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# Comment on "Quantum discord through the generalized entropy in bipartite quantum states"

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**Abstract.** In [X.-W. Hou, Z.-P. Huang, S. Chen, Eur. Phys. J. D **68**, 1 (2014)], Hou et al. present, using Tsallis' entropy, possible generalizations of the quantum discord measure, finding original results. As for the mutual informations and discord, we show here that these two types of quantifiers can take negative values. In the two qubits instance we further determine in which regions they are non-negative. Additionally, we study alternative generalizations on the basis of Rényi entropies.

On an interesting recent paper, Hou et al. [1] introduce generalizations for two quantifiers: mutual information and quantum discord, which they use for the study of quantum correlations in two qubits systems. It is conventionally agreed that the mutual information (MI) quantifies total correlations in bipartite systems. Given a system described by the state $\rho^{ab}$, with subsystems $a$ and $b$, the MI reads

$$I(a:b) := S(\rho^a) + S(\rho^b) - S(\rho^{ab}), \qquad (1)$$

where $\rho^a := \mathrm{Tr}_b \rho^{ab}$ y $\rho^b := \mathrm{Tr}_a \rho^{ab}$ are reduced states associated to our subsystems. $S(\cdot)$ is von Neumann's entropy for a state $\sigma$:

$$S(\sigma) := -\mathrm{Tr}(\sigma \log \sigma). \qquad (2)$$

If one wishes to quantify non-classical correlations, these should be appropriately discriminated from the total ones. A possibility is to compute classical correlations via a classical information measure (CI)

$$C^b(a:b) := S(\rho^a) - \min_{\{\Pi_i\}} \sum_k p_k S(\rho^a_k), \qquad (3)$$

where $\{\Pi_i\}$ is a complete projective measure, local in $b$, and

$$\rho^a_k := \frac{1}{p_k} \mathrm{Tr}_b \left[ (I_a \otimes \Pi_k) \rho^{ab} (I_a \otimes \Pi_k) \right] \qquad (4)$$

is the $a$'s conditional state associated to the outcome $k$ of $b$. Further,

$$p_k := \mathrm{Tr} \left[ (I_a \otimes \Pi_k) \rho^{ab} (I_a \otimes \Pi_k) \right] \qquad (5)$$

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is the corresponding probability. $I_a$ is the identity operator for $a$. Equation (3) quantifies the classical correlations from a $b$-perspective and, analogously, one defines $C^a$. Given that equations (1) and (3) compute quantum and classical correlations, respectively, the discord measure is given by [2]:

$$D^b(a:b) := I(a:b) - C^b(a:b). \qquad (6)$$

Hou et al. [1] generalized these measures replacing von Neumann's entropy by Tsallis' and Rényi's ones ([3–6], and references therein). The $\alpha$-Rényi quantifier is [3]

$$S_\alpha(\sigma) := \frac{\log \mathrm{Tr}\sigma^\alpha}{1-\alpha}, \qquad (7)$$

while Tsallis' counterpart reads [6]

$$S_q(\sigma) := \frac{1 - \mathrm{Tr}\sigma^q}{(q-1)\ln 2}. \qquad (8)$$

Both quantifiers converge to von Neumann's in the limit $\alpha \to 1$ ($q \to 1$). Note that we use always basis-2 logarithms, which slightly modifies the usual definition of $S_q$. Hou et al. [1] replace then $S$ by $S_\alpha$ or $S_q$ in equations (1) and (3), obtaining generalized mutual information measure $I_\alpha$ (RMI) and $I_q$ (TMI). We consequently have generalized classical correlations ($C^b_\alpha$, $C^b_q$) and discords ($D^b_\alpha$, $D^b_q$).

We show below that these last correlation-quantifiers can take negative values, refuting what is conjectured by Hou et al. [1]. Even more, the discord can be different from zero, and even negative, for classical states.
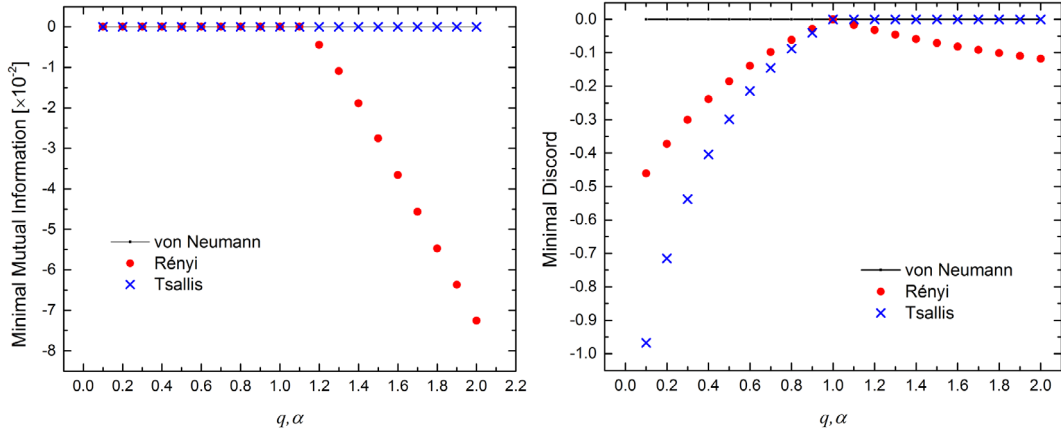
**Fig. 1.** Maximum values of the generalized MI (left) and discord (right), for different $\alpha$s and $q$s.
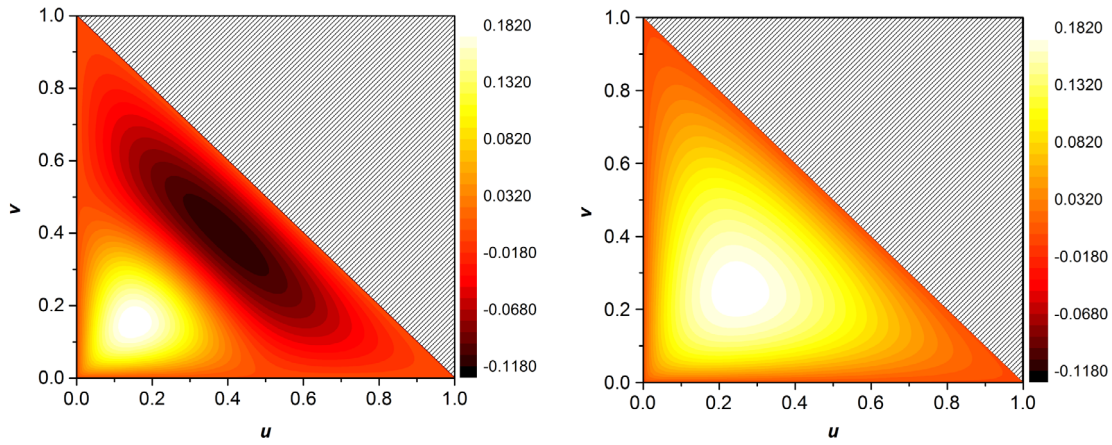


**Fig. 2.** Generalized discords for classical states $\rho_{uv}^{ab}$, with $\alpha = q = 2$. Rényi's discord (left) and Tsallis' one (right). Note the presence of negative values.

*Rank-three classical states of two qubits.* Von Neumann's entropy properties guarantee the positivity of $I$, $C^b$, and $D^b$. We will see that generalized quantifiers do not, in general, share such positivity property.

As an example consider the family of states given below. We focus attention on classical states of range 3 (standard basis).

$$\rho_{uv}^{ab} = \begin{pmatrix} u & 0 & 0 & 0 \\ 0 & v & 0 & 0 \\ 0 & 0 & 1-u-v & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad (9)$$

with $u, v \geq 0$ and $u + v \leq 1$. There exists a non-perturbative, complete and local projective measurement given by the projectors basis $\{|0\rangle\langle 0|, |1\rangle\langle 1|\}$ for the two subsystems. Thus, for the family $\rho_{uv}^{ab}$ one has $D^b(a:b) = 0$ (and $D^a(a:b) = 0$). In Figures 1 and 2, we note that generalized measures can be negative *even for classical states.*Consequently, the ensuing generalized discords cannot discriminate classical in the sense discussed above.

CI turns out to be positive [all $(\alpha, q)$] for the family $\rho_{uv}^{ab}$. It would seem that, for $q > 1$, Tsallis' discord works better, since it is always positive. In the case ($q < 1$, $\alpha < 1$), neither Rényi's nor Tsallis' measures behaves as one would expect for classical states.

*Random states of two qubits.* In this case we compute generalized MI, CI, and discord for different pairs $(\alpha, q)$ so as to estimate the range, in such a plane, for positivity. We considered $10^5$ random states for each of these parameters. Figure 3 plots minima of MI and discord for a given $\alpha$ or $q$.

For 2-qubits states, generalized CI's turned out to be positive for all our states-sample, with $\alpha$ and $q$ ranging in $(0, 1000)$. This makes it credible that the quantifier is positive for all $(\alpha, q)$. Instead, minima for generalized MI and discord reach negative values for all $\alpha \neq 1$ in the Rényi instance, while they are positive in the Tsallis case for $q \geq 1$. This would indicate that Tsallis' entropy is strongly sub-additive (see below). Regretfully enough, the negativity of these discords does not signal classicality. As an example, states that are known to be of a non-classical nature display negative Rényi discord for $\alpha = 2$ (Fig. 4).

*Alternative generalizations.* It is easy to see that the von Neumann-*sub-additivity* (SA) of $S(\rho^{ab})$:

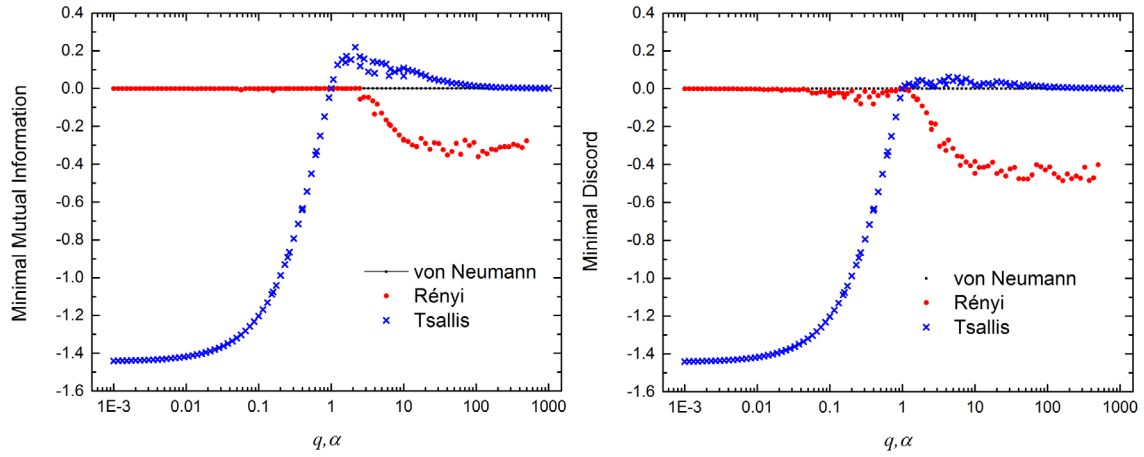$$S(\rho^{ab}) \leq S(\rho^a) + S(\rho^b), \qquad (10)$$

**Fig. 3.** Minima for generalized MI (left) and discord (right), using different values of $\alpha$ and $q$, for a large random sample of states.
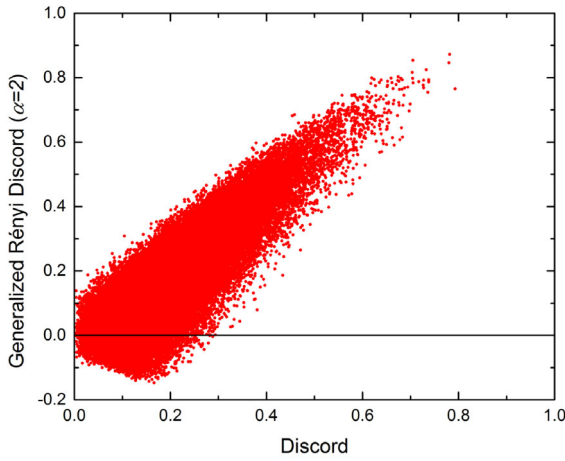


**Fig. 4.** Rényi's discord with $\alpha = 2$ for $10^5$ random states of 2 qubits can be negative for states whose orthodox discord is $\lesssim 0.3$.

is tantamount to MI-positivity and that the *concavity*:

$$S\left(\sum_i p_i\rho_i\right) \geq \sum_i p_iS(\rho_i), \qquad (11)$$

implies that $C^b(a:b)$ and $C^a(a:b)$ are positive measures. Discord positivity is deduced from *strong sub-additivity* (SSA) [2,7]:

$$S(\rho^{abc}) + S(\rho^b) \leq S(\rho^{ab}) + S(\rho^{bc}), \qquad (12)$$

equivalent to the concavity of the conditional entropy $S(a|b) := S(\rho^{ab}) - S(\rho^b)$. In general, generalized entropies do not share these properties for arbitrary values of $\alpha$ and $q$. Rényi's ones are concave in the interval $(0, \alpha^*)$, with $\alpha^* = 1 + \frac{\log 4}{\log(N-1)}$, $N$ being the density matrix range [8,9]. For $\alpha \geq \alpha^*$, $S_\alpha$ is neither convex nor concave. Given $q > 1$, Tsallis' entropy is sub-additive so that the associated mutual information is positive as well, i.e., $I_q \geq 0$ for $q > 1$ [10]. However, for $0 < q < 1$, Tsallis's measure is super-additive for product states while for general

states it is neither sub- nor super-additive [11]. Thus, $I_q$ can adopt negative values for $0 < q < 1$. Rényi's entropies are sub-additive for $\alpha = 0$ and $\alpha = 1$ [12]. For all other $\alpha$-values one can find states for which the associated MI is negative. SSA does not hold in general, save for the von Neumann's instance [13]. For classical states, $S_q$ displays SSA if $q \geq 1$ [14] (there exist particular cases in which $S_\alpha$ also displays SSA, as, for instance, Gaussian states with $\alpha = 2$ [15]). Table 1 details properties of the different entropies.

Concavity and SA are sufficient, but not necessary, to guarantee positivity. In the case of the range 3-classical family $(\rho_{uv}^{ab})$, our numerical results show that Rényi's CI is positive for all $\alpha$, being concave only for $\alpha < 3$. As for discord's positivity, it suffices to demand that

$$I(a:b) \geq \chi(P_a, b), \qquad (13)$$

where $\chi(P_a, b) := S(\rho^a) - S(b|P_a)$ is *Holevo's quantity* associated to the $b$-state conditioned to a POVM measurement of $a$ of operators $P_a$. Coles speaks here of *firm sub-additivity* (FSA), that is less restrictive than SSA. Hierarchically: SSA $\Rightarrow$ FSA $\Rightarrow$ SA [16]. Results for a 2-qubits random simulation (see Fig. 3) would indicate that Tsallis's entropies are FSA for $q \geq 1$, while Rényi's ones are FSA for $\alpha = 1$ and, possibly, for $\alpha = 0$.

In von Neumann's entropic scheme, it is equivalent to define the MI as the relative entropy between the given state and the product of the concomitant reduced states, i.e.,

$$I(a:b) := \min_{\{\sigma^a, \sigma^b\}} S\left(\rho^{ab} || \sigma^a \otimes \sigma^b\right), \qquad (14)$$

**Table 1.** Generalized entropies' properties: concavity, sub-additivity (SA), and strong SA (SSA).

|  | Concavity | SA | SSA |
|---|---|---|---|
| $S$ | ✓ | ✓ | ✓ |
| $S_\alpha$ | $(0, 1]$ | $\{0, 1\}$ | × |
| $S_q$ | $(0, \infty)$ | $[1, \infty)$ | × |

where $S(\rho||\sigma) := -S(\rho) - \text{Tr}(\rho \log \sigma)$ is the relative entropy, and the minimization runs over the set of all completely uncorrelated states. Here, Klein's inequality guarantees the positivity of $I(a : b)$. Equation (14) offers an *alternative* path for generalizing the MI in terms of other entropic measures, different from the one associated to equation (1). This alternative was employed by different authors and is known as the *quantum conditional MI* [17,18]. Different definitions of Rényi's or Tsallis' relative entropies determine distinct alternatives for the conditional MI.

A reasonable idea would then entail to define the generalized mutual information as in equation (14), using some generalized relative entropy or divergence:

$$\tilde{I}_\alpha(a : b) := \min_{\{\sigma^a, \sigma^b\}} S_\alpha\left(\rho^{ab}||\sigma^a \otimes \sigma^b\right). \tag{15}$$

In similar vein, the classical generalized information will be

$$\tilde{C}_\alpha^a(a : b) := \max_{\{\Pi_i\}} \min_{\{\sigma^a, \sigma^b\}} S_\alpha\left(\rho^{ab'}||\sigma^a \otimes \sigma^b\right), \tag{16}$$

where $\rho^{ab'} := \sum_k (I_a \otimes \Pi_k)\rho^{ab}(I_a \otimes \Pi_k)$ is the posterior state to the measurement of $\{\Pi_i\}$ in $b$. The new generalized discord would be given by the difference between these two quantities

$$\tilde{D}_\alpha^a(a : b) := \tilde{I}_\alpha(a : b) - \tilde{C}_\alpha^a(a : b). \tag{17}$$

The positivity of $\tilde{I}$ and $\tilde{C}^a$ will be guaranteed by the positivity of the generalized relative entropies. Noting that our relative entropies fulfill the *data processing inequality*, $\tilde{D}^a$ will be positive as well (see, for instance, [19]). The scheme being advanced here should be the subject further exploration.

Recently, the introduction of a new Rényi's relative entropy, monotonous against general quantum (trace preserving) operations, in the range $1/2 \leq q < \infty$ seem to constitute the most convenient way of computing a states' MI and, a posteriori, to define a new generalized discord quantifier [19–22].

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