M3AS MATHEMATICAL MODELS & METHODS IN APPLIED SCIENCES

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May 8, 2014

Dr. Juan Pablo Agnelli
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Dear Dr. Agnelli,

I am pleased to inform you that your paper, entitled

A Kinetic Theory Approach to the Dynamics of Crowd Evacuation from Bounded Domains

by J.P. Agnelli, F. Colasuonno, and D. Knopoff

has been accepted for publication in "M^{3}AS: Mathematical Models and Methods in Applied Sciences". The paper is scheduled approximately on issue **01(25)2015**

Please, send me the files .tex and .eps (figures) of the paper to the following address: `nicola.bellomo@polito.it`

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Thank you for the collaboration to M^{3}AS!

Sincerely,

Nicola Bellomo

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Mathematical Models and Methods in Applied Sciences

A KINETIC THEORY APPROACH TO THE DYNAMICS OF CROWD EVACUATION FROM BOUNDED DOMAINS*

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A mathematical model of the evacuation of a crowd from bounded domains is derived by a hybrid approach with kinetic and macro features. Interactions at the micro-scale, which modify the velocity direction, are modeled by using tools of game theory and are transferred to the dynamics of collective behaviors. The velocity modulus is assumed to depend on the local density. The modeling approach considers dynamics caused by interactions of pedestrians not only with all the other pedestrians, but also with the geometry of the domain, such as walls and exits. Interactions with the boundary of the domain are non-local and described by games. Numerical simulations are developed to study evacuation time depending on the size of the exit zone, on the initial distribution of the crowd and on a parameter which weighs the unconscious attraction of the stream and the search for less crowded walking directions.

Keywords: Crowd dynamics, complex systems, hybrid model, nonlinear interactions, stochastic games.
AMS Subject Classification: 35Q91, 65M06, 90C90, 91A80.

1. Introduction

In recent decades, interest in the modeling of crowd dynamics has strongly increased due to the large number of applications in engineering and social sciences. The challenging analytical and computational problems that arise from these models have also attracted the attention of many applied mathematicians. Modeling human crowds has to take into account the complexity of living systems, such as the ability of pedestrians to have a strategy and to make decisions accordingly. Most of the

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literature on crowd modeling refers to the micro- and macro-scales. Regarding the microscopic approach it is worth mentioning the social force model [27,25], while some important contributions on macroscopic models are given in [29], in which the crowd is treated as a "thinking fluid", and in [33,34] where the approach of time-evolving measures is introduced. The link between the micro- and the macro-scale is studied in paper [15], while [24] offers an interesting comparison between them. In the middle of these two scales, the kinetic theory approach was initiated in [2] and subsequently developed in [3] for a large system of individuals subdivided into groups moving towards different directions. In the recent paper [17], a hierarchy of kinetic and macroscopic models is derived from a heuristic description at the micro-scale.

The model proposed in [3], valid in unbounded domains, can also take into account a heterogeneous distribution of pedestrians' movement ability, but not their interactions with walls or evacuation from exit doors. It is well understood that the presence of obstacles and walls can modify dynamics in a significant manner. Moreover, it can also produce dangerous situations as shown in the simulations presented in [14] concerning the dynamics on the Jamarat bridge.

In the present paper we propose a new approach suitable to model the dynamics in a domain with boundaries which takes into account interactions with walls and flow through exit doors. Modeling the interactions of pedestrians with walls can be viewed as a statement of boundary conditions for kinetic equations. This topic presents some additional conceptual difficulties with respect to those known in the theory of classical particles [11], where it is possible to implement suitable reflection rules. In fact, individuals feel the presence of walls at a distance and modify their dynamics in order to avoid them. These specific features are taken into account in the present paper, bearing in mind recent studies on non-local interactions for hyperbolic systems [21].

In our model, pedestrians are viewed as active particles, whose micro-states are defined by position and velocity direction. As in [3], the overall state of the system is delivered by a distribution function over the micro-state; equations which model the time and space dynamics of this distribution function are obtained by a balance of particles in an elementary volume of the space of micro-states. The net flow into this volume is due to transport and interactions. To model interaction dynamics, we assume that pedestrians modify their walking direction by a decision strategy which takes into account various inputs, namely the desire to reach the exit and to avoid the wall, the search for less crowded walking directions, and the unconscious attraction of the stream. It is worth noting that, while the first three inputs describe the behavior of a completely rational crowd, the last one models the irrational side of the crowd which is always present but is stronger in panic situations. Interactions are nonlinearly additive, meaning that they produce a global effect which is not given by the sum of all the individual interactions.

Concerning the velocity modulus, we assume that pedestrians adjust their speed according to the level of congestion around them, so that higher densities induce people to reduce their velocity modulus, while in less congested scenarios they can increase their speed.

This work refers to the existing literature reviewed and critically analyzed in [5], and to theoretical tools of game theory [36,37]. Foundations of the kinetic theory approach are reviewed in [4], where various applications in different fields, such as vehicular traffic [7,22], social systems, dynamics in networks and migration phenomena [30] are critically analyzed. Additional useful references are offered by the interpretation of empirical data [10,35,38], investigation of individual behaviors [16,26,32], coupling between pedestrian motion and traffic flow [1,8], design of evacuation strategies [39], and nonlinear interactions in swarms [6].

The paper is organized as follows. In Section 2 we propose a mathematical model for the evacuation of a crowd from a domain with walls and exits, using some tools from game theory to describe interactions between individuals and with the geometry of the environment. Section 3 is devoted to numerical results, where four case studies are presented in order to test the predictive ability of the model and to calculate evacuation time under different scenarios. Finally, Section 4 proposes a critical analysis looking at research perspectives.

2. A Mathematical Model

Methods of the kinetic theory lead to models which describe the state of the overall system by a one-particle distribution function, over the microscopic state of each particle, given by position and velocity.

For a system composed by a large number of pedestrians distributed inside a bounded domain $\Sigma \subset \mathbb{R}^2$, the distribution function is given by $f = f(t, \mathbf{x}, \mathbf{v})$ for all $t \geq 0$, $\mathbf{x} \in \Sigma$, $\mathbf{v} \in D_{\mathbf{v}}$, where $\mathbf{x} \in \Sigma$ denotes position, $\mathbf{v} \in D_{\mathbf{v}}$ velocity, and $D_{\mathbf{v}} \subset \mathbb{R}^2$ the velocity domain. Under suitable integrability conditions, $f(t, \mathbf{x}, \mathbf{v})d\mathbf{x}d\mathbf{v}$ represents the number of individuals who, at time t, are located in the infinitesimal rectangle $[x, x + dx] \times [y, y + dy]$ and have a velocity belonging to $[v_1, v_1 + dv_1] \times [v_2, v_2 + dv_2]$, where $\mathbf{x} = (x, y)$ and $\mathbf{v} = (v_1, v_2)$. If polar coordinates are used, as in [3], the velocity is given by the modulus $v = (v_1^2 + v_2^2)^{\frac{1}{2}}$ and the direction θ, thus the distribution function writes $f = f(t, \mathbf{x}, v, \theta)$.

The approach of [3] used discrete variables for the individual velocity states v and θ. This assumption is motivated by the need of taking into account the granular feature of pedestrian dynamics that does not justify the assumption of the kinetic theory concerning the continuity of the distribution function over the micro-variable. The validity of the model proposed in [3] is limited to dynamics in unbounded domains, while this present paper is focused on the evacuation dynamics from domains with boundaries and exits. This problem presents some non-trivial difficulties related to the fact that pedestrians not only interact with other pedestrians, but also need to look for the exit area and to avoid walls, where in special dangerous situations they might be even squeezed on. Therefore, the modeling of interactions between individuals and walls has to be properly developed. This topic has been exhaustively studied in the field of mathematical kinetic theory. For instance, in [11] the relation between the distribution function of the particles hitting and leaving the wall is modeled by a linear map. This approach cannot be applied to the physical system under consideration since pedestrians are well aware of the presence of walls and modify their trajectory to avoid them.

Bearing all above in mind, the hallmarks of the approach proposed in the present paper are the following:

- Each type of environment where pedestrians move has specific features which can facilitate or hinder the motion. The characteristic speed, at which a certain dynamics occurs, depends on the quality of the environment. Both large dimensions and good quality of the path conditions increase the said speed, while for instance walking through a narrow corridor or climbing stairs is more difficult and decreases the characteristic speed. The quality of the domain where pedestrians move is assessed by a parameter $\alpha \in [0,1]$, where $\alpha = 0$ corresponds to the worst quality which forces pedestrians to reduce the velocity modulus and to stop, while the value $\alpha = 1$ corresponds to the best quality, which contributes to keep high speeds.
- Pedestrians modify their walking direction by a decision strategy which takes into account four different inputs: the will to reach the exit and to avoid the walls, the preference for less congested areas and the unconscious attraction of the stream.
- The variable θ, that identifies the velocity direction, attains discrete values in the interval $[0, 2\pi)$.
- Pedestrians adjust their velocity modulus according to the level of congestion around them.

The assumption that pedestrians behave homogeneously in adjusting their velocity modulus and heterogeneously in looking for the preferred direction is a simplification of physical reality although it is confirmed by observations, and some conceivable improvements are discussed in the last section.

2.1. *Representation of the system*

As stated above, $\Sigma \subset \mathbb{R}^2$ is the bounded domain where pedestrians move. We assume that the boundary $\partial\Sigma$ includes the exit zone $E \subset \partial\Sigma$, while the remaining part of the boundary constitutes the wall $W \subset \partial\Sigma$. It is worth noticing that E could be the finite union of disjoint sets, i.e. the domain may have more than one exit. We consider, for simplicity, a convex domain, as shown in Fig. 2.1.1, therefore the presence of internal obstacles is not included in our treatment.

In the present paper, we propose a new approach considering a representation with continuous–discrete hybrid features. More in details, $\mathbf{x} = (x, y)$ is supposed to be a continuous variable defined over Σ, θ is a discrete variable, heterogeneously distributed among active particles, supposed to take values in the set

$$I_\theta = \left\{ \theta_i = \frac{i-1}{n} 2\pi \,:\, i = 1, \ldots, n \right\}.$$

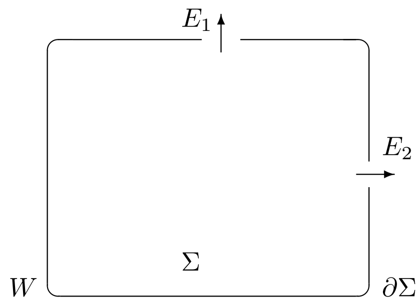


Fig. 2.1.1. Geometry of the bounded domain Σ with boundary $\partial\Sigma = W \cup E_1 \cup E_2$.

In relation to the velocity modulus, v is modeled as a continuous deterministic variable which evolves in time and space according to macroscopic effects determined by the overall dynamics. This choice is motivated by the fact that in practical situations the speed of pedestrians depends mainly on the level of congestion around them. A specific model will be proposed later consistently with empirical data.

Therefore, due to the deterministic nature of the variable v, the kinetic type representation is delivered by the *reduced* distribution function

$$f(t, \mathbf{x}, \theta) = \sum_{i=1}^{n} f_i(t, \mathbf{x})\delta(\theta - \theta_i),$$

where $f_i(t, \mathbf{x}) = f(t, \mathbf{x}, \theta_i)$ represents the active particles that, at time t and position $\mathbf{x}$, move with direction θ_i, and δ denotes the Dirac delta function.

Here and in what follows, dimensionless quantities are used. In details, we define the following reference quantities:

- L, the diameter of the domain Σ,
- V_M, the highest velocity modulus that a fast pedestrian can reach in low density and optimal environmental conditions,
- a reference time T given by L/V_M,
- ρ_M, the maximal admissible number of pedestrians per unit area.

In this way, we define the following dimensionless variables: position $\hat{\mathbf{x}} = \mathbf{x}/L$, time $\hat{t} = t/T$, velocity modulus $\hat{v} = v/V_M$ and distribution function $\hat{f} = f/\rho_M$, although hats will be omitted to simplify notation.

Due to the normalization of f, and of each f_i, the dimensionless local density is obtained by summing the distribution functions over the set of directions:

$$\rho(t, \mathbf{x}) = \sum_{i=1}^{n} f_i(t, \mathbf{x}).$$

Concerning the velocity distribution, we suppose that the velocity modulus of particles depends formally on the local density, i.e. $v = v[\rho](t, \mathbf{x})$, where square brackets are used to denote that v depends on ρ in a functional way, for instance on ρ and on its gradient.

The modeling approach takes into account interactions of active particles with all other particles and with the environment. The derivation of the mathematical structure can be obtained by a suitable balance of particles in an elementary volume of the space of microscopic states, considering the net flow into such volume due to transport and interactions:

$$\partial_t f_i(t, \mathbf{x}) + \mathrm{div}_{\mathbf{x}}(\mathbf{v}_i[\rho](t, \mathbf{x}) f_i(t, \mathbf{x})) = \mathcal{J}_i[f](t, \mathbf{x}), \qquad i = 1, \ldots, n, \tag{2.1}$$

where $\mathbf{v}_i[\rho] = v[\rho](\cos\theta_i, \sin\theta_i)$ and the left-hand term models the transport of particles, while $\mathcal{J}_i[f]$ represents the net balance for those particles that move with direction θ_i due to interactions. In the next subsections, a detailed analysis of each of the terms in Eq. (2.1) is performed, in order to obtain a specific model.

2.2. *Modeling interactions*

This subsection is devoted to the modeling of interactions that results in the specification of the right-hand side of Eq. (2.1). Interactions involve three types of particles: ***test particles*** with micro-state $(\mathbf{x}, \theta_i)$ which are representative of the whole system; ***field particles*** with micro-state $(\mathbf{x}, \theta_k)$, whose presence triggers the interactions of the candidate particles; ***candidate particles*** with micro-state $(\mathbf{x}, \theta_h)$, which can reach in probability the state of the test particles after individual based interactions with field particles or with the environment. In what follows we refer to an i-particle to mean a pedestrian moving with direction θ_i. As we shall see in this subsection, two types of interactions are considered, those between candidate and field particles and those between candidate particles and the environment where the dynamics occurs, that is with its geometrical and qualitative properties. In this way, the right-hand side of Eq. (2.1) can be decomposed as $\mathcal{J}_i = \mathcal{J}_i^P + \mathcal{J}_i^G$, where $\mathcal{J}_i^P$ and $\mathcal{J}_i^G$ refer to each of the two said types of interactions, respectively. The expressions of $\mathcal{J}_i^P$ and $\mathcal{J}_i^G$ are given explicitly in Section 2.4.

2.2.1. *Towards the selection of the desired direction*

Let us now consider the dynamics that generates changes in the direction of movement of active particles. In this paragraph we describe how the decision process, which consists in selecting the desired direction for each pedestrian, is modeled.

We assume that not only binary interactions induce a change in the state of a particle, but pedestrians make a decision according to a combination of different causes, detailed below.

- *Trend to move toward the exit.* During an evacuation, pedestrians try to reach the exit walking through the shortest path. Hence, given a candidate particle at the point $\mathbf{x}$, we define its distance to the exit as

$$d_E(\mathbf{x}) = \min_{\mathbf{y} \in E} \|\mathbf{x} - \mathbf{y}\|, \tag{2.2}$$

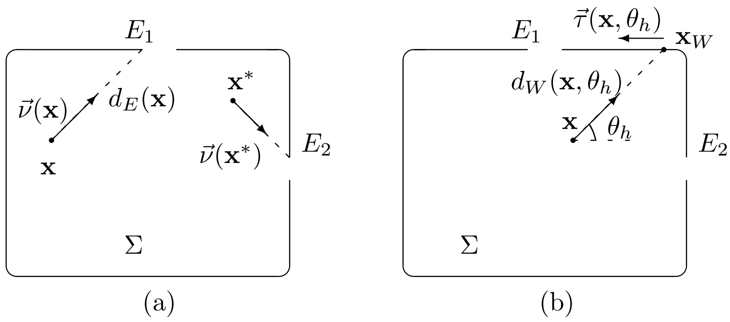


Fig. 2.2.1. (a) We denote the distance to the exit of a particle located in $\mathbf{x}$ by $d_E(\mathbf{x})$ and the vector pointing from $\mathbf{x}$ to the exit by $\vec{\nu}(\mathbf{x})$. (b) A particle in $\mathbf{x}$ moving with direction θ_h is expected to collide the wall in $\mathbf{x}_W$, then it computes the tangent direction to the wall that would take it toward the exit.

and we consider the unitary vector $\vec{\nu}(\mathbf{x})$, pointing from $\mathbf{x}$ to the exit, see Fig. 2.2.1(a).

- *Trend to avoid the collision with walls.* If a particle at $\mathbf{x}$ moves with direction θ_h, and if this direction does not point it toward the exit, then it is expected to collide (unless it changes direction) with the wall at a point $\mathbf{x}_W(\mathbf{x}, \theta_h)$, which is located at a distance $d_W(\mathbf{x}, \theta_h)$ from the particle, as shown in Fig. 2.2.1(b). Then a suitable direction should be chosen in order to prevent particles to hit the wall. In accordance with this aim, we select the unitary tangent vector $\vec{\tau}(\mathbf{x}, \theta_h)$ to $\partial\Sigma$ at $\mathbf{x}_W$ that points in the direction that would let a pedestrian get closer to the exit.
- *Tendency to move towards less congested areas.* In order to facilitate its movement, a candidate particle at $\mathbf{x}$, moving with direction θ_h, decides to change direction by considering the presence of less congested areas. This is achieved by choosing the direction that gives the minimal directional derivative of the density at the point $\mathbf{x}$. This direction is denoted by the unitary vector $\vec{\gamma}(\theta_h, \rho)$.
- *Tendency to follow the stream.* Binary interactions, at each time t and position $\mathbf{x}$, involve test, candidate, and field particles. A candidate particle modifies its state, in probability, into that of the test particle, due to interactions with field particles, while the test one loses its state as a result of these interactions. This dynamics is inserted in the model in order to take into account the fact that a (candidate) pedestrian interacting with a (field) pedestrian may decide to follow him/her. We define the unitary vector $\vec{\sigma}_k = (\cos\theta_k, \sin\theta_k)$ to describe the movement of the field k-particle.

Let us observe that the first two types of dynamics are related to purely geometric aspects of the domain, meaning that candidate particles take into account the presence of doors or walls but without caring about other pedestrians' behavior. Conversely, the other two effects take into consideration that people's behavior is strongly affected by that of the others. In fact, on the one hand a candidate particle is capable to scan its surroundings in order to choose, at each moment and position, a proper direction that will prevent it to move into congested areas, while on the other hand the interaction with a field particle will try to bring its direction closer to that of the latter.

The modeling approach is based on the following assumptions:

(A1) The trend to the exit increases as particles get closer to it.
(A2) Particles are subject to a stronger influence to avoid the wall as they get closer to it.
(A3) The tendency to look for less congested areas depends proportionally on the local density.
(A4) The tendency to follow the stream also depends proportionally on the local density.

Notice that the effects related to assumptions (A3)-(A4) compete with each other. In other words, higher densities will induce a higher tendency to look for less congested areas but at the same time to follow the stream. We introduce a parameter $\varepsilon \in [0, 1]$ that reinforces one effect or the other according to the particular situation to be modeled. The value $\varepsilon = 0$ corresponds to the situation in which only the research of less congested areas is considered, while $\varepsilon = 1$ corresponds to the situation in which only the tendency to follow the stream is taken into account.

2.2.2. *Interaction terms and selection of the desired direction*

Pedestrians change their velocity direction according to a not purely deterministic decision which is taken by considering all the said effects. This feature can be efficiently modeled by a stochastic point of view, since two pedestrians are not expected to react in the same way when facing a certain particular situation. More precisely, at each interaction, each pedestrian is assumed to play a game whose payoff is the updating of his/her direction accordingly to his/her strategy.

Dynamics induced by the shape of the environment. This type of dynamics is modeled by means of the following two interaction terms.

- The *interaction rate* $\mu[\rho]$ models the frequency of interactions between candidate particles and the boundary of the domain. We suppose that μ decreases with local density, since the lower this quantity is, the easier is for pedestrians to realize about the presence of walls and doors. Following this idea, we assume $\mu[\rho] \sim 1 - \rho$.

- The *transition probability* $\mathcal{A}_h(i)$ is the probability that an h-candidate particle adjusts its direction into that of the test particle θ_i, induced by the presence of walls and exit areas. This term satisfies the following constraint:

$$\sum_{i=1}^{n} \mathcal{A}_h(i) = 1 \quad \text{for all } \ h \in \{1, \dots, n\}.$$

The modeling approach takes into account that particles change direction, in probability, only to an adjacent clockwise or anticlockwise direction in the discrete set I_θ. This means that a candidate h-particle may eventually end up into the states $h-1$, $h+1$ or remain in the state h. We point out that, in the case $h=1$, we set $\theta_{h-1} = \theta_n$ and, in the case $h=n$, we set $\theta_{h+1} = \theta_1$.

The set of all transition probabilities $\mathcal{A} = \{\mathcal{A}_h(i)\}_{h,i=1,\dots,n}$ forms the so-called *table of games* that models the game played by active particles interacting with the geometry of the environment.

According to assumptions (A1)-(A2), we define the vector

$$\vec{\omega}_G(\mathbf{x}, \theta_h) = (1 - d_E(\mathbf{x}))\, \vec{\nu}(\mathbf{x}) + (1 - d_W(\mathbf{x}, \theta_h))\vec{\tau}\,(\mathbf{x}, \theta_h),$$

whose direction θ_G is the ***geometrical preferred direction***, meaning the ideal direction that a pedestrian should take in order to reach the exit and avoid the walls in an optimal way.

Since directions are discretized, an h-particle will update its direction by choosing among the three allowed outputs θ_{h-1}, θ_h and θ_{h+1} the closest to θ_G. The compact form of the table of games $\mathcal{A}$ is given by

$$\mathcal{A}_h(i) = \beta_h(\alpha)\delta_{s,i} + (1 - \beta_h(\alpha))\delta_{h,i}, \quad i = h-1, h, h+1,$$

where

$$s = \underset{j \in \{h-1, h+1\}}{\arg\min} \{d(\theta_G, \theta_j)\},$$

$$d(\theta_*, \theta^*) = \begin{cases} |\theta_* - \theta^*| & \text{if } |\theta_* - \theta^*| \le \pi, \\ \\ 2\pi - |\theta_* - \theta^*| & \text{if } |\theta_* - \theta^*| > \pi, \end{cases}$$

$\delta_{j,i}$ denotes the Kronecker delta function, and the coefficient β_h, proportional to the parameter α, is introduced to describe the fact that even in the case that the geometrical preferred direction θ_G is close to θ_h, a transition may occur, and the more distant the two states are, the more probable is this transition:

$$\beta_h(\alpha) = \begin{cases} \alpha & \text{if } d(\theta_h, \theta_G) \ge \Delta\theta, \\ \\ \alpha \frac{d(\theta_h, \theta_G)}{\Delta\theta} & \text{if } d(\theta_h, \theta_G) < \Delta\theta, \end{cases}$$

where $\Delta\theta = 2\pi/n$. Notice that if $\theta_G = \theta_h$, then $\beta_h = 0$ and $\mathcal{A}_h(h) = 1$, meaning that a pedestrian keeps the same direction, at least due to the geometrical effects, with probability 1.

Dynamics induced by interactions between pedestrians. In this case, the interaction dynamics can be modeled as follows:

- The *interaction rate* $\eta[\rho]$, which defines the number of binary encounters per unit time. It is meaningful to use the same idea developed in the case of vehicular traffic [7], by increasing the interaction rate with increasing local density. One possibility [3,4], among several conceivable ones, is to take $\eta[\rho] = \eta_0\rho$, where η_0 is the rate, to be measured by experimental data, corresponding to the spatially homogeneous case at low densities. It is worth stressing that, differently from the case of classical particles, this rate is not related to the relative velocity, but to the sensitivity of particles to surrounding pedestrians.
- The *transition probability* $\mathcal{B}_{hk}(i)[\rho]$ is the probability that a candidate particle modifies its direction into that of the test particle θ_i induced by a decision that combines the research of less congested areas and the interaction with a field particle moving with direction θ_k. The square brackets denote the dependence on the density ρ. This term satisfies:
$$\sum_{i=1}^{n} \mathcal{B}_{hk}(i)[\rho] = 1 \quad \text{for all } \; h,k \in \{1,\dots,n\},$$
and for all densities ρ.

Concerning the vacuum direction $\vec{\gamma}$, we have to consider how pedestrians react according to their perception of the density around them, and the game consists in the choice of the less congested direction among the three admissible ones. This direction can be computed for a candidate h-pedestrian situated in $\mathbf{x}$, by taking
$$\ell = \ell(\theta_h, \rho(t,\mathbf{x})) = \underset{j\in\{h-1,h,h+1\}}{\arg\min} \; \{\partial_j \rho(t,\mathbf{x})\},$$
where $\partial_j\rho$ denotes the derivative of ρ in the direction θ_j. In this way, we have $\vec{\gamma}(\theta_h,\rho) = (\cos\theta_\ell, \sin\theta_\ell)$.

According to assumptions (A3)-(A4) and recalling that $\vec{\sigma}_k$ denotes the vector pointing in the direction of the field particle, we define the vector
$$\vec{\omega}_P(\theta_h,\theta_k,\rho) = \varepsilon\, \vec{\sigma}_k + (1-\varepsilon)\vec{\gamma}(\theta_h,\rho),$$
where the subscript P stands for *pedestrians*, and the direction θ_P of $\vec{\omega}_P$ is the ***interaction-based preferred direction***, obtained as a weighted combination between the trend to follow the stream and the tendency to avoid crowded zones. Then, we propose the following table of games:
$$\mathcal{B}_{hk}(i)[\rho] = \beta_{hk}(\alpha)\rho\delta_{r,i} + (1-\beta_{hk}(\alpha)\rho)\delta_{h,i}, \quad i = h-1, h, h+1,$$

where r and β_{hk} are defined as in the previous case

$$r = \underset{j\in\{h-1,h+1\}}{\arg\min} \{d(\theta_P, \theta_j)\},$$

$$\beta_{hk}(\alpha) = \begin{cases} \alpha & \text{if } d(\theta_h, \theta_P) \geq \Delta\theta, \\ \alpha \frac{d(\theta_h,\theta_P)}{\Delta\theta} & \text{if } d(\theta_h, \theta_P) < \Delta\theta. \end{cases}$$

2.3. *Modeling the velocity modulus*

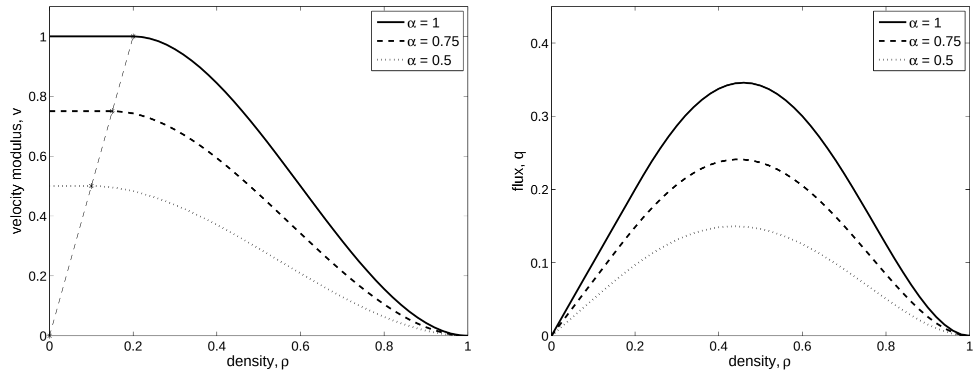
The decay or increase of the velocity modulus depends on the interactions between pedestrians. A simplification, that can contribute to reduce the technical complexity, consists in modeling this type of dynamics simply by a heuristic model suitable to link velocity modulus to local density conditions. In details, pedestrians adjust their velocity modulus according to the level of congestion around them. In this sense, we refer to some important contributions related to flux limiters [22,40], while dependence of pedestrians' speed on the local density is an important way to control overcrowding [12,13,24,39].

We assume that the maximal reachable dimensionless velocity modulus $v_m = v_m(\alpha)$ depends linearly on the quality of the environment, in such a way that $v_m(0) = 0$ –any movement is hindered– and $v_m(1) = 1$ –the maximal speed can be reached. The dependence of the velocity modulus on the local density proposed in this paper is motivated by the fundamental diagrams for crowd flow, critically analyzed in [38]. The main idea is that the maximal speed $v_m(\alpha)$ is kept under low density conditions (free flow regime), up to a certain critical density $\rho_c(\alpha)$ that can be experimentally measured. For values of ρ greater than ρ_c, the velocity modulus decreases to zero (slowdown zone). In the slowdown zone, we choose a polynomial-type dependence of the velocity modulus on the local density, see Fig. 2.3.

A refinement of this approach can be achieved by introducing the concept of perceived density, first proposed (in the case of vehicular traffic) by De Angelis [20], who suggested that, in the presence of positive density gradients, drivers perceive a density which is higher than the real one, while the opposite situation occurs in the case of negative gradients. Accordingly, using a flux limiting approach, the following definition can be given:

$$\rho_i^p[\rho](t,\mathbf{x}) = \rho(t,\mathbf{x}) + \frac{\partial_i\rho(t,\mathbf{x})}{\sqrt{1+\partial_i\rho(t,\mathbf{x})^2}}[(1-\rho(t,\mathbf{x}))H(\partial_i\rho(t,\mathbf{x})) + \rho(t,\mathbf{x})H(-\partial_i\rho(t,\mathbf{x}))], \tag{2.3}$$

where $\rho_i^p[\rho](t,\mathbf{x})$ is the perceived density in the direction θ_i, ∂_i denotes the directional derivative along this direction, and H is the Heaviside function, i.e. $H(x) = 1$ if $x \geq 0$, and $H(x) = 0$ if $x < 0$. Various definitions of the perceived density are proposed and numerically compared in [9].



(a) (b)

Fig. 2.3.1. (a) Dependence of the dimensionless velocity modulus v on the dimensionless density ρ for different values of the parameter α representing the quality of the environment. In the free flow zone ($\rho \leq \rho_c(\alpha) = \alpha/5$) pedestrians move with the maximal velocity modulus $v_m(\alpha) = \alpha$ allowed by the environment. In the slowdown zone ($\rho > \rho_c(\alpha)$) pedestrians have a velocity modulus which is here heuristically modeled by the third order polynomial joining the points $(\rho_c(\alpha), v_m(\alpha))$ and $(1, 0)$ and having horizontal tangent in such points. (b) Dependence of the dimensionless flux $q = v\rho$ on the dimensionless density ρ.

2.4. *The mathematical model*

We have already introduced the necessary concepts in order to specify each of the terms of Eq. (2.1), letting us introduce the resulting mathematical model

$$\begin{aligned}\partial_t f_i(t,\mathbf{x}) + \mathrm{div}_{\mathbf{x}}(\mathbf{v}_i[\rho](t,\mathbf{x}) f_i(t,\mathbf{x})) &= \mathcal{J}_i[f](t,\mathbf{x}) = \mathcal{J}_i^G[f](t,\mathbf{x}) + \mathcal{J}_i^P[f](t,\mathbf{x}) \\ &= \mu[\rho(t,\mathbf{x})] \left(\sum_{h=1}^{n} \mathcal{A}_h(i) f_h(t,\mathbf{x}) - f_i(t,\mathbf{x}) \right) \\ &\quad + \eta[\rho(t,\mathbf{x})] \left(\sum_{h,k=1}^{n} \mathcal{B}_{hk}(i)[\rho] f_h(t,\mathbf{x}) f_k(t,\mathbf{x}) - f_i(t,\mathbf{x})\rho(t,\mathbf{x}) \right) \end{aligned} \tag{2.4}$$

for $i = 1, \ldots, n$. As already stated, the term $\mathcal{J}_i^G$ represents the difference between the gain and the loss of particles moving with direction θ_i due to geometrical effects, while $\mathcal{J}_i^P$ accounts for the balance due to binary interactions among particles. It is worth mentioning that both $\mathcal{J}_i^G$ and $\mathcal{J}_i^P$ contribute independently to the search of the desired direction, and the relative importance of each one is somehow weighted through the interaction rates μ and η which depend on the local density. In particular, low densities favor the selection of the geometrical desired direction, since $\mu \to 1$, $\eta \to 0$ as $\rho \to 0$. Meanwhile, high levels of congestion makes difficult to follow the optimal geometrical path, since $\mu \to 0$, $\eta \to 1$ as $\rho \to 1$.

3. Numerical results

This section presents some sample simulations focused on the computation of evacuation time from a room Σ with walls and exits. In accordance with this aim, we select four case studies in order to analyze the role of the exit size, of the initial crowd distribution and of the parameter ε.

Simulations are obtained by solving Eq. (2.4), endowed with given initial conditions $f_i(0, \mathbf{x}) = \phi_i(\mathbf{x})$ for $\mathbf{x} \in \Sigma$ and $i = 1, \dots, n$, while boundary conditions are not explicitly imposed, but are induced by the non-local action over the particles given by the term $\mathcal{J}_i^G$. Indeed, pedestrians are induced to avoid walls through the effect of vectors $\vec{\nu}$ and $\vec{\tau}$ and even in the extreme case of a pedestrian reaching a point $\mathbf{x} \in W \subset \partial\Sigma$ the geometrical preferred direction induces him/her to move with tangent direction to the wall.

We obtain the numerical solution of this problem by using the same numerical scheme described in Section 6 of paper [3]. This scheme consists in splitting the problem into a set of sub-problems, such that for each of them a simpler and more practical algorithm is available [28]. In our case, Eq. (2.1) is decomposed into a two-dimensional transport equation and one non-homogeneous ordinary differential equation over time that models interactions. The numerical method is then formed by solving the transport equation by a first order upwind scheme, and the temporal ODE using Euler's method.

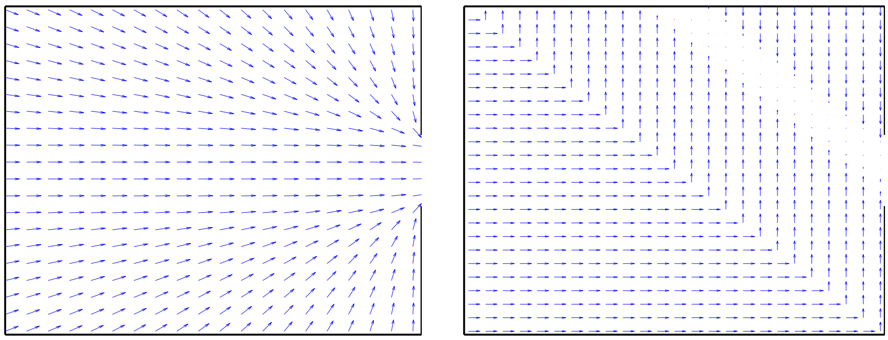
In what follows we consider a square domain of side length $10\,m$, with one or two exit doors located on the right side. Simulations are performed taking the quality of the environment $\alpha = 1$ and interaction rates $\mu = 1 - \rho$ and $\eta = \rho$. We consider eight different velocity directions in the discrete set

$$I_\theta = \left\{ \theta_i = \frac{i-1}{8} 2\pi \,:\, i = 1, \dots, 8 \right\},$$

and the velocity modulus is assumed to depend on the perceived density (2.3), as described in Subsection 2.3. Dimensionless quantities are obtained by referring the spatial coordinates with respect to the diameter $L = 10\sqrt{2}\,m$ of Σ, while density and velocity modulus are referred to $\rho_M = 7\,ped/m^2$ and $V_M = 2\,m/s$, respectively, according to experimental data presented in [10]. The reference time $T_M = 5\sqrt{2}\,s$ follows from these values.

The dynamics is based on a proper modeling of the geometrical preferred direction given by the vectors $\vec{\nu}$ and $\vec{\tau}$. Regarding to $\vec{\nu}$, we consider that the influence of the door over each point $\mathbf{x} \in \Sigma$ depends not only on the closest point joining $\mathbf{x}$ with the door (as it is used in the definition of $d_E(\mathbf{x})$ in Eq. (2.2)), but also on the whole amplitude of the exit. More precisely, we consider two vectors: one pointing to the closest point of the door and the other pointing to its center. Then, the resulting vector $\vec{\nu}$ is obtained by a convex combination of these two vectors with coefficients depending on the y-coordinate of $\mathbf{x}$, as shown in Fig. 3.0.1(a). This choice is made in order to avoid queue formation, a phenomena that is observed when $\vec{\nu}$ is simply defined as the vector pointing to the closest point of the door.

On the other hand, the modeling of $\vec{\tau}$ is based on the idea that pedestrians perceive the presence of the walls and try to choose an appropriate direction in order to thwart a collision with them. Figure 3.0.1(b) shows $\vec{\tau}(\mathbf{x}, \theta_h)$ for direction $h = 8$. In all the cases, we assume that $\vec{\tau}$ is the tangent vector to the boundary which let pedestrians get closer to the exit.



(a) (b)

Fig. 3.0.1. (a) The smooth vector field $\vec{\nu}(x)$ is defined taking into account that the pedestrian at a point $\mathbf{x}$ looks at the whole exit zone, even though, during the decision process, the point of the door which is closest to him/her weighs more than other points. (b) The vector field $\vec{\tau}(\mathbf{x}, \theta_8)$ is defined considering the tangent vector to the wall expected to be collided and points in the direction that would let a pedestrian get closer to the exit.

3.1. *On the role of the size of the exit zone*

The aim of this case study is to analyze the dependence of evacuation time on the size of the exit door, from a square room of side length $10\,m$. The problem is solved numerically for different sizes of the exit zone chosen in the range from $1.2\,m$ to $4.1\,m$. The door is located in the right side of the room and is centered in the middle point of this side. Approximately 45 pedestrians are initially distributed into two circular clusters moving one against the other with opposite directions θ_3 and θ_7, as shown in Fig. 3.1.1(a). Motivated by the results given in Section 3.3, simulations are performed with $\varepsilon = 0.4$. The evacuation dynamics is qualitatively the same for all sizes of the door. A sample situation is shown in Fig. 3.1.1(a)-(d) which refers to the case of a $2.6\,m$ long door.

Figure 3.1.1(f) shows how evacuation time increases with decreasing size of the exit. One can observe that for sufficiently large sizes, evacuation time does not change significantly. Indeed, this result suggests that for a given number of pedestrians initially occupying the room, there is a minimal exit size above which evacuation time cannot be considerably reduced.

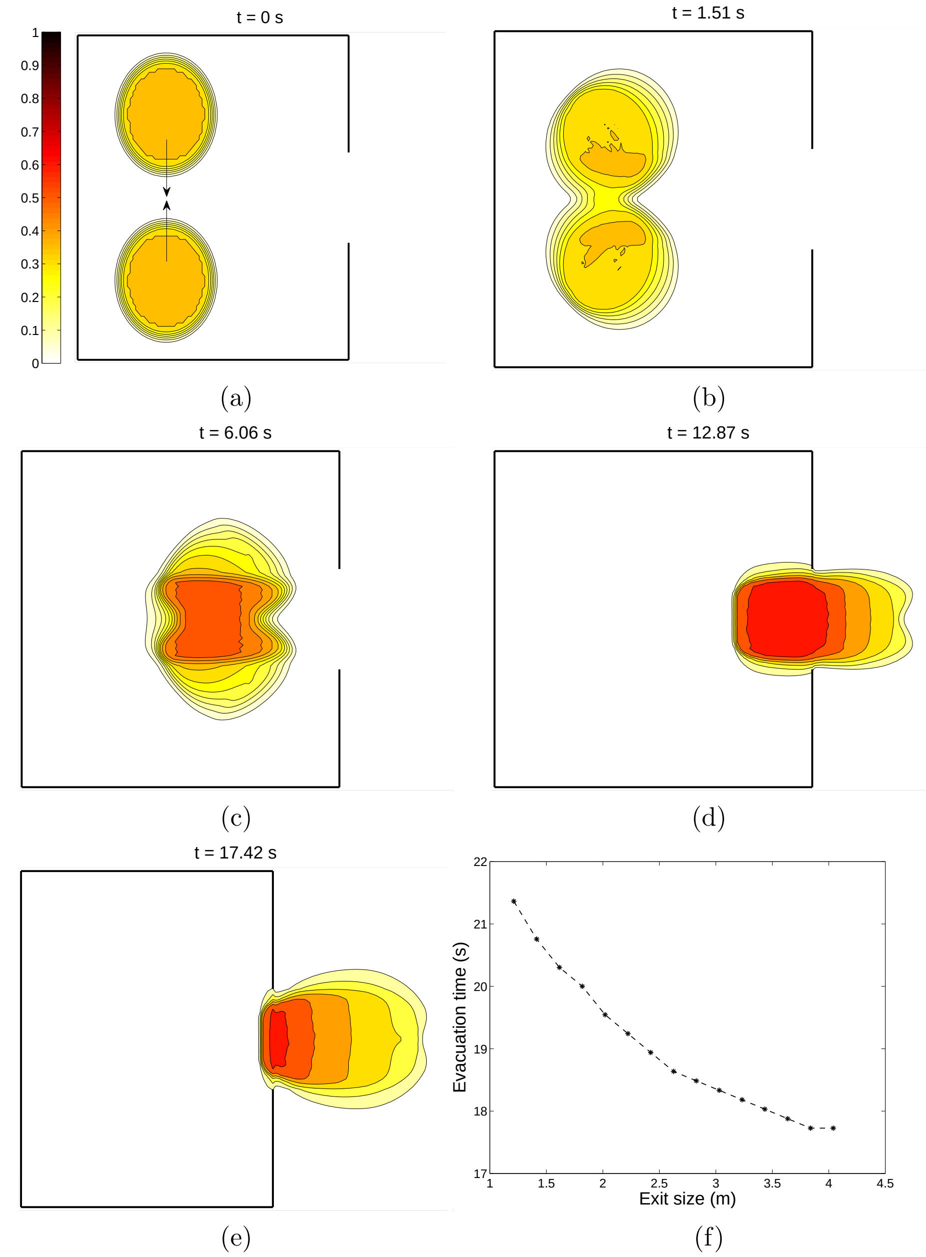
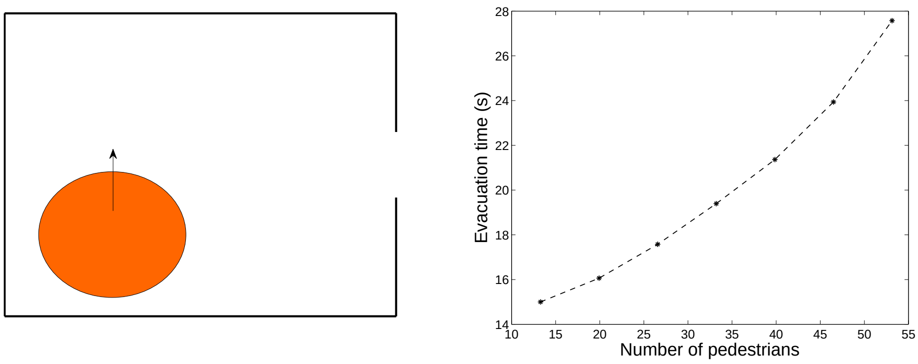


Fig. 3.1.1. Case study 1. Dependence of evacuation time on the size of the exit door from a square room of side length $10\,m$. (a) Initial distribution of about 46 people grouped into two clusters moving one against the other with opposite directions θ_3 and θ_7, exit size $2.6\,m$ and $\varepsilon = 0.4$. (b)-(e) Evacuation progress for $t = 0\,s$, $1.51\,s$, $6.06\,s$, $12.87\,s$, $17.42\,s$, respectively. (f) Evacuation times for different sizes of the exit door.

3.2. *On the role of initial conditions*

Let us now investigate the relationship between evacuation time and the number of pedestrians inside the room. With this purpose, we consider a fixed area inside the room initially occupied by the crowd moving with direction θ_3, see Fig. 3.2.1(a), and simulations are performed considering different initial constant densities ρ in this area, varying in the interval $[0.2, 0.8]$, corresponding to a number of people ranging from 13 to 53 approximately. Here again, we take $\varepsilon = 0.4$ and a fixed door size of $2\,m$. Then, evacuation time is computed and the results are shown in Fig. 3.2.1(b). The model reproduces the experimental evidence that evacuation of more people requires more time.



(a) (b)

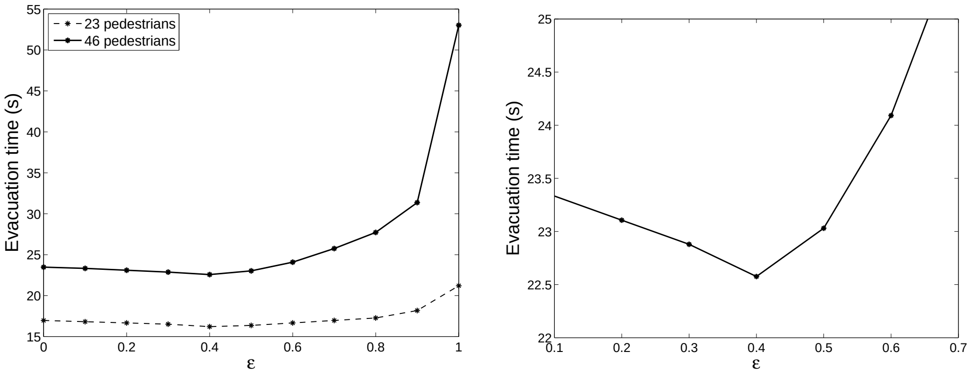
Fig. 3.2.1. Case study 2. (a) Pedestrians are initially distributed in a circular shape crowd moving with direction θ_3. (b) Evacuation times for different initial densities, i.e. different number of pedestrians in the room. In all cases the door size is $2\,m$ and $\varepsilon = 0.4$. The model reproduces the fact that evacuation of more people requires more time.

3.3. *On the role of the parameter ε*

This case study is selected in order to study how evacuation time is affected by the parameter ε that weighs the vacuum and stream effects. This parameter can be interpreted as a measure of the level of panic.

We study the dependence of evacuation time on ε starting with two different initial conditions. First, pedestrians are distributed in a circular region, with constant density $\rho = 0.35$, which corresponds approximately to 23 pedestrians. Then, pedestrians are distributed in the same circular region, but with constant density $\rho = 0.70$, that corresponds approximately to 46 pedestrians. In both situations, the crowd initially moves with direction θ_3 and the exit door of length $2\,m$ is centered in the middle point of the right side of the room, as in the previous case study, see

Fig.3.2.1(a). Simulations are performed varying the value of ε in the interval $[0, 1]$. The results are shown in Fig.3.3.1.



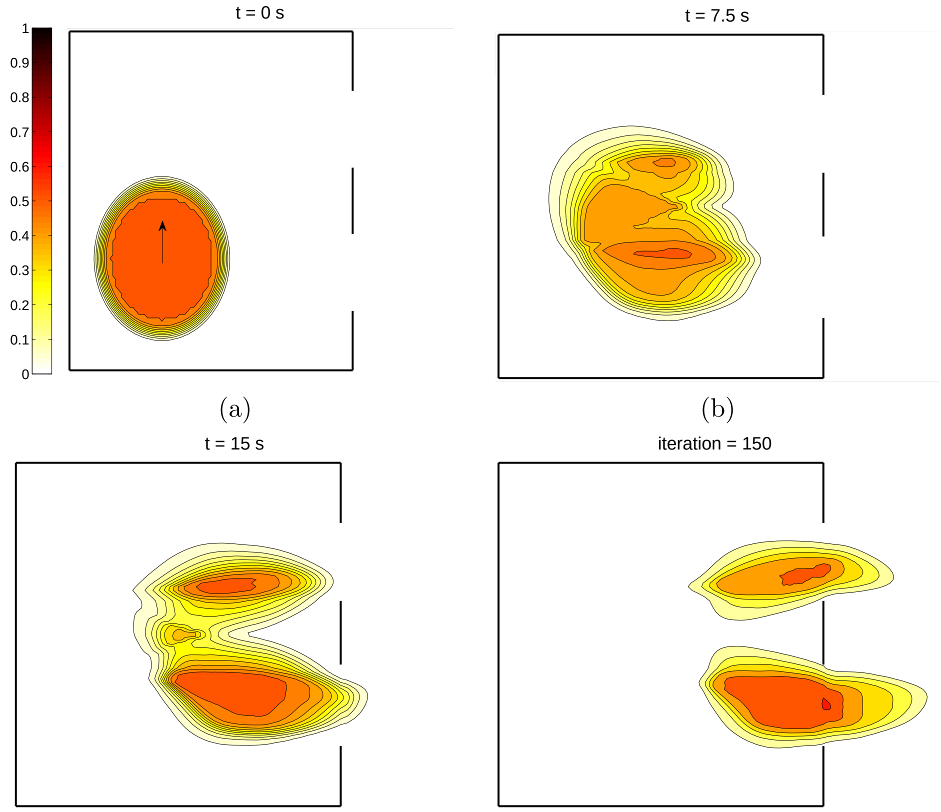
(a) (b)

Fig. 3.3.1. Case study 3. (a) Dependence of evacuation time on the parameter ε that weighs the vacuum and stream effects. Two different initial conditions are considered, with approximately 23 and 46 pedestrians. In both situations the crowd is initially distributed as in Fig.3.2.1(a) and moves with direction θ_3. A $2\,m$ exit door is centered in the middle point of the right side of the room. We observe that the minimum in both cases is attained at $\varepsilon = 0.4$. (b) Zoom of the figure in (a) for the situation with approximately 46 pedestrians. Here it is clearer that the minimum evacuation time is attained at $\varepsilon = 0.4$.

For both initial conditions, we observe that neither in $\varepsilon = 0$ nor in $\varepsilon = 1$ the minimum evacuation time is reached. Simulations show that the optimal value of ε is approximately 0.4. A possible interpretation of this fact is that in the case in which only the research of less congested areas is considered, pedestrians will take a longer path before approaching the exit. On the other hand, the pure trend to follow the stream leads to higher levels of congestion and therefore to a reduction of the velocity modulus.

3.4. *Two doors*

In the case of two doors, we investigate the ability of the model to let pedestrians decide actively which is the most convenient way to the exit, by taking into account not only the proximity to each door, but also the less crowded path. Simulations are performed with an initial crowd moving with direction θ_3. The square room has side length $10\,m$ and on the right side two identical doors of length $2.2\,m$ are placed, symmetrically with respect to the center of the side. The distance separating the doors is $1\,m$, see Fig.3.4.1(a). The value of ε used for this example is again 0.4.



(a) (b) (c) (d)

Fig. 3.4.1. Case study 4. Evacuation from a square room of side length $10\,m$, with two doors of size $2.2\,m$. In the simulations, the value of the parameter ε is 0.4. (a) Initial distribution of a cluster with approximately 53 pedestrians moving with direction θ_3. (b)-(d) Evolution of the evacuation process. Pedestrians choose the most convenient way to the exit by taking into account not only the proximity to each door but also the less crowded path.

4. Critical Analysis and Perspectives

The mathematical model proposed in this paper has been shown to be able to describe some interesting features of crowd dynamics in domains with walls and exits. Indeed, a number of case studies have been considered in order to test the ability of the model to reproduce qualitative behaviors regarding evacuation phenomena through the detection of evacuation time under different scenarios.

Further developments may include some issues concerning possible improvements of the modeling approach by revisiting interactions at the micro-scale, taking advantage of some recent studies on the dynamics of vehicles on roads [7,22]. Below are some ideas for further research to be developed within *ad hoc* programs.

- The modeling of the velocity modulus is not only consistent with the physics of the system, but also flexible, since the dependence of the maximum reachable velocity modulus v_m on α enables the model to depict different scenarios. On the other hand, an alternative approach can be obtained by discretizing the velocity modulus and considering interactions between slow and fast walkers, conditioned by the local density and α. In this way heterogeneity and granularity in the velocity modulus can be properly described. Papers [7,19] offer guidelines for this type of dynamics in the case of traffic flow.

- A typical problem related to crowd modeling is the granular nature of the space variable, since the distance between pedestrians can span from small to large values. In [22], in the case of vehicular traffic, both velocity and space variables are discretized in order to take into account all the granular features of the system. The same approach can be developed also in the case of crowd dynamics, although it presents the additional difficulty of a space variable of a higher dimension.
- Another point to be considered is the presence of various types of pedestrians who may pursue diverse strategies, for instance different exit zones, or who may differ in their walking abilities, which may range from persons with a disability to fast walkers. The approach induced by these specific cases can develop the idea already delivered in [3] that the whole population should be subdivided into different functional subsystems corresponding to the features already mentioned.
- If the evacuation process goes through heterogeneous paths including halls, corridors, stairs, etc., it is also necessary to take into account the specific features of each element constituting the overall environment.
- A detailed analysis of the role of the visibility zone of a pedestrian can be developed. In order to determine its shape, it is necessary to take into account the geometry of the environment, for example walls and obstacles that can favor or obstruct pedestrians' visual field. In addition, the size of the domain can be determined by local density conditions. For instance, a person interacts with a certain fixed density of pedestrians rather than with all those who are in a fixed domain, see [6].

Finally, it is worth stressing that some of the tools developed in this paper can partially contribute to the modeling of different self-propelled particles, such as cells [31] and swarms [18,23]. Of course, the physics is different and suitable changes and additional work are required.

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