The unexplored implications of Lange's 1938 model: dynamics and the role of the marginal propensity to save

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Introduction

Oskar Lange's 1938 paper "The Rate of Interest and the Optimum Propensity to Consume" is widely recognized as one of the earliest mathematical studies of Keynes's *General Theory*. In light of its analytical content, it has been associated with the IS-LL model proposed by John Hicks in 1937 (Schumpeter, 1981; Ackley, 1961; Robbins, 1968; MacDowell, 1957; Young, 1987; Darity & Young, 1995)

Building on Lange's model, Samuelson (1941) proceeded to devise a Neo-Keynesian dynamic system[1]. However, Lange's and Kalecki's quotations of one another's work in the late 1930s reveal that the potential implications, in terms of dynamic analysis of Lange's model, are far from being totally explored[2]. The present article tries to shed light on this overlooked issue through a deeper investigation of the implications in terms of the dynamics of Lange's 1938 contribution, which are entirely lacking in the original[3].

To this end, we reconstruct Lange's 1930s' general beliefs about capitalist dynamics (expressed in a series of articles published between 1934 and 1942), in order to highlights its most outstanding features. On this 'philological' basis, we attempt to provide a consistent formalization of Lange's dynamic model, whose properties, because they are focused both on

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the time-lag between investment orders and actual investment, and the implication of Lange's non-linear investment function, differ significantly from those proposed by Samuelson.

Section 1 discusses Lange's early reflection on dynamics. Section 2 focuses on the assumptions in Lange's original model. Section 3, using some additional assumptions, outlines the necessary and sufficient conditions for the generation of self-sustaining cycles and briefly compares Lange's dynamic model and Kalecki's 1939 model.

1. The foundations of Lange's endogenous dynamics

Lange manifested his interest in dynamics very early in his career. In his 1935 article "Marxian Economics and Modern Economic Theory", he posited a methodological premise to trade cycle and growth theorizing. He asserted that the methodologically correct approach to business fluctuations is to explain precisely the evolution of the economy as resulting from "within" the economic process, in a capitalist society.

For that reason, Lange denied that "*Modern Economic Theory*" provided a satisfactory description of the intrinsic development of capitalist economies[4]. Its fault lay in investigation the economic process "under a system of constant data" which completely ignored both the characteristics of the data (which became the object of economic statistics) and changes in the data (the object of economic history). This approach was likely to consolidate the view that capitalist economies were *naturally* stable[5].

Sarcastically, Lange wrote:

> It was very generally held among "bourgeois" economists both at the beginning of the twentieth century and in the years preceding 1929, that the economic stability of

> Capitalism was increasing and that business fluctuations were becoming less and less intense. (Lange, 1935, p. 190)

Lange thought that the only way to eschew such bias was to define the assumptions related to the institutional framework within which the economic process operates. His thesis was that Marxian economics could offer an adequate framework both to understand this process[6] and to translate an institutional datum (i.e. the existence of a capitalist society) into the "language" of economics. Its merit would lie in it assuming the existence of two social classes: the owners of the means of production, and the owner of labor power:

> … the consequences of the additional institutional' datum which distinguishes Capitalism from other forms of exchange economy, i.e. the existence of a class of people who do not possess any means of production, is scarcely examined. Now, Marxian economics is distinguished by making the specification of this additional institutional datum *the very corner-stone of its analysis*, thus discovering the clue to the peculiarity of the Capitalist system by which it differs from other forms of exchange-economy. (Lange, 1935, p. 192, emphasis added)

In other words, Lange was persuaded that *Capital* (meant as a social relation of production) was the key concept for satisfactorily treatment of the problem of capitalist dynamics. However, he was aware that this social relation had a dual dimension in dealing with both the struggle between capitalist and workers, and the competition among the capitalists themselves. Therefore he adopted a dual perspective on the interaction between the dynamics of income distributive shares, investment and technological change.

In the first place, he recalled the essentials of Marx's analysis of the business cycle (Lange, 1935, Section 8)[7] according to which a high rate of accumulation implies an increase in both employment and real wages which eventually reduces the profit rate. Fierce competition

drives firms to resort to more capital-intensive techniques. Once this tendency spreads throughout the economy, technical progress is accompanied by a surge in unemployment followed by the fall in real wages, which eventually restores the profit rate:

> For Capitalism creates, according to Marx, its own surplus population (industrial reserve army) through technical progress, replacing workers by machines. The existence of the surplus population created by technical progress prevents wages from rising so as to swallow profits. Thus technical progress is necessary to maintain the capitalist system and the dynamic nature of the capitalist system, which explains the constant increase of the organic composition of capital, is established. (Lange, 1935, p. 199)

It is thus the interaction between the real wage dynamics and the dynamics of investment that explain some of intrinsic instability of capitalism. Notwithstanding this, Lange did not fail to emphasize that an assessment of unstable economic development does not mean that cycles and growth are automatic.

> Of course, the necessity of the fact that labour-saving technical innovations are always available at the right moment cannot be deduced by economic theory and in this sense the "necessity" of economic evolution cannot be proved. But Marxian economics does not attempt to prove this. All it establishes is that the capitalist system cannot maintain itself without such innovations. And this proof is given by an economic theory which shows that profit and interest on capital can exist only on account of the instability of a certain datum i.e. the technique of production, and that it would necessarily disappear the moment further technical progress proved impossible. (Lange, 1935, pp. 199-200)

In the second place, in a series of parallel works written between 1934 and 1937, Lange emphasizes the pivotal role played by the unequal distribution of income in a capitalist economy, which in turn is related directly to the *"irrational"* accumulation of capital. Since the distribution of income is an historical datum which occurs "*independently of the*

requirements of the maximisation of social welfare" (Lange, 1937, p.123), in a capitalist economy the amount of saving will be determined independent of the demand for investment. In fact, it will be a result of both social habits and the historical distribution of income (few income classes are unable to save at all). As a consequence, the accumulation of capital will necessarily be affected in a prejudicial way[8]:

> …saving is ... in the present economic order determined only partly by pure utility considerations, and *the rate of saving is affected much more by the distribution of incomes, which is irrational from the economist's point of view*. (Lange, 1937, p.127, emphasis in original)

Thus Lange clearly identifies the two nodes of the problem that must be unraveled in the analysis of capitalist dynamics. It is worth elaborating here how the savings rate dynamics is related to the dynamics of the income distributive share. Since the real wage cannot increase above the subsistence level, workers' savings are assumed to remain negligible relative to capitalists' savings. As a consequence, the higher the profit share, the higher will be the savings rate. This means that any innovation that strengthens the profit share will be accompanied by a rise in the savings rate. Since technical progress is a mere historical datum, it is not possible to impose a different savings rate from the rate determined by the operation of the cycle (which, unfortunately, is *"irrational"*).

Of course, Lange was aware that the dynamics of these critical economic "data" was likely to interact with extra-economic factors and this became the field of his application of historical materialism:

> … the full evolution of Capitalism in all its concreteness cannot be explained by a theory of economic evolution alone. It can be explained only by a joint use of both economic

> theory and the theory of historical materialism. The latter is an inseparable part of the Marxian analysis of Capitalism. (Lange, 1935, p. 201)

In turn, Lange's endorsement of Marx's materialist conception of history, had a crucial implication for dynamics. Since Lange evidently was persuaded that "The production of ideas, of conceptions, of consciousness, is at first directly interwoven with the material activity and the material intercourse of men, the language of real life" (Marx & Engels, 1975, p. 35), he claimed that time lags should in any case be considered the ultimate determinants of fluctuations and instability. At best, they could be considered the *phenomenal form* of deeper changes in the economic and institutional data. Lange denied neither their existence nor their theoretical relevance on certain occasions, he claimed simply that they could not, alone, suffice to produce a dynamic theory. It is for this reason that Lange rejected the analysis based on the "Cobweb Theorem", for example, which, according to him, rested mainly on the existence of time lags:

> These theories deduce the impossibility of an equilibrium … from the very nature of the adjustment mechanism, but they cannot deduce theoretically the changes of data responsible for the trend on which the fluctuations due to the process of adjustment are superimposed. (Lange, 1935, n.2 pp. 192-193)

Given such premises, it is worth considering to what extent Lange believed that the analytical apparatus created by Keynes might be a useful tool to cope with the problem of dynamics. In our view, Lange's 1938 paper "The Rate of Interest and the Optimum Propensity to Consume" sheds some light on this issue.

2. Lange's 1938 static model

Lange developed in explicit mathematical form the meaning of the Keynesian system. It stressed three fundamental relationships: (1) the consumption function relating consumption to income, and for generality, to the interest rate as well; (2) the marginal efficiency of capital relating net investment to the interest rate and to the level of consumption (as for a level of capital equipment fixed for the short period under investigation); (3) the schedule of liquidity preference relating the existing amount of money to the interest rate and the level of income[9].

Lange makes some assumptions similar to those made by Hicks (1937). First, Lange's analysis is static. Since net investment varies, so does total capital, which influences output, investment, and savings. But these effects are ignored and the stocks of production factors and technology are treated as constant[10]. Second, like Hicks, Lange extended his model by allowing both investment and saving to depend on real income as well as the rate of interest[11].

> Mr. Keynes treats investment and expenditure on consumption as two independent quantities and thinks that total income can be increased indiscriminately by expanding either of them. But it is a common place which can be read in any textbook of economics that the demand for investment goods is derived from the demand for consumption goods. The real argument of the under consumption theories is that investment depends on the expenditure on consumption and, therefore, cannot be increased without an adequate increase of the later, *at least in a capitalist economy where investment is done for profit*. (Lange, 1938, p. 23, emphasis added).

By introducing level of consumption as an argument in the investment function Lange departs from Keynes's analysis of long term expectations in Chapter 12 of the *General Theory.* Keynes argued that long-term expectations were merely exogenous, which amounts

to assuming that investment is insensitive to current levels of output consumption or national income. In this respect, Lange's model really departs from Keynes's theory[12].

In light of this premise, Lange's model can be summarized by the following four equations.

$$\frac{M}{P} = L(i, y) \qquad L_i \leq 0, L_y \geq 0 \qquad (1)$$

$$c = \phi(y, i) \qquad 0 < \phi_y < 1 \quad \phi_i \gtreqless 0 \qquad (2)$$

$$I = F(i, c) \qquad F_i < 0\,, F_{\,c} > 0 \qquad (3)$$

$$y \equiv c + I \qquad (4)$$

where M is the amount of money held by individuals or the real value of cash balances, y is total real income, i is the interest rate, c is total expenditure on consumption per unit of time, and I is investment per unit of time. According to Lange, M, y, c and I are measured in wage units. Once the amount of money M (in wage units) is given, these four equations determine the four unknowns c, I, y and i. Alternatively, i can be assumed to be given (i.e. set by the banking system) and M can be assumed to be endogenous.

In this case, the process of determination of the rate of interest is depicted by Lange in three diagrams. The first represents the relation between the demand for cash balances and the rate of interest. The quantity of money (in wage-units) is measured on the axis OM and the rate of interest on the axis Oi, yielding a family of liquidity preference curves - one for each level of total income (measured in wage-units). The greater the total income the higher positioned is the corresponding curve. We have a second family of curves (for each rate of interest) representing the relation between income and expenditure on consumption. Income is measured along Oy and expenditure on consumption along Oc. The relation between

investment and the rate of interest is represented by the third graph. Measuring investment per unit of time along the OI axis and the rate of interest along the Oi axis we have a family of curves indicating investment corresponding to each value of the interest rate. These curves represent the marginal net return (marginal efficiency) of each amount of investment per unit of time. It is important to note that there is a separate curve for each level of expenditure on consumption. The greater the expenditure on consumption the higher the position of the corresponding curve.

Having constructed his tool, Lange then determines interest rate, level of consumption and investment in the economy. With a given amount of money, M_0 and a given initial level of income, say y, equation 1 gives us a rate of interest of i_0. With y and i_0 given, equation (2) determines total consumption, C_0, and equation (3) provides the level of investment, i_0.

If we find that the sum of total consumption and investment precisely equals total income – and equation (4) is confirmed, if not we must start on a process of adjustment until an equilibrium position in the economy is established.

> This process of *mutual adjustment* goes on until the curves in our three diagrams have reached a position compatible with each other and with the quantity of money given, i.e., until equilibrium is attained. (Lange, 1938, p. 17 – emphasis in original).

However, it should be noted that Lange was evidently aware that his apparently pure static model could be translated into a dynamic model of cyclical fluctuation. In a cryptic note he explicitly states that in presence of time lags, the result will be not only disequilibrium but also cyclical fluctuations:

> If this process of adjustment involves a time lag of a certain kind, a cyclical fluctuation, instead of equilibrium, is the result. Cf. Kalecki 'A theory of business cycle'…. (Lange, 1938, n1, p. 17)

In other words, although his article exclusively targets static analysis of the theory of interest and output, Lange seems to recommend an exhaustive approach to this topic should include dynamics as well. In fact, he explicitly cites Kalecki's well-known article, thus implicitly suggesting a complementary as well as a consistent appendix to his own contribution. Not coincidentally, his conclusion to the article, Lange points out that static investigation of how the optimum propensity to save (or to consume) is attained, is only part of the question, for:

> In a society where the propensity to save is determined by the individuals, there are no forces at work which keep it automatically at its optimum and it is well possible, as the underconsumption theorists maintain, that there is a tendency to exceed it. (Lange, 1938, p. 32)

Unfortunately, Lange's 1938 article does not provide any further information on the role and characteristics of dynamics with respect to the change in the propensity to save.

In the following development, we try to shed some light on this point by attempting a formal reconstruction coherent with Lange's broad beliefs on dynamics.

3. The dynamization of Lange's model: an interpretation

In order both to render effective and to simplify our exposition, we start with a general IS-LM model. First, it should be noted that such a general IS-LM model can be deduced easily from Lange's equations (1) to (4) on condition only that the equality (4) is assumed to be an equation (Lange emphasizes that it is an identity)[13],[14].

Second, in the analysis presented here we adopt Kalecki's lag structure. In Kalecki's 1937 paper, the time lag results from the fact that investment decisions (e.g., investment goods

orders) and the actual investment activity they cause do not coincide. In the short run, the stock of capital, *K*, and the level of investment, *I*, are given. This formalizes, in a simple manner, Kalecki's notion that current investment does not depend on current economic variables due to the existence of lags. Over the longer run, however, we assume that investment changes according to

$$dI/dt = \theta [I^d - I] \quad (5)$$

where I^d refers to the desired investment. In the long run, the stock of capital changes according to

$$dK/dt = I - D, \quad (6)$$

where *D* is capital depreciation. We use the more standard assumption that depreciation is proportional to the stock of capital, so that

$$D = \delta K, \quad (7)$$

where $\delta < 1$, is the rate of depreciation. Substituting (7) into (6) we obtain

$$dK/dt = I - \delta K. \quad (8)$$

In contrast to this, it is assumed that the money market adjusts very quickly through fluctuations in the interest rate and that the money market is balanced for all t. Solving the money market equation for i, we obtain:

$$i = i(y, P, M) \quad (9)$$

where:

$$i_y(\equiv \partial i/\partial y) = -L_y/L_i > 0,$$

$$i_P(\equiv \partial i/\partial P) = -M/(P^2 L_i) > 0$$

$$i_M(\equiv \partial i/\partial M) = 1/L_r P < 0$$

The essential dynamic feature that enables the model to display cyclical behavior is introduced by a Kaleckian assumption about the longer term shifting of the investment function[15]. A change in the capital stock will have a parametric influence on the position of the investment function. If net investment is high, the capital stock increases and, for a given level of consumption, investment is lower. The desired investment function is thus:

$$I^d = F(c(y,i), i(y,P,M), K) \tag{10}$$

where:

$$F_K(\equiv \partial F/\partial K) < 0$$

Substituting this investment function in equation (9), using equation (6), we get

$$\frac{dI}{dt} = \theta(F(\phi(y,i), i(y,P,M), K) - I) \tag{11}$$

The dynamics for this model therefore are given by equations (8) and (11). To examine the local dynamics, we calculate the Jacobian matrix for this dynamic system, which is given by:

$$J = \begin{bmatrix} \theta\left(\dfrac{F_c\phi_y + F_i i_y}{1-\phi_y} - 1\right) & \theta F_K \\ 1 & -\delta \end{bmatrix}$$

The trace and determinant of this matrix are:

$$T = -\left[\delta + \theta\left(1 - \tfrac{F_c\phi_y + F_i i_y}{1-\phi_y}\right)\right] \tag{12}$$

$$D = \theta\left[-\delta\left(\tfrac{F_c\phi_y + F_i i_y}{1-\phi_y} - 1\right) - F_K\right] \tag{13}$$

Since the sufficient conditions for stability of the system are *Tr* (***J***) < 0 and *Det* (***J***) > 0, the condition $F_c\phi_y + F_i i_y < 1 - \phi_y$ is sufficient to ensure stability, and hence, dampened

oscillations. However, that it is not necessary is evidenced by the fact that $F_c\phi_y + F_i i_y = 1 - \phi_y$ still satisfies the stability condition. Explosive oscillations will occur if *Tr* (***J***) > 0 while *Det* (***J***) > 0. If $F_c\phi_y + F_i i_y > 1 - \phi_y$, and θ is sufficiently large, so that θ ≥ δ [$1 - \phi_y/(F_c\phi_y + F_i i_y - (1 - \phi_y))$], the trace condition will be violated and persistent or explosive oscillations become possible. Only when $T(J) = 0$ are we assured of harmonic oscillation and a stable cycle. Thus the stable oscillations depend on the reaction coefficients $\delta, F_c, \phi_y, F_i, i_y$ and the time lag represented by θ.

The long-run equilibrium values for *I* and *K*, obtained by setting d*I*/d*t*=0 and d*K*/d*t*=0 in equations (8) and (11), are given by:

$$I = \frac{\frac{F_c\phi_y + F_i i_y}{1-\phi_y}}{1-\frac{F_c\phi_y + F_i i_y}{1-\phi_y} - F_K\delta} \qquad (14)$$

and:

$$K = \frac{1}{\delta}\,\frac{\frac{F_c\phi_y + F_i i_y}{1-\phi_y}}{1-\frac{F_c\phi_y + F_i i_y}{1-\phi_y} - F_K\delta} \qquad (15)$$

Equations (4) and (14) imply that the long-run equilibrium level of y is given by:

$$y = \frac{1}{1-\phi_y}\,\frac{\frac{F_c\phi_y + F_i i_y}{1-\phi_y}}{1-\frac{F_c\phi_y + F_i i_y}{1-\phi_y} - F_K\delta} \qquad (16)$$

An increase in F_c and in ϕ_y increases the long-run equilibrium values of these variables by increasing effective demand due to an increase in capitalist consumption and investment, while an increase in F_K and δ reduce it by reducing investment demand, by increasing the depressive effect of capital stock on it, and by reducing the steady state level of investment.

To further analyze the dynamic behavior of this model we use the phase diagram in Figure 1 which measures the levels of the two state variables, *I* and *K*, on the two axes. The isocline for d*K*/d*t* = 0 is seen, from equation (8), to be given by the equation:

$$I = \delta K,$$

whose slope is equal to:

$$\left.\frac{dK}{dI}\right|_{\dot{K}=0} = \frac{1}{\delta}$$

which is a positively-sloped straight line. Equation (12) shows that *K* rises below this line and falls above it, explaining the direction of the vertical arrows. The isocline for $dI/dt = 0$ is given by:

$$I = -\frac{F_K K}{1 - \frac{F_c \phi_y + F_i i_y}{1 - \phi_y}}$$

whose slope is equal to:

$$\left.\frac{dK}{dI}\right|_{\dot{I}=0} = -\frac{1 - \frac{F_c \phi_y + F_i i_y}{1 - \phi_y}}{F_K}$$

This shows that to the left of this isocline *I* is rising and to its right it is falling, explaining the direction of the horizontal arrows in the figure. If $F_c \phi_y + F_i i_Y$ is lower than 1-ϕ_y, this yields a downward-sloping straight. As depicted in Figure 1, the cyclical behavior is necessarily dampened.

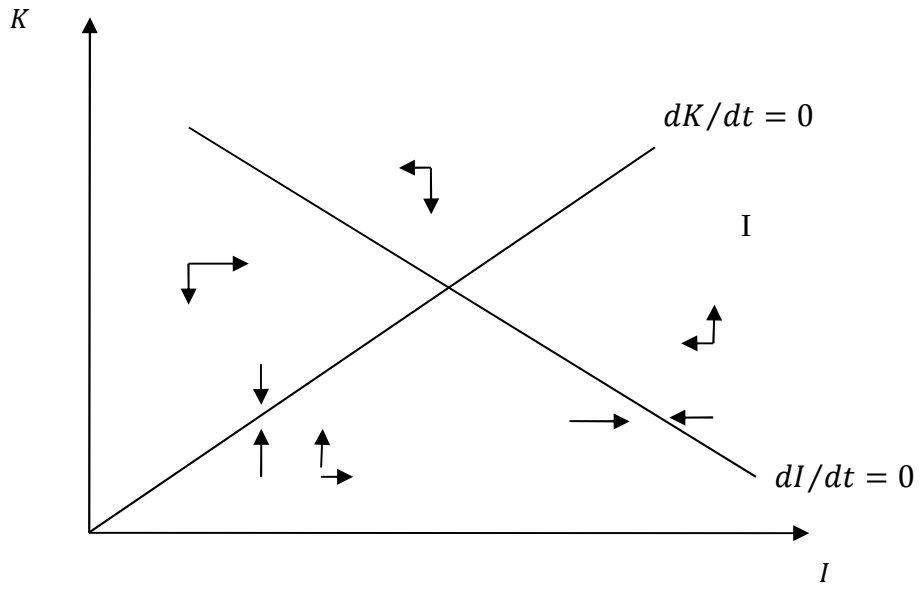

Figure 1. Damped cycles

If we have $F_c\phi_y + F_i i_Y > 1 - \phi_y$, the d*I*/d*t*=0 isocline becomes positively sloped. In this case, we can distinguish between two possibilities. In one, in which the d*I*/d*t*=0 isocline is flatter than the d*K*/d*t*=0 isocline, we have $\left(\frac{F_c\phi_y + F_i i_Y}{1-\phi_y} - 1\right) > \frac{F_K}{\delta}$, which implies that the determinant condition for stability is violated. This means that the dynamics of the system are saddlepoint-unstable, and there are no cycles. In the other case in which the d*I*/d*t*=0 isocline is steeper than the d*K*/d*t*=0 isocline, the determinant condition is satisfied, and the dynamics of *I* are unstable (leading *I* away from its null-cline) while those for *K* are stable. The result is cycles: explosive cycles if the trace becomes positive, and dampened ones if it is negative. Since the trace condition can be written as $\theta\left(\frac{F_c\phi_y + F_i i_Y}{1-\phi_y} - 1\right) < \delta$, it is more likely to be satisfied the longer the investment lag or the smaller is θ.

When cycles do occur, their occurrence is related to both the slow adjustment of desired investment to actual investment and to the negative effect of capital stock on desired

investment. If θ is infinitely large so that adjustment is very rapid, the economy will always be on the d*I*/d*t* = 0, so that adjustment to the long-run equilibrium will be smooth as long as the determinant condition is satisfied. If the time-lag between the investment decisions and the corresponding income is large relative to the rate at which the amount of equipment is increasing, that is, the rate of investment decisions can continue to fall even below what corresponds to replacement, simply because the fall in income lags behind. Thus introducing a time-lag between the investment decision and the corresponding income, according to Lange, explains a cyclical movement even if the underlying situation is stable; although, in order that the cycle is not highly dampened (i.e., that it does not peter out too quickly in the absence of new disturbing factors), we need to assume that the effect of current investment on total equipment is relatively large, such that the equipment added during the period of the time lag has a considerable influence on the profit rate, and hence on the investment decision.

Let us now show how changes in the coefficients of the models are likely to generate a self-sustained cycle. If for extreme values of the level of investment, $F_c\phi_y + F_i i_y$ is lower than $1 - \phi_y$ and for normal values of investment $F_c\phi_y + F_i i_y$ is higher than $1 - \phi_y$, the trace will be positive for very high and for very low levels of investment, and negative for normal levels of investment. Mathematically, changes in the sign of the trace allow the generation of self-sustaining cycles. This is illustrated in Figure 2 by the fact that the $dI/dt = 0$ curve is decreasing for low and high levels of investment, and increasing for normal values of capital stock. In that case, the economy never reaches a stationary equilibrium. We saw earlier that the shape of the isocline for $dI/dt = 0$ because F_K is negative (negative capacity effect), depends upon the value of $\frac{F_c\phi_y + F_i i_Y}{1-\phi_y}$

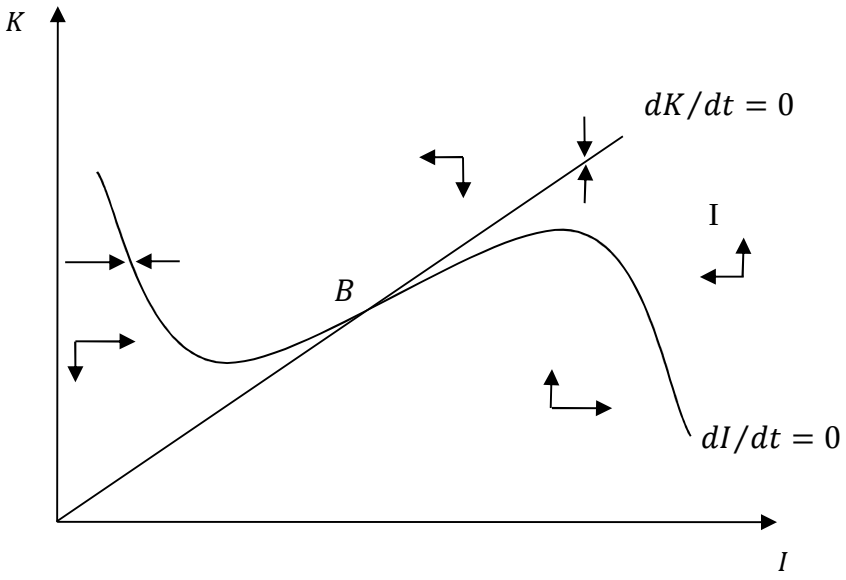

Figure 2. Self-sustained cycles

Figure 2 shows that at very low and very high values of , the isocline is negatively-sloped, this corresponds to the areas where the savings function is steeper than the investment function. However, for normal values of I, the isocline is positively-sloped, which corresponds to the region where the investment curve is steeper than the savings curve. Above the isocline, $dI/dt < 0$, and investment falls; below the isocline, $dI/dt > 0$, and investment rises. The directional arrows indicate these tendencies.

When the d*I*/d*t*=0 isocline flatter than the d*K*/d*t*=0 line at its intersection, the determinant of the Jacobian matrix is positive and limit cycles can exist. Indeed, if the trace condition is violated, that is, if $F_c\phi_y + F_i i_y > 1 - \phi_y$ is sufficiently large, the equilibrium at *B* is unstable, and trajectories close to it will push the economy towards the limit cycle. The combination of investment lags and the non-linear investment curve is sufficient to produce limit cycles.

When there is no investment lag, and θ is infinitely large and the economy is always on the dI/dt=0 isocline, the economic dynamics are particularly interesting. The underlying dynamics, with economy always on the dI/dt=0 line, implies that there will be a catastrophic drop from the high to the lower equilibrium. Notice that during this catastrophic fall in output is driven solely by the fast multiplier dynamic, the slower-moving capital dynamic is inoperative since, in moving from a high to a low equilibrium, capital is constant. Therefore we show that cycles can exist without investment lags.

It should be noted that Lange's dynamic model offers an interesting variant of Kalecki's model (1939). Kalecki's theory explained the business cycle in terms of fluctuations in the marginal return on investment, resulting from both the accumulation/decumulation of capital and from the effect of investment on income. Kalecki suggests expressing the level of investment decision as a S-shaped function of income. This argument was justified absolutely by the evolution of expectation elasticity, assumed to be inelastic for extremes values of output and elastic for normal values. Indeed Kalecki suggests that entrepreneurs are assumed to be cycle-conscious, and hence more cautious in their investment decisions following a prolonged boom or a prolonged slump than at the beginning one of these cycles. However Kalecki assumes the multiplier coefficient remains constant during the entire course of the cycle, thereby relying entirely on the non-linearity of the investment function to show the intrinsic instability of capitalism. It is precisely this assumption that Lange and Kalecki disagreed about:

> There is no a priori reason why this [relationship between national income and investment] should be a straight line. Mr. Kalecki takes it to be a straight line because his statistical investigation has yielded a linear relationship for the United States in the period I924-35 (pp. 73 and 136). Since we do not know whether this relationship is linear in

> other years or in other countries, this restricts unnecessarily the generality of Mr. Kalecki's argument. (Lange, 1941, pp. 284-85)

Unlike Kalecki, Lange was prone to ascribe a critical role to the change in the savings rate[16]. Following a Marxian line of reasoning, it was natural for him (as we saw in Section 1) to base his cycle analysis on the dynamics of income distributive shares and eventually the dynamics of the saving rate. Once Lange's model becomes dynamic, we can see that the savings rate has a major influence on the properties of stability in the economy. This is because, in Lange's model, the savings rate determines the slope of both the savings and the investment curves. In paying more attention to the saving rate, Lange was less bound than his Polish countryman to breaking with Marx's income distribution analysis. More generally, one can state quite legitimately that Lange's analysis of capitalist dynamics was closer to Marx's view (in Volume 2 of *Capital*), according to which equilibrium is an event possible in theory but almost unattainable in practice, at least in a capitalist society.

Concluding Remarks

The investigation in this paper clarifies the dynamic structure of Lange's 1938 model and makes clear the role played the marginal propensity to consume. The main result of this study is a recognition that the marginal propensity to save may, by means of the investment, determine the stability property of the economy in a particular way. If there is a unique stationary equilibrium and the standard macro condition is satisfied globally, the unique equilibrium will be stable. If there are multiple equilibriums due to varying marginal propensities to save, some will be stable and others will be unstable and self-sustained cycles become possible. This highlights two prominent ideas in Lange's view of dynamics developed in the 1930s, which can be accommodated in this framework. First, it becomes possible to

demonstrate the intrinsic instability of capitalism. Second, by relating income distribution to saving, one of Marx's main ideas about cycles and growth is encapsulated.

[1] Samuelson (1941) referred to Hicks (1937), Meade (1937) and Lange (1938) and resorted to a dynamic formulation in which the money market but not the goods market adjusts instantaneously. His stability analysis was based on both differences and differential equations.

[2] Lange's article contained a sibylline reference to Kalecki's business cycle theory (1937) which was immediately rebuffed by Kalecki's statement (in 1939 and, later, in 1966) that Lange's work was ultimately concerned with the capitalist *dynamics*.

[3] Young (2008) focuses on some overlooked aspects of the interrelationships between the early mathematical models in Keynes's General Theory and the IS-LM dynamic approach.

[4] Lange, however, recognized that the economic equilibrium approach, insofar as it precludes institutional data, has the merit of being abstract and, therefore, universal (since its basic notions hold true in any kind of economic system, included a socialist one). Consequently, it provides "*a scientific basis*" for current administrations of the economy in many respects, such as prices, market-structure, or the allocation of resources.

[5] It is for this reason that Lange believed that exogenous theories of the business cycle were not a great help since they try to explain undampened cycles by assuming the existence of causes outside the economic process. The "theories of moving equilibrium" were useless because they: *"… explain only the reaction of the economic system to a given continuous change of data. … [which] is determined statistically but is not an object of theoretical analysis"* (Lange, 1935, p. 192).

[6] On the other hand, Lange notes that a purely dynamic analysis would be "*a poor basis*" for solving more "ordinary" problems, such as monopoly prices, distribution of productive resources, etc. Therefore Lange concludes that a correct method of investigation presupposes both statics and dynamics, as clearly shown by business cycle theories.

[7] However the analytical tools used by Lange were quite different from the traditional Marxist approach because of the former's firm refusal of the labour theory of value: "*In the Marxian system the labour theory of values serves also to demonstrate the exploitation of the working class under Capitalism, i.e. the difference between the personal distribution of income in a capitalist economy and in an 'einfache Warenproduktion'. It is this deduction from the labour theory of value which makes the orthodox Marxist stick to it. But the same fact of exploitation can also be deduced without the help of the labour theory of value"* (Lange, 1935, n3 p.195).

Lange's rejection of the Marxian labor theory of value is expressed, even clearer, in the Appendix to "On the Economic Theory of Socialism" (1937).

[8] In a series of seminars held in 1942 and published posthumously in 1987, Lange emphasized that in a capitalist economy saving and investment decisions were uncoordinated because they were taken by different people, based on different and independent grounds. Any decision to save depended on both the *"saving habits of the population"* and the distribution of income since people with small incomes cannot save anything. Investment, instead, depends partly on the rate of interest (which, in turn, depends on banking policy, i.e. on the banks' expectations about the safety of the investment) and partly on the profit expected by the entrepreneur who makes the investment, based on poor and volatile expectations, because all that was known was the current state of the market and any anticipation of the future became *"a purely haphazard type or even … subject to the quite erratic influences of mass psychology"* (Lange, 1942, p. 15).

[9] For a detailed analysis see Kowalik (1964, 1994, 2008) and Lampa (2011)

[10] "It also ought to be observed that the investment function holds only for a given capital equipment and for a given distribution of the expenditure for consumption between the different industries" (Lange, 1938, p. 13).

[11] The argument is close to that raised by Hicks who wrote: *"Surely there is every reason to suppose that an increase in the demand for consumers' goods, arising from an increase in employment, will often directly stimulate an increase in investment, at least as soon as an expectation develops that the increased demand will continue. If this is so, we ought to include* I *[national income] in the second equation [investment function], though it must be confessed that the effect of* I *on the marginal efficiency of capital will be fitful and irregular"* (Hicks, 1937, p. 156).

[12] Kregel (1976, pp. 215-17) notes however that when dealing with money wages dynamics, and general policy, Keynes admitted that long-term expectations could be affected by current events.

[13] One could add that also the shape of the LM curve is the result of a theoretical assumption, whereas in Lange it was *"empirically determined"*. See: Boianovski, M., 2004, pp. 106-107.

[14] The IS curve is therefore given by the equation:

$$Y = \phi(y, i) + F\big(\phi(y, i)\big)$$

whose slope is

$$\left.\frac{di}{dy}\right|_{IS} = \frac{1 - \phi_y - F_C\phi_y}{\phi_i + F_C\phi_i + F_i}$$

[15] Lange (1941) attached great importance to this assumption.

[16] On the other hand, Kalecki disagreed with Lange because Lange did not distinguish between investment and investment *decisions* in order to explain time lags, and also because he ignored the effect of investment on the capital stock. (Kalecki, 1939, pp. 139-140) quoted in Toporowski (2012), p. 3.

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Abstract

Oskar Lange's 1938 work "The Rate of Interest and the Optimum Propensity to Consume", is generally recognized as being one of the early static mathematical models inspired by Keynes's *General Theory*. Starting from Lange's model, Samuelson (1941) proceeded to devise a Neo-Keynesian dynamic system. Although this is acknowledged to be a seminal contribution, its potential dynamic implications remain largely unexplored and are the subject of the present article. Section 1 clarifies Lange's 1930s' view of dynamics. Section 2 centers on Lange's 1938 static model. Section 3 provides a dynamic analysis of Lange's 1938 model that allows for cyclical solutions, and discusses the role of the propensity to consume followed by a brief comparison of Lange's dynamic model and Kalecki's 1939 business cycle analysis.

Keywords: Lange; Kalecki; Marx; cycles; stability; non-linearity; marginal propensity to save