



On dB spaces with nondensely defined multiplication operator and the existence of zero-free functions



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ABSTRACT

In this work we consider de Branges spaces where the multiplication operator by the independent variable is not densely defined. First, we study the canonical selfadjoint extensions of the multiplication operator as a family of rank-one perturbations from the viewpoint of the theory of de Branges spaces. Then, on the basis of the obtained results, we provide new necessary and sufficient conditions for a real, zero-free function to lie in a de Branges space.

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1. Introduction

In this paper we deal with some properties of the class of de Branges spaces (dB spaces) characterized by the fact that the operator S of multiplication by the independent variable is not densely defined. We recall that a de Branges space \mathcal{B} is a Hilbert space of entire functions which can be defined by means of an Hermite–Biehler function $e(z)$ (for details see Section 2). As it is well known, when the domain of S , denoted $\text{dom}(S)$, is not dense, its codimension equals one [3, Theorem 29]. In particular, in such case $\text{dom}(S)$ is orthogonal to one of the associated functions

$$s_\beta(z) := \frac{i}{2} [e^{i\beta} e(z) - e^{-i\beta} e^\#(z)] = \sin \beta s_{\pi/2}(z) + \cos \beta s_0(z), \quad (1)$$

where $\beta \in [0, \pi)$. As we recall in Section 3, this family of functions is in one-to-one correspondence with the set of canonical (that is, within \mathcal{B}) selfadjoint extensions S_β of S . Moreover, the function $s_\beta(z)$ that is

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orthogonal to $\text{dom}(S)$ is the only one of this family that belongs to \mathcal{B} . Without loss of generality we shall henceforth assume that this occurs for $s_0(z)$ (otherwise one can always perform a change of parametrization).

We begin by looking at the operator S and its canonical selfadjoint extensions. The main result here is [Theorem 3.4](#), where we render the selfadjoint operator extensions of S as a family of rank-one perturbations of $S_{\pi/2}$ along the function $s_0(z) \in \mathcal{B}$, viz.,

$$S_\beta = S_{\pi/2} - \frac{\cot \beta}{\pi} \langle s_0(\cdot), \cdot \rangle_{\mathcal{B}} s_0(z), \quad \beta \in (0, \pi). \tag{2}$$

Generically speaking, a formula of this sort is known to be valid from the abstract theory of rank-one perturbations of relations with deficiency indices $(1, 1)$; see for instance [\[4\]](#). However, we derive [\(2\)](#) exclusively from the properties of functions in dB spaces and the family $s_\beta(z)$, $\beta \in [0, \pi)$. We believe that this derivation yields further insight on the interplay between the functions $s_\beta(z)$ and the corresponding selfadjoint extensions of S . In passing, we note that the selfadjoint extension S_0 is not itself an operator but rather a (multi-valued) linear relation; see [\(7\)](#) below.

Eq. [\(2\)](#) suggests studying whether $s_0(z)$ is a generating vector of a selfadjoint extension of S . For a definition of generating vector we refer the reader to [\[1, Section 69, Definition 1\]](#). [Theorem 3.5](#) asserts that $s_0(z)$ is a generating vector for $S_{\pi/2}$, and therefore, for all of the selfadjoint extensions of S with the exception of S_0 .

With these results at hand, we tackle the question of whether a dB space of the class considered in this work has a zero-free function. The existence of a real zero-free function in a dB space (or more generally, in certain spaces of functions associated with it) has been studied in great detail; see for instance [\[8,11,12\]](#). From the point of view of Krein’s theory of entire operators [\[7\]](#), if $g(z) \in \mathcal{B}$ is zero-free then it is an entire gauge for S , viz., it satisfies

$$\mathcal{B} = \text{ran}(S - wI) \dot{+} \text{span}\{g(z)\}, \quad \forall w \in \mathbb{C}. \tag{3}$$

In [Theorem 4.1](#) we show that a real zero-free function of the form $s_\beta(z)/j_\beta(z)$ is in \mathcal{B} , where $j_\beta(z)$ is any real entire function whose zero-set coincides with that of $s_\beta(z)$, if and only if

$$\frac{1}{j_\beta(z)} = \sum_{k=1}^{\infty} \frac{c_k}{z - x_k},$$

where $\{c_k\}_{k \in \mathbb{N}}$ satisfies

$$\sum_{k=1}^{\infty} |c_k|^2 \left| \frac{s'_\beta(x_k)}{s_0(x_k)} \right| < \infty.$$

[Theorem 4.1](#) does not hold for $\beta = 0$. This case is treated apart in [Theorem 4.2](#), where specific necessary and sufficient conditions, for a real zero-free function of the form $s_0(z)/j_0(z)$ to be in \mathcal{B} , are given. This characterization is based on the fact, elaborated in [Remark 4](#), that every zero-free function in \mathcal{B} is a generating vector for some, hence every, selfadjoint operator extension of S .

According to [\[12, Theorem 3.2\]](#), if there exists a real zero-free function in the dB space, then this function is unique up to a multiplicative real constant. Therefore, in this case, all the functions $s_\beta(z)/j_\beta(z)$, $\beta \in [0, \pi)$, are basically the same one. Thus, [Theorems 4.1 and 4.2](#) give two different characterizations of dB spaces with nondensely defined multiplication operator and having zero-free functions. Note also that, since [\(3\)](#) means that S is an entire operator, each of these theorems provides necessary and sufficient conditions for the operator S to be entire.

It is worth remarking that the characterizations of dB spaces with zero-free functions given by [Theorems 4.1 and 4.2](#) differ from all the characterizations already known, viz., from the one stated by de Branges [\[3, Theorem 25\]](#) and those found, with diverse degree of generalization, in [\[8,11,12\]](#).

By the end of this note we briefly address the question of how the generating vector $s_0(z)$ and the entire gauge $g(z)$ are related. [Proposition 4.4](#) is a simple observation on a connection between these two functions within the dB space.

2. Preliminaries

In what follows by a dB space we will always mean a de Branges Hilbert space.

The usual definition of a dB space starts from an Hermite–Biehler function, that is, an entire function $e(z)$ satisfying $|e(z)| > |e(\bar{z})|$ for all $z \in \mathbb{C}^+$. Then, the dB space generated by $e(z)$ is defined as

$$\mathcal{B}(e) := \{f(z) \text{ entire} : f(z)/e(z), f^\#(z)/e(z) \in H_2(\mathbb{C}^+)\},$$

where $H_2(\mathbb{C}^+)$ is the Hardy space

$$H_2(\mathbb{C}^+) := \left\{ f(z) \text{ is holomorphic in } \mathbb{C}^+ : \sup_{y>0} \int_{\mathbb{R}} |f(x+iy)|^2 dx < \infty \right\};$$

here \mathbb{C}^+ denotes the open upper half-plane. We also use the standard notation $f^\#(z) := \overline{f(\bar{z})}$. The linear space $\mathcal{B}(e)$ equipped with the inner product

$$\langle g, f \rangle_{\mathcal{B}} := \int_{\mathbb{R}} \frac{\overline{g(x)}f(x)}{|e(x)|^2} dx \quad (4)$$

is a Hilbert space [\[10, Theorem 2.2\]](#).

There are alternative definitions of a dB space; see for instance [\[10, Proposition 2.1\]](#) and [\[3, Chapter 2\]](#). It is also possible to define a de Branges space without relying on a given Hermite–Biehler function [\[3, Problem 50\]](#). Moreover, a given dB space can be generated by different Hermite–Biehler functions [\[2, Theorem 1\]](#).

By definition, a dB space has a reproducing kernel, that is, there exists a function $k(z, w)$ that belongs to \mathcal{B} for all $w \in \mathbb{C}$ and satisfies the property $\langle k(\cdot, w), f(\cdot) \rangle_{\mathcal{B}} = f(w)$ for all $f(z) \in \mathcal{B}$. Moreover, $k(w, z) = \overline{k(z, w)}$ and $\overline{k(\bar{z}, w)} = k(z, \bar{w})$ [\[3, Theorem 23\]](#).

One important operator in a dB space is the operator of multiplication by the independent variable,

$$\text{dom}(S) = \{f(z) \in \mathcal{B} : zf(z) \in \mathcal{B}\}, \quad (Sf)(z) := zf(z).$$

This operator is symmetric, closed, regular, and has deficiency indices $(1, 1)$. Its domain has codimension 1 or 0, depending on whether one (and in that case, only one) of the functions $s_\beta(z)$ is within \mathcal{B} or none is [\[3, Theorem 29\]](#).

To any de Branges space there corresponds a so-called space of associated functions [\[3, Section 25\]](#). This space can be succinctly defined by

$$\text{assoc } \mathcal{B} := \mathcal{B} + z\mathcal{B}$$

(see [\[5, Lemma 4.5\]](#)). Within $\text{assoc } \mathcal{B}$ lies the distinguished family of entire functions $s_\beta(z)$ defined by [\(1\)](#). Generically, $s_\beta(z) \in \text{assoc } \mathcal{B} \setminus \mathcal{B}$. As already mentioned, this family of functions is in bijective correspondence

with the set of canonical selfadjoint extension of S (see (6) and (7) below). From its definition, it follows that $s_\beta(z)$ is real (that is, it satisfies $\overline{s_\beta(z)} = s_\beta(\bar{z})$), it can also be verified that this function has only simple zeros and its zero-set coincides with the spectrum of the corresponding selfadjoint extension S_β .

The reproducing kernel can be written in terms of the functions $s_\beta(z)$. In particular [8, Section 2],

$$k(z, w) = \begin{cases} \frac{s_{\pi/2}(z)s_0(\bar{w}) - s_0(z)s_{\pi/2}(\bar{w})}{\pi(z - \bar{w})}, & z \neq \bar{w}, \\ \frac{1}{\pi}[s'_{\pi/2}(z)s_0(z) - s_{\pi/2}(z)s'_0(z)], & z = \bar{w}. \end{cases} \tag{5}$$

3. Selfadjoint extensions of S

Since we are assuming that the multiplication operator S in \mathcal{B} is not densely defined, one of the functions $s_\beta(z)$ necessarily belongs to \mathcal{B} . As already mentioned, we may suppose that this happens for $\beta = 0$. Consequently, $\text{dom}(S)^\perp = \text{span}\{s_0(z)\}$ [3, Theorem 29]. The selfadjoint operator extensions of S , corresponding to $\beta \in (0, \pi)$, can be described as follows [5, Propositions 4.5 and 6.1] (cf. [11, Proposition 3.8]),

$$\text{dom}(S_\beta) = \left\{ g(z) = \frac{s_\beta(w)f(z) - s_\beta(z)f(w)}{\sin \beta(z - w)}, f(z) \in \mathcal{B}, w \in \mathbb{C} \right\}, \tag{6a}$$

$$(S_\beta g)(z) = zg(z) + \frac{1}{\sin \beta} f(w)s_\beta(z), \tag{6b}$$

while the remainder selfadjoint extension of S is given by the linear relation

$$S_0 = \{ (g(z), zg(z) + cs_0(z)) : g(z) \in \text{dom}(S), c \in \mathbb{C} \}; \tag{7}$$

clearly $\text{dom}(S_0) = \text{dom}(S)$.

Lemma 3.1. *Assume $s_0(z) \in \mathcal{B}$. Then $\text{dom}(S_\beta) = \text{dom}(S_{\pi/2})$ for all $\beta \in (0, \pi)$. Furthermore,*

$$(S_\beta g)(z) = (S_{\pi/2}g)(z) + \frac{\cos \beta}{\sin \beta} f(w)s_0(z), \tag{8}$$

for all $g(z) \in \text{dom}(S_\beta)$ and where $f(z)$ is related to $g(z)$ by (6a).

Proof. Consider $g(z) \in \text{dom}(S_\beta)$. By (6a),

$$g(z) = \frac{s_\beta(w)f(z) - s_\beta(z)f(w)}{\sin \beta(z - w)}$$

for some $f(z) \in \mathcal{B}$. Using (1) this can be written as

$$g(z) = \frac{s_{\pi/2}(w)f(z) - s_{\pi/2}(z)f(w)}{z - w} + \frac{\cos \beta}{\sin \beta} \frac{s_0(w)f(z) - s_0(z)f(w)}{z - w}. \tag{9}$$

The first term above belongs to $\text{dom}(S_{\pi/2})$ due to (6a), hence the second one belongs to \mathcal{B} . Moreover, since $s_0(z) \in \mathcal{B}$, the numerator of the second term is in \mathcal{B} and, therefore, it is in $\text{ran}(S - wI)$. Thus, the second term in (9) lies in $\text{dom}(S)$. Taking into account that $\text{dom}(S) \subset \text{dom}(S_{\pi/2})$, one concludes that there exists $n(z) \in \mathcal{B}$ such that

$$\frac{\cos \beta}{\sin \beta} \frac{s_0(w)f(z) - s_0(z)f(w)}{z - w} = \frac{s_{\pi/2}(w)n(z) - s_{\pi/2}(z)n(w)}{z - w}. \tag{10}$$

The fact that both terms in (9) belong to $\text{dom}(S_{\pi/2})$ shows that $\text{dom}(S_{\beta}) \subseteq \text{dom}(S_{\pi/2})$. Since in the argument above we can switch the roles of S_{β} and $S_{\pi/2}$, we in fact have $\text{dom}(S_{\beta}) = \text{dom}(S_{\pi/2})$.

Since the numerator of the l.h.s. of (10) lies in \mathcal{B} , it also does the numerator of the r.h.s. As $s_{\pi/2}(z) \notin \mathcal{B}$, necessarily $n(w) = 0$, that is, $n(z) \in \text{ran}(S - wI)$. Let $h(z) := f(z) + n(z)$. Then, resorting to (1) once again, we obtain

$$\begin{aligned} (S_{\beta}g)(z) &= zg(z) + \frac{1}{\sin \beta} f(w)s_{\beta}(z) \\ &= zg(z) + f(w)s_{\pi/2}(z) + \frac{\cos \beta}{\sin \beta} f(w)s_0(z) \\ &= zg(z) + h(w)s_{\pi/2}(z) - n(w)s_{\pi/2}(z) + \frac{\cos \beta}{\sin \beta} f(w)s_0(z), \end{aligned}$$

which yields (8). \square

The following assertion does not depend on assuming that a function $s_{\beta}(z)$ is in \mathcal{B} (that is, it holds on any dB space).

Lemma 3.2. For every $s_{\beta}(z)$, $\beta \in [0, \pi)$, and $h(z) \in \text{dom}(S)$,

$$\int_{-\infty}^{\infty} \frac{\overline{s_{\beta}(x)}h(x)}{|e(x)|^2} dx = 0.$$

Proof. Let x_0 be a zero of $s_{\beta}(z)$. Then, by (6), $k(z) := s_{\beta}(z)/(z - x_0)$ is an eigenfunction of S_{β} with eigenvalue x_0 . Therefore,

$$\int_{-\infty}^{\infty} \frac{\overline{s_{\beta}(x)}h(x)}{|e(x)|^2} dx = \int_{-\infty}^{\infty} \frac{\overline{k(x)}(x - x_0)h(x)}{|e(x)|^2} dx = \langle k(\cdot), (S - x_0I)h(\cdot) \rangle_{\mathcal{B}} = 0,$$

where the last identity follows after realizing that $k(z) \in \ker(S^* - x_0I)$ and the multi-valued part of S^* equals $\text{span}\{s_0(z)\}$ [4]. \square

Lemma 3.3. Let $s_0(z) \in \mathcal{B}$. For $g(z) \in \text{dom}(S_{\beta})$, $f(z) \in \mathcal{B}$ and $w \in \mathbb{C}$ related to each other by (6a), one has $\langle s_0(\cdot), g(\cdot) \rangle_{\mathcal{B}} = -\pi f(w)$.

Proof. Let us start by considering (9). As already mentioned, the second term of this identity lies in $\text{dom}(S)$, so

$$\begin{aligned} \langle s_0(\cdot), g(\cdot) \rangle_{\mathcal{B}} &= \int_{-\infty}^{\infty} \frac{\overline{s_0(x)}s_{\pi/2}(w)f(x) - \overline{s_0(x)}s_{\pi/2}(x)f(w)}{|e(x)|^2(x - w)} dx \\ &= -\pi \langle k(\cdot, w), f(\cdot) \rangle_{\mathcal{B}} \\ &\quad + \int_{-\infty}^{\infty} \frac{\overline{s_{\pi/2}(x)}}{|e(x)|^2} \left[\frac{s_0(w)f(x) - s_0(x)f(w)}{x - w} \right] dx, \end{aligned}$$

where the fact that the functions $s_{\beta}(z)$ are real has been used. In the second term, the expression between squared brackets lies in $\text{dom}(S)$ so by Lemma 3.2 this term equals zero. Obviously the first term equals $-\pi f(w)$ so the assertion is proven. \square

Theorem 3.4. Assume $s_0(z) \in \mathcal{B}$. Then the set of canonical selfadjoint operator extensions of S is given by $\text{dom}(S_\beta) = \text{dom}(S_{\pi/2})$,

$$S_\beta = S_{\pi/2} - \frac{\cot \beta}{\pi} \langle s_0(\cdot), \cdot \rangle_{\mathcal{B}} s_0(z), \tag{11}$$

for $\beta \in (0, \pi)$.

Proof. The assertion follows straightforwardly from Lemmas 3.1 and 3.3. \square

The previous discussion generalizes effortlessly if one assumes that $s_\gamma(z) \in \mathcal{B}$, $\gamma \in [0, \pi)$. For $\beta \in [\gamma, \gamma + \pi)$, it is easy to see that

$$s_\beta(z) = \sin(\beta - \gamma) s_{\gamma+\pi/2}(z) + \cos(\beta - \gamma) s_\gamma(z),$$

so instead of (11) one has

$$S_\beta = S_{\gamma+\pi/2} - \frac{\cot(\beta - \gamma)}{\pi} \langle s_\gamma(\cdot), \cdot \rangle_{\mathcal{B}} s_\gamma(z),$$

now for $\beta \in (\gamma, \gamma + \pi)$.

Now we turn to the proof of the fact that $s_0(z)$ is generating element (see [1, Section 69]) of $S_{\pi/2}$.

Theorem 3.5. Assume that $s_0(z) \in \mathcal{B}$. Then, for every $\beta \in (0, \pi)$, $s_0(z)$ is a generating vector for the operator S_β .

Proof. Since $s_0(z) \in \mathcal{B}$, one has, on the basis of (6a) and (6b), that

$$\begin{aligned} (S_{\pi/2} - wI)^{-1} s_0(z) &= \frac{1}{s_{\pi/2}(w)} \frac{s_{\pi/2}(w) s_0(z) - s_{\pi/2}(z) s_0(w)}{z - w} \\ &= -\frac{\pi}{s_{\pi/2}(w)} k(z, \bar{w}) \end{aligned}$$

for all $w \notin \text{spec}(S_{\pi/2})$. Hence, taking into account that (see [11, Section 3])

$$k(z, \bar{w}) \in \ker(S^* - wI) \quad \text{for any } w \in \mathbb{C}, \tag{12}$$

one verifies

$$\overline{\text{span}_{w \in \text{spec}(S_0)} \{(S_{\pi/2} - wI)^{-1} s_0(z)\}} = \mathcal{B}.$$

Thus, $s_0(z)$ is a generating element for $S_{\pi/2}$, but then, it can be derived from (11), that $s_0(z)$ is a generating vector for S_β with $\beta \in (0, \pi)$. \square

Remark 1. Alternatively, $s_0(z)$ is a generating vector for S_β , $\beta \in (0, \pi)$, because it has a nonzero projection to each eigenspace of S_β . Indeed, this follows from (12) and the fact that the eigenvalues of S_β with $\beta \in (0, \pi)$ never intersect the zeros of $s_0(z)$. In passing, it is also clear that $s_0(z)$ is not a generating vector for S_0 .

4. On the existence of a zero-free function

Let $E_\beta(t)$ be the resolution of the identity of S_β , $\beta \in (0, \pi)$. Define the family of spectral functions

$$m_\beta(t) := \langle s_0(\cdot), E_\beta(t)s_0(\cdot) \rangle_{\mathcal{B}} = \sum_{x_n < t} \frac{|s_0(x_n)|^2}{\|k(\cdot, x_n)\|_{\mathcal{B}}^2}, \quad \{x_n\}_{n \in \mathbb{N}} = \text{spec}(S_\beta).$$

Since $s_0(z)$ is a generating element of S_β , $\beta \in (0, \pi)$, one can consider the family of canonical isometries $U_\beta : L_2(\mathbb{R}, m_\beta) \xrightarrow{\text{onto}} \mathcal{B}$ (cf. [1, Section 69, Theorem 2]) given by

$$(U_\beta f)(z) := f(S)s_0(z) = \sum_{x_n \in \text{spec}(S_\beta)} \frac{f(x_n)\langle k(\cdot, x_n), s_0(\cdot) \rangle_{\mathcal{B}}}{\|k(\cdot, x_n)\|_{\mathcal{B}}^2} k(z, x_n). \tag{13}$$

Theorem 4.1. *Assume $s_0(z) \in \mathcal{B}$ and fix $\beta \in (0, \pi)$. Let $j_\beta(z)$ be any real entire function with simple zeros exactly at $\{x_n\}_{n \in \mathbb{N}} = \text{spec}(S_\beta)$. The zero-free function $s_\beta(z)/j_\beta(z)$ is in \mathcal{B} if and only if the reciprocal of the function $j_\beta(z)$ can be decomposed as follows,*

$$\frac{1}{j_\beta(z)} = \sum_{k=1}^{\infty} \frac{c_k}{z - x_k}, \tag{14}$$

where $\{c_k\}_{k \in \mathbb{N}}$ satisfies

$$\sum_{k=1}^{\infty} |c_k|^2 \left| \frac{s'_\beta(x_k)}{s_0(x_k)} \right| < \infty \tag{15}$$

and the convergence in (14) is uniform on compact subsets of $\mathbb{C} \setminus \text{spec}(S_\beta)$.

Proof. We begin by proving the necessity of the condition for $\beta = \pi/2$. Since $s_0(z)$ is a generating vector for the operator $S_{\pi/2}$, for every $f \in L_2(\mathbb{R}, m_{\pi/2})$, $f(S)s_0(z)$ is an element of \mathcal{B} , and any vector in \mathcal{B} can be written in this way. Using the properties of the reproducing kernel and

$$\|k(\cdot, x_n)\|_{\mathcal{B}}^2 = \langle k(\cdot, x_n), k(\cdot, x_n) \rangle_{\mathcal{B}} = -\frac{1}{\pi} s'_{\pi/2}(x_n) s_0(x_n), \tag{16}$$

which is obtained from (5), one can rewrite the action of $U_{\pi/2}$ as follows

$$(U_{\pi/2} f)(z) = -\pi \sum_{x_n \in \text{spec}(S_{\pi/2})} \frac{f(x_n)k(z, x_n)}{s'_{\pi/2}(x_n)}.$$

Suppose that $s_{\pi/2}(z)/j_{\pi/2}(z)$ is in \mathcal{B} ; then there is a function $f \in L_2(\mathbb{R}, m_{\pi/2})$ such that

$$\begin{aligned} \frac{s_{\pi/2}(z)}{j_{\pi/2}(z)} &= -\pi \sum_{x_n \in \text{spec}(S_{\pi/2})} \frac{f(x_n)}{s'_{\pi/2}(x_n)} k(z, x_n) \\ &= - \sum_{x_n \in \text{spec}(S_{\pi/2})} \frac{f(x_n)s_{\pi/2}(z)s_0(x_n)}{s'_{\pi/2}(x_n)(z - x_n)}, \end{aligned} \tag{17}$$

where we have used (5). Hence, one has

$$\frac{1}{j_{\pi/2}(z)} = - \sum_{x_n \in \text{spec}(S_{\pi/2})} \frac{f(x_n)s_0(x_n)}{s'_{\pi/2}(x_n)(z - x_n)},$$

where the series converges uniformly on compacts of $\mathbb{C} \setminus \text{spec}(S_{\pi/2})$ since (17) converges in \mathcal{B} . By setting

$$c_n = - \frac{f(x_n)s_0(x_n)}{s'_{\pi/2}(x_n)}$$

one establishes the necessity of the condition.

Let us now prove that the condition is sufficient for $\beta = \pi/2$. For any $n \in \mathbb{N}$, define

$$a_n := \frac{c_n s'_{\pi/2}(x_n)}{s_0(x_n)}$$

and substitute it into (14) to obtain

$$\frac{1}{j_{\pi/2}(z)} = \sum_{n=1}^{\infty} \frac{a_n s_0(x_n)}{s'_{\pi/2}(x_n)(z - x_n)}.$$

Therefore, using (5) and (16), one has

$$\begin{aligned} \frac{s_{\pi/2}(z)}{j_{\pi/2}(z)} &= \sum_{n=1}^{\infty} \frac{a_n s_{\pi/2}(z)s_0(x_n)}{s'_{\pi/2}(x_n)(z - x_n)} \\ &= \sum_{n=1}^{\infty} \frac{a_n \langle k(\cdot, x_n), s_0(\cdot) \rangle}{\|k(\cdot, x_n)\|^2} k(z, x_n) \end{aligned} \tag{18}$$

for any $z \in \mathbb{C}$. By definition of the numbers $\{a_n\}_{n \in \mathbb{N}}$ there is a function $f \in L_2(\mathbb{R}, m_{\pi/2})$ such that $f(x_n) = a_n$ for all $n \in \mathbb{N}$. Thus, (18) means that $s_{\pi/2}(z)/j_{\pi/2}(z) = (U_{\pi/2}f)(z) \in \mathcal{B}$.

Once the assertion has been proven for $\beta = \pi/2$, one uses [11, Lemmas 3.3 and 3.4] to finish the proof. \square

Remark 2. A. Baranov pointed out to us that Theorem 4.1 for $\beta = \pi/2$ can be proven by expanding the function $s_{\pi/2}(z)/j_{\pi/2}(z)$ with respect to the orthonormal basis $k(z, x_n)/\|k(\cdot, x_n)\|$ (with $\{x_n\}_{n \in \mathbb{N}} = \text{spec}(S_{\pi/2})$), thus obviating the use of a generating vector.

Remark 3. If (14) and (15) hold, and additionally we suppose that

$$|c_n|(1 + |x_n|) \geq \left| \frac{s_0(x_n)}{s_{\pi/2}(x_n)} \right| \quad \text{for all } n \in \mathbb{N},$$

then, due to a theorem by Krein [9, Lecture 16, Theorem 3], the function $s_{\beta}(z)/j_{\beta}(z)$ is in the Cartwright class.

Remark 4. Clearly, if a zero-free function belongs to \mathcal{B} , then it is a generating vector for S_{β} with $\beta \in [0, \pi)$, since it has a nonzero projection onto every eigenspace (cf. Remark 1).

By using the fact that $s_0(z)/j_0(z)$ is a generating vector for any selfadjoint extension whenever $s_0(z)/j_0(z) \in \mathcal{B}$, we prove the following assertion which gives a different set of necessary and sufficient conditions for a zero-free function to be in \mathcal{B} .

Theorem 4.2. Assume $s_0(z) \in \mathcal{B}$ and let $j_\beta(z)$ be defined as in [Theorem 4.1](#). If the function $s_0(z)/j_0(z)$ is in \mathcal{B} , then, for all $\beta \in (0, \pi)$,

$$\frac{1}{j_0(t)} \in L_2(\mathbb{R}, m_\beta) \quad \text{and} \quad \frac{s_0(z)}{j_0(z)} = \left(U_\beta \frac{1}{j_0} \right)(z). \tag{19}$$

Conversely, if there exists a set $\mathcal{I} \subset (0, \pi)$ having an accumulation point and such that

$$\frac{1}{j_0(t)} \in L_2(\mathbb{R}, m_\beta) \quad \forall \beta \in \mathcal{I} \quad \text{and} \quad \left(U_\beta \frac{1}{j_0} \right)(z) = \left(U_{\beta'} \frac{1}{j_0} \right)(z) \quad \forall \beta, \beta' \in \mathcal{I},$$

then $s_0(z)/j_0(z)$ is in \mathcal{B} .

Proof. Assume that $s_0(z)/j_0(z) \in \mathcal{B}$. Define the spectral functions

$$\tilde{m}_\beta(t) := \langle s_0(\cdot)/j_0(\cdot), E_\beta(t)s_0(\cdot)/j_0(\cdot) \rangle_{\mathcal{B}},$$

and the isometries \tilde{U}_β from $L_2(\mathbb{R}, \tilde{m}_\beta)$ onto \mathcal{B} such that $(\tilde{U}_\beta f)(z) := f(S_\beta) \frac{s_0(z)}{j_0(z)}$. Since the function $g(t) \equiv 1$ lies in $L_2(\mathbb{R}, \tilde{m}_\beta)$ for all $\beta \in (0, \pi)$ we have the first part of [\(19\)](#). Moreover, taking into account [\(13\)](#), one has

$$\frac{s_0(z)}{j_0(z)} = (\tilde{U}_\beta g)(z) = \sum_{x_n \in \text{spec}(S_\beta)} \frac{s_0(x_n)}{j_0(x_n) \|k(\cdot, x_n)\|_{\mathcal{B}}^2} k(z, x_n) = \left(U_\beta \frac{1}{j_0} \right)(z)$$

for every $\beta \in (0, \pi)$. For the converse part of the assertion, consider the function $r(z) = (U_\beta \frac{1}{j_0})(z)$ for all $\beta \in \mathcal{I}$. It is straightforward to verify that $r(x_n) = s_0(x_n)/j_0(x_n)$ for $x_n \in \text{spec}(S_\beta)$. Now, since \mathcal{I} has an accumulation point, [\[6, Chapter 7, Theorem 3.9\]](#) implies that the entire functions $r(z)$ and $s_0(z)/j_0(z)$ coincide in a set having accumulation points. \square

In [\[11, Proposition 3.9\]](#) (see also [\[12, Theorem 3.2\]](#)), necessary and sufficient conditions for a function to be in \mathcal{B} are given in terms of the spectra of two selfadjoint extensions of S . Two of these conditions imply that the products below are convergent

$$h_\beta(z) := \begin{cases} \lim_{r \rightarrow \infty} \prod_{|b_k| \leq r} (1 - \frac{z}{b_k}) & \text{if } 0 \notin \text{spec}(S_\beta), \\ z \lim_{r \rightarrow \infty} \prod_{0 < |b_k| \leq r} (1 - \frac{z}{b_k}) & \text{otherwise,} \end{cases}$$

for any $\beta \in [0, \pi)$. Moreover, the unique real zero-free function in \mathcal{B} (up to a multiplicative real constant) is $s_\beta(z)/h_\beta(z)$. Therefore, one arrives at the following straightforward conclusion.

Proposition 4.3. Let $s_0(z)$ be an element of \mathcal{B} . If $s_\beta(z)/j_\beta(z) \in \mathcal{B}$, then $j_\beta(z) = h_\beta(z)$ up to a multiplicative real constant. On the other hand, if $j_\beta(z)$ is decomposed as in [\(14\)](#) with the sequence $\{c_n\}_{n \in \mathbb{N}}$ satisfying [\(15\)](#), then $j_\beta(z) = h_\beta(z)$ (up to a multiplicative real constant).

Remark 5. Assuming that \mathcal{B} is decomposed as in [\(3\)](#) (equivalently that there is a zero-free function in \mathcal{B}), the unique real zero-free function is nothing but the unique real entire gauge (up to a multiplicative real constant).

In order to clarify the connection between the gauge and the function $s_0(z)$, let us define the operator $f(S)$ as the operator in \mathcal{B} given by

$$\text{dom}(f(S)) := \{g(z) \in \mathcal{B} : f(z)g(z) \in \mathcal{B}\}, \quad (f(S)g)(z) := f(z)g(z).$$

Clearly this definition is consistent with the notion of a function of an operator. Moreover, the following assertion immediately follows from it.

Proposition 4.4. *If there is a zero-free function in \mathcal{B} , then $s_0(z)/h_0(z) \in \text{dom}(h_0(S))$, $s_0(z) \in \text{dom}((1/h_0)(S))$, and*

$$\left(h_0(S)\frac{s_0}{h_0}\right)(z) = s_0(z), \quad \left(\frac{1}{h_0}(S)s_0\right)(z) = \frac{s_0(z)}{h_0(z)}.$$

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