Pilot wave approach to the NRT nonlinear Schrödinger equation

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HIGHLIGHTS

- We develop the pilot wave representation of a non-linear Schrödinger equation based on the non-extensive thermostatistics.
- We study in detail the pilot wave model associated with exact solutions of the NRT equation.
- We show that for q-plane wave solutions of the Bohmian velocity field is q-invariant.

ARTICLE INFO

ABSTRACT

We investigate the main features of a pilot wave representation of the nonlinear Schrödinger equation recently advanced by Nobre, Rego-Monteiro, and Tsallis (NRT) on the basis of Tsallis q-thermo-statistical formalism. The present approach sheds new light upon the NRT equation. In particular, it suggests a new point of view concerning an apparent inconsistency concerning the physical meaning of its q-plane wave solutions.

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1. Introduction

1.1. Preliminaries

Recently, Nobre, Rego-Monteiro and Tsallis [1,2] have advanced an interesting new version of the nonlinear Schrödinger equation, an intriguing proposal that can be regarded as part of a program to investigate non-linear versions of some of the basic equations of physics, a research venue that registers significant activity [3,4]. Different non-linear versions of the Schrödinger equation have found applications in several research areas like fiber optics or water waves [4]. The most popular of these versions involves a cubic nonlinearity in the wave function. In quantal scenarios the nonlinear Schrödinger equation usually determines the behavior of a single-particle's wave function that, in turn, provides an effective, mean-field description of a quantum many-body system. We can refer to the Gross–Pitaevskii equation, applied to the study of

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Bose–Einstein condensates [5]. The cubic nonlinear term appearing in such a GP equation describes the short-range inter-actions between the constituents of the condensate. The nonlinear Schrödinger equation for the system's (effective) single-particle wave function is determined by assuming a Hartree–Fock-like form for the global many-body wave function, with a Dirac's delta form for the inter-particle potential. The Nobre–Rego Monteiro–Tsallis (NRT) equation differs from previous nonlinear Schrödinger equations because it has its non-linearity within the Laplacian appearing in the kinetic energy term. Various features of the NRT equation have been the subject of recent research [2,6–8].

The NRT equation can be traced to the thermo-statistical formalism that revolves around the Tsallis S_q non-additive, power-law information measure, whose manifold applications involve variegated systems and processes. This Tsallis-approach has been the focus of intense attention in the last 20 years (as just an example we can mention [9–15], and references therein). For our present purposes we note that the S_q entropy has proved to be useful in diverse problems of quantum physics [16–24].

The one-dimensional NRT equation governing the field $\Phi(x, t)$ ("wave function") for a particle of mass m is of the form [1,2],

$$i\hbar \frac{\partial}{\partial t} \left[\frac{\Phi(x,t)}{\Phi_0} \right] = -\frac{1}{2-q} \frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} \left[\frac{\Phi(x,t)}{\Phi_0} \right]^{2-q}, \tag{1}$$

where the scaling constant Φ_0 guarantees an appropriate physical normalization for the different terms appearing in the equation, while i is the imaginary unit, $\hbar$ is the Planck constant, and q is a real parameter constituting the most typical feature of Tsallis' formalism [10].

From Ref. [1] one infers that the wave equation (1) admits time dependent solutions possessing the "q-plane wave" form,

$$\Phi(x, t) = \Phi_0 [1 + (1 - q)i(kx - wt)]^{\frac{1}{1-q}}, \tag{2}$$

with k and w real parameters having, respectively, dimensions of inverse length and inverse time (that is, k can be regarded as a wave number and w as a frequency). In the limit $q \rightarrow 1$, the q-plane waves (2) become the plane wave solutions $\Phi_0 \exp(-i(kx - wt))$ of the standard, linear Schrödinger equation describing a free particle of mass m [1]. In what follows we employ $\Psi(x, t) = \frac{\Phi(x,t)}{\Phi_0}$.

1.2. Bohm's pilot wave model of quantum mechanics

The de Broglie–Bohm theory, also called the pilot-wave theory, Bohmian mechanics, or the quantum theory of motion, is an interpretation of quantum theory of great current interest [25–45]. In reviewing 100 years of quantum physics, Tegmark and Wheeler considered it one of the most significant items in its history [29]. Among the interesting aspects of Bohm's theory that have attracted the attention of researchers in recent years we can mention its applications to quantum cosmology [30–35] and its relevance for the study of quantum chaos [36] and the quantum measurement problem [37–39]. Entanglement [40], thermal equilibrium of quantum systems [41], and the dynamics of quantum particles with position dependent effective mass [42] have also been recently discussed within Bohm's approach. Last, but certainly not least, Bohm's dynamics constitutes the basis of an intriguing extension of quantum mechanics proposed by Valentini that leads to specific astrophysical and cosmological predictions that might be observationally tested [43,44].

Bohm's approach involves a wave function, but the particles constituting the system are assumed to have well defined positions in configuration space. The time-evolution of the particles' configuration is governed by the wave function using a "guiding" equation, but the temporal evolution of the wave function is determined by Schrödinger's equation. Knowledge about the positions of the system's particles is limited to a probability density in configuration space, given by the squared modulus of the wave function. Bohm's model yields a possible solution for the measurement problem, since the outcome of an experiment is registered by the configuration of the particles of the experimental apparatus after the experiment is finished. In this sense, the Bohmian approach emphasizes an essential aspect of the measurement process, which is that "…*all experiments and certainly all measurements in physics are in the last analysis essentially kinematic, for they are ultimately based on observations of the position of a particle or of a pointer on a scale as a function of time*" [46]. It can be shown that the Bohmian description of the measurement process is fully compatible with Born's rule in standard quantum mechanics. In fact, the experimental predictions of Bohm's theory are the same as those of the standard quantum mechanical formalism (see, however, Valentini's Bohm-based proposal for an extension of quantum theory [43,44]). The de Broglie–Bohm formalism expresses in explicit fashion the basic quantal non-locality. Indeed, the Bell inequalities were inspired by Bell's reaction to the work of Bohm [26].

Let us emphasize that the Bohmian dynamics incorporates the whole Hilbert space formalism of the standard formulation of quantum mechanics. For instance, a quantum particle has an associated wave function $\Psi(\mathbf{r}, t)$ obeying the Schrödinger equation of motion. The new ingredient of the Bohmian approach is that a quantum particle is also endowed with a definite position $\mathbf{r}$ whose time evolution is governed, as in classical mechanics, by the equation $d\mathbf{r}/dt = \mathbf{p}/m = \mathbf{v}$, where $\mathbf{p}$ is the particle's linear momentum and $\mathbf{v}$ is the particle's velocity. The position $\mathbf{r}$ of a particle is an example of a hidden variable. The result of a position measurement is regarded as predetermined, albeit not predictable. Indeed, this is what happens with any kind of measurements since all measurements at some stage manifest themselves in the position of particles [26].

The Bohmian dynamics differs from standard classical dynamics in a fundamental aspect. The linear momentum $\mathbf{p}$ and the velocity $\mathbf{v}$ are not free variables anymore. Instead, they are determined by the particle's wave function Ψ and position

r. Indeed, the particle's motion is governed by a velocity field $\mathbf{v}(\mathbf{r}, t)$ which is, in turn, determined by a time dependent solution of Schrödinger's equation $\Psi(\mathbf{r}, t)$ through,

$$\mathbf{v}(\mathbf{r}, t) = -\frac{i\hbar}{2m}\left[\Psi^{-1}(\mathbf{r}, t)\nabla(\Psi(\mathbf{r}, t)) - \Psi^{*-1}(\mathbf{r}, t)\nabla(\Psi^{*}(\mathbf{r}, t))\right]. \tag{3}$$

In Bohm's model a quantum particle has a definite position **r** that evolves in time according to the evolution equation

$$\frac{d\mathbf{r}}{dt} = \mathbf{v}(\mathbf{r}, t), \tag{4}$$

where the velocity field $\mathbf{v}(\mathbf{r}, t)$ is given by (3). However, when one prepares a state of the particle, one has no control over or knowledge of the particular value adopted by **r**. One only knows that the probability density $\rho(\mathbf{r}, t)$ associated with the different possible particle's positions is given by

$$\rho(\mathbf{r}, t) = |\Psi(\mathbf{r}, t)|^2. \tag{5}$$

The velocity field (4) for Bohm's particle leads to the Liouville-like continuity equation,

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\mathbf{v}\rho) = 0, \tag{6}$$

for the probability density ρ. It follows from form (3) of the velocity field, the continuity equation (6), and Schrödinger's equation, that, if at the starting time $t = 0$ the ensemble of initial positions for the particle is given by (5), then the (time-dependent) density it given by (5) at all times t.

1.3. Our goal

In this paper we wish to construct Bohm's pilot wave representation of the Nobre–Rego-Monteiro–Tsallis (NRT) non-linear Schrödinger equation and study its main properties, with the idea of gaining some deeper insight into the NRT workings, as the main parameter q determining the form of the NRT equation departs from unity.

2. Basic properties of the NRT equation

To start with, it is convenient to review basic NRT properties. Our q-plane wave solutions (2) propagate at a constant velocity $c = w/k$, without shape-modification (thus displaying soliton-like behavior). In contrast to the $q \to 1$ case (standard plane waves), the solutions for $q \neq 1$ lack a spatially constant modulus. For instance, in the one-dimensional case (setting $\psi = \Phi/\Phi_0$), we have

$$|\psi(x, t)|^2 = \left[1 + (1 - q)^2(kx - wt)^2\right]^{\frac{1}{1-q}}, \tag{7}$$

which corresponds, for $1 < q < 3$, to a normalizable q-Gaussian centered at $x = wt/k$. Accordingly, the q-plane wave solution describes a phenomenon characterized by a certain degree of spatial localization. In Ref. [2] one finds a field-theoretical approach to the NRT equation. It is shown there that the equation can be derived from a variational principle. Also, the NRT equation is seen to be formally related to the nonlinear Fokker–Planck equation, with a diffusion term depending on a power of the density. Evolution equations like the nonlinear Fokker–Planck equation, and their relations with the q-thermostatitical formalism, have received much attention recently [47–55]. The formal resemblance between the NRT-Schrödinger equation and the nonlinear Fokker–Planck notwithstanding, major differences between these two types of equation exist. For example, the nonlinear Fokker–Planck equation does not admit q-plane wave solutions of form (2), that propagate without shape-change.

An interesting property of solutions (2) deserves mention [1]: they are consistent with the celebrated de Broglie relations [56],

$$E = \hbar w,$$
$$p = \hbar k, \tag{8}$$

connecting, respectively, energy with frequency and momentum with wave number. q-plane wave (2) satisfies Eq. (1) if and only if the parameters w and k comply with the relation,

$$w = \frac{\hbar k^2}{2m}, \tag{9}$$

which, in conjunction with (8), leads to the standard relation between linear momentum and kinetic energy,

$$E = \frac{p^2}{2m}. \tag{10}$$

This in turn suggests that the q-plane waves (2) represent particles of mass m with kinetic energy $\hbar w$ and momentum $\hbar k$ [1]. After these considerations, we are ready for tackling our Bohm association.

3. Pilot wave approach to the NRT equation

We begin by introducing here the NRT equation in N dimensions, a generalization of Eq. (1) whose form is

$$i\hbar \frac{\partial \Psi(\mathbf{r}, t)}{\partial t} = -\left(\frac{1}{2-q}\right) \frac{\hbar^2}{2m} \nabla^2 \left(\Psi^{2-q}(\mathbf{r}, t)\right), \tag{11}$$

where $\mathbf{r} \in R^N$ and ∇^2 is the N-dimensional Laplacian. Consider now the probability density $\rho(\mathbf{r}, t) = |\Psi(\mathbf{r}, t)|^2 = \Psi^*(\mathbf{r}, t) \Psi(\mathbf{r}, t)$. Introducing the velocity field $\mathbf{v}_q$,

$$\mathbf{v}_q(\mathbf{r}, t) = -\left(\frac{i}{2-q}\right) \frac{\hbar}{2m} \left[\frac{1}{\Psi(\mathbf{r}, t)} \nabla(\Psi^{2-q}(\mathbf{r}, t)) - \frac{1}{\Psi^*(\mathbf{r}, t)} \nabla(\Psi^{*2-q}(\mathbf{r}, t))\right], \tag{12}$$

it follows from the NRT equation that the probability density $\rho(\mathbf{r}, t)$ satisfies the probability-balance (continuity-like) equation,

$$\frac{\partial \rho(\mathbf{r}, t)}{\partial t} + \nabla \cdot \left[\rho(\mathbf{r}, t)\mathbf{v}_q(\mathbf{r}, t)\right] = F_q(\mathbf{r}, t), \tag{13}$$

where,

$$F_q(\mathbf{r}, t) = -\left(\frac{i}{2-q}\right) \frac{\hbar}{2m} \left[\nabla\Psi^*(\mathbf{r}, t) \cdot \nabla\Psi^{2-q}(\mathbf{r}, t) - \nabla\Psi(\mathbf{r}, t) \cdot \nabla\Psi^{*2-q}(\mathbf{r}, t)\right]. \tag{14}$$

Eqs. (12)–(14) admit a Bohmian pilot wave interpretation. The associated Bohm particle has position $\mathbf{r}$ and moves according to the deterministic equation

$$\frac{d\mathbf{r}}{dt} = \mathbf{v}_q(\mathbf{r}, t), \tag{15}$$

where $\mathbf{v}_q$ is given by (12). As in the case of the standard Schrödinger equation, one only knows the probability for $\mathbf{r}$ to have different values, which is given by $\rho = |\Psi|^2$. Now, the main differences between the pilot wave representation of the NRT scenario and the one associated with the usual linear Schrödinger equation are two. On the one hand, the form of the dependence upon the wave function of the velocity field $\mathbf{v}_q$ is not the one corresponding to the standard case. On the other hand, the continuity-like equation (13) for the NRT Bohmian dynamics involves a new field $F_q(\mathbf{r}, t)$ which can be construed as describing sources and sinks of probability density. In the NRT scenario the Bohmian particles describe orbits $\mathbf{r}(t)$ which are solutions of (15), and have a probability of being created or destroyed which is proportional to F_q.

Eqs. (12) and (14) can be recast, alternatively, as

$$\mathbf{v}_q(\mathbf{r}, t) = -\frac{i\hbar}{2m} \left[\Psi^{-q}(\mathbf{r}, t)\nabla(\Psi(\mathbf{r}, t)) - \Psi^{*-q}(\mathbf{r}, t)\nabla(\Psi^*(\mathbf{r}, t))\right], \tag{16}$$

plus

$$F_q(\mathbf{r}, t) = -\frac{i\hbar}{2m} \left[\Psi^{1-q} \nabla\Psi^*(\mathbf{r}, t) \cdot \nabla\Psi(\mathbf{r}, t) - \Psi^{*1-q} \nabla\Psi(\mathbf{r}, t) \cdot \nabla\Psi^*(\mathbf{r}, t)\right]. \tag{17}$$

The probability current associated with continuity-like equation (13) is,

$$\mathbf{J}_q = \rho(\mathbf{r}, t)\mathbf{v}_q(\mathbf{r}, t) = -\frac{i\hbar}{2m} \left[\Psi^*\Psi^{1-q}(\mathbf{r}, t)\nabla(\Psi(\mathbf{r}, t)) - \Psi\Psi^{*1-q}(\mathbf{r}, t)\nabla(\Psi^*(\mathbf{r}, t))\right]. \tag{18}$$

Finally, it follows form continuity equation (13) that the time derivative of the global norm is given by,

$$\frac{d}{dt} \int \rho(\mathbf{r}, t) \, d\mathbf{r} = \int F_q(\mathbf{r}, t) \, d\mathbf{r}. \tag{19}$$

From Eqs. (17) and (18) we appreciate the fact that, if $q \to 1$, then we recover the well-known probability current $J = \rho(\mathbf{r}, t)\mathbf{V}_1(\mathbf{r}, t)$ for the standard case and the associated (vanishing) density of sources and sinks, which are given by

$$\mathbf{J} = -\frac{i\hbar}{2m} \left[\Psi^*(\mathbf{r}, t)\nabla(\Psi(\mathbf{r}, t)) - \Psi(\mathbf{r}, t)\nabla(\Psi^*(\mathbf{r}, t))\right], \tag{20}$$

and,

$$F_1(\mathbf{r}, t) = -\frac{i\hbar}{2m} \left[\nabla\Psi^*(\mathbf{r}, t) \cdot \nabla\Psi(\mathbf{r}, t) - \nabla\Psi(\mathbf{r}, t) \cdot \nabla\Psi^*(\mathbf{r}, t)\right] = 0. \tag{21}$$

In the next section we are going to consider, as illustrations of the pilot wave representation of the NRT equation, the Bohmian dynamics associated with two exact, analytical solutions of the NRT equation.
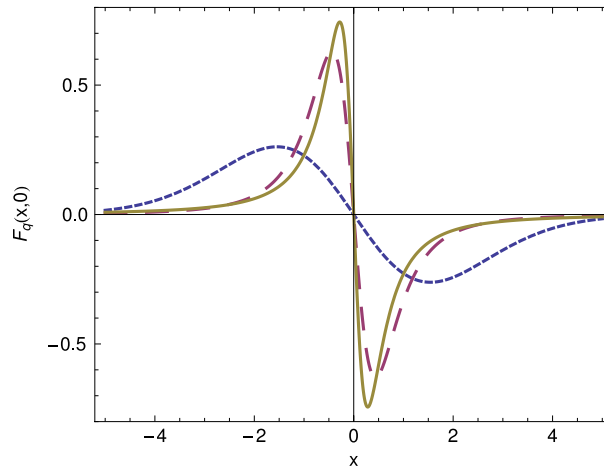
Fig. 1. $F_q(x, 0)$ as a function of x for several values of the parameter q, i.e., $q = 1.1$, 1.9, and 2.7. Increasing values of q correspond to increasing (absolute) extreme values of F_q. Units employed correspond to $\hbar = 1$, $m = 1$ and $\omega = 1$.

4. Pilot wave representation and exact solutions of the NRT equation

There are only two known exact, analytical, time dependent solutions of the NRT equations: the q-plane wave solutions advanced by Nobre, Rego-Monteiro and Tsallis in Ref. [1] and the q-Gaussian solutions investigated Curilef and two of us in Ref. [6] (in point of the fact, the q-plane wave solutions can be obtained as a particular, limit case of the q-Gaussian ones [6]).

4.1. The q-plane wave solutions

NRT-equation (11) admits q-plane waves as solutions, that in N dimensions acquire the appearance

$$\Psi(\mathbf{r}, t) = [1 + (1 - q)\mathrm{i}(\mathbf{k} \cdot \mathbf{r} - \omega t)]^{\frac{1}{1-q}}, \tag{22}$$

with $\mathbf{r}, \mathbf{k} \in R^N$ and $1 < q < 3$, provided that

$$\hbar\omega = \frac{\hbar^2 \mathbf{k}^2}{2m}. \tag{23}$$

The q-plane waves are arguably the most fundamental solutions of the NRT equations. As already mentioned, Eq. (23) admits a direct dynamical interpretation in terms of the de Broglie relations, suggesting that the q-plane wave solutions represent particles with momentum $\mathbf{p} = \hbar\mathbf{k}$ and kinetic energy $E = \mathbf{p}^2/2m$.

Inserting Eq. (22) into Eqs. (12) and (14) we find that

$$\mathbf{v}_q(\mathbf{r}, t) = \frac{\hbar\mathbf{k}}{m}, \tag{24}$$

and

$$F_q(\mathbf{r}, t) = \frac{\hbar}{m} \left(\frac{(1 - q)\mathbf{k}^2(\mathbf{k} \cdot \mathbf{r} - \omega t)}{1 + (1 - q)^2(\mathbf{k} \cdot \mathbf{r} - \omega t)^2} \right) |\Psi(\mathbf{r}, t)|^2. \tag{25}$$

In the one dimensional case we have

$$F_q(x, t) = \frac{\hbar k^2}{m}(1 - q)(kx - \omega t) \left(1 + (1 - q)^2(kx - \omega t)^2 \right)^{\frac{q}{1-q}}, \tag{26}$$

with

$$k = \sqrt{2m\omega}. \tag{27}$$

In Fig. 1 we plot the density of sources and sinks of probability $F_q(x, t)$ as a function of x for several values of the parameter q, at $t = 0$. This density is compared with the probability density itself, $\rho = |\Psi|^2$ in Fig. 2, for $q = 2.9$ and $t = 0$. $F_q(x, t)$ is depicted as a function of both x and t in Fig. 3.

Remarkably, the Bohmian velocity field (24) associated with the q-plane wave solutions is an invariant structure not depending on the value of the Tsallis' parameter q. That is, all the q-plane wave solutions have a velocity field identical to the one associated with the standard (exponential) plane wave solutions of the linear Schrödinger equation.
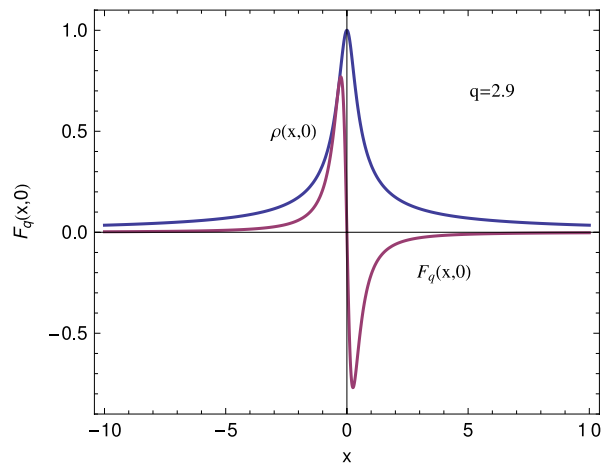
Fig. 2. $F_q(x, 0)$ and $\rho(x, 0)$ as a function of x for a fixed value of q. Units employed correspond to $\hbar = 1$, $m = 1$ and $\omega = 1$.
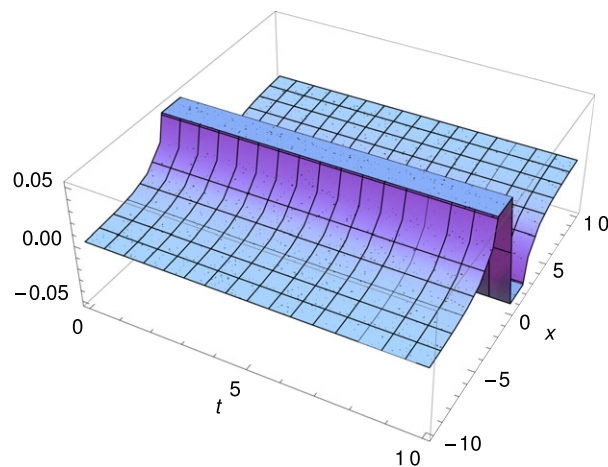

Fig. 3. Behavior of $F_q(x, t)$ as a function of x and t for a fixed value of q. Units employed correspond to $\hbar = 1$, $m = 1$ and $\omega = 1$.

The pilot wave representation of the q-plane wave solutions of the NRT equation highlights another interesting aspect of these solutions. In contrast to what happens with the linear ($q = 1$) case, for $q \neq 1$ the density F_q of sources and sinks of probability does not vanish. This means that, in spite of the fact that the global normalization of the q-plane wave solutions is constant in time, the probability density $\rho = |\psi|^2$ does not satisfy a local continuity equation. In other words, probability density does not obey a local conservation law. On the contrary, probability density is constantly (and everywhere) being locally "created" and "destroyed" according to the density F_q of sources and sinks of probability. Notice that for the q-plane wave solutions the density $F_q(x, 0)$ is an odd function of the space coordinate x. This means that the sources are exactly "compensated" by the sinks, in the sense that $\int F_q(x, 0)dx = 0$. This is, of course, fully consistent with the fact that in the case of the q-plane wave solutions the global norm is preserved by the NRT dynamics.

As indicated by NRT, the q-plane wave solutions are compatible with the de Broglie relations, suggesting that these solutions represent the (non-relativistic) motion of a free particle of mass m, linear momentum $p = \hbar k$, and kinetic energy $E = \frac{p^2}{2m}$. This interpretation suggests that the velocity of the particle is $p/m = \hbar k/m$. However, the q-plane wave solutions constitute localized, soliton-like solutions that move "rigidly" with velocity given (in our one-dimensional example) by $\omega/k = \hbar k/2m$. Therefore, the velocity derived from the free-particle dynamical interpretation does not coincide with the manifest translation velocity of the localized "particle-like" solutions given by the q-plane waves. This discrepancy demands an explanation, because the localized character of the q-plane wave solutions is one of their main features suggesting that they represent "particles". It is thus remarkable that this discrepancy disappears within the pilot wave representation of the NRT equation. Indeed, we can see from Eq. (24) that the velocity of the Bohmian particles is precisely equal to the one that is dynamically derived from the de Broglie relations. Within the pilot wave representation the velocity of the q-plane wave solutions is not the physical velocity of the particle (which is given by the velocity field (24)). The velocity of the q-plane wave is only the velocity of the ensemble probability density describing the available knowledge regarding the position of the Bohm particle. This velocity is equal to one half of the particle's velocity. This velocity reduction is due to the field F_q

describing the density of sources and sinks of probability. As the statistical ensemble describing the Bohmian particle moves, the presence of the field F_q leads to an effective reduction of the ensemble's velocity.

4.2. The q-Gaussian solution

We are now going to apply the ideas developed above to the case of the exact, time dependent solutions to the NRT equation based upon a q-Gaussian wave packet ansatz. This ansatz was proposed in Ref. [6] and one has the q-wave packet expressed via

$$\Psi(x, t) = \left[1 - (1 - q)(ax^2 + bx + c)\right]^{\frac{1}{1-q}}, \tag{28}$$

where a, b, and c are appropriate (complex) time dependent coefficients. From Eq. (16) we can calculate the velocity field, obtaining

$$v_q = -\frac{\hbar}{m} \operatorname{Im}(2ax + b). \tag{29}$$

It was proved in Ref. [6] that the q-Gaussian wave constitutes a solution to the NRT equation, provided that the coefficients a, b, and c are of the form

$$a(t) = \left(\frac{(3 - q)i\hbar t}{m} + \alpha\right)^{-1}, \tag{30}$$

$$b(t) = \beta \left(\frac{(3 - q)i\hbar t}{m} + \alpha\right)^{-1} \equiv \beta a(t), \tag{31}$$

and

$$c(t) = a(t)^{\frac{1-q}{3-q}} \left[\frac{a(t)^{\frac{q-1}{3-q}}}{1 - q} + \frac{\beta^2 a(t)^{2/(3-q)}}{4} + \gamma - \frac{1}{1 - q}\right], \tag{32}$$

where α, β, and γ are integration constants determined by the initial conditions (for details and discussions see Ref. [6]). Let α be real and β purely imaginary (that is, $\beta = i\operatorname{Im}(\beta)$). Accordingly, Eqs. (29), (30), and (31) easily yield

$$v_q = \frac{c_0 + c_1 xt}{1 + c_2 t^2}, \tag{33}$$

with

$$c_0 = \frac{i\hbar\beta}{m\alpha}, \tag{34}$$

$$c_1 = \frac{2(3 - q)\hbar^2}{m^2\alpha^2}, \tag{35}$$

$$c_2 = \left(\frac{(3 - q)\hbar}{m\alpha}\right)^2. \tag{36}$$

The equation of motion we are interested in becomes

$$v_q = \frac{dx(t)}{dt} \equiv \dot{x}(t), \tag{37}$$

that can be written as a first-order differential equation of the form

$$\dot{x}(t) - \frac{c_1 t}{1 + c_2 t^2} x(t) - \frac{c_0}{1 + c_2 t^2} = 0, \tag{38}$$

whose general solution is

$$x(t) = (1 + c_2 t^2)^{\frac{c_1}{2c_2}} \left[x_0 + c_0 t \, {}_2F_1\left(\frac{1}{2}, 1 + \frac{c_1}{2c_2}; \frac{3}{2}; -c_2 t^2\right)\right], \tag{39}$$

where $x_0 = x(t = 0)$ is the initial condition and ${}_2F_1$ is a hypergeometric function [57]. The orbits $x(t)$ corresponding to various initial conditions are depicted in Fig. 6, for different values of q and $\beta = 0$.

It follows from Eq. (17) that in the case of the q-Gaussian solutions of the NRT equation the density of sources and sinks is given by,

$$F_q = -\frac{\hbar}{m}(1 - q)|2ax + b|^2 \rho(x, t)^q \operatorname{Im}\left(ax^2 + bx + c\right). \tag{40}$$
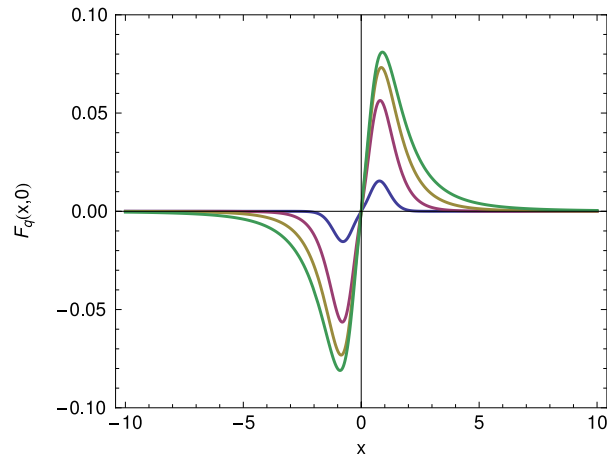
Fig. 4. $F_q(x, 0)$ as a function of x for the following q-values: 1.1, 1.6, 2.1, 2.6. Increasing values of q correspond to increasing (absolute) extreme values of F_q. Units employed correspond to $\hbar = 1$ and $m = 1$.
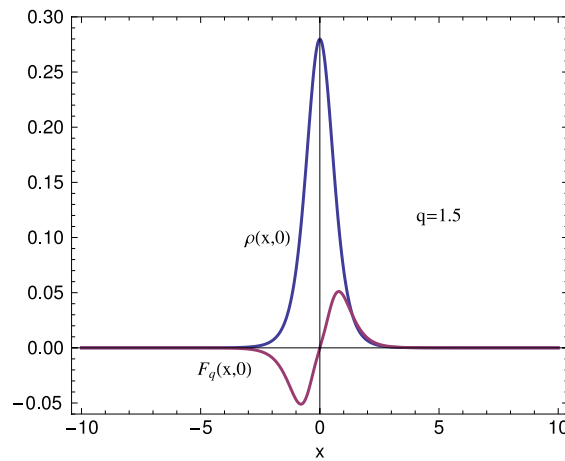

Fig. 5. $F_q(x, 0)$ and $\rho_q(x, 0)$ as a function of x. Units employed correspond to $\hbar = 1$ and $m = 1$.

Illustrative examples of the probability density ρ_q and of the density F_q of sources and sinks corresponding to q-Gaussian solutions of the NRT equation are given in Figs. 4–5. The parameters used in these figures are $\alpha = 1$, $\gamma = 1$, $\beta = i$, $t = 0$ and $\hbar = m = 1$. In Fig. 4 the density F_q is given as a function of the spatial coordinate x for different values of the Tsallis parameter q. In Fig. 5 the densities ρ and F_q are depicted together for $q = 1.5$. It transpires from Figs. 4–5 that the qualitative features of the density of sources and sinks of probability associated with the q-Gaussian solutions are similar to those corresponding to the q-plane wave solutions.

It is interesting to consider the special case $\beta = 0$. This corresponds to q-Gaussian solutions with no "overall motion", the center of the wave packet remaining motionless at the origin. In this case the motion of the Bohmian particles takes into account the spreading of the wave packet (similar to the spreading of a Gaussian wave packet solution of the linear Schrödinger equation). The behavior of the Bohmian particles is given by,

$$x(t) = x_0 \left[1 + \left(\frac{(3 - q)\hbar t}{m\alpha} \right)^2 \right]^{\frac{1}{3-q}}. \tag{41}$$

We see that at large times the motion of the Bohmian particle is asymptotically described by a power law, $x \sim t^{\frac{2}{3-q}}$ with a q-dependent exponent.

Some Bohmian trajectories corresponding to the q-Gaussian solutions are plotted in Fig. 6, for various values of q, including the linear case given by $q = 1$. We observe that the general qualitative aspect of the Bohmian trajectories corresponding to the q-Gaussian solutions is the same for the different q-values considered. In all cases the Bohmian orbits are displayed in a fan-like way. There is a fixed point solution corresponding to $x(t) = 0$ (for all times t) as well as solutions where $x(t)$ is either a monotonously increasing or a monotonously decreasing function of t, depending on whether the initial value $x(0)$ is larger or smaller than 0.
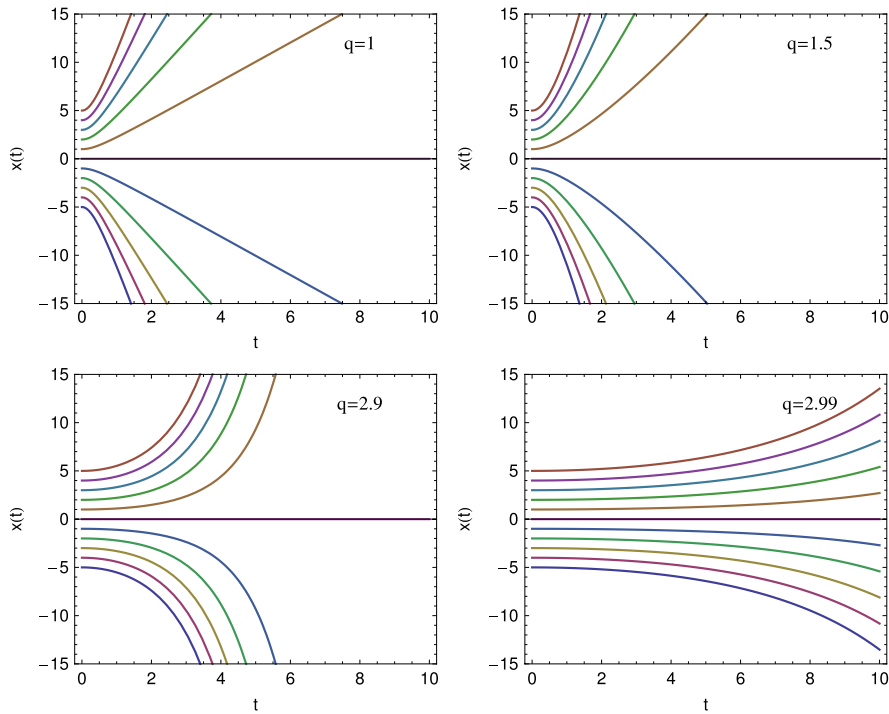
Fig. 6. Behavior of $x(t)$ as a function of t for several values of the parameter q, $\beta = 0$, and different initial conditions x_0 within the ranges $[-5, 5]$. Units employed correspond to $\hbar = 1$ and $m = 1$.

We see in Fig. 6 that for increasing values of q remaining close to $q = 1$ the trajectories starting from different initial conditions tend to depart from each other faster for the higher the value of q. On the other hand, for increasing values of q approaching $q = 3$ the trajectories tend (for not too large values of t) to stay at a constant distance from each other. In the limit $q \to 3$ the trajectories approach the form $x(t) = x_0$ (that is, the quantum particle tends to stay at its initial location). This limit corresponds to the "frozen" solutions to the NRT equation discussed in Ref. [6]. Note that for $q < 3$, no matter how close q is to 3, the trajectories will eventually depart from each other according to the power law already mentioned. This is related to the fact that the limits $q \to 3$ and $t \to \infty$ of the right hand side of (41) do not commute.

Let us consider now the limit $q \to 1$ of the velocity field and the Bohmian trajectories corresponding to the q-Gaussian solutions. In the limit $q \to 1$ we have $c_1 = c_2 = \delta$ and ${}_2F_1\left(\frac{1}{2}, \frac{\delta}{1+2\delta}; \frac{3}{2}; -\delta t^2\right) = 1/\sqrt{1 + \delta t^2}$. When $q \to 1$ the q-Gaussian solution of the NRT equation reduces to a Gaussian wave packet solution of the linear Schrödinger equation with dispersion at $t = 0$ given by $\sigma_0 = \alpha^{1/2}/2$. Therefore, recalling expression (35) for c_1, and taking into account that $c_0 = v_0$ is the velocity of the Bohmian particle at $t = 0$ (see Eq. (38)), one obtains the velocity field,

$$v(x, t) = v_0 + \frac{\hbar^2 t (x - v_0 t)}{\left(4m^2\sigma_0^4 + \hbar^2 t^2\right)}, \tag{42}$$

and the particle's trajectories,

$$x(t) = v_0 t + x_0 \sqrt{1 + \left(\frac{\hbar t}{2m\sigma_0^2}\right)^2}, \tag{43}$$

recovering the Bohmian orbits corresponding to the Gaussian wave packet solution of the standard Schrödinger equation (see Ref. [25]).

5. Conclusions

We have investigated Bohm's pilot wave approach to the Nobre–Rego Monteiro–Tsallis nonlinear Schrödinger equation by deducing it and examining its main properties in analytically tractable cases, including the one corresponding to the q-plane wave solutions. This approach sheds new light on basic aspects of the NRT equation. In particular, it provides an explanation for an apparent inconsistency of the free-particle interpretation of the q-plane wave solutions: the dynamical velocity associated with these solutions, which is derived from de Broglie relations, does not coincide with their translation velocity. On the other hand, when one considers the pilot wave representation, the physical velocity of the Bohmian particles coincides with the dynamical one obtained from the de Broglie relations.

Remarkably, the Bohmian velocity field associated with the q-plane wave solutions constitutes an invariant structure that is independent of the value of the Tsallis parameter q. As already mentioned, this velocity field is fully consistent with the de Broglie relations. This reinforces the interpretation of the q-plane wave solutions as describing dynamical states of a free particle with well defined momentum and energy.

The pilot wave representation of the NRT equation provides a new point of view with regard to the non-conservation of the norm associated with the NRT dynamics. The pilot wave approach highlights the fact that even for those solutions that preserve the global norm $\int |\Psi|^2 d\mathbf{r}$ (such as the q-plane wave solutions) the norm is actually not preserved locally. The density $\rho = |\Psi|^2$ is constantly being "created" and "destroyed" locally. In this sense, time dependent solutions of the NRT equation that preserve the global norm exhibit some similarity with traveling-wave solutions to reaction–diffusion equations. It would be interesting to investigate the behavior of overlap measures between pairs of solutions of the NRT equation, and to explore the possibility of formulating H-like theorems associated with the NRT dynamics [58]. Finally, it is worth mentioning that the mathematical formalism associated with the pilot wave representation of the standard Schrödinger equation has been suggested as a tool for modeling complex dynamical processes outside the field of physics (for instance, economic processes. See Ref. [45]). The essential non-linearity associated with the NRT equation might increase the versatility of the pilot wave formalism as a modeling tool for complex systems.

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