r), c_1(Y_r)\rangle = r^2 \left(\sum_{k=1}^{\infty} \ \gamma^{2k}\right) = r^2 \dfrac{\gamma^2}{1-\gamma^2} < \infty$.
- $(Y_r^* Y_r)_{22} = \langle f_2(Y_r), c_2(Y_r)\rangle = r^2\gamma^2 + \sum_{k=1}^{\infty} \ \gamma^{2k} = r^2\gamma^2 + \dfrac{\gamma^2}{1-\gamma^2} < \infty$.

- Inductively for each $n \geq 3$:

$$(Y_r^*Y_r)_{nn} = \langle f_n(Y_r), c_n(Y_r)\rangle = r^2\gamma^{2(n-1)} + (n-2)\gamma^{2(n-2)} + \sum_{k=n-1}^{\infty} \gamma^{2k}$$
$$= r^2\gamma^{2(n-1)} + (n-2)\gamma^{2(n-2)} + \frac{\gamma^{-2+2n}}{1-\gamma^2} < \infty.$$

Then,

$$tr(Y_r^*Y_r) = tr(Y_r^2) = \sum_{n=1}^{\infty}(Y_r^*Y_r)_{nn}$$
$$= r^2\frac{\gamma^2}{1-\gamma^2} + r^2\gamma^2 + \frac{\gamma^2}{1-\gamma^2} + \sum_{n=3}^{\infty}\left[r^2\gamma^{2(n-1)} + (n-2)\gamma^{2(n-2)} + \frac{\gamma^{2n-2}}{1-\gamma^2}\right]$$
$$= r^2\frac{\gamma^2}{1-\gamma^2} + r^2\gamma^2 + \frac{\gamma^2}{1-\gamma^2} + \frac{r^2\gamma^4}{1-\gamma^2} + \frac{\gamma^2(1+\gamma^2)}{(1-\gamma^2)^2} < \infty.$$

The operator $-iZ_r$ fulfills the conditions of minimality in Theorem 2 stated in the Preliminaries and has been studied in [2]. Therefore,

$$\|[Y_r]\| = \inf_{D\in\mathcal{D}(\mathcal{K}(\mathcal{H})^{ah})} \|Y_r + D\| = \|Y_r + D_0\| = \|Z_r\|.$$

Moreover, the diagonal operator D_0 is the unique minimal diagonal (bounded, but non-compact) operator for Y_r. Since $D_0b - bD_0 = 0$, then $x = Y_rb - bY_r \in (T\mathcal{O}_A)_b$ and

$$\|x\|_b = \|Z_rb - bZ_r\|_b = \inf_{D\in\mathcal{D}(\mathcal{K}(\mathcal{H})^h)} \|Y_r + D\| = \|[Y_r]\| = \|Z_r\| < \|Y_r + D\|$$

for all $D \in \mathcal{D}(\mathcal{K}(\mathcal{H})^{ah})$. In other words, there is no compact minimal lifting for x in this case.

The following proposition is a characterization of the unitary Fredholm group in terms of operators in $\mathcal{K}(\mathcal{H})^{ah}$.

Proposition 4. *$w \in \mathcal{U}_c(\mathcal{H})$ if and only if there exists $X \in \mathcal{K}(\mathcal{H})^{ah}$ such that $w = e^X$.*

Proof. Given $w \in \mathcal{U}_c(\mathcal{H})$, by Lemma 2.1 in [1] there exists $X \in \mathcal{K}(\mathcal{H})^{ah}$ such that $w = e^X$. On the other hand, consider $X \in \mathcal{K}(\mathcal{H})^{ah}$ and the series expansion of e^X

$$e^X = 1 + X + \frac{1}{2}X^2 + \frac{1}{3!}X^3 + \dots$$
$$= 1 + X\left[1 + \frac{1}{2}X + \frac{1}{3!}X^2 + \dots\right] = 1 + K\ ,\ K \in \mathcal{K}(\mathcal{H}).$$

Additionally, $(e^X)^* = e^{-X}$ and $(e^X)^*e^X = e^X(e^X)^* = e^{X-X} = I$. Then, $e^X \in \mathcal{U}_c(\mathcal{H})$. □

Remark 5. Even if $Z \notin \mathcal{K}(\mathcal{H})^{ah}$, e^Z may belong to $\mathcal{U}_c(\mathcal{H})$. Indeed, let $X_0 \in \mathcal{K}(\mathcal{H})^{ah}$, then $Z = X_0 + 2\pi iI \notin \mathcal{K}(\mathcal{H})^{ah}$ but

$$e^{X_0+2\pi iI} = e^{X_0} \in \mathcal{U}_c(\mathcal{H}).$$

For Z_r as in (3.2) define the uniparametric curve β by

$$\beta(t) = e^{tZ_r} b e^{-tZ_r} \ , \ t \in \left[-\frac{\pi}{2\,\|Z_r\|}, \frac{\pi}{2\,\|Z_r\|} \right]. \tag{3.4}$$

To prove that β is a curve in $\mathcal{O}_A$, we introduce first the next result.

Lemma 6. *Let Z_r be the operator defined in (3.2). Then for each $t \in \mathbb{R}$, there exist $z_t \in \mathbb{C}$, $|z_t| = 1$ and $U(t) \in \mathcal{U}_c(\mathcal{H})$ such that*

$$e^{tZ_r} = z_t U(t).$$

Proof. Let $\alpha = -i \lim_{n\to\infty} d_n = \frac{i}{1-\gamma}$. Then $e^{tZ_r + \alpha I t} = e^{tZ_r} e^{t\alpha I}$. Observe that $e^{t\alpha I} = e^{t\alpha} I$. Thus

$$e^{tZ_r} = e^{-t\alpha} e^{tZ_r + t\alpha I} = e^{-t\alpha} e^{tY_r + tD_0 + t\alpha I},$$

with $e^{-t\alpha} \in \mathbb{C}$, $|e^{-t\alpha}| = 1$ for every $t \in \mathbb{R}$. Moreover, $D_0 + \alpha I \in \mathcal{D}(\mathcal{K}(\mathcal{H})^{ah})$, since it is a bounded diagonal and

$$\left| (D_0 + \alpha I)_{jj} \right| = \left| -\frac{1-\gamma^{j-2}}{1-\gamma} - \frac{\gamma^j}{1-\gamma^2} + \frac{1}{1-\gamma} \right| = \left| \frac{\gamma^{j-2}}{1-\gamma} - \frac{\gamma^j}{1-\gamma^2} \right| \to 0$$

when $j \to \infty$. Therefore, since $tZ_r + t\alpha I \in \mathcal{K}(\mathcal{H})^{ah}$ for every $t \in \mathbb{R}$ then $U(t) = e^{tZ_r + t\alpha I} \in \mathcal{U}_c(\mathcal{H})$ and

$$e^{tZ_r} = z_t U(t), \text{ with } z_t = e^{t\alpha} \in \mathbb{C}. \qquad \square$$

Remark 7. For any minimal lifting $Z \in \mathcal{B}(\mathcal{H})^{ah}$ of $x = [Y, b]$, the curve $\kappa(t) = e^{Zt} b e^{-Zt}$ has minimal length over all the smooth curves in $\mathcal{P} = \{uAu^* : \ u \in \mathcal{U}(\mathcal{H})\}$ that join $\beta(0) = b$ and $\beta(t)$, with $|t| \le \frac{\pi}{2\|Z\|}$ (Theorem II in [4]). Since $\mathcal{O}_A \subseteq \mathcal{P}$, then for each $t_0 \in \left[-\frac{\pi}{2\|Z\|}, \frac{\pi}{2\|Z\|}\right]$ follows that

$$\begin{aligned} \mathrm{L}(\kappa) &= \inf\{\mathrm{L}(\chi) : \chi \subset \mathcal{P}, \chi \text{ is smooth}, \chi(0) = b \text{ and } \chi(t_0) = \beta(t_0)\} \\ &\le \inf\{\mathrm{L}(\chi) : \chi \subset \mathcal{O}_A, \chi \text{ is smooth}, \chi(0) = b \text{ and } \chi(t_0) = \beta(t_0)\} \\ &= \mathrm{dist}(b, \beta(t_0)). \end{aligned}$$

Using the previous remark and Lemma 6 we can prove the following theorem.

Theorem 8. *Let $A = u\mathrm{Diag}\left(\{\lambda_i\}_{i\in\mathbb{N}}\right) u^*$, with $u \in \mathcal{U}_c(\mathcal{H})$ and $\{\lambda_i\}_{i\in\mathbb{N}} \subset \mathbb{R}$ such that $\lambda_i \ne \lambda_j$ for each $i \ne j$. Let $b = \mathrm{Diag}\left(\{\lambda_i\}_{i\in\mathbb{N}}\right) \in \mathcal{O}_A$ and the parametric curve β defined in (3.4). Then β satisfies:*

1. $\beta(t) = e^{t(Z_r + \frac{i}{1-\gamma} I)} b e^{-t(Z_r + \frac{i}{1-\gamma} I)}$, *which means that $\beta(t) \in \mathcal{O}_A$ for every t.*
2. $\beta'(0) = x = Y_r b - bY_r = Z_r b - bZ_r \in (T\mathcal{O}_A)_b$.
3. *β has minimal length between all smooth curves in $\mathcal{O}_A$ joining b with $\beta(t_0)$, for every $t_0 \in \left[-\frac{\pi}{2\|Z_r\|}, \frac{\pi}{2\|Z_r\|}\right]$. That is*

$$\begin{aligned} \mathrm{L}\left(\beta|_{[0,t_0]}\right) &= \inf\{\mathrm{L}(\chi) : \chi \textit{ is smooth}, \chi(0) = b \textit{ and } \chi(t_0) = \beta(t_0)\} \\ &= \mathrm{dist}(b, \beta(t_0)). \end{aligned}$$

4. $\mathrm{L}\left(\beta|_{[0,t_0]}\right) = |t_0|\, \|x\|_b$, *for each $t_0 \in \left[-\frac{\pi}{2\|Z_r\|}, \frac{\pi}{2\|Z_r\|}\right]$.*

Proof.

1. By Lemma 6, if $U(t) = e^{tZ_r + t\frac{i}{1-\gamma}I}$, then β can be rewritten as

$$\beta(t) = z_t U(t) b (z_t U(t))^* = z_t \overline{z_t} U(t) b U^{-1}(t)$$
$$= U(t) b U^{-1}(t) = e^{t(Z_r + \frac{i}{1-\gamma}I)} b e^{-t(Z_r + \frac{i}{1-\gamma}I)}$$

and $U(t) \in \mathcal{U}_c(\mathcal{H})$ for each $t \in \mathbb{R}$. Follows that $\beta(t) \in \mathcal{O}_A$ for every $t \in \mathbb{R}$.

2. $\beta'(0) = e^{tZ_r} [Z_r, b] e^{-tZ_r}\big|_{t=0}$.
3. Observe that $\|Z_r\| = \|[Y_r]\|_{\mathcal{B}(\mathcal{H})^{ah}/\mathcal{D}(\mathcal{B}(\mathcal{H}))^{ah}}$ and Z_r is (the unique) minimal lifting of $x = [Y_r, b]$ in $\mathcal{B}(\mathcal{H})$. Then, the result is a direct consequence of Remark 7.
4. Observe that $\mathrm{L}(\beta) = \int_0^{t_0} \|\beta'(t)\|_{\beta(t)} \, dt = t_0 \|Y_r b - bY_r\|_b$. Indeed,

$$\|\beta'(t)\|_{\beta(t)} = \left\|Z_r e^{tZ_r} b e^{-tZ_r} - e^{tZ_r} b Z_r e^{-tZ_r}\right\|_{\beta(t)} = \left\|e^{tZ_r} [Z_r, b] e^{-tZ_r}\right\|_{\beta(t)}$$
$$= \left\|z\overline{z} U(t) [Z_r, b] U^{-1}(t)\right\|_{\beta(t)} = |z|^2 \left\|U(t) [Z_r, b] U^{-1}(t)\right\|_{\beta(t)}$$
$$= \left\|U(t) [Z_r, b] U^{-1}(t)\right\|_{U(t)bU^{-1}(t)} = \|Z_r b - bZ_r\|_b$$
$$= \|Y_r b - bY_r\|_b = \|x\|_b \,,$$

where the equality $\left\|U(t) [Z_r, b] U^{-1}(t)\right\|_{U(t)bU^{-1}(t)} = \|Z_r b - bZ_r\|_b$ holds due to the unitary invariance of the Finsler norm. □

Summarizing, if $Z_\alpha = Z_r + \frac{i}{1-\gamma} I \in \mathcal{K}(\mathcal{H})^{ah}$, we obtained that the parametric curve given by

$$\pi_b \circ (e^{tZ_\alpha}) = e^{tZ_\alpha} b e^{-tZ_\alpha}$$

has minimal length between elements of $\mathcal{O}_A$. Nevertheless, the operator Z_α is not a minimal element in its class (recall that $[Z_r] = \{Z_r + D : D \in \mathcal{D}(\mathcal{K}(\mathcal{H})^{ah}) = [Y_r]\}$). On the other hand,

$$e^{tZ_\alpha} b e^{-tZ_\alpha} = e^{tZ_r} b e^{-tZ_r}$$

and Z_r is minimal, but it does not belong to $\mathcal{K}(\mathcal{H})^{ah}$. We conclude with the following comment.

Remark 9. Let $b \in \mathcal{O}_A$, $b = \mathrm{Diag}\left(\{\lambda_i\}_{i\in\mathbb{N}}\right)$ such that $\lambda_i \neq \lambda_j$ for each $i \neq j$. Then, there exist minimal length curves of the form $\rho(t) = e^{tZ} b e^{-tZ}$ in $\mathcal{O}_A$ such that they join b with other points of the orbit, but with $Z \in \mathcal{K}(\mathcal{H})^{ah}$ and $\|Z\| > \|[Z]\|_{\mathcal{K}(\mathcal{H})^{ah}/\mathcal{D}(\mathcal{K}(\mathcal{H})^{ah})}$.

4. Approximation with minimal length curves of matrices

There are two main objectives in this section: the first is to build two sequences of minimal matrices which approximate Z_r and $Z_r + \frac{i}{1-\gamma} I$ in the strong operator topology (SOT) and in the operator norm, respectively. The second objective is to find a family of minimal length curves of matrices which approximates the curve β defined in (3.4).

Let Y_r be the anti-Hermitian compact operator defined in (3.3) and consider the following decomposition

$$Y_r = rL + Y^{[1]}, \text{ where } L = i\begin{pmatrix} 0 & \gamma & \gamma^2 & \gamma^3 & \cdots \\ \gamma & 0 & 0 & 0 & \cdots \\ \gamma^2 & 0 & 0 & 0 & \cdots \\ \gamma^3 & 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}. \tag{4.1}$$

Let D_0 be the diagonal bounded operator defined in (3.3). If $r \geq \frac{\left\|Y^{[1]}+D_0\right\|}{\|c_1(L)\|}$, then $Z_r = rL + Y^{[1]} + D_0$ is minimal.

Let us consider for each $n \in \mathbb{N}_{\geq 3} = \{n \in \mathbb{N} : n \geq 3\}$ the orthogonal projection P_n over the space generated by $\{e_1, \ldots, e_n\}$. We define the following finite rank operators

$$Y_n = r_n P_n L P_n + P_n Y^{[1]} P_n, \tag{4.2}$$

with $r_n \in \mathbb{R}_{>0}$ for each $n \in \mathbb{N}$. For each $n \in \mathbb{N}_{\geq 3}$ we define the diagonal operator $D_n = i\text{Diag}\left(\{d_k^{(n)}\}_{k\in\mathbb{N}_{\geq 3}}\right)$ uniquely determined by the conditions:

1. $d_1^{(n)} = 0$;
2. $\langle c_1(Y_n + D_n), c_j(Y_n + D_n)\rangle = 0$, for each $j \in \mathbb{N}$, $j \neq 1$;
3. $d_k^{(n)} = 0$, for every $k > n$.

Thus, each $d_k^{(n)}$ is determined for every $n \in \mathbb{N}_{\geq 3}$ as

$$\begin{cases} d_2^{(n)} = -\sum_{j=1}^{n-2} \gamma^{2j} < 0 & \\ d_k^{(n)} = -\sum_{j=0}^{k-3} \gamma^j - \sum_{j=0}^{2n-2k-2} \gamma^{k+2j} < 0 & \text{if} \quad k < n \\ d_n^{(n)} = -\sum_{j=0}^{k-3} \gamma^j < 0 & \\ d_k^{(n)} = 0 & \text{for all} \quad k > n. \end{cases} \tag{4.3}$$

The proof is by induction over the indices k for every $n \in \mathbb{N}_{\geq 3}$. Observe that the choice of each $d_k^{(n)}$ is independent of the parameter r_n.

The following lemma will be used to prove the minimality of each $Y_n + D_n$ for a fixed r_n.

Lemma 10. *Let $Y_n = r_n P_n L P_n + P_n Y^{[1]} P_n$ and D_n as defined in (4.2) and (4.3) for each $n \in \mathbb{N}_{\geq 3}$, respectively. Then*

$$\sup_{n\in\mathbb{N}_{\geq 3}} \left\|P_n Y^{[1]} P_n + D_n\right\| < \infty.$$

Proof. Fix $n \in \mathbb{N}_{\geq 3}$. Since $\sup_{n\in\mathbb{N}} \left|d_n^{(n)}\right| \leq \|D_0\|$, for D_0 the diagonal operator defined in (3.3), then

$$\left\|P_n Y^{[1]} P_n + D_n\right\| \leq \left\|P_n Y^{[1]} P_n\right\| + \|D_n\| \leq \|P_n\|^2 \left\|Y^{[1]}\right\| + \sup_{1\leq k\leq n} \left|d_k^{(n)}\right|$$
$$\leq \left\|Y^{[1]}\right\| + \left|d_n^{(n)}\right| \leq \left\|Y^{[1]}\right\| + \sup_{n\in\mathbb{N}_{\geq 3}} \left|d_n^{(n)}\right| \leq \left\|Y^{[1]}\right\| + \|D_0\| < \infty. \quad \square$$

As a consequence of this lemma, there exists a constant $M_0 \in \mathbb{R}_{>0}$ such that:

$$M_0 = \max \left\{ \sup_{n \in \mathbb{N}_{\geq 3}} \left\| P_n Y^{[1]} P_n + D_n \right\|, \left\| Y^{[1]} + D_0 \right\| \right\}. \tag{4.4}$$

Now we can prove the minimality of each $Y_n + D_n$ for all $n \in \mathbb{N}$.

Proposition 11. *Let $Y_n = r_n P_n L P_n + P_n Y^{[1]} P_n$ and D_n as defined in (4.2) and (4.3) for each $n \in \mathbb{N}_{\geq 3}$, respectively. Consider the constant M_0 as in (4.4) and define $r_n = \frac{M_0}{\|c_1(P_n L P_n)\|}$. Then for each $n \in \mathbb{N}_{\geq 3}$ the operator $Y_n + D_n$ is minimal in $\mathcal{K}(\mathcal{H})^{ah}/\mathcal{D}(\mathcal{K}(\mathcal{H})^{ah})$, that is*

$$\|[Y_n]\| = \inf_{\tilde{D} \in \mathcal{D}(\mathcal{K}(\mathcal{H})^{ah})} \left\| Y_n + \tilde{D} \right\| = \|Y_n + D_n\| = M_0.$$

Proof. Fix $n \in \mathbb{N}_{\geq 3}$. Without loss of generality, we can consider $Y_n + D_n$ as an $n \times n$ matrix. Then

- $d_1^{(n)} = 0$;
- $\langle c_1(Y_n + D_n), c_j(Y_n + D_n) \rangle = 0$, for each $j \in \mathbb{N}$, $2 \leq j \leq n$;
- $\|c_1(Y_n + D_n)\| = r_n \|c_1(P_n L P_n)\| = M_0 \geq \left\| P_n Y^{[1]} P_n + D_n \right\|$.

As an $n \times n$ matrix, D_n is the unique minimal diagonal operator for Y_n (see Theorem 8 in [5]). Since

$$\inf_{D \in \mathcal{D}(\mathcal{K}(\mathcal{H})^{ah})} \|Y_n + D\| = \min_{\tilde{D} \in \mathcal{D}(M_n(\mathbb{C})^{ah})} \left\| Y_n + \tilde{D} \right\|,$$

follows that

$$\|[Y_n]\| = \|Y_n + D_n\|. \qquad \square$$

Observe that the norm of the minimal operator $Y_n + D_n$ is M_0 for every $n \in \mathbb{N}_{\geq 3}$.

Remark 12. For every $n \in \mathbb{N}_{\geq 3}$

$$\inf_{D \in \mathcal{D}(\mathcal{K}(\mathcal{H})^{ah})} \|Y_n + D\| = \min_{D' \in \mathcal{D}(M_n(\mathbb{C})^{ah})} \|Y_n + D'\| = \|Y_n + D_n\|,$$

but there is no uniqueness of the $D' \in \mathcal{D}(\mathcal{K}(\mathcal{H})^{ah})$ that attain the minimum. Moreover, every block operator of the form $C_n = \begin{pmatrix} D_n & 0 \\ 0 & D_c \end{pmatrix}$, with D_c diagonal and such that $\|D_c\| \leq \|c_1(Y_n)\|$ satisfies

$$\|Y_n + C_n\| = \max \{\|Y_n + D_n\|; \|D_c\|\} = \|Y_n + D_n\| = \|[Y_n]\|.$$

Reconsider the operator $Y_r = rL + Y^{[1]}$ fixing $r = \frac{M_0}{\|c_1(L)\|}$. Note that

$$\frac{\left\| Y^{[1]} + D_0 \right\|}{\|c_1(L)\|} \leq r < \infty$$

where the last inequality holds due to Lemma 10. Then, $Z_r = Y_r + D_0$ satisfies the hypothesis of Theorem 2 and is a minimal operator with D_0, the unique (non-compact) bounded diagonal operator such that

$$\|[Y_r]\| = \inf_{D \in \mathcal{D}(\mathcal{K}(\mathcal{H})^{ah})} \|Y_r + D\| = \|Z_r\|.$$

Moreover,

$$\|[Y_r]\| = \|c_1(Z_r)\| = \|c_1(Y_r)\| = M_0.$$

Therefore,

$$\|[Y_r]\| = \|[Y_n]\|\,, \text{ for all } n \in \mathbb{N}_{\geq 3}. \tag{4.5}$$

The following result relates Y_r with Y_n.

Proposition 13. *Let Y_r be the operator defined in (4.1) and $\{Y_n\}_{n=3}^{\infty}$ the family of finite rank operators defined in (4.2). If M_0 is the real constant defined in (4.4) such that $r = \frac{M_0}{\|c_1(L)\|}$ and $r_n = \frac{M_0}{\|c_1(P_n L P_n)\|}$ for each $n \in \mathbb{N}_{\geq 3}$, are fixed. Then*

1. $\lim_{n\to\infty} r_n = r$.
2. $Y_n \to Y_r$ *when $n \to \infty$ in the operator norm.*

Proof.

1. Since $\|c_1(P_nLP_n)\| = \left(\sum_{i=1}^{n-1} \gamma^{2i}\right)^{\frac{1}{2}}$ and $\|c_1(L)\| = \left(\sum_{i=1}^{\infty} \gamma^{2i}\right)^{\frac{1}{2}}$, follows that $\lim_{n\to\infty} r_n = r$.
2. $$\begin{aligned} \|Y_r - Y_n\| &= \left\|rL + Y^{[1]} - r_n P_n L P_n - P_n Y^{[1]} P_n\right\| \\ &\leq \|rL \pm r_n L - r_n P_n L P_n\| + \left\|Y^{[1]} - P_n Y^{[1]} P_n\right\| \\ &\leq |r - r_n|\,\|L\| + |r_n|\,\|L - P_n L P_n\| + \left\|Y^{[1]} - P_n Y^{[1]} P_n\right\| \to 0 \end{aligned}$$

when $n \to \infty$, since L and $Y^{[1]}$ are Hilbert–Schmidt operators and $r_n \to r$. □

Observe that the numerical sequence $\{d_k^{(n)}\}_{n\in\mathbb{N}_{\geq 3}}$ defined in (4.3) converges to d_k when $n \to \infty$, for each $k \in \mathbb{N}$

$$\begin{cases} d_2^{(n)} \searrow -\sum_{j=1}^{\infty} \gamma^{2j} = -\frac{\gamma^2}{1-\gamma^2} = d_2, \\ d_k^{(n)} \searrow -\sum_{j=0}^{k-3} \gamma^j - \sum_{j=0}^{\infty} \gamma^{k+2j} = \frac{1-\gamma^{k-2}}{\gamma-1} - \frac{\gamma^k}{1-\gamma^2} = d_k, \forall k \geq 3. \end{cases}$$

As a consequence, the sequence of diagonal operators $\{D_n\}_{n\in\mathbb{N}_{\geq 3}}$ converges SOT to the unique best approximant (non-compact) diagonal $D_0 \in \mathcal{D}(\mathcal{B}(\mathcal{H}))$ for Y_r.

Proposition 14. *Let Y_r be the operator defined in (3.3) and D_0 the unique bounded diagonal operator such that $\|[Y_r]\|_{\mathcal{K}(\mathcal{H})^{ah}/\mathcal{D}(\mathcal{K}(\mathcal{H})^{ah})} = \|Y_r + D_0\|$. Let $\{D_n\}_{n\in\mathbb{N}_{\geq 3}}$ be the sequence of finite rank diagonal operators defined in (4.3). Then*

$$D_n \to D_0 \text{ SOT when } n \to \infty.$$

Proof. $\{D_n - D_0\}_{n\in\mathbb{N}_{\geq 3}}$ is a bounded family of $\mathcal{B}(\mathcal{H})$ and

$$(D_n - D_0)\,(e_k) = d_k^{(n)} - d_k \to 0$$

when $n \to \infty$ for every e_k that belongs to the fixed orthonormal basis. Then standard arguments of operator theory imply that $D_n \to D_0$ SOT when $n \to \infty$ (see [3]). □

Observe that Propositions 13 and 14 imply that $\lim_{n\to\infty} Y_n + D_n = Z_r$ SOT. Since $D_n \in \mathcal{K}(\mathcal{H})^{ah}$ for all $n \in \mathbb{N}_{\geq 3}$ and $D_0 \notin \mathcal{D}(\mathcal{K}(\mathcal{H})^{ah})$, the convergence cannot be in the operator norm. To establish the second main result of this section we prove first the convergence in the operator norm of $Y_n + D_n + \alpha I$ to $Z_r + \alpha I$, for a particular $\alpha \in \mathbb{R}$.

Proposition 15. *Let $Y_r, D_0, \{Y_n\}_{n\in\mathbb{N}_{\geq 3}}, \{D_n\}_{n\in\mathbb{N}_{\geq 3}}, \{P_n\}_{n\in\mathbb{N}_{\geq 3}}$ be the operators and sequence of operators defined previously in (4.1), (4.2) and (4.3). Then*

$$Y_n + D_n + \frac{i}{1-\gamma} P_n \to Y_r + D_0 + \frac{i}{1-\gamma} I,$$

in the operator norm when $n \to \infty$.

Proof. Let $\epsilon > 0$, then

$$\begin{aligned}&\left\| Y_r + D_0 + \frac{i}{1-\gamma} I - Y_n - D_n - \frac{i}{1-\gamma} P_n \right\| \\ &\leq \|Y_r - Y_n\| + \left\| D_0 + \frac{i}{1-\gamma} I - D_n - \frac{i}{1-\gamma} P_n \right\|.\end{aligned}$$

By Proposition 13, there exists $n_1 \in \mathbb{N}$ such that $\|Y_r - Y_n\| < \epsilon$, for all $n \geq n_1$. Focus on the second term. For each $n \in \mathbb{N}_{\geq 3}$

$$\begin{aligned}\left\| D_0 + \frac{i}{1-\gamma} I - D_n - \frac{i}{1-\gamma} P_n \right\| &= \sup_{k\in\mathbb{N}} \left| d_k + \frac{1}{1-\gamma} - d_k^{(n)} - \left(\frac{1}{1-\gamma} P_n\right)_{kk} \right| \\ &= \max\left\{ \max_{1\leq k\leq n} \left| \sum_{j=n}^{\infty} \gamma^{2j-k} \right| ; \sup_{k>n} \left| d_k + \frac{1}{1-\gamma} \right| \right\}.\end{aligned}$$

By Proposition 14, $\max_{1\leq k\leq n} \left|\sum_{j=n}^{\infty} \gamma^{2j-k}\right|$ and $\sup_{k>n} \left| d_k + \frac{1}{1-\gamma}\right|$ converge to 0 when $n \to \infty$. Then, there exists $n_2 \in \mathbb{N}$ such that for each $n \geq n_0$

$$\max\left\{ \max_{1\leq k\leq n} \left| \sum_{j=n}^{\infty} \gamma^{2j-k} \right| ; \sup_{k>n} \left| d_k + \frac{1}{1-\gamma} \right| \right\} < \epsilon.$$

Finally, if $n_0 = \max\{n_1; n_2\}$ follows that

$$n \geq n_0 \Rightarrow \left\| Y_r + D_0 + \frac{i}{1-\gamma} I - Y_n - D_n - \frac{i}{1-\gamma} P_n \right\| < 2\epsilon,$$

which means that $Y_n + D_n + \frac{i}{1-\gamma} P_n$ converges to $Y_r + D_0 + \frac{i}{1-\gamma} I$ when $n \to \infty$ in the operator norm. □

In the above proof we also obtained that $\{D_n + \frac{i}{1-\gamma} P_n\}_{n\in\mathbb{N}_{\geq 3}}$, which is a sequence of finite rank operators, converges in the operator norm to $D_0 + \frac{i}{1-\gamma} I \in \mathcal{D}(\mathcal{K}(\mathcal{H})^{ah})$. Even though $Y_n + D_n + \frac{i}{1-\gamma} P_n$ and $Y_r + D_0 + \frac{i}{1-\gamma} I$ are not minimal operators, they are useful to construct minimal length curves in the unitary orbit of A. We will use the operators $Y_n + D_n + \frac{i}{1-\gamma} P_n$ to construct a sequence of minimal length curves that converge to β defined in (3.4).

The first result in this direction is the convergence of the sequence of exponential curves in $\mathcal{O}_A$.

Proposition 16. *Let $b \in \mathcal{O}_A$ and $\varsigma_n(t) = e^{tZ_n} b e^{-tZ_n}$, a sequence of curves in $\mathcal{O}_A$ with $t \in \mathbb{R}$ and $\{Z_n\}_{n\in\mathbb{N}} \subset \mathcal{K}(\mathcal{H})^{ah}$ such that $\|Z_n - Z\| \to 0$ when $n \to \infty$. If we define $\varsigma(t) = e^{tZ} b e^{-tZ}$, then*

$$\varsigma_n \to \varsigma$$

uniformly in the operator norm when $n \to \infty$ for any interval $[t_1, t_2] \subset \mathbb{R}$.

Proof. Let $\epsilon > 0$.

$$\begin{aligned}\|\varsigma_n(t) - \varsigma(t)\| &\le \left\|e^{tZ_n} b e^{-tZ_n} - e^{tZ} b e^{-tZ_n}\right\| + \left\|e^{tZ} b e^{-tZ_n} - e^{tZ} b e^{-tZ}\right\| \\ &\le \left\|\left(e^{tZ_n} - e^{tZ}\right) b e^{-tZ_n}\right\| + \left\|e^{tZ} b \left(e^{-tZ_n} - e^{-tZ}\right)\right\| \\ &\le \left(\left\|e^{tZ_n} - e^{tZ}\right\| + \left\|e^{-tZ_n} - e^{-tZ}\right\|\right) \|b\| .\end{aligned}$$

It is known that the exponential map $exp : \mathcal{K}(\mathcal{H})^{ah} \to \mathcal{U}_c(\mathcal{H})$ is Lipschitz continuous in compact sets of $\mathcal{K}(\mathcal{H})$, then there exists $n_0 \in \mathbb{N}$ such that

$$\text{for all } n \ge n_0 \Rightarrow \begin{cases} \left\|e^{tZ_n} - e^{tZ}\right\| < \frac{\epsilon}{\|b\|} \\ \left\|e^{-tZ_n} - e^{-tZ}\right\| < \frac{\epsilon}{\|b\|}, \end{cases}$$

for each t in a closed interval $[t_1, t_2] \subset \mathbb{R}$. Therefore

$$\|\varsigma_n(t) - \varsigma(t)\| < \epsilon$$

for each $n \ge n_0$ and $t \in [t_1, t_2]$, which implies that $\varsigma_n \to \varsigma$ uniformly in the operator norm in that interval. □

If we consider the sequence $\{Y_n + D_n + \frac{i}{1-\gamma} P_n\}_{n\in Ngt}$ and use Proposition 15 then

$$Y_n + D_n + \frac{i}{1-\gamma} P_n \to Y_r + D_0 + \frac{i}{1-\gamma} I$$

in the operator norm when $n \to \infty$. Define for each $n \in \mathbb{N}_{\ge 3}$ and $t_0 \in \mathbb{R}$ the curves parametrized by

$$\beta_n(t) = e^{t(Y_n + D_n + \frac{i}{1-\gamma} P_n)} b e^{-t(Y_n + D_n + \frac{i}{1-\gamma} P_n)},\ t \in [0, t_0]. \tag{4.6}$$

Observe that these can be considered as matricial type curves.

Theorem 17. *Let A and $b \in \mathcal{O}_A$ as in Theorem 8. Let $\{\beta_n\}_{n\in\mathbb{N}_{\ge 3}}$ be the sequence of curves defined in (4.6), and β be the curve defined in (3.4). Then, for each $n \in \mathbb{N}_{\ge 3}$*

1. $\begin{cases} \beta_n(0) = b \\ \beta_n'(0) = Y_n b - b Y_n \in (T\mathcal{O}_A)_b. \end{cases}$.
2. $\beta_n(t) = e^{t(Y_n + D_n)} b e^{-t(Y_n + D_n)}$ *for all t, since $\frac{i}{1-\gamma} P_n$ commutes with $Y_n + D_n$.*
3. *For each $t_0 \in \left[-\frac{\pi}{2\|[Y_n]\|}, \frac{\pi}{2\|[Y_n]\|}\right] = \left[-\frac{\pi}{2M_0}, \frac{\pi}{2M_0}\right]$ holds that*

$$\mathrm{L}\left(\beta_n|_{[0,t_0]}\right) = |t_0|\, \|[Y_n]\| = |t_0|\, M_0 = \mathrm{L}\left(\beta|_{[0,t_0]}\right).$$

4. $\beta_n : [0, t_0] \to \mathcal{O}_A$ *with* $t_0 \in \left[-\frac{\pi}{2M_0}, \frac{\pi}{2M_0}\right]$ *is a minimal length curve in* $\mathcal{O}_A$.
5. $\beta'_n(0) \to \beta'(0)$ *in the norm* $\|.\|_b$ *of* $(T\mathcal{O}_A)_b$.

Moreover, by Proposition 16, $\beta_n \to \beta$ *uniformly in the operator norm in the interval* $\left[-\frac{\pi}{2M_0}, \frac{\pi}{2M_0}\right]$.

Proof. The proof of items (1), (2), (3) is analogous to the proof in Theorem 8. The equality $\|[Y_n]\| = M_0 = \|[Y_r]\|$ is due to Proposition 11.

Since for each $n \in \mathbb{N}_{\geq 3}$ fixed $Y_n + D_n$ is a minimal compact operator, by Theorem I in [4] β_n is a minimal length curve between all curves in $\mathcal{O}_A$ joining $\beta_n(0) = b$ and $\beta_n(t)$ with $|t| \leq \frac{\pi}{2\|Y_n + D_n\|}$. Then (4) is proved.

We proceed to prove (5): fix $\epsilon > 0$. Then there exists $n_0 \in \mathbb{N}$ such that if $n \geq n_0$ then $\|Y_n - Y_r\| < \epsilon$. Therefore,

$$\begin{aligned}\|\beta'_n(0) - \beta'(0)\|_b &= \inf\left\{\|Z\| : Z \in \mathcal{K}(\mathcal{H})^{ah},\ [Z, b] = (Y_n - Y_r)\, b - b\,(Y_n - Y_r)\right\} \\ &= \inf_{D \in \mathcal{D}(\mathcal{K}(\mathcal{H})^{ah})} \|Y_n - Y_r + D\| \leq \|Y_n - Y_r\| < \epsilon\end{aligned}$$

for each $n \geq n_0$. Then $\|\beta'_n(0) - \beta'(0)\|_b \to 0$ when $n \to \infty$. □

Therefore, we obtained a minimal length curve $\beta \subset \mathcal{O}_A$ that can be uniformly approximated by minimal curves of matrices $\{\beta_n\}$. Nevertheless, β does not have a minimal compact lifting, although each β_n has at least one minimal matricial lifting.

5. Bounded minimal operators $Z + D$ with $Z \in \mathcal{K}(\mathcal{H})$ and non-compact diagonal D

Let Y_r, D_0 be the operators defined in (3.3). To establish the equality $\beta(t) = e^{Y_r + D_0 + \frac{i}{1-\gamma} I}\, b\, e^{-(Y_r + D_0 + \frac{i}{1-\gamma} I)}$ in Theorem 8 the following properties were essential:

1. $D_0 + \frac{i}{1-\gamma} I \in \mathcal{D}(\mathcal{K}(\mathcal{H})^{ah})$ and
2. $\frac{i}{1-\gamma} I$ commutes with Z_r and b but $\frac{i}{1-\gamma} I \notin \mathcal{K}(\mathcal{H})$.

This can be generalized.

Proposition 18. *Let* $Z \in \mathcal{K}(\mathcal{H})^{ah}$ *and suppose that there exists* $D_1 \in \mathcal{D}(\mathcal{B}(\mathcal{H})^{ah})$ *such that*

$$\|[Z]\|_{\mathcal{K}(\mathcal{H})^{ah}/\mathcal{D}(\mathcal{K}(\mathcal{H})^{ah})} = \|Z + D_1\|$$

and D_1 *is not compact. If there exists* $\lambda \in i\mathbb{R}$ *such that* $\lim_{j\to\infty} (D_1)_{jj} = \lambda$, *then the curve*

$$\chi(t) = e^{t(Z + D_1 - \lambda I)}\, b\, e^{-t(Z + D_1 - \lambda I)}$$

has minimal length between all the smooth curves in $\mathcal{O}_A$ *joining* b *with* $\chi(t_0)$, *for* $t_0 \in \left[-\frac{\pi}{2\|Z\|}, \frac{\pi}{2\|[Z]\|}\right]$.

Proof. First observe that $Re\,((D_1)_{jj}) = 0$ for each $j \in \mathbb{N}$, since $D_1 \in \mathcal{D}(\mathcal{B}(\mathcal{H})^{ah})$. Then,

$$\lim_{j\to\infty} (D_1)_{jj} = \lambda$$

and $\lambda \neq 0$ since D_1 is not compact. Therefore, using functional calculus and Proposition 6 in [2]

$$\|Z + D_1 - \lambda I\| = \max\{|-\|[Z]\| - |\lambda||\,;\, \|[Z]\| - |\lambda|\} > \|[Z]\|\,.$$

Also $D_1 - \lambda I \in \mathcal{D}(\mathcal{K}(\mathcal{H})^{ah})$, since $|(D_1 - \lambda I)_{jj}| = |(D_1)_{jj} - \lambda| \to 0$, when $j \to \infty$. Then, $Z + D_1 - \lambda I$ is not minimal in $\mathcal{K}(\mathcal{H})^{ah}/\mathcal{D}(\mathcal{K}(\mathcal{H})^{ah})$ but the curve parameterized by

$$\chi(t) = e^{t(Z+D_1-\lambda I)} b e^{-t(Z+D_1-\lambda I)} \in \mathcal{O}_A$$

has minimal length, as χ is equal to the curve $\delta(t) = e^{t(Z+D_1)} b e^{-t(Z+D_1)}$, which has minimal length in the homogeneous space $\{uAu^* : u \in \mathcal{U}(\mathcal{H})\}$ (Theorem II in [4]). Therefore χ has minimal length in $\mathcal{O}_A$. $\square$

Given $Z \in \mathcal{K}(\mathcal{H})^{ah}$, it is not true that every diagonal operator D_1 such that $Z + D_1$ is minimal fulfills the condition

$$\exists \lambda \in i\mathbb{R} \text{ such that } \lim_{j\to\infty} (D_1)_{jj} = \lambda.$$

Indeed, consider the following operator

$$Z_0 = i \begin{pmatrix} 0 & -\delta & \gamma & -\delta^2 & \gamma^2 & -\delta^3 & \gamma^3 & \cdots \\ -\delta & 0 & \gamma & -\delta^2 & \gamma^2 & -\delta^3 & \gamma^3 & \cdots \\ \gamma & \gamma & 0 & -\delta^2 & \gamma^2 & -\delta^3 & \gamma^3 & \cdots \\ -\delta^2 & -\delta^2 & -\delta^2 & 0 & \gamma^2 & -\delta^3 & \gamma^3 & \cdots \\ \gamma^2 & \gamma^2 & \gamma^2 & \gamma^2 & 0 & -\delta^3 & \gamma^3 & \cdots \\ -\delta^3 & -\delta^3 & -\delta^3 & -\delta^3 & -\delta^3 & 0 & \gamma^3 & \cdots \\ \gamma^3 & \gamma^3 & \gamma^3 & \gamma^3 & \gamma^3 & \gamma^3 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \text{ with } \gamma, \delta \in (0,1). \tag{5.1}$$

It is easy to prove that Z_0 is a Hilbert Schmidt operator.

Let $D_0' = i\mathrm{Diag}\left(\{d_n'\}_{n\in\mathbb{N}}\right)$ the unique bounded diagonal operator such that

$$\langle c_1(Z_0), c_n(Z_0 + D_0')\rangle = 0, \ \forall \ n \in \mathbb{N}. \tag{5.2}$$

Simple calculations show that the condition (5.2) implies that $\{d_n'\}_{n\in\mathbb{N}}$ satisfies the following:

- $d_1' = 0$
- $d_2' = \frac{\delta^3}{1-\delta^2} + \left(\frac{\gamma^2}{\delta}\right)\frac{1}{1-\gamma^2}$ and for every even, with $k \in \mathbb{N}$, $k > 1$

$$d_{2k}' = \left(\sum_{j=1}^{k-1} \delta^j\right) - \left(\sum_{j=1}^{k-1} \gamma^j\right) + \frac{\delta^{k+2}}{1-\delta^2} + \left(\frac{\gamma^2}{\delta}\right)^k \frac{1}{1-\gamma^2}.$$

- $d_3' = \delta - \frac{\gamma^3}{1-\gamma^2} - \left(\frac{\delta^4}{\gamma}\right)\frac{1}{1-\delta^2}$ and for every odd, with $k \in \mathbb{N}$, $k > 1$

$$d_{2k-1}' = \left(\sum_{j=1}^{k-1} \delta^j\right) - \left(\sum_{j=1}^{k-2} \gamma^j\right) - \frac{\gamma^{k+1}}{1-\gamma^2} - \left(\frac{\delta^2}{\gamma}\right)^k \frac{\gamma}{1-\delta^2}.$$

If $\gamma^2 \le \delta$ and $\delta^2 \le \gamma$ both sequences, $\{d_{2k}'\}_{k\in\mathbb{N}}$ and $\{d_{2k-1}'\}_{k\in\mathbb{N}}$, are convergent.

If $Z_0^{[1]}$ is the operator Z_0 defined in (5.1) but with zeros in its first column and row and $r = \frac{\left\|Z_0^{[1]}+D_0'\right\|}{c_1(Z_0)}$ then $r(Z_0 - Z_0^{[1]}) + Z_0^{[1]} + D_0'$ is a minimal operator by Theorem 2 and D_0' is the unique bounded minimal diagonal operator for $r(Z_0 - Z_0^{[1]}) + Z_0^{[1]}$. Also, if we fix the conditions $\gamma^2 = \delta$ and $\delta^2 < \gamma$ then

$$\lim_{k\to\infty} d'_{2k} = \frac{\delta}{1-\delta} - \frac{\gamma}{1-\gamma} + \frac{1}{1-\gamma^2} \text{ and } \lim_{k\to\infty} d'_{2k-1} = \frac{\delta}{1-\delta} - \frac{\gamma}{1-\gamma},$$

which implies that $\{(D'_0)_{nn}\}_{n\in N}$ has no limit. We call these diagonals "oscillant" in the sense that the sequence $\{\langle De_n, e_n\rangle\}_{n\in\mathbb{N}}$ has at least two different limits.

Observe that an approximation to Z_0 by matrices can be built as the one done in section 4. Consider for each $n \in \mathbb{N}_{\geq 5} = \{n \in \mathbb{N} : n \geq 5\}$ the orthogonal projection P_n and define the following finite rank operators

$$Z_n = r_n P_n(Z_0 - Z_0^{[1]})P_n + P_n Z_0^{[1]} P_n, \tag{5.3}$$

with $r_n \in \mathbb{R}_{>0}$ for each $n \in \mathbb{N}_{\geq 5}$. For each $n \in \mathbb{N}_{\geq 5}$ we define a diagonal operator $D'_n = i\mathrm{Diag}\left(\{d_l^{(n)'}\}_{l\in\mathbb{N}}\right)$ uniquely determined as

1. $d_1^{(n)'} = 0$;
2. $\langle c_1(Z_n + D'_n), c_j(Z_n + D'_n)\rangle = 0$, for each $j \in \mathbb{N}$, $j \neq 1$;
3. $d_l^{(n)'} = 0$, for every $l > n$.

Then, $d_l^{(n)'}$ is determined for every $n \in \mathbb{N}_{\geq 5}$ as follows.

- If $l \leq n$ and l is even, $l = 2k$, then

$$d_2^{(n)'} = \left(\sum_{j=1}^{\lfloor\frac{n-2}{2}\rfloor} \delta^{2j+1}\right) + \frac{\sum_{j=1}^{\lfloor\frac{n-1}{2}\rfloor} \gamma^{2j}}{\delta},$$

$$\begin{aligned} d_{2k}^{(n)'} &= \left(\sum_{j=1}^{k-1} \delta^j\right) - \left(\sum_{j=1}^{k-1} \gamma^j\right) + \left(\sum_{j=1}^{\lfloor\frac{n-2k}{2}\rfloor} \delta^{2j+k}\right) + \\ &+ \frac{\sum_{j=k}^{\lfloor\frac{n-1}{2}\rfloor} \gamma^{2j}}{\delta^k} \quad, \text{ if } 1 < k \leq \frac{n-1}{2} \end{aligned} \tag{5.4}$$

  If n is odd and $k = \frac{n+1}{2}$ then $d_{2k}^{(n)'} = d_{n-1}^{(n)'}$ is obtained by the above formula (5.4) without its third term. If n is even and $k = \frac{n}{2}$ then $d_{2k}^{(n)'} = d_n^{(n)'}$ is obtained by the above formula (5.4) without its third and fourth terms.
- If $l \leq n$ and l is odd, $l = 2k - 1$, then

$$d_3^{(n)'} = \delta - \left(\sum_{j=2}^{\lfloor\frac{n-1}{2}\rfloor} \gamma^{2j-1}\right) - \frac{\sum_{j=2}^{\lfloor\frac{n}{2}\rfloor} \delta^{2j}}{\gamma},$$

$$\begin{aligned} d_{2k-1}^{(n)'} &= \left(\sum_{j=1}^{k-1} \delta^j\right) - \left(\sum_{j=1}^{k-2} \gamma^j\right) - \left(\sum_{j=k}^{\lfloor\frac{n-1}{2}\rfloor} \gamma^{2j-k+1}\right) - \\ &- \frac{\sum_{j=k}^{\lfloor\frac{n}{2}\rfloor} \delta^{2j}}{\gamma^{k-1}}, \text{ if } \ 2 < k \leq \frac{n-1}{2} \end{aligned} \tag{5.5}$$

  If n is odd and $k = \frac{n+1}{2}$ then $d_{2k-1}^{(n)'} = d_n^{(n)'}$ is obtained by the above formula (5.5) without its third and fourth terms. If n is even and $k = \frac{n}{2}$ then $d_{2k-1}^{(n)'} = d_{n-1}^{(n)'}$ is obtained by the above formula (5.5) without its third term.

- If $l > n$ then

$$d_l^{(n)'} = 0 \tag{5.6}$$

The proof is by induction over the indices k for every $n \in \mathbb{N}_{\geq 5}$. Observe the following:

1. The choice of each $d_k^{(n)'}$ is independent from the parameter r_n.
2. $\lim_{n\to\infty} d_2^{(n)'} = \left(\sum_{j=1}^{\infty} \delta^{2j+1}\right) + \frac{\sum_{j=1}^{\infty} \gamma^{2j}}{\delta} = d_2'$.
3. $\lim_{n\to\infty} d_{2k}^{(n)'} = \left(\sum_{j=1}^{k-1} \delta^j\right) - \left(\sum_{j=1}^{k-1} \gamma^j\right) + \left(\sum_{j=1}^{\infty} \delta^{2j+k}\right) + \frac{\sum_{j=k}^{\infty} \gamma^{2j}}{\delta^k} = d_{2k}'$.
4. $\lim_{n\to\infty} d_3^{(n)'} = -\left(\sum_{j=2}^{\infty} \gamma^{2j-1}\right) - \frac{\sum_{j=2}^{\infty} \delta^{2j}}{\gamma} = d_3'$.
5. $\lim_{n\to\infty} d_{2k-1}^{(n)'} = \left(\sum_{j=1}^{k-1} \delta^j\right) - \left(\sum_{j=1}^{k-2} \gamma^j\right) - \left(\sum_{j=k}^{\infty} \gamma^{2j-k+1}\right) - \frac{\sum_{j=k}^{\infty} \delta^{2j}}{\gamma^{k-1}} = d_{2k-1}'$.
6. For every $k \in \mathbb{N}$ and for each $n \in \mathbb{N}_{\geq 5}$:

$$d_{2k-1}' \leq d_{2k-1}^{(n)'} \leq d_{2k}^{(n)'} \leq d_{2k}'.$$

   Then, $\|D_0'\| = \sup_{k\in\mathbb{N}}\{\left|d_{2k-1}'\right|; |d_{2k}'|\} \geq \|D_n'\|$.
7. $D_n' \to D_0'$ SOT, since $\mathrm{Diag}\left(\{d_{2k}^{(n)'}\}_{k\in\mathbb{N}}\right) \to \mathrm{Diag}\left(\{d_{2k}'\}_{k\in\mathbb{N}}\right)$ SOT and $\mathrm{Diag}\left(\{d_{2k-1}^{(n)'}\}_{k\in\mathbb{N}}\right) \to \mathrm{Diag}\left(\{d_{2k-1}'\}_{k\in\mathbb{N}}\right)$ SOT.

With the previous properties, there exists $M_1 \in \mathbb{R}_{>0}$ such that:

$$M_1 = \max\left\{\sup_{n\in\mathbb{N}} \left\|P_n Z_0^{[1]} P_n + D_n'\right\|, \left\|Z_0^{[1]} + D_0'\right\|\right\}. \tag{5.7}$$

For any injective $\sigma : \mathbb{N} \to \mathbb{N}$ define the projection

$$P^\sigma = \sum_{k\in\mathbb{N}} e_{\sigma(k)} \otimes e_{\sigma(k)}. \tag{5.8}$$

Thus, the following result is a direct consequence of all previous remarks.

Theorem 19. *Let Z_0, D_0', $Z_n = r_n P_n(Z_0 - Z_0^{[1]})P_n + P_n Z_0^{[1]} P_n$ and D_n' be the operators defined in (5.1), (5.2), (5.3), (5.4), (5.5) and (5.6) for each $n \in \mathbb{N}_{\geq 5}$, respectively. Consider the real constant M_1 as in (5.7) and define $r_n = \frac{M_1}{\left\|c_1\left(P_n(Z_0 - Z_0^{[1]})P_n\right)\right\|}$ for each $n \in \mathbb{N}$ and $r = \frac{M_1}{\left\|c_1\left(Z_0 - Z_0^{[1]}\right)\right\|}$. If*

$$\lambda = \lim_{n\to\infty} d_{2k}', \ \mu = \lim_{n\to\infty} d_{2k-1}',$$

then

1. *$Z_n + D_n'$ is minimal in $\mathcal{K}(\mathcal{H})^{ah}/\mathcal{D}(\mathcal{K}(\mathcal{H})^{ah})$ and*

$$\|[Z_n]\| = \inf_{D\in\mathcal{D}(\mathcal{K}(\mathcal{H})^{ah})} \|Z_n + D\| = \|Z_n + D_n'\| = M_1.$$

2. *If P^{σ_1} and P^{σ_2} are the projections defined in (5.8) for $\sigma_1(k) = 2k$ and $\sigma_2(k) = 2k - 1$, respectively, then*

$$Z_n + D'_n - \lambda P_n P^{\sigma_1} P_n - \mu P_n P^{\sigma_2} P_n \to r(Z_0 - Z_0^{[1]}) + Z_0^{[1]} + D'_0 - \lambda P^{\sigma_1} - \mu P^{\sigma_2}$$

in the operator norm when $n \to \infty$.

Proof.

1. Observe that if D'_n is determined as in (5.4), (5.5) and (5.6) the operator $-i(Z_n + D'_n)$ fulfills the conditions stated in Theorem 2 and

$$\inf_{D \in \mathcal{D}(\mathcal{K}(\mathcal{H})^{ah})} \|Z_n + D\| = \|Z_n + D'_n\| = \|c_1(Z_n + D'_n)\| = M_1.$$

2. Let $\epsilon > 0$. Since Z_0 is compact and $r_n \to r$ then there exists $n_1 \in \mathbb{N}$ such that

$$\left\| Z_n - r(Z_0 - Z_0^{[1]}) + Z_0^{[1]} \right\| < \frac{\epsilon}{2},$$

for each $n \geq n_1$. Similarly than in the case of diagonal with one limit point (see proof of Proposition 15), for each $n \in \mathbb{N}_{\geq 5}$:

$$\begin{aligned} &\|D'_n - \lambda P_n P^{\sigma_1} P_n - \mu P_n P^{\sigma_2} P_n - D'_0 + \lambda P^{\sigma_1} + \mu P^{\sigma_2}\| \\ &= \sup_{l \in \mathbb{N}} \left| d'_l - \lambda \left(P_n P^{\sigma_1} P_n\right)_{ll} - \mu \left(P_n P^{\sigma_2} P_n\right)_{ll} - d_l^{(n)'} - \lambda \left(P^{\sigma_1}\right)_{ll} - \mu \left(P^{\sigma_2}\right)_{ll} \right| \\ &= \max \left\{ \max_{1 \leq l \leq n} \left| d'_l - d_l^{(n)'} \right| ; \sup_{k > n} |d'_{2k} - \lambda| ; \sup_{k > n} \left| d'_{2k-1} - \mu \right| \right\}. \end{aligned} \tag{5.9}$$

Since $\lim_{n\to\infty} d_{2k}^{(n)'} = d'_{2k}$, $\lim_{n\to\infty} d_{2k-1}^{(n)'} = d'_{2k-1}$, $\lim_{n\to\infty} d'_{2k} = \lambda$ and $\lim_{n\to\infty} d'_{2k-1} = \mu$, there exists $n_2 \in \mathbb{N}$ such that the last expression is smaller than $\frac{\epsilon}{2}$ for every $n \geq n_2$. Therefore, it holds that

$$\begin{aligned} &\|Z_n + D'_n - \lambda P_n P^{\sigma_1} P_n - \mu P_n P^{\sigma_2} P_n \\ &\quad - \left[r(Z_0 - Z_0^{[1]}) + Z_0^{[1]} + D'_0 - \lambda P^{\sigma_1} - \mu P^{\sigma_2} \right] \Big\| \\ &\leq \left\| Z_n - r(Z_0 - Z_0^{[1]}) + Z_0^{[1]} \right\| \\ &\quad + \|D'_n - \lambda P_n P^{\sigma_1} P_n - \mu P_n P^{\sigma_2} P_n - D'_0 + \lambda P^{\sigma_1} + \mu P^{\sigma_2}\| < \epsilon \end{aligned}$$

for every $n \geq \max\{n_1; n_2\}$. □

Remark 20. As $r(Z_0 - Z_0^{[1]}) + Z_0^{[1]}$, with Z_0 and r defined previously, there exist other compact operators such that its best bounded diagonal approximant oscillates. Moreover, there exist examples of minimal bounded operators in which the elements on the main diagonal are the union of m subsequences ($m \in \mathbb{N}$) such that each one converges to a different limit. For those m-oscillant operators an analogous result as that of Theorem 19 can be obtained with essentially the same arguments. Nevertheless, the techniques used in Theorems 8 and 17 to prove that the curves constructed in (3.4) and (4.6) belong to $\mathcal{O}_A$ cannot be adapted to the case of an oscillant minimal diagonal for a compact $Z \in \mathcal{K}(\mathcal{H})^{ah}$.

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References

[1] E. Andruchow, G. Larotonda, The rectifiable distance in the unitary Fredholm group, Stud. Math. 196 (2) (2010) 151–178.
[2] T. Bottazzi, A. Varela, Best approximation by diagonal compact operators, Linear Algebra Appl. 439 (2013) 3044–3056.
[3] J.B. Conway, A Course in Operator Theory, AMS, 2000.
[4] C. Durán, L. Mata-Lorenzo, L. Recht, Metric geometry in homogeneous spaces of the unitary group of a C^*-algebra. I. Minimal curves, Adv. Math. 184 (2) (2004) 342–366.
[5] A. Kloubouk, A. Varela, Minimal 3×3 Hermitian Matrices, Publicaciones previas del Instituto Argentino de Matemática, vol. 455, 2012.