Estimation of Weibull Parameters for the Flexural Strength of PMMA-Based Bone Cements

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The wide scatter of data observed in the strength of bone cements based on poly(methyl methacrylate) (PMMA) can be described properly by the two-parameter Weibull function. However, the statistical character of the distribution leads to an uncertainty in the parameters evaluated from a limited number of experiments. This study is concerned with the analysis of the methods of estimation as well as sample size on the estimates of the Weibull parameters. The maximum likelihood method, moments method, and linear regression method were studied. Monte Carlo simulations were carried out in order to assess the influence of the number of specimens tested on the Weibull parameters calculated by the different methods. The number of specimens tested displayed a large influence upon the calculated Weibull modulus. By applying weighing factors to the linear regression method, the standard deviation of Weibull parameters decreased significantly. As a compromise between minimizing both the dispersion of the evaluation method and the experimental effort, the use is suggested of the linear regression method with a minimum number of 20 specimens in a nonweighted analysis and 10 in a weighted analysis.

INTRODUCTION

PMMA-based bone cements are widely used in orthopedics to fix joint replacements into the bone. The main function of the cement is to transfer load from the prosthesis to the bone. Therefore, the effectiveness of surgical cement is viewed in light of its mechanical properties (1). Currently, there are two standards and specifications for the evaluation of bone cements, the ASTM F-451 (2) and the ISO 5833 (3). These standards specify methods for determination of compressive strength, flexural modulus and flexural strength under static conditions. Clinically, the cement is cyclically loaded and would most likely fail in fatigue (4–8). However, owing to the existence of the aforementioned codes of practice, many of the mechanical characterizations have been carried out under static conditions.

In clinical practice, liquid methylmethacrylate monomer is mixed with powdered PMMA polymer in order to obtain a dough capable of being introduced into the cavity of the bone. The mixing results in air entrapment creating pores of variable size. In addition, bone cements usually contain a particulate filler providing radiopacity. Thus, the hardened material contains various defects, such as pores, voids and inclusions due to the presence of additives (1, 9–16) These defects control the initiation of brittle fracture, and their efficiency as crack initiators is dependent on their size and shape. The random distribution of defects leads to a random distribution of the measured values of strength; therefore, the fracture characterization of bone cements should be carried out in the frame of a statistical analysis.

A statistical model commonly used in characterization of brittle materials is that given by Weibull (17), who proposed an empirical formula to relate the probability of failure to the rupture stress. It is given by:

$$P_f(V) = 1 - \exp\left[-\left(\frac{\sigma - \sigma_u}{\sigma_0}\right)^m\right] \tag{1}$$

where P_f is the failure probability, σ is the fracture strength, σ_u is the strength below which $P_f = 0$ and σ_0 and m are the Weibull parameters, the scale parameter and the Weibull modulus respectively. On a normalized scale, a higher m would lead to a steeper function and then a lower dispersion of fracture strength.

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Setting $\sigma_u = 0$ results in an overestimate of the probability of failure, which is desirable for a conservative approach. In this case, the two-parameter Weibull function is obtained as follows:

$$P_f(V) = 1 - \exp\left[-\left(\frac{\sigma}{\sigma_0}\right)^m\right] \tag{2}$$

Knowledge of the Weibull parameters of a material leads to a complete characterization of the statistical variation in fracture strength. In addition, it brings out the effect of different processing parameters or subsequent treatments through comparison of the Weibull distribution for the different cases. A limit of the description of the fracture behavior of brittle materials by the Weibull model is due to the statistical character of the distribution itself, which leads to an uncertainty in the parameters obtained by evaluation from a limited number of experiments. The true values of the Weibull parameters are obtained only for an infinite number of samples. For any smaller number, only an estimate but not the true value can be achieved. In practice it is possible to test only a limited number of specimens, so, it is relevant to find out the method of evaluation that results in the most accurate estimation of the Weibull parameters.

The aim of the present work was to compare different methods of evaluation as well as to study the influence of the number of specimens tested on the calculated Weibull parameters. In order to achieve this purpose, Monte Carlo simulations were used to characterize the statistical properties of three evaluation methods: maximum likelihood method, moments method, and linear regression. The Weibull parameters of a theoretical distribution of known m and σ_0 values were estimated for data produced by Monte Carlo techniques. The comparison of the calculated values with the known values of the theoretical distribution permits an assessment of the accuracy of each method.

A similar investigation was carried out, but rather than using computer-generated data, actual experimental results of flexural strength values measured in four-point bending were employed. Flexural testing is considered an appropriate measure of the strength of the cement because it combines elements of compression, tension and shear, which probably closely mimic *in vivo* stresses than does either compression or tension testing. The uncertainty on the Weibull modulus and mean strength when only a part of the total batch is used for the determination of these parameters was analyzed in a commercial bone cement. The minimum number of specimens to be tested in order to obtain reliable Weibull parameters was assessed.

MATERIALS AND METHODS

Materials and Mechanical Testing

Flexural studies were performed on three different preparations of acrylic bone cements. Standard viscosity (SV) and low viscosity (LV) cements (Subiton Laboratories, Buenos Aires, Argentina) were employed. Each dose of cement consists of 40 g of PMMA powder and 20 g of liquid monomer. The liquid component is composed of 19.76 g MMA monomer, 0.24 g N,N-dimethyl-*p*-toluidine and 18–20 ppm hydroquinone. The solid component is composed of 36 g PMMA and 4 g $BaSO_4$ to impart radiopacity to the cement. Standard viscosity and low viscosity cements differ in the amount of benzoyl peroxide present in the powder phase.

Manual mixing in accordance with manufacturer recommendations was performed in a bowl for 0.5 min. The cement dough was poured into a mold and the mix was allowed to cure at atmospheric pressure. The mold consisted of two rectangular glass plaques spaced by a rubber cord and held together with clamps. An additional set of specimens of the standard viscosity cement was prepared under external pressure. The cement dough was poured into a steel mold and a pressure equal to 0.2 MPa was applied for 15 min while the cement mass was polymerizing. After 15 min, samples were removed from the molds and machined to produce bars with dimensions 3.3 mm × 10 mm cross section and 90 mm in length. The machined specimens were placed into a 37°C water bath for 48 hrs as stated by the ISO 5833 protocol. The bars were loaded to failure in four-point bending and the maximum outer-fiber stress was calculated as follows

$$B = \frac{3Fa}{bd^2}$$

where B is the bending strength, F is the force at break, b and d are the width and the thickness of the specimen respectively, and a is the distance between loading points (20 mm). The distance between support points was 60 mm. The samples were tested using an Instron testing machine, Model 4467, at a deflection rate of 5 mm/min. The cements tested are summarized in *Table 1*.

Estimation of the Weibull Parameters

There are different approaches for estimating the two Weibull parameters from experimental data. The methods usually employed are as follows.

Maximum Likelihood Method

In this approach, values for the two parameters m and σ_0 are sought that result in a Weibull function which describes the experimental data that are most likely. The probability that for an estimated set of Weibull parameters, the experimental results would have occurred, is maximized by the following equation:

Table 1. Number of Specimens Tested for Each Bone Cement.

Cement	N
LV cured at atmospheric pressure (I)	72
SV cured at atmospheric pressure (II)	58
LV cured under external pressure (III)	40

$$\frac{N}{m} + \sum_{i=1}^{N} \ln \sigma_i - N \frac{\sum_{i=1}^{N} \sigma_i^m \ln \sigma_i}{\sum_{i=1}^{N} \sigma_i^m} = 0 \qquad (3)$$

in which N is the number of specimens tested and σ_i is the flexural strength of the specimen i. Only the parameter m and the experimental data appear in *Eq 3*, which was solved by the iterative method of Successive Substitutions.

Method of Moments

A set of data or a distribution may be reduced to a few numbers through calculating its moments. The first moment results in the mean value, and the standard deviation may be calculated from the second moment of a distribution. The probability density function of the Weibull distribution is given by:

$$f(\sigma) = \frac{m}{\sigma_0}\left(\frac{\sigma}{\sigma_0}\right)^{m-1} \exp\left[-\left(\frac{\sigma}{\sigma_0}\right)^m\right] \qquad (4)$$

In the method of moments it is assumed that the mean and variance of the experimental data equal those of the whole distribution, i.e., infinite number of samples. Putting $f(\sigma)$ of the Weibull distribution in the general definition of average and variance of a distribution, the first two moments of the Weibull distribution can be calculated.

The mean value is given by:

$$\bar{\sigma} = \sigma_0 \Gamma\left(1 + \frac{1}{m}\right) \qquad (5)$$

and the variance is:

$$S^2 = \sigma_0^2\left[\Gamma\left(1 + \frac{2}{m}\right) - \Gamma\left(1 + \frac{1}{m}\right)^2\right] \qquad (6)$$

where Γ is the gamma function. The standard deviation is the square root of the variance. Thus, the coefficient of variation for the Weibull distribution is:

$$C_{\mathrm{var},\sigma} \equiv \frac{S_\sigma}{\bar{\sigma}} = \frac{\left[\Gamma\left(1 + \frac{2}{m}\right) - \Gamma\left(1 + \frac{1}{m}\right)^2\right]^{1/2}}{\Gamma\left(1 + \frac{1}{m}\right)} \qquad (7)$$

Setting the mean and variance of the experimental data in *Eq 7*, the parameter σ_0 drops, and *Eq 7* becomes a function of m only and can be solved for m using an iterative procedure. The Newton-Rhapson method was used in the present work.

Least Square Analysis of Weibull Function

Linear regression is a special case of the least-squares method and it is the most common way of analyzing strength data. Taking the logarithm of *Eq 2* twice gives a linear equation:

$$\ln \ln\left[\frac{1}{1 - P_f}\right] = m \ln \sigma_f - m \ln \sigma_0 \qquad (8)$$

with a slope equal to m and a y-intercept equal to $-m \ln \sigma_0$. The σ_f values are the experimentally determined fracture stresses. The set of test results is converted into an experimental probability distribution. This is done by ordering the results from lowest stress to rupture to highest. The ith result in the set of N samples is assigned a cumulative probability of failure P_{fi}. Since the true value of P_f for each σ_f is not known, it has to be estimated. Three of the most common estimators for the probability of fracture are (18–24).

$$P_{f1} = \frac{i - 0.5}{N}\ (a); \quad P_{f2} = \frac{i}{N + 1}\ (b); \quad P_{f3} = \frac{i - 0.3}{N + 0.4}\ (c) \qquad (9)$$

The method of least squares is built on the hypothesis that the optimum description of a set of data is one that minimizes the weighted sum of squares of deviation of the data from the fitting function. When applying the linear least squares analysis to *Eq 8*, it is assumed that the error is constant or randomly distributed. However, by using the theory of propagation of errors, previous workers pointed out that values of $\ln \ln [1/(1 - P_f)]$ in *Eq 8* should be weighted according to their uncertainty (18, 22). Hence, in order to compare the anaysis performed with and without weighting factors, the following weight functions were used.

$$W_1 = 1 \qquad (10)$$

$$W_2 = \lfloor (1 - P_{fi}) \ln(1 - P_{fi})^2 \rfloor \qquad (11)$$

$$W_3 = 3.3P_{fi} - 27.5 \lfloor 1 - (1 - P_{fi})^{0.025} \rfloor \qquad (12)$$

Probably the best method of evaluation is least squares estimation using *Eq 2* directly. The nonlinear fitting of the Weibull equation with experimental data was made using the following procedure. Tables of the probability of failure, P_{fi}, calculated with a selected estimator from *Eqs 9 (a)–(c)* and the corresponding flexural strength, σ_i, were build up from every experimental set of data. An initial set of the parameters m and σ_0 was selected. Predicted P_{fi} values were generated using *Eq 2* and the following summation for the selected set of parameters was calculated:

$$S = \sum_{i=1}^{N} [P_i - P_{i,c}]^2$$

A minimum of the S function was searched using an optimization program that operates over the adjustable parameters: m and σ_0

Monte Carlo Simulation

The statistical properties of different estimators and methods of analysis were studied by Monte Carlo simulation techniques. It was assumed a fictitious material for which the fracture stresses follow a Weibull distribution of known parameters. That is, the exact values of both parameters m and σ_0 of the mother population were known: m_∞ and $\sigma_{0\infty}$. A random generator procedure was used to generate N real numbers in the interval [0, 1], which were taken as the fracture probability. Each fracture probability was set in *Eq 2*, where the known values m_∞ and $\sigma_{0\infty}$ were already set, and solved for σ_f. So, a set of N fictitious fracture stresses was obtained, which was then treated as experimental results. The Weibull function of this set was evaluated using the methods described above. The sample size, N, was increased progressively from 4 to 100 in order to study the influence of the number of specimens on the estimated Weibull parameters. By repeating the above procedure many times, the statistical behavior of failure probability estimators and methods of analysis were assessed. The procedure was repeated 1000 times for each method and each sample size N in order to ensure statistical convergence of the results. Some simulations were done with 1500 repetitions; however, the comparison of the results revealed that 1000 repetitions are representative.

RESULTS AND DISCUSSION

Monte Carlo Analysis of Theoretical Weibull Distribution

In practice, the evaluation of the Weibull parameters is performed from a limited number of test specimens and it is relevant to find out which method of evaluation results in the most accurate estimation. In order to compare the methods described previously and to assess the influence of the sample size on the Weibull parameters, Monte Carlo simulations were carried out. Simulations for samples having different number of specimens were performed for arbitrary chosen true values of the Weibull parameters m_∞ and $\sigma_{0\infty}$ equal to 7, 10, 15 and 45, 50, 55 respectively. Results for m and σ_0 values equal to 7 and 55 respectively are depicted in *Figs. 1–4*.

Figure 1 shows the calculated m value against the sample size N for the moments and maximum likelihood methods. It is seen that for $N > 10$, the maximum likelihood estimation results on average in a higher overestimation of the m value compared with the moments method. The average value of the estimators is extremely sensitive to sample size, particularly for the range $N < 30$, and it approaches the true value m_∞ with increasing number of specimens. The results in *Fig. 1* show that if N random fracture stresses of this material are chosen and then the Weibull modulus is determined, it will definitely result in m_∞ as a consequence of the estimation procedure. This is because the true parameters of a distribution are known only when an infinite number of samples are tested. Monte Carlo simulations performed by the moments method for m values of 7, 10 and 15 at different sample sizes are shown in *Fig. 2*. It is seen that the bias increases with increasing Weibull modulus, which is in agreement with results reported by previous workers (22). Conversely, the maximum likelihood method was demonstrated to be independent of the m value.

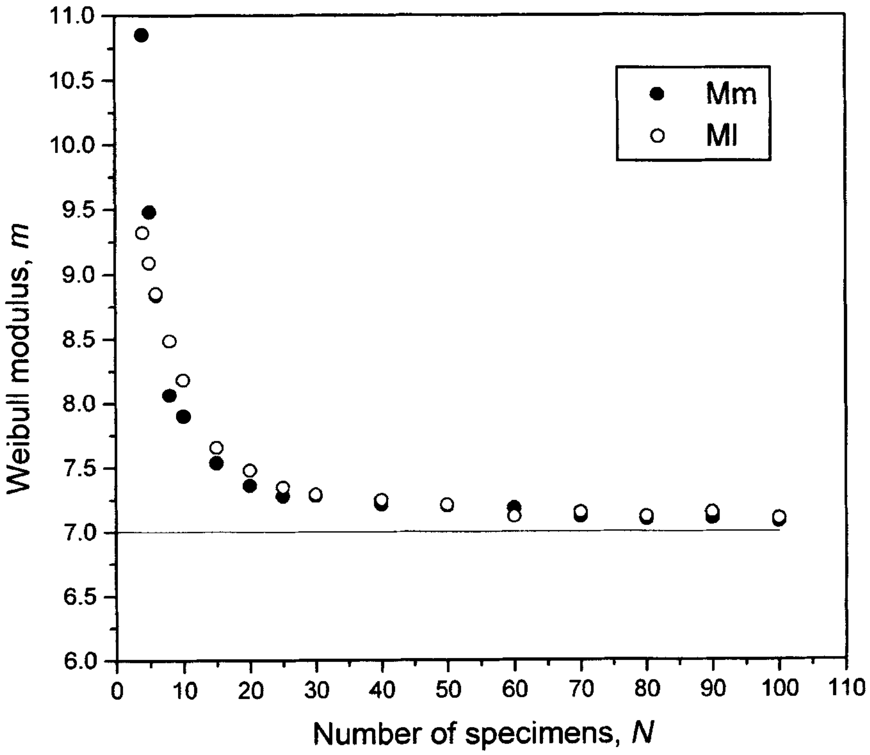

Fig. 1. Weibull modulus as a function of sample size calculated from the moments (Mm) and maximum likelihood (Ml) methods. The solid line is the set m value.

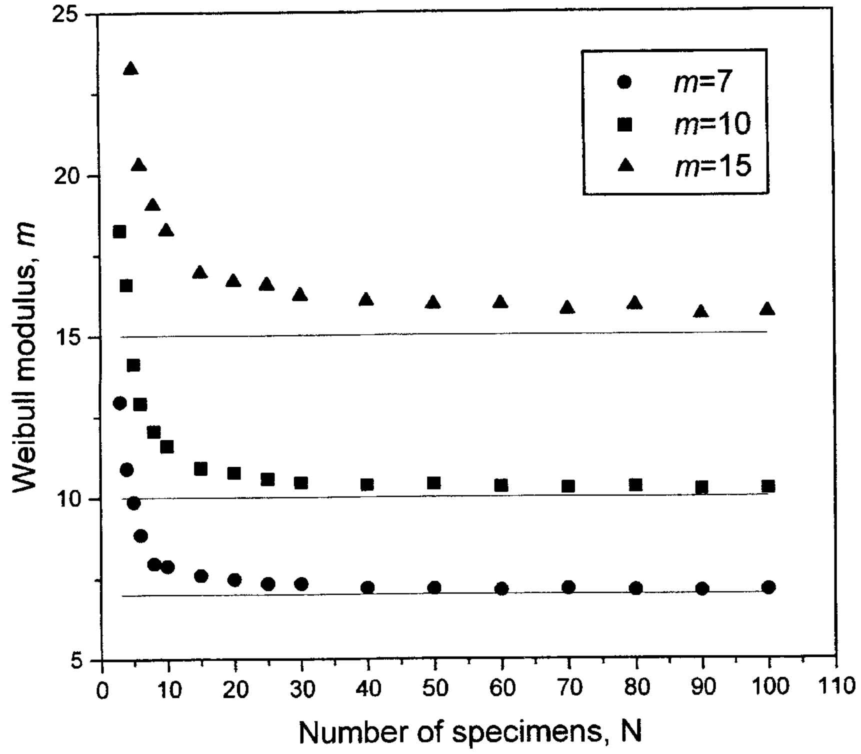

Fig. 2. Weibull modulus as a function of sample size calculated from the moments method for different m values.

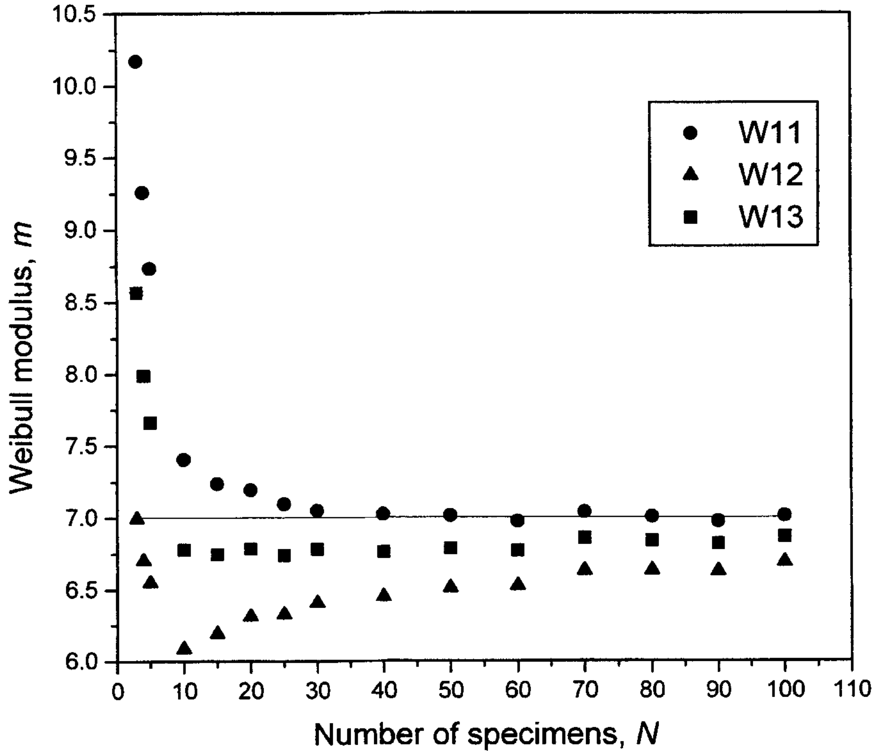

Fig. 3. Weibull modulus as a function of estimator and sample size calculated from linear regression using the failure probability in Eqs 9 and the weight function W_1.

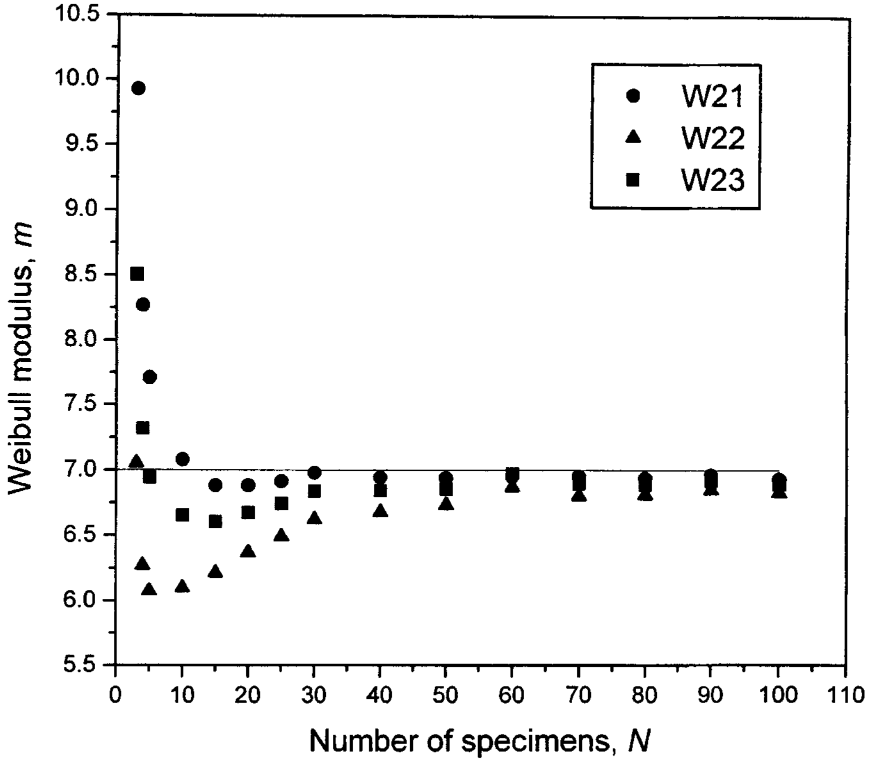

Fig. 4. Weibull modulus as a function of estimator and sample size calculated from linear regression using the failure probability in Eqs 9 and the weight function W_2.

Figures 3–5 show the results of the simulation carried out by the linear regression method given by *Eq 8*. Since for this analysis the true failure probability for each fracture stress is not known, a prescribed function has to be used to calculate its value. The three estimators for fracture probability given in *Eqs 9 a–c* were used in each simulation procedure. The influence of the weight functions W_1, W_2 and W_3 given in *Eqs 10–12* was also assessed. Results are presented in terms of W_{ij}, which denotes the i weight function from *Eqs 10–12* along with the j estimator from *Eqs 9*. For example, W_{21} represents the weight function W_2 and the estimator P_{f1}.

Figure 3 corresponds to the linear regression analysis performed without weighting factors, i.e., W_1 equal to 1, for the three definitions for the failure probability. Similarly to the moments and maximum likelihood methods, the average estimate is extremely sensitive to sample size for the range $N < 30$. This method converges to the true value when the estimator $P_{f1} = (i - 0.5)/N$ is used. However, with the definitions $P_{f2} = i/(N + 1)$ and $P_{f3} = (i - 0.3)/(N + 0.4)$, which underestimate the m value for all sample sizes, does not converge to the true value m_∞. The definition of probability of fracture $P_{f2} = i/(N + 1)$ results in the least acceptable outcome. For a nonweighted analysis, the estimator $P_{f3} = (i - 0.3)/(N + 0.4)$ is the least biased for $N < 20$ while for $N > 20$ the definition $P_{f1} = (i - 0.5)/N$ results in better estimator. For sample sizes of 30 or greater, the predicted modulus is essentially independent of sample size when the estimator $P_{f1} = (i - 0.5)/N$ is used. These results indicate that the use of a large sample size does not satisfactorily compensate for the use of a poor estimator. Even a sample size of 100 is not sufficient if the estimator $P_{f2} = i/(N + 1)$ is used. On the other hand, by comparing the method of moments (*Figs. 1–2*) with the results shown for the linear regression with the definition $P_{f1} = (i - 0.5)/N$, it emerges that there are no statistical advantages in using the method of moments.

The effectiveness of applying weighting factors in linear regression analysis was assessed by Monte Carlo simulations. *Figures 4–5* show the results of applying the weighting factors W_2 and W_3 to the three estimators for the failure probability given in *Eqs 9*. The results in *Figs. 4–5* indicate that the use of weighting factors in linear regression with the definition $P_{f1} = (i - 0.5)/N$ reduces the bias in determining m. The m value approaches the true value m_∞ for $N > 10$, indicating an important reduction in the number of samples to be tested compared with the nonweighted analysis. The weight functions applied to $P_{f3} = (i - 0.3)/(N + 0.4)$ markedly improve the results for $N > 30$. However, the bias increases for $N < 30$ compared with a nonweighted analysis. In contrast, for the definition $P_{f2} = i/(N + 1)$ there is no significant gain for $N < 60$. Comparison of the behavior of the weight factors W_2 and W_3 shows that both weight functions result in a

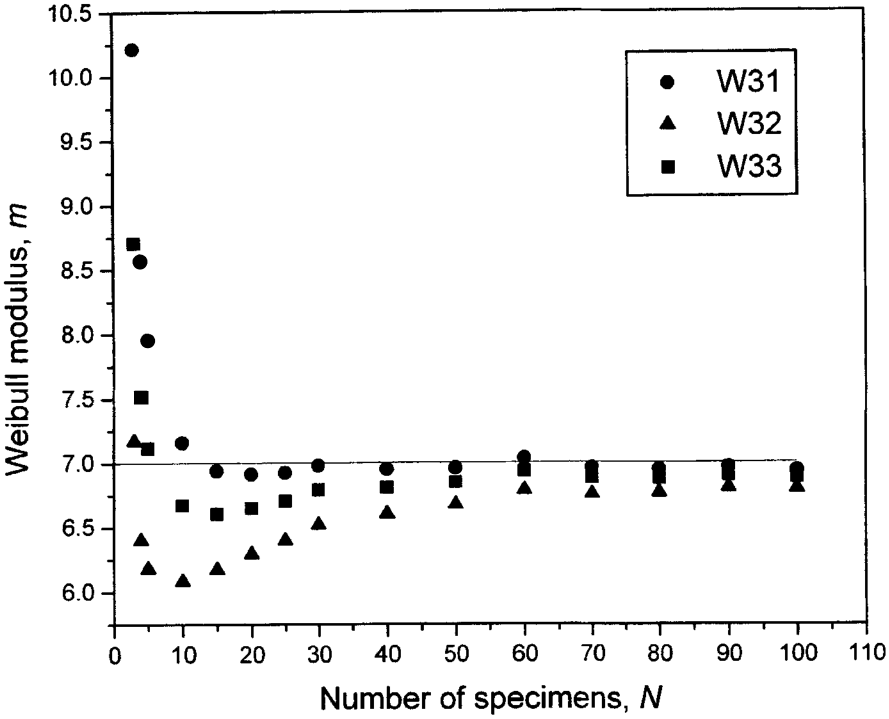

Fig. 5. Weibull modulus as a function of estimator and sample size calculated from linear regression using the failure probability in Eqs 9 and the weight function W_3.

similar trend, so the choice of definition is not important. The plots represented in *Figs. 3–5* were all determined for an arbitrary value of $m = 7$. Similar plots performed for m values equal to 10 and 15 indicated that the dispersion of the parameter m is independent of the value assigned to m_∞ for all three evaluation methods. Therefore, the results are valid for all values of m_∞.

The accuracy of the evaluation procedure is described by the variation coefficient, C_{var}, which is a common measure for the breath of a distribution. It is defined for the parameter m as the standard deviation divided by the mean value. *Figure 6* shows the variation coefficient as a function of the sample size computed for the maximum likelihood, the moments method and the linear regression without weighting factors and the definition $P_{f1} = (i - 0.5)/N$. Comparison of the behavior of C_{var} for the three different definitions of failure probability demonstrated that there is no significant difference among the definitions. Thus only the trend using $P_{f1} = (i - 0.5)/N$ is illustrated in *Fig. 6*. Similarly to the average value of the Weibull modulus, C_{var} decreases with increasing sample size, i.e. the more samples are measured, the more accurate the results. The fitted polynomial goes through zero for $N = \infty$, i.e. the true value of the parameter m is obtained only for an infinite number of samples. The curvature of all curves decreases continuously with increasing the number of specimens. Therefore, up to about 30 specimens, there is a high gain in accuracy for each additional sample. From about 30 samples upwards, the gain in precision decreases with increasing sample size. It is clear from *Fig. 6* that the C_{var} value of the maximum likelihood evaluation procedure is lower than that of the linear regression method for the whole range of sample size. Since the maximum likelihood method leads to the least dispersion, i.e., best reproducibility for all sample sizes, the use of this method was recommended by previous authors (20, 21). *Figure 7* shows the trend of the C_{var} value for the linear regression method for a weighted analysis. The linear regression without weighting factors and the maximum likelihood methods are shown for comparison in the same plot. It is observed that C_{var} decreases significantly with the use of weighting factors and approaches to the values of the maximum likelihood method. Hence, in order to have an equal measure of accuracy, a smaller number of samples is required if the linear regression with weighting factors is used.

Results presented in *Figs. 1–7* suggest that a minimum number of 30 specimens is required to have an acceptable degree of accuracy in obtaining the Weibull modulus by the maximum likelihood method and moment method. The linear least squares method is preferred for its simplicity, rather than more complex methods involving direct curve fitting. If linear regression in an analysis without weighting factors is to be chosen as the evaluation method, the definitions $P_{f3} = (i - 0.3)/(N + 0.4)$ for $N < 20$ or $P_{f1} = (i - 0.5)/N$ for $N > 20$ result in the best approach to the true value m_∞. On the other hand, if the evaluation is performed by linear regression with weighting factors, the

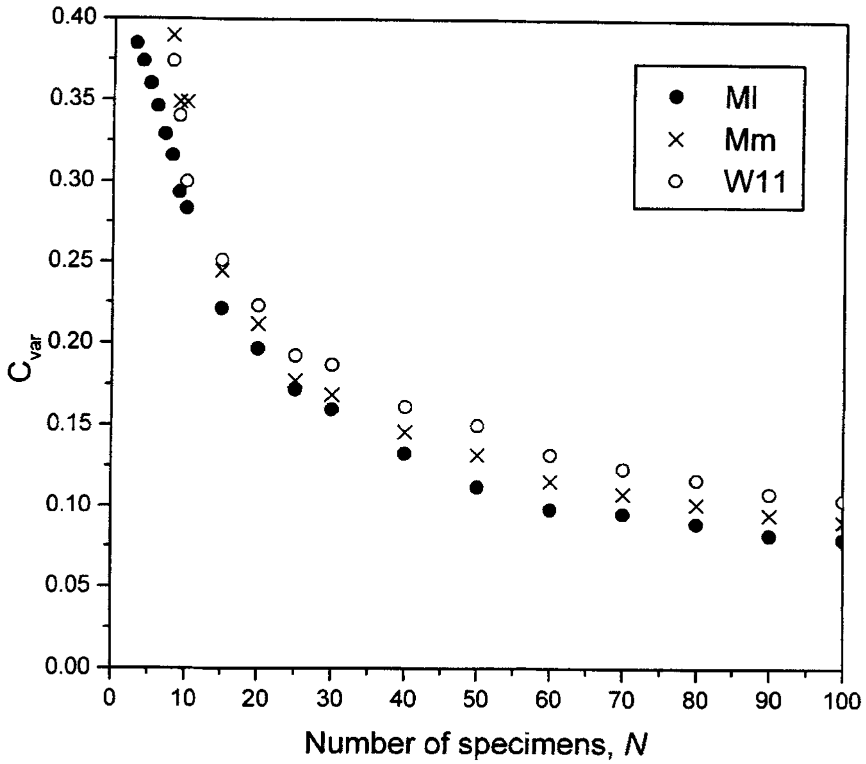

Fig. 6. The variation coefficient of the Weibull modulus as a function of sample size calculated from the moments (Mm), maximum likelihood method (Ml) and linear regression without weighting factors (W11).

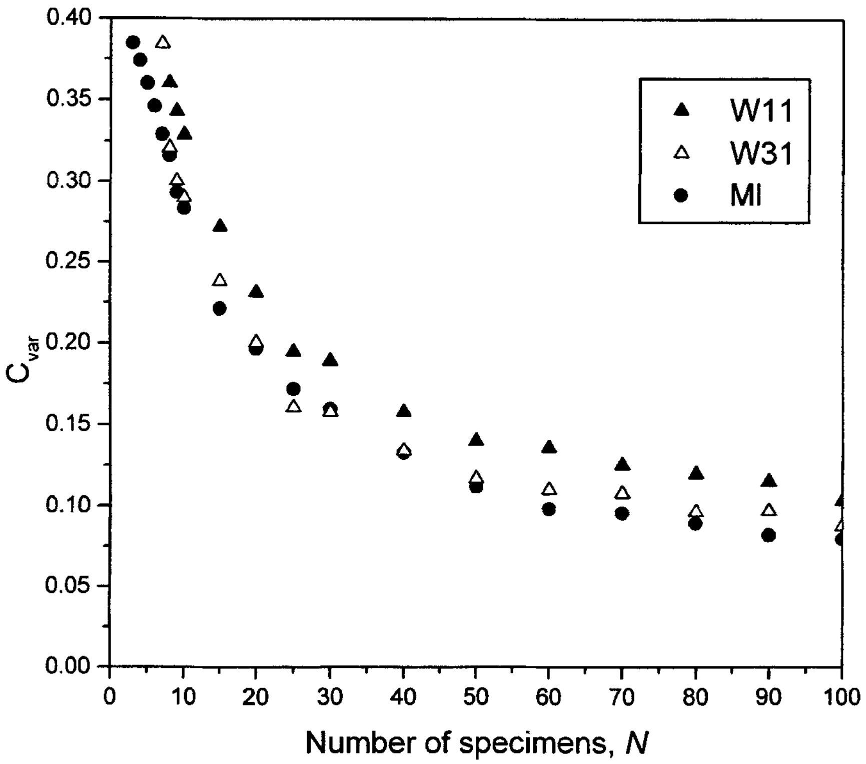

Fig. 7. The variation coefficient of the Weibull modulus as a function of sample size calculated from the maximum likelihood method (Ml) and linear regression with and without weighting factors (W31 and W11).

most successful estimator seems to be $P_{f1} = (i - 0.5)/N$ for which the Weibull modulus values stabilize at a lower sample size. While the nonweighted analysis requires a minimum set of 30 specimens for the evaluation of the Weibull modulus, in the weighted analysis the m value stabilizes at a sample size of about 10 specimens.

Considering the parameter σ_0, it can be expected that as for the parameter m an increasing sample size results in decreasing error in estimation of σ_0. *Figure 8* shows the normalized σ_0 value as a function of the sample size calculated for the linear regression method, moments method and the method of maximum likelihood. The behavior of the three failure probability estimators is similar to that observed in the evaluation of m Clearly, σ_0 may be determined with a higher degree of accuracy than m. This is consistent with results reported by previous authors who found that compared with the m determination no significant variation occurs in determining the scale parameter σ_0 (20, 21).

The uncertainty of the mean strength when only a part of the total batch is used for its determination was evaluated generating subsets of different sizes by Monte Carlo simulation. *Figure 9* shows the relative uncertainly against the number of samples for different Weibull modulus values. It is observed that the scattering is sensitive to both the number of specimens and the Weibull modulus. For a material having an m value equal to 20, the scattering is not in excess of 5% for sets of five samples. However, for materials having a Weibull modulus equal to 7 and 10, which are typical values for bone cements, the number of specimens to test in order to obtain a 5% scattering increases to 15 and 25 respectively. Sets of 10 samples lead to an uncertainly of about 10% on the estimated mean strength with regard to its value calculated from the whole batch. *Figure 10* corresponds to the simulation carried out for an m value equal to 7 and different σ_0 values. Contrary to the trend observed when varying the Weibull modulus, the scattering in the mean strength is almost unchanged by varying the σ_0 parameter.

Analysis of Experimental Results

A similar investigation was carried out, but rather than using computer-generated data, actual experimental results were employed. Although the actual modulus and scale parameters are not known *a priori* in this case, a good estimate of these parameters can be made by using a large sample size. The number of specimens tested for each cement are summarized in *Table 1* and the Weibull plots of the data using $P_{f1} = (i - 0.5)/N$ are shown in *Fig. 11*. The regression coefficients of the flexural strength data in *Fig. 11* indicate that the cements tested are satisfactorily described by the Weibull model. To investigate the effect of sample size on Weibull modulus, subsets of the fracture stress data were selected in groups of 4, 5, 10, 15, 20 and so on, up to N. The subsets were selected randomly from the full set of experimental results by

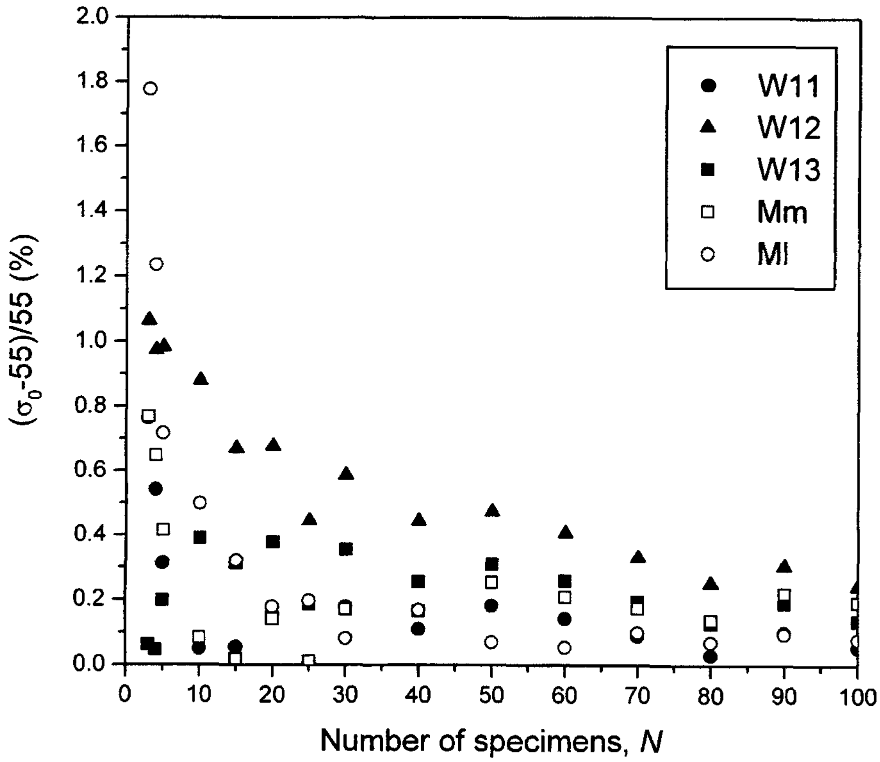

Fig. 8. Normalized scale parameter as a function of sample size calculated from the moments (Mm), maximum likelihood method (Ml) and linear regression without weighting factors.

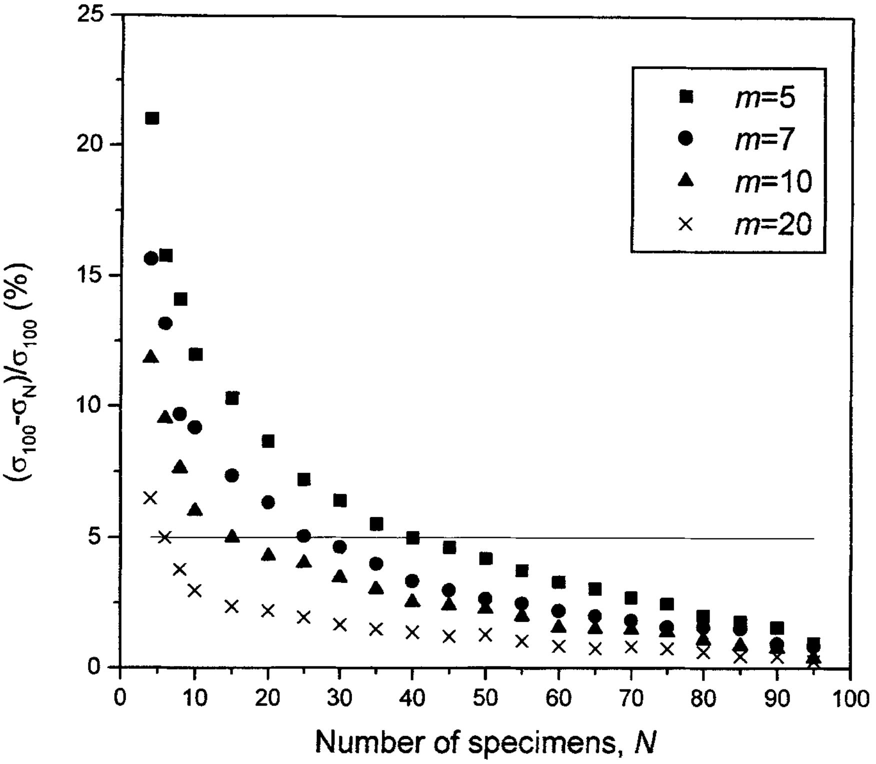

Fig. 9. Scattering of mean strength as a function of sample size and Weibull modulus.

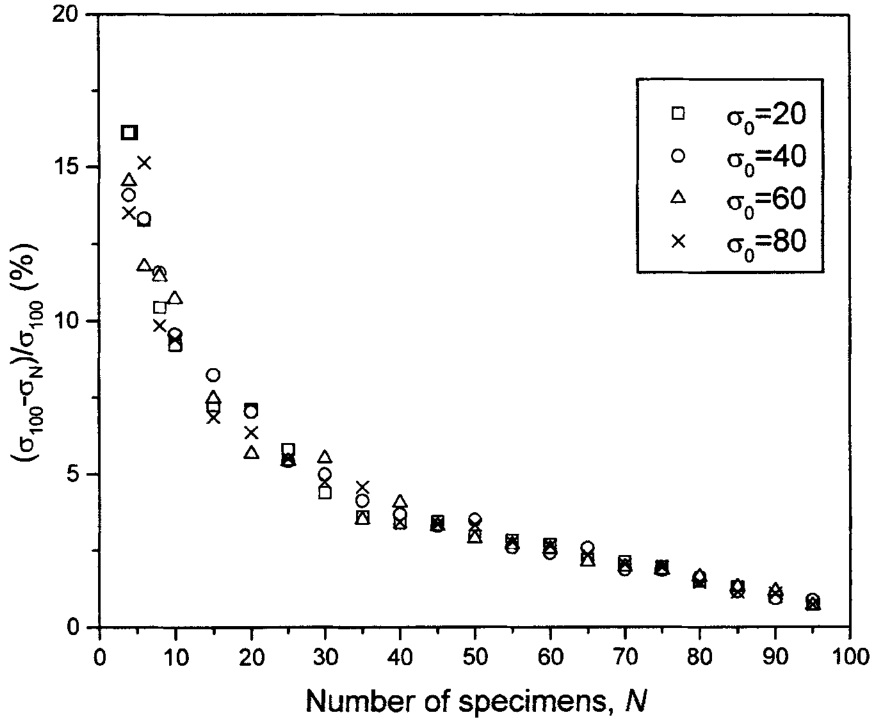

Fig. 10. Scattering of mean strength as a function of sample size and σ_0.

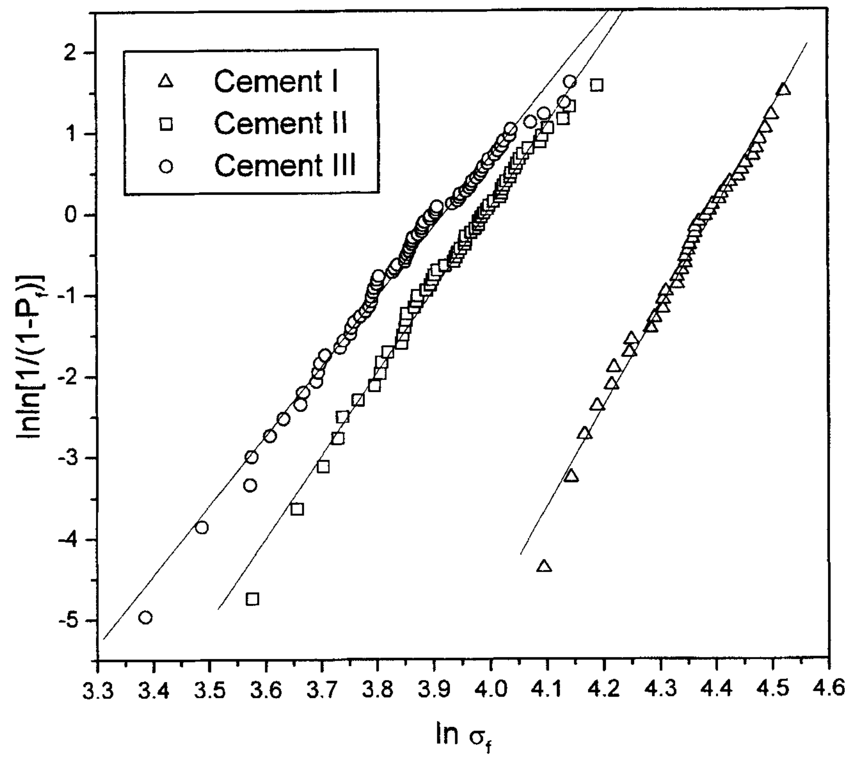

Fig. 11. Weibull plots for the cements tested.

assigning a computer-generated random number to each specimen of the set. Each subset was treated as an independent set of data. The procedure was repeated 100 times for each subset. The Weibull modulus for each subset was obtained by linear least squares fitting using the estimators P_{f1} and P_{f3} and the weight functions W_1 and W_3.

Figure 12 shows the averages and standard deviations of m calculated for the cement I as a function of the sample size. The standard deviation, as expected, decreases markedly as the number of specimens increases. It is worth noting that the standard deviation for a sample size of 20 using the definition P_{f1} in a nonweighted analysis is approximately 16% of the mean. This restriction should be considered when determining the Weibull modulus on small sets.

The influence of the subset size on the m values for the three cements tested is shown in *Figs. 13–15*. The solids lines are the m values calculated by each method from the total number of specimens. Unlike the theoretical results, the m values obtained from experimental results stabilize at lower sample sizes. It can be seen that the definition P_{f3} yields lower Weibull modulus values and reaches an asymptotic value at sample sizes much higher than the definition P_{f1}. For sample sizes lower than about 15 specimens in a nonweighted analysis, the estimator P_{f3} is the least biased. However, in contrast to P_{f1}, for $N < 15$ the bias is not improved by the use of weighting factors. This trend was displayed by the three cements and is in agreement with the previous theoretical results. The most successful estimator seems to be P_{f1}, for which the predicted modulus is essentially independent of sample size for a sample size of 20 in a nonweighted analysis and about 10 in a weighted analysis. The use of this estimator was recommended on the basis of Monte Carlo simulations, which is verified by these experimental results.

As discussed previously, the direct least squares analysis of *Eq 2* should yield the best estimate of the Weibull parameters. The study of the statistical properties for this method by using 1000 samples for each sample size is much more complicated than the linear least squares analysis. Therefore, the nonlinear least square analysis was not studied in the corresponding complete way as for the other methods of analysis. However, from a limited number of samples it is possible to compare this method with the methods of analysis described earlier. *Table 2* summarizes the m values of each cement calculated for the total number of specimens. From the comparison of the m values calculated by each method, it emerges that the linear analysis using weighting factors yields the best agreement to the direct nonlinear fitting, which gives the best description of the sample.

From the practical point of view, the method of evaluation is selected as a compromise between the accuracy of the estimation and the experimental effort. Based on the results obtained, it may be concluded that a nonweighted linear regression with the definition P_{f3}

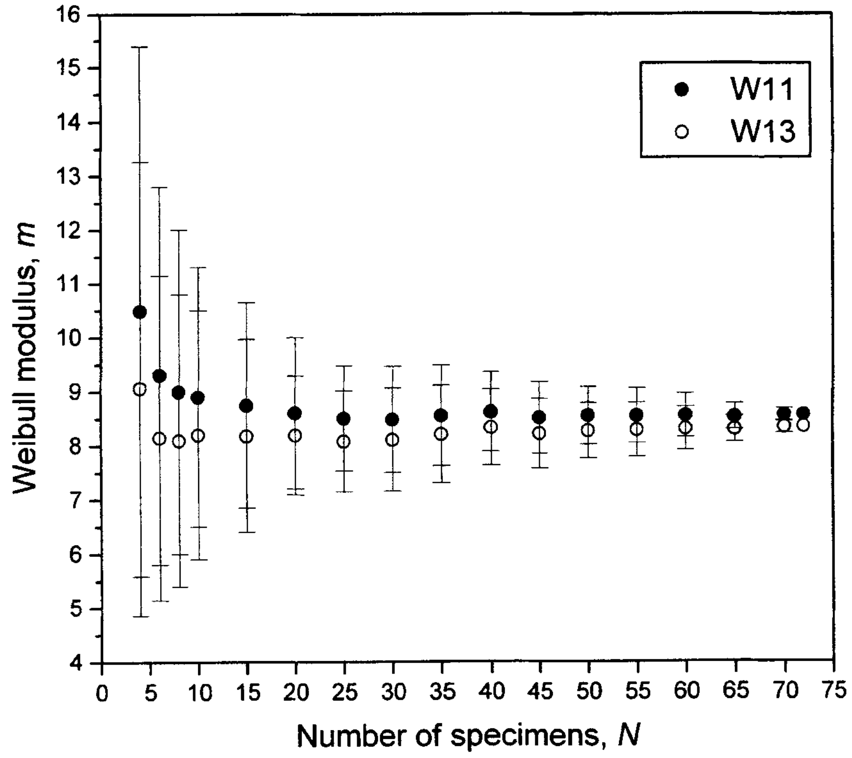

Fig. 12. Weibull modulus and its standard deviation as a function of sample size for the Cement I.

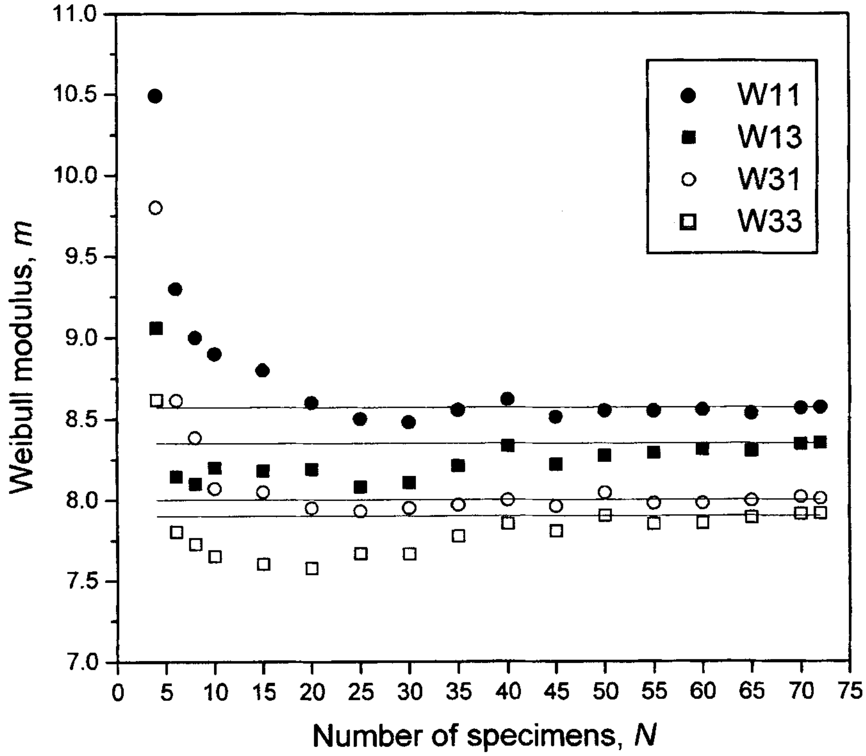

Fig. 13. Weibull modulus as a function of sample size calculated from the definitions P_{f1} and P_{f2} and the weight functions W_1 and W_3 for Cement I.

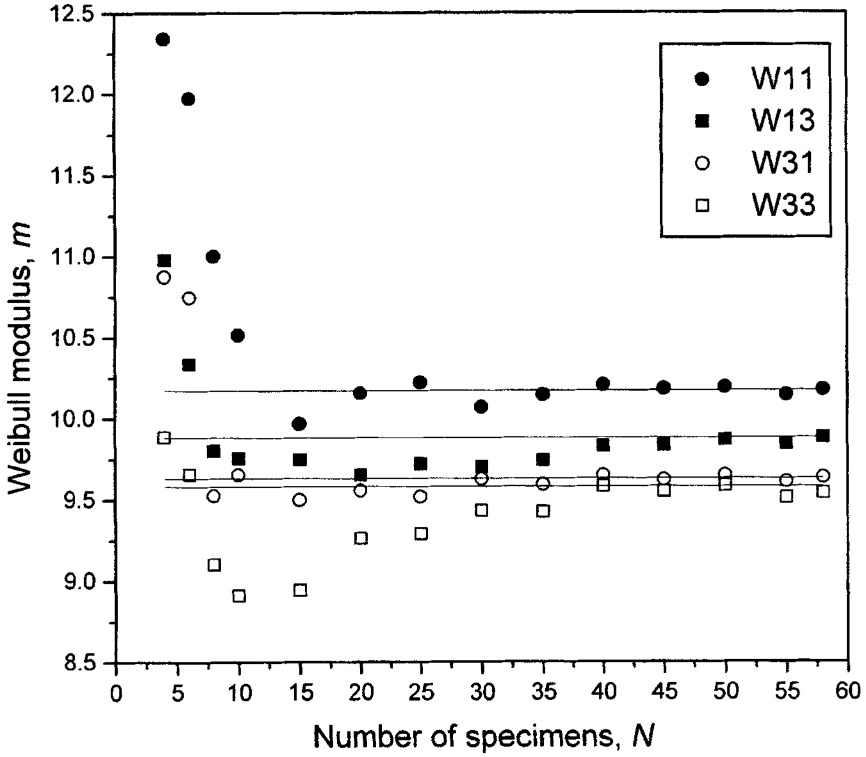

Fig. 14. Weibull modulus as a function of sample size calculated from the definitions P_{f1} and P_{f2} and the weight functions W_1 and W_3 for Cement II.

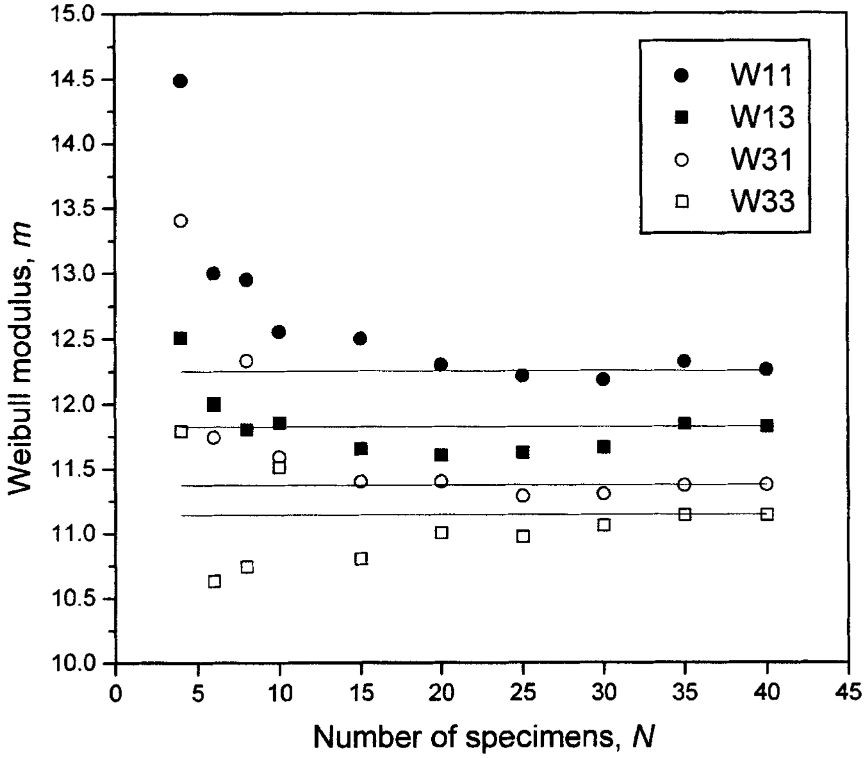

Fig. 15. Weibull modulus as a function of sample size calculated from the definitions P_{f1} and P_{f2} and the weight functions W_1 and W_3 for Cement III.

Table 2. Weibull Modulus for the Total Number of Specimens Calculated by Linear Least Squares With and Without Weighting Factors and Direct Least Squares (DLS).

Cement	$W_1\ P_{f1}$	$W_1\ P_{f3}$	$W_3\ P_{f1}$	$W_3\ P_{f3}$	DLF P_{f1}	DLS P_{f3}
I	8.57	8.35	8.00	7.91	8.15	8.08
II	10.17	9.88	9.63	9.58	9.75	9.65
III	12.25	11.82	11.37	11.14	11.47	11.30

should be used if a limited number of specimens are available. For more accurate Weibull modulus values, the definition P_{f1} should be applied. In this case, a sample size of 20 specimens is adequate in a non-weighted analysis and a sample size of 10 specimens in a weighted analysis. On the other hand, as shown in *Fig. 7*, the standard deviation of m decreases significantly with the use of weighting factors, so this analysis should be preferred for determining Weibull parameters.

CONCLUSIONS

Monte Carlo simulations were used to characterize three evaluation methods of Weibull parameters: linear regression, moments method and maximum likelihood method. The sample size has a large influence upon how well the mother population is described for all methods studied. From the comparison of the moments and maximum likelihood methods using the linear regression method and the definition $P_{f1} = (i - 0.5)/N$, it emerges that there are no statistical advantages in using the first two methods. Although the dispersion of the m value is the lowest for the maximum likelihood method it decreases significantly by applying weighting factors in the linear regression method. In addition, the use of weighting factors in the linear analysis yields the best agreement to the direct nonlinear fitting, which gives the best description of the sample.

The analysis of experimental results suggests that as a compromise between minimizing both the dispersion of the estimator on one hand and the experimental effort on the other hand, it is suggested to use the linear regression method with the definition $P_{fi} = (i - 0.5)/N$ and a sample size of 20 in a nonweighted analysis and 10 in a weighted analysis. For sample sizes lower than 10, the estimator P_{f3} in a nonweighted analysis yields the least biased results.

Monte Carlo analysis demonstrates that the scattering of the mean strength is sensitive to both the number of specimens and the Weibull modulus. Conversely, σ_0 showed no influence on the mean strength value. For a material having an m value equal to 20 the scattering is less than 5% for sets of five samples. However, for m values in the range 7–10, which are typical values for bone cements, the number of specimens to test in order to obtain a 5% scattering increases to 15 and 25 respectively. Sets of 10 samples lead to an uncertainly of about 10% on the estimated mean strength with regard to its value calculated from the whole batch.

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