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A Note on the Continuability of Solutions of a Perturbed Second Order Nonlinear Differential Equation of Liénard Type

Research Article

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Abstract: In this note we study the continuability of the solutions of a Liénard type equation with forcing term under suitable assumptions.

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1. Introduction

In this note we consider the equation

$$x'' + f(x)x' + a(t)g(x)h(x') = b(t, x, x'), \tag{1}$$

where $a : \mathbf{I} \to \mathbb{R}_+$, $b : \mathbf{I} \times \mathbb{R}^{\mathbf{2}} \to \mathbb{R}$, $f : \mathbb{R} \to \mathbb{R}_+$, $g : \mathbb{R} \to \mathbb{R}$ and $h : \mathbb{R} \to \mathbb{R}_+$ are continuous functions in their arguments and the function $a \in C^1$ satisfying $0 < a \leq a(t) < +\infty$, with $\mathbb{R} := (-\infty, +\infty)$, $\mathbb{R}_+ := (\mathbf{0}, +\infty)$ and $I := [t_0, +\infty)$ it is not excluded $t_0 \equiv 0$.

We shall determine sufficient conditions for continuability of solutions of equation (1). Also we include some remarks and examples to illustrate our results, which improve some of the previously obtained results, some preliminary work in this direction are [7], [16] and [30] and references cited therein.

The solutions of equation (1) are *bounded* if there exists a constant $k > 0$ such that $|x(t)| < k$ for all $t \geq t_0 > 0$ for some t_0. By *continuable* we mean a solution which is defined on a half line I.

In the last forty decades, many authors have investigated the Liénard equation

$$x'' + f(x)x' + g(x) = 0. \tag{2}$$

They have examined some qualitative properties of the solutions. The book of Sansone and Conti [30] contains an almost complete list of papers dealing with these equation as well as a summary of the results published up to 1960. The book of

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Reissig, Sansone and Conti [25] updates this list and summary up to 1962. The list of the papers which appeared between 1960 and 1970 is presented in the paper of John R. Graef [8]. Among the papers which were published in the last years we refer to the following ones [2–5, 10–14, 18, 19, 23, 25–27, 31–38].

If in (1) we make $a(t)\equiv 1$, $b(t,x,x')\equiv 0$, and $h(x')=1$, it is clear that equation (1) becomes equation (2) so, every qualitative result for the equation (1) produces a qualitative result for (2).

2. Results

Let $F(x)=\int_0^x f(s)ds$, $G(x)=\int_0^x g(s)ds$ and $H(y)=\int_0^y \frac{r}{h(r)}dr$. Our main assumptions are:

(i) $xg(x)>0$, $x\neq 0$

(ii) There exists a continuous function $u: I\to\mathbb{R}$ such that $|b(t,x,\acute{x})|\leq u(t)$, $u\in L^1(t_0,+\infty)$.

(iii) There exists non negative constant M such that $\frac{|y|}{h(y)}\leq MH(y)$ for $|y|\geq 1$ and $\underset{|y|\to\infty}{Lim}\ H(y)=\infty$. We write the equation (1) as the system

$$\begin{aligned} x' &= y,\\ y' &= -f(x)y-a(t)g(x)h(y)+b(t,x,y). \end{aligned} \tag{3}$$

For any function $M:\mathbf{I}\to\mathbb{R}$ we let $M(t)_+=\max\{M(t),0\}$ and $M(t)_-=\max\{-M(t),0\}$ so that $M(t)=M(t)_+-M(t)_-$. We first give a continuability result for (1).

Theorem 2.1. *Under the conditions i)-iii) above, any solution of equation of (1) is continuable to the right of its initial t-value, i.e. all solutions of (1) can be defined for all t.*

Proof. Let $x(t)$ be a solution of (1) with initial t-value t_0 in I. Suppose, on the contrary, that $x(t)$ can not be continued past the finite time $T>t_0$, $T\in I$. It suffices to show that $x(t)$ remains bounded as t approaches T from the left. We define

$$V(t,x,y)=G(x)+H(y)+C,$$

where C is a non negative constant. From definitions of $G(x)$ and $H(y)$ it follows that

$$V(t,0,0)=0 \ and\ V(t,x,y)>0\ for\ all\ x,y\neq 0.$$

Calculating the time derivative of Liapunov function $V(t,x,y)$, we find

$$V(t,x,y)=-\frac{1}{a(t)}\frac{y^2}{h(y)}f(x)+\frac{b(t,x,\acute{x})}{a(t)}\frac{y}{h(y)}-\frac{\acute{a}(t)}{a(t)}H(y).$$

Applying the assumptions i)-iii) we have

$$V(t,x,y)\leq\frac{MB(t)}{a}H(y)+\frac{a'_+(t)}{a}H(y)=\left[\frac{MB(t)}{a}+\frac{a'_+(t)}{a}\right]H(y).$$

Thus, we have

$$V(t,x,y)\leq m(t)H(y),$$

where

$$m(t)=\frac{MB(t)}{a(t)}+\frac{a'_+(t)}{a(t)}.$$

Now let $K = \int_{t_0}^{T} m(t)dt$ for some positive constant K, so by Gronwall's inequality we have

$$H(y) \leq V(t,x,y) \leq [V(t_0,x_0,y_0)+K] \exp \int_{t_0}^{t} m(s)ds \tag{4}$$

Hence, we see $H(y)$ remains bounded as $t \to T_-$ and so y is bounded on $[t_0,T]$. An integration of first equation of (3) shows that x(t) is also bounded on $[t_0,T]$ contradicting the assumption that $x(t)$ is a solution of (1) with finite escape time. This completes the proof. □

Remark 2.2. *It is clear that this result can be written in terms of the system (3).*

Remark 2.3. *To obtain another boundedness result for solutions of (1) we impose additional conditions on*

$$\int_{t_0}^{\infty} \frac{B(t)}{a(t)}dt, \tag{5}$$

and

$$\int_{t_0}^{\infty} \frac{a'_+(t)}{a(t)}dt, \tag{6}$$

and not use the boundedness of $a(t)$.

Remark 2.4. *Notice that (6) implies that a(t) is bounded from above.*

Now we will prove a boundedness theorem for solution $(x(t), y(t))$ of system (3) by using a modification of the technique of the previous result. In addition to the given assumptions we suppose that:

i´) $f(x) \geq f_0 > 0$ for all $x \in \mathbb{R}$, $xg(x) > 0$, $x \neq 0$ and $\underset{|y|\to\infty}{Lim}\ G(x) = \infty$.

ii´) $\frac{y^2}{h(y)} \leq MH(y) + N_1$, $\frac{|y|}{h(y)} \leq MH(y) + N_2$ for all $y \in \mathbb{R}$.

iii´) There are continuous functions $r_i : I \to I$, $i = 1,2$ such that $|b(t,x,y)| \leq r_1(t) + r_2(t)\,|y|$ for all $(t,x,y) \in Ix\mathbb{R}^2$.

Theorem 2.5. *If assumptions i´)-iii´) hold then for each solution $x(t)$ of equation (1), with initial t-value $t_0 \in I$, $x\acute{}(t)$ is bounded on I, if in addition $\underset{|y|\to\infty}{Lim}\ H(y) = \infty$ holds then $x(t)$ is bounded also on I.*

Proof. Let x(t) be a continuable solution of (1) with initial t-value t_0. Multiplying equation (1) by $\frac{x\acute{}(t)}{h(x\acute{}(t))}$ and integrating on $[t_0,t] \subset [t_0,T)$, we get

$$\begin{aligned} H(y) - H(y_0) + \int_{t_0}^{t} \left[(f(x(s))x\acute{}(s))^2 + \left(\frac{x\acute{}(s)}{h(x\acute{}(s))} \right)^2 \right] dt + \\ +a\left[G(x) - G(x_0)\right] \leq \int_{t_0}^{t} b(s,x(s),x\acute{}(s)) \frac{x\acute{}(s)}{h(x\acute{}(s))} dt. \end{aligned}$$

Using assumptions i ´)-iii ´) the above inequality can be written in the form

$$\begin{aligned} H(y) - H(y_0) + a\left[G(x) - G(x_0)\right] \leq \\ \leq \int_{t_0}^{t} \left[M(-f_0 + r_2) + Mr_1\right] H(y(s))dt + \int_{t_0}^{t} \left[-f_0N_2 + r_1N_1 + r_2N_2\right] dt. \end{aligned} \tag{7}$$

define V as

$$V(t,x,y) = aG(x(t)) + H(y(t))$$

the inequality (7) takes the form

$$V(t,x,y) \leq V(t_0) + m_0 + \int_{t_0}^{t} M(s)V(s,x(s),y(s))dt.$$

where $m_0 = \int_{t_0}^{t}[-f_0N_2 + r_1N_1 + r_2N_2]\,dt$ and $M(t) = [M(-f_0 + r_2) + Mr_1]$. By using the Gronwall-Bellman inequality there is a constant C, depending on $r_1(t)$ and $r_2(t)$ but not on $V(t,x(t),y(t))$ such that

$$V(t,x(t),y(t)) \leq (V(t_0) + m_0)\,C.$$

From this it follow that $G(x(t))$ is bounded for $t \geq t_0$. The boundedness of $x(t)$ follows from i ´) and the boundedness of $x\'(t)$ is easily obtained from the first equation (3). □

Remark 2.6. *It is very easy to give examples of equations in which the conclusions of Theorem are not valid, breached some of the conditions of the theorem. So we have that if $f \equiv 0$ and iii´) is violated, the equation $x\'+ e^{2t}x = x\'$ has the bounded solution $x(t) = sin(e^t)$, with an unbounded derivative. Under assumptions $f(x) \geq f_0 > 0$ for some positive constant f_0, the class of equation (1) is not very large, but if this condition is not fulfilled, we can exhibit equations that have unbounded solutions, for example $x\' - xx\' + 2x = 2x(3-x)$ has the unbounded solution $x(t) = e^{2t}$.*

Remark 2.7. *If, instead of the assumptions iii´) one assumes $r_1(t)$ and $r_2(t)$ integrable on I,the conclusion of Theorem 2.5 is false, as example*

$$x\' + \frac{x^2+2}{x^2+1}x\' + \frac{x}{1+x^2} = \frac{(t+1+\frac{2}{t})x}{1+t^2}$$

which has $x = t$ as a solution.

From Theorem 2.1 and Theorem 2.5, we have the following result.

Corollary 2.8. *If, in addition to assumptions of Theorem 2 we consider that $H(y) \to \infty$ as $|y| \to \infty$, then all solutions of (1) are bounded.*

3. Conclusion.

In this section we present some observations that demonstrate the prominence of our results on several well-known results of the literature.

Remark 3.1. *In [23] the authors consider the following second order non-autonomous and non-linear differential equation x" + a(t)g(x,x')x' + b(t)h(x) = p(t,x,x'); where a, b, g, h and p are continuous functions. For the cases p(t,x,x')=0 and p(t,x,x')≠ 0, respectively, the authors establish sucents conditions under which the zero solution is globally asymptotically stable and every solution and its derivative are bounded. It is clear that these results are no contradicting with our results.*

Remark 3.2. *Tunç studied the boundedness of the solutions of equation ((r(t)x')' + ϕ(t, x, x')x' + p(t)f(x) = q(t, x, x'); under assumption ϕ(t; x; x')≥ 0 for all (t; x; x')∈Ix$\mathbb{R}^2$. It is clear that the*

above remarks still valid.

Remark 3.3. *Our results are consistent with those of Baker of [1], because him studied the (p(t)x')' + ϕ(t, x, x')x' + p(t)f(x) = 0 with ϕ(t, x, x')≥0.*

Remark 3.4. *Our results cover the Theorems I and III of [16] refer to the equation y" + c(t)f(y)y' + a(t)b(y) = 0 obtained from (1) if f(t, x, x') = c(t)f(y) and h(x')≡1. Kroopnick obtained the above results considering c(t) 0 and $f(y) > 0$. This remark is valid in the case of equation y" + c(t)f(y)y' + a(t)b(y) = e(t); studied in [17] under the same assumptions on c and f.*

Remark 3.5. *In [9] the authors obtained qualitative results of the solutions of equation (a(t)x')' + h(t, x, x') + q(t)f(x)g(x') = e(t, x, x') when h(t, x, x') = f(t, x, x')x', a(t) = p(t), f(x) = g(x) and g(x') = h(x'), the Theorems 2.1 and 2.5 and Corollary 2.8 of this paper have contact points with our results under milder conditions on function e(t, x, x').*

Remark 3.6. *A. Castro and R. Alonso [6] considered the special case*

$$x'' + h(t)x' + x = 0; \tag{8}$$

of equation (1) under condition h∈C^1(I) and h(t)≥ $b > 0$. Further, they required that the condition ah´(t) + 2h(t) = 4a be fulfilled, and obtained various results on the stability of the trivial solution of (17). Taking into account the previous results we can obtain the stability of trivial solution of (17) considering instead of (7) and (11) $C_1 \leq 1$ and $\int_{t_0}^{\infty} w(t)dt < 1$.

Remark 3.7. *Other results obtained for special cases of the equation (1) and which are improved by foregoing theorems and corollary are [12, 18, 21, 22, 26–29, 32].*

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