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q-generalization of quantum phase-space representations

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HIGHLIGHTS

- Usual quantum phase space representations are generalized to a nonextensive environment.
- Analytical expressions for Wigner functions, Husimi functions, and P-ones are obtained.
- The behavior of the concomitant Tsallis entropy is investigated.

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ABSTRACT

We generate a family of phase space, thermal coherent-state's representations, within the framework of Tsallis' Generalized Statistical Mechanics and study their properties. Our protagonists are q-gaussian distributions. We obtain analytical expressions for the most important representations, namely, the P-, Husimi-, and Wigner ones. The behavior of the associated Tsallis entropy is investigated. It is shown that q-values close to two provide the best performance.

1. Introduction

Quantum phase space distributions (QPSD) constitute a subject of permanent interest, that revolves around coherent states. QPSD are useful tools because they allow one to evaluate expectation values using quantum phase space distributions in a manner that resembles the classical one [1]. R.J. Glauber [2] and E.C.G. Sudarshan [3] have introduced the phase space P-distribution [1]. There are other feasible functions, such as the Husimi–Kano Q-function, which is obtained as the expectation value of the density operator in a coherent state [4], and the celebrated Wigner distribution, with which the whole game started in the thirties [5]. The different phase space distributions refer to specific choices of creation–destruction operators' ordering. The Q-, Wigner, or P-distributions are associated with antinormal, symmetric, and normal ordering, respectively [1].

The q-gaussian is a probability distribution emerging from the q-generalization of the central limit theorem and is a generalization of the ordinary gaussian probability distribution [6]. The usual distribution is recovered in the limit $q \to 1$. The q-gaussian has been applied to statistical mechanics, geology, anatomy, astronomy, economics, finance, and machine learning. The distribution is often useful because of its heavy tails in comparison to the gaussian, for $1 < q < 3$. In these heavy tail zones, the distribution is equivalent to the Student's t-distribution. A q-gaussian form often arises for systems that are non-extensive.

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The purpose of this work is to generate a family of phase space quantum distributions within the structures of Tsallis' Statistical Mechanics [7–9] and investigate their properties and usefulness. For this reason, we employ q-gaussian distributions, which are useful tools to characterize complex systems including long-range correlations, multifractality and non-gaussian distributions with asymptotic power law behavior [10].

In order to facilitate comprehension, this paper is organized as follows. In Section 2, we collect some basic concepts referring to q-gaussian distributions and present its construction in the phase space representation. In Section 3 we review some notions on P, Q, and Wigner's representations. With these tools, in Section 4 we suggest a definition of generalized P-distribution as q-gaussians and we analyze the validity of this proposal. Sections 5 and 6 are devoted to obtain the generalized Q and Wigner functions, respectively. In Section 7 we calculate the Shannon and Tsallis entropy for the generalized P-function. Finally, Section 8 is devoted to concluding remarks.

2. Mathematical tools: *q*-gaussians

The q-gaussian probability distribution function, $G_q(\delta; x)$ introduced by Tsallis [6], is the q-generalization of the gaussian distribution. It reads

$$G_q(\delta; x) = C_q \frac{\sqrt{\delta}}{\sqrt{\pi}} e_q(-\delta x^2), \tag{1}$$

with q a real parameter and $e_q(x)$ standing for the q-exponential function:

$$e_q(x) = [1 + (1-q)x]_+^{\frac{1}{1-q}}, \tag{2}$$

with the notation $[y]_+ = \max(y, 0)$ and obviously $\lim_{q\to 1} e_q(x) = e_1(x) = \mathrm{e}^x$. Furthermore, C_q is a normalization constant that satisfies

$$C_q^{-1} = \frac{\sqrt{\delta}}{\sqrt{\pi}} \int_{-\infty}^{+\infty} \mathrm{d}x\, e_q(-\delta x^2), \tag{3}$$

so that

$$\begin{aligned} C_q &= \frac{(3-q)\sqrt{1-q}\,\Gamma\left(\frac{3-q}{2(1-q)}\right)}{2\,\Gamma\left(\frac{1}{1-q}\right)} && : -\infty < q < 1, \\ C_q &= 1 && : q = 1, \\ C_q &= \frac{\sqrt{q-1}\,\Gamma\left(\frac{1}{q-1}\right)}{\Gamma\left(\frac{3-q}{2(q-1)}\right)} && : 1 < q < 3. \end{aligned} \tag{4}$$

For $q = 1$, $G_q(\delta, x)$ becomes the traditional gaussian distribution

$$G_1(\delta; x) = \frac{\sqrt{\delta}}{\sqrt{\pi}} \mathrm{e}^{-\delta x^2}, \quad \delta > 0. \tag{5}$$

If $q < 1$, this density vanishes for $|x| > 1/\sqrt{(1-q)\delta}$. For $q > 3$, normalization is not possible because the associated integral diverges. Also, for $q \le 5/3$ the variance is finite, and for $5/3 \le q < 3$, the variance diverges. The aspect of the q-gaussian distribution is illustrated in Fig. 1.

In two dimensions, the q-gaussian distribution adopts the appearance

$$G_q(\delta; x, y) = C'_q \frac{\sqrt{\delta_x}\sqrt{\delta_y}}{\pi} e_q(-(\delta_x x^2 + \delta_y y^2)). \tag{6}$$

For $\delta_x = \delta_y = \delta$, and $\rho = \sqrt{x^2 + y^2}$, G_q becomes

$$G_q(\delta; \rho) = C'_q \frac{\delta}{\pi} e_q(-\delta \rho^2); \tag{7}$$

with C'_q a normalization factor that only depends on q, such that

$$C_q'^{-1} = \delta \int \frac{\mathrm{d}^2\rho}{\pi}\, e_q(-\delta\rho^2). \tag{8}$$

We have found $C'_q = 2 - q$ for all $q < 2$-values. Thus, G_q is defined for $q < 2$, otherwise the integral diverges and the function cannot be normalized. Finally, we get

$$G_q(\delta; \rho) = (2-q)\frac{\delta}{\pi} e_q(-\delta\rho^2); \quad q < 2. \tag{9}$$

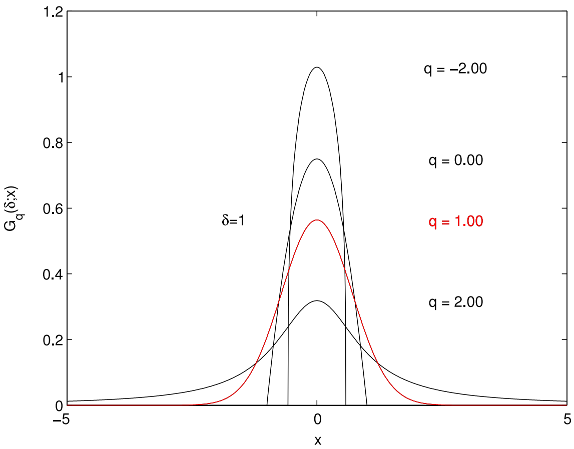

Fig. 1. q-gaussian distribution for some q-values. We take $\delta = 1$.

3. Basic concepts about quantum distributions in phase space

We begin this section with some comments concerning density operators. For a pure state $|\alpha\rangle$, the corresponding density operator is

$$\hat{\rho} = |\alpha\rangle\langle\alpha|. \tag{10}$$

The most important basis that we will employ in our treatment is that of coherent states $|\alpha\rangle$ [2]. In this sense, it is possible to cast the density operator in a diagonal manner, i.e., as a superposition of the projection operators (10), provided that an overcomplete basis is used [3]. This is known as the P-representation [3]

$$\hat{\rho} = \int \frac{d^2\alpha}{\pi}\, P(\alpha)\, |\alpha\rangle\langle\alpha|, \tag{11}$$

where the function $P(\alpha)$ plays a role analogous to a probability density for the distribution of values of α over the complex plane. We see that a central role is assigned to the function P. The system evolves as prescribed by the evolution of the P distribution function. The normalization property of the density operator requires that $P(\alpha)$ obeys the normalization condition [2]

$$\mathrm{Tr}\,\hat{\rho} = \int \frac{d^2\alpha}{\pi}\, P(\alpha) = 1. \tag{12}$$

The standard coherent states $|\alpha\rangle$, that we will use, are those of the harmonic oscillator, which are eigenstates of the annihilation operator $\hat{a}$, with complex eigenvalues α, satisfying $\hat{a}|\alpha\rangle = \alpha|\alpha\rangle$ [2]

$$|\alpha\rangle = e^{-|\alpha|^2/2} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}}\, |n\rangle, \tag{13}$$

where the Fock states $\{|n\rangle\}$ are a complete orthonormal set of eigenstates of the Hamiltonian $\hat{H} = \hbar\omega\left(\hat{n} + 1/2\right)$, in which the number operator is $\hat{n} = \hat{a}^\dagger\hat{a}$, and whose spectrum of energy is $E_n = (n + 1/2)\hbar\omega$, $n = 0, 1, \ldots$. Also, the states $|\alpha\rangle$ are normalized, i.e., $\langle\alpha|\alpha\rangle = 1$, and they provide us with a resolution of the identity operator

$$\int \frac{d^2\alpha}{\pi}\, |\alpha\rangle\langle\alpha| = 1, \tag{14}$$

which is a completeness relation for the coherent states [2]. Accordingly, the expectation value of an observable $\hat{A}$ in such representation is given by [11]

$$\langle\hat{A}\rangle_P = \mathrm{Tr}(\hat{\rho}\hat{A}) = \int \frac{d^2\alpha}{\pi}\, P(\alpha)\, \langle\alpha|\hat{A}|\alpha\rangle. \tag{15}$$

In this context, the average particle-number acquires a simple form that, according to Eq. (11), can be cast in the fashion [2]

$$\langle\hat{n}\rangle_P = \mathrm{Tr}(\hat{\rho}\hat{a}^\dagger\hat{a}) = \int \frac{d^2\alpha}{\pi}\, P(\alpha)\, |\alpha|^2, \tag{16}$$

indicating that the average particle number is the mean squared absolute value of α.

Now, from (11) we can calculate the diagonal matrix elements of $\hat{\rho}$ by multiplying on the left and right by the Fock state $|n\rangle$, so that

$$\rho_{nn} = \langle n|\hat{\rho}|n\rangle = \int \frac{d^2\alpha}{\pi} P(\alpha)\, |\langle n|\alpha\rangle|^2, \tag{17}$$

or, making use of Eq. (13)

$$\rho_{nn} = \frac{1}{n!} \int \frac{d^2\alpha}{\pi} P(\alpha)\, |\alpha|^{2n}\, e^{-|\alpha|^2}, \tag{18}$$

an important expression that we will use in following section for checking out the validity of our procedure. In addition to the P-function, there exists an infinite family of alternative distribution functions, but we will concentrate efforts only on the so-called Q-function, and on the celebrated Wigner's W-function. For an arbitrary density operator, the connection between Q-and P- functions is given by [12]

$$Q(\alpha) = \int \frac{d^2z}{\pi} P(z)\, e^{-|z-\alpha|^2}, \tag{19}$$

and in this case, the antinormal-ordered average of $\hat{n}$ in the Q-representation is

$$\langle \hat{n}\rangle_Q = \int \frac{d^2\alpha}{\pi} Q(\alpha)\, |\alpha|^2 - 1. \tag{20}$$

From P we can get W using the transformation [12]

$$W(\alpha) = 2\int \frac{d^2z}{\pi} P(z)\, e^{-2|\alpha-z|^2}, \tag{21}$$

and the mean value of $\hat{n}$, for symmetric ordered operators, is now [12]

$$\langle \hat{n}\rangle_W = \int \frac{d^2\alpha}{\pi} W(\alpha)\, |\alpha|^2 - \frac{1}{2}. \tag{22}$$

Independently of the representation chosen, the three mean values coincide with the quantum average $\langle \hat{n}\rangle = \mathrm{Tr}(\hat{\rho}\,\hat{n})$, this is, $\langle \hat{n}\rangle = \langle \hat{n}\rangle_P = \langle \hat{n}\rangle_Q = \langle \hat{n}\rangle_W$.

3.1. Thermal state

For a thermal state of harmonic oscillator, for which the density operator is of the form prescribed by the canonical ensemble's celebrated distribution, we have [13]

$$\hat{\rho} = (1 - e^{-\beta\hbar\omega})\, e^{-\beta\hbar\omega\hat{a}^\dagger\hat{a}}, \tag{23}$$

where $\beta = 1/k_BT$ and T is the temperature, while the corresponding thermal diagonal elements of $\hat{\rho}$ read

$$\rho_{nn} = (1 - e^{-\beta\hbar\omega})\, e^{-\beta\hbar\omega n}. \tag{24}$$

The quantum phase space distributions for this typical density operator are the following gaussian expressions [12]

$$P(\alpha) = \frac{1}{\langle \hat{n}\rangle} \exp\left(-\frac{|\alpha|^2}{\langle \hat{n}\rangle}\right), \quad \text{for } P\text{-functions}, \tag{25}$$

$$Q(\alpha) = \frac{1}{\langle \hat{n}\rangle + 1} \exp\left(-\frac{|\alpha|^2}{\langle \hat{n}\rangle + 1}\right), \quad \text{for } Q\text{-functions}, \tag{26}$$

$$W(\alpha) = \frac{1}{\langle \hat{n}\rangle + 1/2} \exp\left(-\frac{|\alpha|^2}{\langle \hat{n}\rangle + 1/2}\right), \quad \text{for } W\text{-functions}, \tag{27}$$

while the average particle-number is given by [2]

$$\langle \hat{n}\rangle = \frac{e^{-\beta\hbar\omega}}{1 - e^{-\beta\hbar\omega}}, \tag{28}$$

which depends on the temperature T.

4. P_q-phase space representation

4.1. P_q-escort distribution

We propose here to introduce a possible q-generalization of the phase space probability distribution $P(\alpha)$, that one can write according to Eq. (9), in phase space variables, as follows:

$$p_q(\alpha) = (2-q)\delta\, e_q(-\delta|\alpha|^2), \tag{29}$$

i.e., the phase space q-gaussian distribution $p_q(\alpha)$. The normalization property is

$$\int \frac{d^2\alpha}{\pi}\, p_q(\alpha) = 1, \tag{30}$$

where for the inverse variance one has $\delta = 1/\langle\hat{n}\rangle = e^{\beta\hbar\omega} - 1$. If we take the limit $q \to 1$ in Eq. (29) we immediately obtain

$$p_1(\alpha) = P(\alpha) = \delta\, e^{-\delta|\alpha|^2}, \tag{31}$$

which recovers the expression (25).

However, $p_q(\alpha)$ is not the only function that goes over to $P(\alpha)$ in the limit in which q tends to unity. Any q-escort distribution of $p_q(\alpha)$ shares this property, as we will show next. Let us formally introduce now the escort distribution of order q associated to the basic distribution $p_q(\alpha)$ [14]

$$P_q(\alpha) = \frac{p_q(\alpha)^q}{\int \frac{d^2\alpha}{\pi}\, p_q(\alpha)^q}, \tag{32}$$

so that Eq. (29) yields

$$P_q(\alpha) = \delta\, e_q(-\delta|\alpha|^2)^q, \tag{33}$$

provided $\delta > 0$, for all temperatures T and $q > 1$. In addition, the inverse q-variance of the distribution P_q is by definition

$$\sigma_q^{-1} = \int \frac{d^2\alpha}{\pi}\, P_q(\alpha)\, |\alpha|^2, \tag{34}$$

so that performing the pertinent integral we get

$$\sigma_q^{-1} = \frac{1}{(2-q)\delta}, \tag{35}$$

with the condition that $1 < q < 2$ and $\delta > 0$. Thus, $P_q(\alpha)$, in terms of σ_q, can be also written as

$$P_q(\alpha) = \frac{\sigma_q}{(2-q)}\, e_q\left(-\frac{\sigma_q\,|\alpha|^2}{(2-q)}\right)^q. \tag{36}$$

When q goes to unity, $\sigma_1 = \delta$ and then $P_1(\alpha) = P(\alpha)$, as anticipated at the beginning of this section.

4.2. Generalized density operator

We pass now to generalize Eq. (11) by proposing a certain q-dependent density operator $\hat{\rho}_q$ that we suppose admits the following diagonal coherent state representation

$$\hat{\rho}_q = \int \frac{d^2\alpha}{\pi}\, P_q(\alpha)\, |\alpha\rangle\langle\alpha|. \tag{37}$$

Note that (37) reduces to (11) when q tends to unity, since $\hat{\rho}_1 = \hat{\rho}$ and $P_1(\alpha) = P(\alpha)$. With the aim of checking the validity of (37), we evaluate the matrix elements of the density operator $\hat{\rho}_q$ in the basis $\{|n\rangle\}$. One has to satisfy

$$\rho_{nn}^{(q)} = \langle n|\hat{\rho}_q|n\rangle = \int \frac{d^2\alpha}{\pi}\, P_q(\alpha)\, |\langle n|\alpha\rangle|^2. \tag{38}$$

By using Eqs. (13) and (33) one has then to tackle

$$\rho_{nn}^{(q)} = \langle n|\hat{\rho}|n\rangle = \frac{\delta}{n!}\int \frac{d^2\alpha}{\pi}\, e_q(-\delta|\alpha|^2)^q\, e^{-|\alpha|^2}\, |\alpha|^{2n}. \tag{39}$$

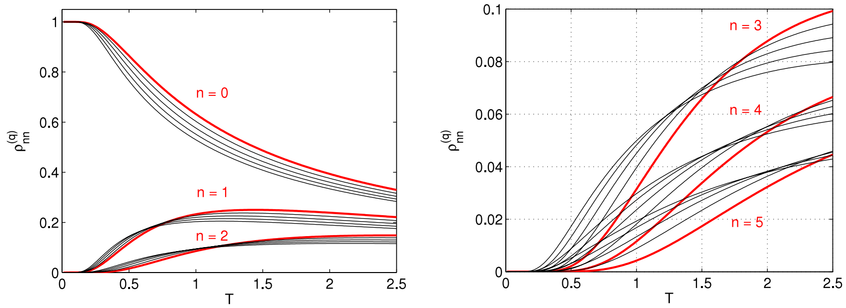

Fig. 2. $\rho_{nn}^{(q)}$ in terms of temperature T for $q = 1.00$ (red), $q = 1.20$, 1.40, 1.60, and 1.80, (black). We take $n = 0, 1, 2$ (left) and $n = 3, 4, 5$ (right). (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

In order to deal with the above integral, we appeal to an integral representation [15], based on the definition of the Euler Gamma function. This is

$$\int_0^\infty \mathrm{d}t\, t^{\nu-1}\, \mathrm{e}^{-t\eta} = \eta^{-\nu}\, \Gamma(\nu), \tag{40}$$

for $\Re(\eta) > 0$ and $\Re(\nu) > 0$ (see for instance Ref. [16, p. 342]). In our case we need to write

$$g_q^{\;q/(1-q)} = \frac{1}{\Gamma\left(\frac{q}{q-1}\right)} \int_0^\infty \mathrm{d}t\, t^{\frac{q}{q-1}-1}\, \mathrm{e}^{-tg_q}, \tag{41}$$

with the restrictions that (i) $g_q = (1-(1-q)x)^{1/(1-q)}$ be positive and (ii) $q > 1$ [17]. Introducing the integral representation into Eq. (39) one arrives at

$$\rho_{nn}^{(q)} = \frac{\delta}{n!\,\Gamma\left(\frac{q}{q-1}\right)} \int_0^\infty \mathrm{d}t\, t^d \mathrm{e}^{-t} \int \frac{\mathrm{d}^2\alpha}{\pi}\, |\alpha|^{2n}\, \mathrm{e}^{-\gamma_q(t)|\alpha|^2}, \tag{42}$$

with

$$d = \frac{1}{q-1}, \tag{43}$$

and

$$\gamma_q(t) = 1 - (1-q)\delta t. \tag{44}$$

The solution of the integral in the above expression is

$$\int \frac{\mathrm{d}^2\alpha}{\pi}\, |\alpha|^{2n}\, \mathrm{e}^{-\gamma_q(t)|\alpha|^2} = \Gamma(n+1)\, \gamma_q(t)^{-1-n}, \tag{45}$$

with $\Gamma(n+1) = n!$. Substituting (45) into (42), we arrive at the equation we are interested in

$$\rho_{nn}^{(q)} = \frac{\delta}{\Gamma\left(\frac{q}{q-1}\right)} \int_0^\infty \mathrm{d}t\, \gamma_q(t)^{-1-n}\, t^d \mathrm{e}^{-t}, \tag{46}$$

with the restrictions $1 < q < 2$ and $\delta > 0$. Eq. (46) complies with the normalization condition $\mathrm{Tr}\hat{\rho}_q = \sum_n \rho_{nn}^{(q)} = 1$. The formal solution of (46) is

$$\rho_{nn}^{(q)} = \delta\,[(q-1)\delta]^{q/(1-q)}\, U\left(\frac{q}{q-1}; \frac{q}{q-1} - n; \frac{1}{(q-1)\delta}\right), \tag{47}$$

where $U(a; b; z)$ is the confluent hypergeometric function of the second kind or also, in an alternative notation, ${}_1F_1(a; b; z)$ [18]. Illustrative examples of the diagonal matrix elements of $\hat{\rho}_q$, as a function of temperature T for several values of n and the parameter q, are given in Figs. 2 and 3.

4.3. Remarks on the hypergeometric functions

Special functions play an important role in mathematical physics, being generally employed to simplify the original problem by transforming its mathematical description from a rather involved form into a much simpler one. The physical

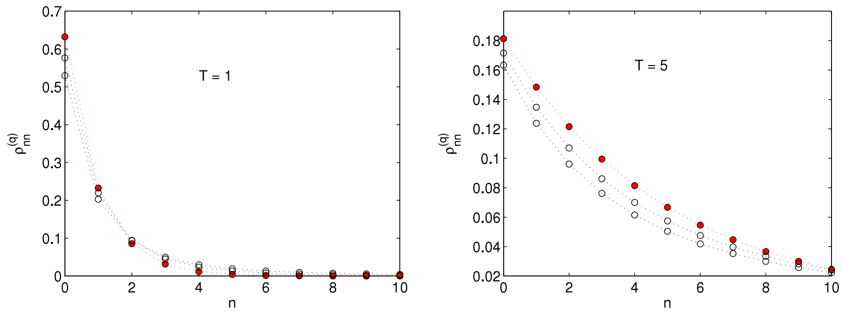

Fig. 3. $\rho_{nn}^{(q)}$ as a function of n, for $q = 1.00$ (red), 1.40 and 1.80 (black circles). We take $T = 1$ (left) and $T = 5$ (right). Units employed correspond to $\hbar\omega/k_B = 1$. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

stage is thereby made more transparent and the solutions of the simplified problem tractable, with their most relevant qualitative features represented in terms of the pertinent parameters. Indeed, a large amount of basic research is available that studies the differential equations obeyed by special functions. Emphasis is placed on connections with problems emerging in all kinds of physical theories.

The hypergeometric and confluent hypergeometric functions (h.f. and c.h.f. respectively) are of special great interest, being used with regard to almost all the solutions of exactly solvable problems in quantum mechanics. As examples one may cite those solutions related to the linear and p-dimensional harmonic oscillator, hydrogen-like, Pöschl–Teller, Woods–Saxon, Hulthen, Morse, Eckart, and Scarf potentials, amongst others. In these circumstances, the standard mechanism takes into account appropriate transformations of the concomitant variables and functions from the Schrödinger into an hypergeometric or confluent hypergeometric equation (h.e. and c.h.e. for the last two respectively) [19,20].

4.4. Mean value of $\hat{n}$

We generalize the mean value of $\hat{n}$ given by Eq. (16) as follows:

$$\langle \hat{n} \rangle_q = \int \frac{d^2\alpha}{\pi} P_q(\alpha)\, |\alpha|^2, \tag{48}$$

so that in the limit $q \to 1$ we recover $\langle \hat{n} \rangle_1 = \langle \hat{n} \rangle$. Replacing P_q given by (33) into Eq. (48) we then find

$$\langle \hat{n} \rangle_q = \sigma_q^{-1} = \frac{1}{\delta(2-q)}, \tag{49}$$

provided that $1 < q < 2$ and $\delta > 0$. Further, the quantum mean value of the particle-number $\hat{n}$ is defined as

$$\langle \hat{n} \rangle_q = \mathrm{Tr}(\hat{\rho}_q\, \hat{n}) = \sum_n n\, \rho_{nn}^{(q)} = \frac{1}{\delta(2-q)}, \tag{50}$$

where we have used the fact that

$$\sum_{n=0}^{\infty} \frac{n}{[1-(1-q)\delta t]^{1+n}} = \frac{1}{\delta^2(q-1)^2 t^2}, \tag{51}$$

together with the definition of the Euler Gamma function (40), and the Gamma function's relation $\Gamma[n] = (n-1)\Gamma[n-1]$ for all positive integers n. One arrives at the identity

$$\frac{\Gamma\left(\frac{2-q}{q-1}\right)}{\Gamma\left(\frac{q}{q-1}\right)} = \frac{(q-1)^2}{2-q}, \tag{52}$$

that we use to demonstrate (50). Therefore, via Eqs. (50) and (53), one is in a position to assert that

$$\langle \hat{n} \rangle_q = \mathrm{Tr}(\hat{\rho}_q\, \hat{n}) = \int \frac{d^2\alpha}{\pi} P_q(\alpha)\, |\alpha|^2. \tag{53}$$

Hence, by inspection of Eqs. (36) and (49), we note that $P_q(\alpha)$ can also be rewritten as

$$P_q(\alpha) = \frac{1}{(2-q)\langle \hat{n} \rangle_q}\, e_q\left(-\frac{|\alpha|^2}{(2-q)\langle \hat{n} \rangle_q}\right)^q, \tag{54}$$

which is an obvious generalization of Eq. (25). If q tends to unity, then $\langle \hat{n} \rangle_1 = \langle \hat{n} \rangle = 1/\delta$. We illustrate in Fig. 4 the behavior of $\langle \hat{n} \rangle_q$ in terms of the temperature, for different values of the parameter q. In the next section, we are going to calculate the associated Q_q-phase space function.

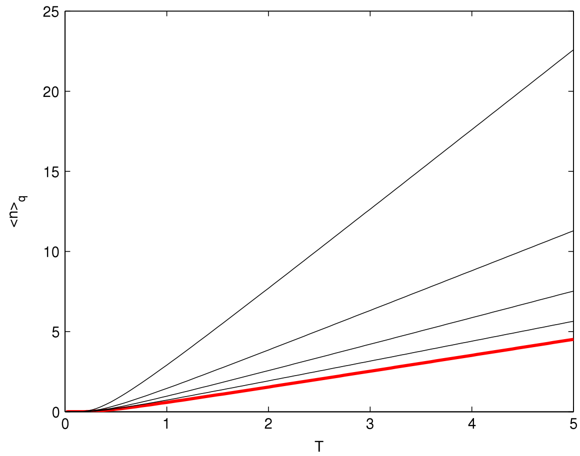

Fig. 4. $\langle \hat{n} \rangle_q$ as a function of T for $q = 1.00$ (red), 1.20, 1.40, 1.60, 1.80 (black). Units employed correspond to $\hbar\omega/k_B = 1$. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

5. Q_q-phase space representation

From Eq. (37), it is possible to obtain the relationship between Q_q and P_q with the help of the overlap between the states $|\alpha\rangle$ and $|z\rangle$, given by $|\langle\alpha|z\rangle|^2 = \exp(-|z-\alpha|^2)$, with $|z-\alpha|^2 = |z|^2 + |\alpha|^2 - \alpha^* z - z^*\alpha$. One has

$$Q_q(\alpha) = \int \frac{d^2 z}{\pi} P_q(z)\, e^{-|z-\alpha|^2}, \tag{55}$$

that we baptize as the q-function $Q_q(\alpha)$. In the limit $q \to 1$ we recover the usual expression, where $Q_1(\alpha) = Q(\alpha)$ and $P_1(\alpha) = P(\alpha)$. Introducing the escort distribution (33) into (55), we immediately find

$$Q_q(\alpha) = \delta \int \frac{d^2 z}{\pi}\, e_q(-\delta|z|^2)^q\, e^{-|z-\alpha|^2}. \tag{56}$$

By recourse to the integral representation (41), and integrating on the variable z with the help of

$$\int \frac{d^2 z}{\pi}\, e^{-\gamma_q(t)|z|^2}\, e^{z\alpha^* + z^*\alpha} = \gamma_q(t)^{-1}\, e^{\gamma_q(t)^{-1}|\alpha|^2}, \tag{57}$$

we finally get the q-function, whose form is

$$Q_q(\alpha) = \frac{\delta}{\Gamma\left(\frac{q}{q-1}\right)} \int_0^\infty dt\, \gamma_q(t)^{-1}\, t^d\, e^{-t}\, e^{-\frac{\gamma_q(t)-1}{\gamma_q(t)}|\alpha|^2}, \tag{58}$$

where we remember that $\gamma_q(t)$ was defined previously in Eq. (44). It is easy to check that this q-function is normalized in accordance with

$$\int \frac{d^2\alpha}{\pi}\, Q_q(\alpha) = 1. \tag{59}$$

The mean value of $\hat{n}$ (antinormal-ordered average) is given by (20), so that, inserting (58) into (20) we again find that

$$\langle \hat{n} \rangle_q = \frac{1}{\delta(2-q)}, \tag{60}$$

where we have made use to the integral

$$\int_0^\infty dt\, \gamma_q(t)\, t^{d-2} e^{-t} = (1 + (d-1)\delta(q-1))\,\Gamma(d-1), \tag{61}$$

together with the relationships (52) and (57).

6. W_q-phase space representation

The W_q-function can be obtained from the P_q-function from the generalized relation [12]

$$W_q(\alpha) = 2 \int \frac{d^2 z}{\pi} P_q(z)\, e^{-2|\alpha-z|^2}, \tag{62}$$

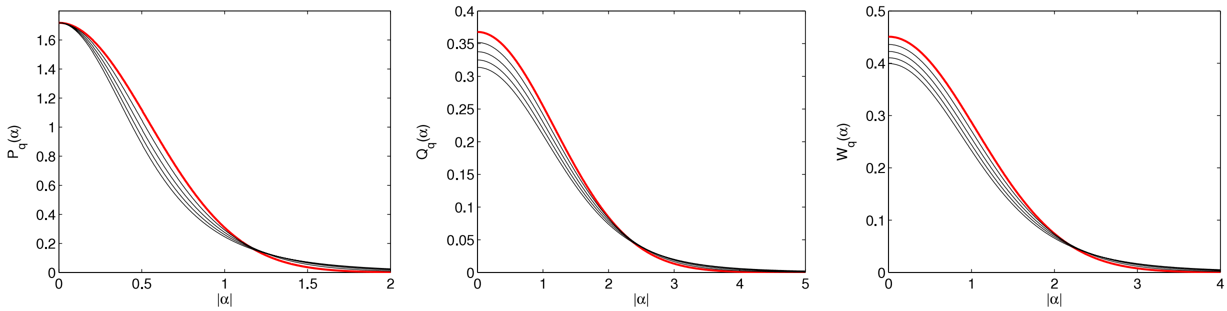

Fig. 5. P_q distribution (left), Q_q distribution (center) and W_q distribution for $q = 1.00$ (red), 1.20, 1.40, 1.60 and 1.80 (black). (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

that recovers its usual form (21) when $q \to 1$. Following the same procedure as in the previous section we reach

$$W_q(\alpha) = \frac{2\delta}{\Gamma\left(\frac{q}{q-1}\right)} \int_0^\infty \mathrm{d}t\, (1+\gamma_q(t))^{-1}\, t^d\, \mathrm{e}^{-t}\, \mathrm{e}^{-\frac{2(\gamma_q(t)-1)}{1+\gamma_q(t)}|\alpha|^2}, \tag{63}$$

with $\gamma_q(t)$ defined in expression (44). The mean value of $\hat{n}$ for symmetric ordered operators (whose form is given by (22)) is found as in the previous sections. Thus, inserting (63) into (22) we obtain

$$\langle \hat{n} \rangle_q = \frac{1}{\delta(2-q)}, \tag{64}$$

which is seen to be the same result for our three representations, that coincide with the quantum average as well. In Fig. 5 we appreciate the behavior of the P_q, Q_q, and W_q distributions as a function of $|\alpha|$, for several values of q.

7. Entropy in phase space

7.1. Shannon entropy

For a continuous probability distribution function $f(x)$, the Shannon entropy is [21]

$$S_1[f] = -\int_{-\infty}^{\infty} f(x) \ln f(x) \mathrm{d}x. \tag{65}$$

In the case of the 2-D PDF $P_q(\alpha)$, Eq. (29), the entropy becomes

$$S_1[P_q(\alpha)] = -\int \frac{\mathrm{d}^2\alpha}{\pi} P_q(\alpha) \ln P_q(\alpha). \tag{66}$$

One finds analytically that

$$S_1[P_q] = q - \ln(\delta), \tag{67}$$

which depends on T through $\delta = 1/\langle \hat{n} \rangle = \mathrm{e}^{\beta\hbar\omega} - 1$. We appreciate the fact that the critical temperature at which the entropy becomes negative decreases when q increases (see Fig. 6). Since, ideally, one would wish $T_c = 0$, one could speak of a certain superiority of the q-generalized *phase space distribution* over the ordinary one (at $q = 1$).

7.2. Tsallis entropy

The Tsallis entropy is a generalization of the standard Boltzmann–Gibbs entropy. It was introduced in 1988 by Constantino Tsallis [7–9], in the form (for a continuous probability distribution function $f(x)$)

$$S_\kappa = \frac{1}{\kappa - 1}\left(1 - \int \mathrm{d}x f(x)^\kappa\right), \tag{68}$$

where $f(x)$ is a probability density function and κ any real number. In the limit $\kappa \to 1$ one recovers the Shannon S_1 entropy. The analysis of the properties of Tsallis entropy for escort distributions has been exhaustively performed, for one variable, in Ref. [22].

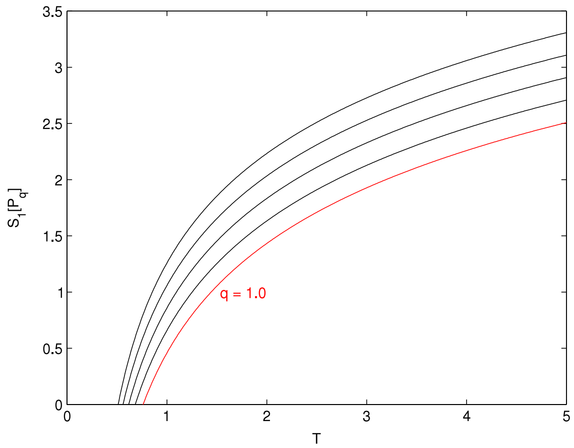

Fig. 6. $S_1[P_q]$ vs. T, for $q = 1.00$ (red) and $q = 1.20, 1.40, 1.60$, and 1.80 (black). Units employed correspond to $\hbar\omega/k_B = 1$. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

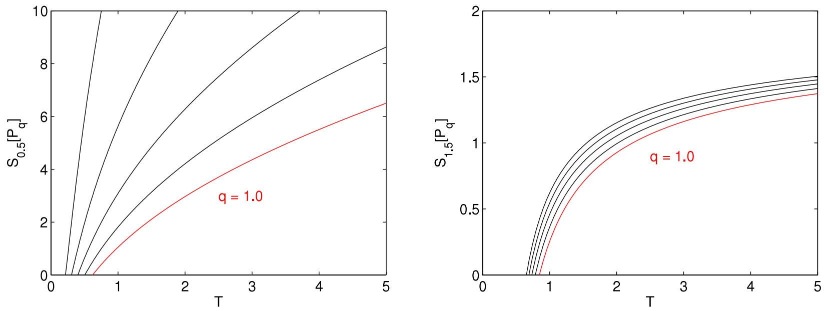

Fig. 7. $S_\kappa(P_q)$ vs. T for $\kappa = 0.50$ (left) and $\kappa = 1.50$ (right). Black lines correspond to $q = 1.00, 1.20, 1.40$, and 1.60. Units employed correspond to $\hbar\omega/k_B = 1$. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

Here we are interested in a Tsallis's entropy expressed in phase space variables. One has

$$S_\kappa[P_q] = \frac{1}{\kappa - 1}\left(1 - \int \frac{d^2\alpha}{\pi}\, P_q(\alpha)^\kappa\right), \tag{69}$$

where κ denotes a real parameter. Integrating over all phase space, we obtain

$$S_\kappa[P_q] = \frac{1}{\kappa - 1}\left(1 - \frac{\delta^{\kappa-1}}{1 + q(\kappa - 1)}\right), \tag{70}$$

with $\kappa > q - 1$ and $q > 1$. We can see in Fig. 7, that the smaller the κ is, the lower the critical temperature. Thus, Tsallis entropy S_κ with $\kappa < 1$ allows for a wider range of temperatures with positive entropy than Shannon's one.

We note that the entropy $S_\kappa \geq 0$, whenever $T \geq T_c$, where the critical temperature is

$$T_c = \frac{\hbar\omega/k_B}{\ln[1 + e_\kappa(q)]}; \quad \kappa \neq 1, \tag{71}$$

and

$$T_c = \frac{\hbar\omega/k_B}{\ln(1 + e^q)}; \quad \kappa = 1. \tag{72}$$

We can see how the quantity T_c depends on both q and κ in Fig. 8.

8. Conclusions

In this paper we have studied a possible generalization of quantum distributions in phase space to a nonextensive scenario by proposing, as a first step in such a direction, a q-gaussian distribution as a feasible P-function, that we call P_q. Thus, instead of the P-function for the harmonic oscillator for a thermal state we have a q-gaussian function P_q for a system that reduces to the harmonic oscillator when the parameter q tends to unity. There is a whole family of q-distributions that

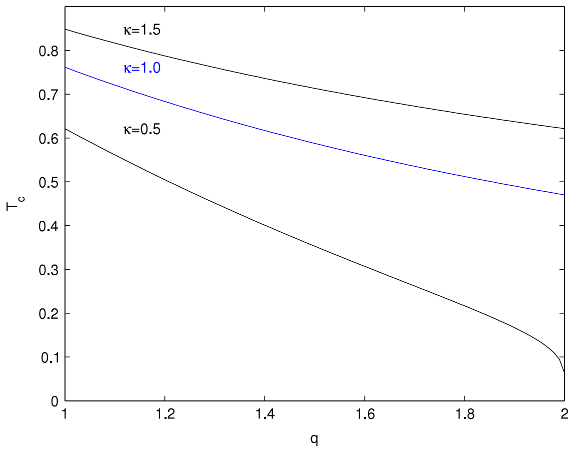

Fig. 8. T_c vs. q for $\kappa = 0.50$, 1.00 and 1.50. Units employed correspond to $\hbar\omega/k_B = 1$.

tend to $P(\alpha)$ in this limit, whose members are the well-known escort distributions of order q. The range of validity of this formulation es $1 < q < 2$. This generalized function P_q, which is analytically tractable, allows us to calculate the other two generalized quantum distributions, Q_q-and W_q-functions, respectively, by means of transformations frequently used in this kind of approach.

Our results are expressed in terms of confluent hypergeometric functions, frequent in diverse areas of theoretical physics. A connection between these functions and a Tsallis environment is reported in Ref. [23].

In addition, we have shown that the definition of P_q is consistent with the associated diagonal representation of the generalized density operator $\hat{\rho}_q$. Furthermore, we also noted that our three distributions do lead to the correct quantum average of the particle number operator.

Finally, we have calculated the entropy for an escort distribution in phase space, discovering that the entropy is negative if the temperature is smaller than a critical value. This last quantity depends on the parameter q and reaches its minimum value for q approaching the value 2, and the corresponding maximum at $q = 1$. Nonextensive entropy exhibits, in a way of speaking, "larger positivity regions" when the parameter κ decreases.

Our approach sheds new light on possible future applications to statistical nonextensive systems.

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References

[1] W.P. Schleich, Quantum Optics in Phase Space, Wiley-VCH Verlag, Berlin, 2001.
[2] R.J. Glauber, Phys. Rev. 131 (1963) 2766.
[3] E.C.G. Sudarshan, Phys. Rev. Lett. 10 (1963) 277.
[4] K. Husimi, Proc. Phys. Math. Soc. Japan 22 (1940) 264.
[5] E.P. Wigner, Phys. Rev. 40 (1932) 749.
[6] C. Tsallis, Braz. J. Phys. 39 (2009) 337.
[7] C. Tsallis, J. Stat. Phys. 52 (1988) 479.
[8] C. Tsallis, et al., in: S. Abe, Y. Okamoto (Eds.), Nonextensive Statistical Mechanics and its Applications, Springer-Verlag, Heidelberg, 2001.
[9] C. Tsallis, Introduction to Nonextensive Statistical Mechanics, Springer-Verlag, Heidelberg, 2009.
[10] S. Picoli Jr., R.S. Mendes, L.C. Malacarne, R.P.B. Santos, Braz. J. Phys. 39 (2009) 468.
[11] C. Brif, Y. Ben-Aryeh, Quantum Opt. 6 (1994) 391.
[12] H.J. Carmichael, Statistical Methods in Quantum Optics, Vol. I, Springer, Berlin, 2010.
[13] F. Reif, Fundamentals of Statistical and Thermal Physics, McGraw Hill, New York, 1965.
[14] C. Beck, F. Schlögl, Thermodynamics of Chaotic Systems: An Introduction, Cambridge University Press, Cambridge, 1993.
[15] C. Tsallis, in: J.L. Morán-López, J.M. Sanchez (Eds.), New Trends in Magnetism, Magnetic Materials and Their Applications, Plenum Press, New York, 1994, p. 451.
[16] I.S. Gradshteyn, I.M. Ryzhik, Table of Integrals, Series and Products, sixth ed., Academic Press, San Diego, 2000.
[17] A.R. Plastino, A. Plastino, Phys. Lett. A 193 (1994) 251.
[18] M. Abramowitz, I.A. Stegun (Eds.), Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables, Dover, New York, 1965.
[19] S. Flugge, Practical Quantum Mechanics, Springer, Berlin, 1971.
[20] N. Breton, J.S. Dyas, and H. Quevedo (Eds.), in: Proceedings of the Fourth Mexican School on Gravitation and Mathematical Physics, Membranes 2000, Huatulco Oax, Mexico.
[21] C.E. Shannon, W. Weaver, The Mathematical Theory of Communication, University of Illinois Press, Urbana, Chicago, 1949.
[22] R.P. Di Sisto, S. Martinez, R.B. Orellana, A.R. Plastino, A. Plastino, Physica A 265 (1999) 590.
[23] P. Jizba, H. Kleinert, P. Haenerc, Physica A 388 (2009) 3503.