Granular flow through an aperture: Influence of the packing fraction

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For the last 50 years, the flow of a granular material through an aperture has been intensely studied in gravity-driven vertical systems (e.g., silos and hoppers). Nevertheless, in many industrial applications, grains are horizontally transported at constant velocity, lying on conveyor belts or floating on the surface of flowing liquids. Unlike fluid flows, that are controlled by the pressure, granular flow is not sensitive to the local pressure but rather to the local velocity of the grains at the outlet. We can also expect the flow rate to depend on the local density of the grains. Indeed, vertical systems are packed in dense configurations by gravity, but, in contrast, in horizontal systems the density can take a large range of values, potentially very small, which may significantly alter the flow rate. In the present article, we study, for different initial packing fractions, the discharge through an orifice of monodisperse grains driven at constant velocity by a horizontal conveyor belt. We report how, during the discharge, the packing fraction is modified by the presence of the outlet, and we analyze how changes in the packing fraction induce variations in the flow rate. We observe that variations of packing fraction do not affect the velocity of the grains at the outlet, and, therefore, we establish that flow-rate variations are directly related to changes in the packing fraction.

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I. INTRODUCTION AND BACKGROUND

Because of its obvious practical relevance, the flow of granular media through an aperture has been intensely studied in the last 50 years in vertical gravity-driven systems (e.g., silos and hoppers) [1–11]. The discharge of a silo through an orifice can present three regimes: a continuous flow, an intermittent flow, or a complete blockage due to arching [12–14].

In the continuous flow regime, the *mass flow rate* $Q_\mathrm{m} \equiv dM/dt$ (i.e., the mass M flowing out per unit time t) is generally satisfactorily given by the so-called Beverloo's law [1,15]: $Q_\mathrm{m} = C\rho_{3\mathrm{D}}\sqrt{g}(A - k\,D)^{5/2}$ where A is the diameter of the opening (assumed here to be circular), $\rho_{3\mathrm{D}}$ the bulk density of the granular sample, g the acceleration due to gravity, and D the diameter of the granules, whereas k and C are empirical, dimensionless, constants. Beverloo's law thus points out a value $A_\mathrm{c} \equiv k\,D$ of the aperture size A at which the flow rate is expected to vanish. Therefore, instead of A, the effective aperture $A_\mathrm{eff} \equiv A - k\,D$ is to be considered. The value of k has been found to be independent of the size D of the grains and to take values ranging from 1 to 3 depending on the grains and container properties [16]. Nevertheless, some works [12,17] claim that the only plausible value for k is 1. It should also be noted that a recent work [18] states that k is just a fitting parameter with no clear physical meaning as the authors found clogging of the flow for apertures $A > k\,D$. In the jamming regime, the jamming probability has been shown to be controlled by the ratio A/D of the aperture size to the grain diameter [13,14,19–22].

In many industrial applications, however, granular materials are transported horizontally, lying on conveyor belts [23] or floating on the surface of flowing liquids [24–26]. In a two-dimensional (2D) configuration, or similarly for slit-shaped apertures, one expects Beverloo's law to be $Q_\mathrm{m} = C\rho_{2\mathrm{D}}\sqrt{g}\,(A - k\,D)^{3/2}$ [15]. Recent works considered the discharge of a dense packing of disks driven through an aperture by a conveyor belt. For large apertures ($A/D \geqslant 6$), the flow rate is continuous throughout the discharge. In this case, the number of discharged disks N depends linearly on time t, and the flow rate $Q \equiv dN/dt$ (i.e., the number N of disks flowing out per unit time t) obeys

$$Q = C\left(\frac{4}{\pi D^2}\right) V\,(A - k\,D), \tag{1}$$

where $k \simeq 2$ and the constant C reduces to the packing fraction [27]. Indeed, $\pi D^2/4$ is the surface area of one disk so that $C(4/\pi D^2)$ is the number of grains per unit surface, which, multiplied by the belt velocity and by the size of the aperture, gives an estimate of the number of disks flowing out per unit time. Note that Eq. (1) is equivalent to the 2D Beverloo's law in which the typical velocity $\sqrt{g\,A_\mathrm{eff}}$, understood as the typical velocity of the grains at the outlet, is replaced by the belt velocity V. It predicts that the dimensionless flow rate $Q^* \equiv QD/V$ is independent of V and increases linearly with the dimensionless aperture size A/D. It is interesting to note that this empirical law was demonstrated to be valid for small apertures $A/D < 6$, even if the system is likely to jam and deviations from linearity might be expected [27]. Indeed, in 3D configurations, a marked deviation from the 5/2 Beverloo's scaling has been observed for very small apertures [12]. Moreover, these previous works show that, unlike fluid flows, granular flows are not governed by the pressure, but rather controlled by the velocity of the grains at the outlet [27,28]. The latter does not necessarily depend on the stress conditions in the outlet region as proven by the experimental fact that, in gravity-driven systems, the typical velocity at the outlet is $\sqrt{g\,A}$, independent of the pressure. These observations were corroborated in vertical gravity-driven systems [29].

Even if Beverloo's law has been intensively discussed, the influence of the packing fraction, i.e., the ratio of the area occupied by grains over the total available area, has only been partially considered. However, it is expected that the flow rate can be altered by the packing fraction of the grains aside from their velocity. On the one hand, vertical granular systems are usually gravity packed in dense configurations, except in situations where inflow rate is controlled [30,31], and little effect of the packing fraction is expected in usual conditions. But, on the other hand, in horizontal configurations the packing fraction can explore a large range of values, and one can expect significant changes in the flow rate. Ahn *et al.* studied granular flow rate in vertical silos filled under different conditions, which, as a consequence, lead to different values of packing fraction [32]. However, aiming at relating flow-rate variations to changes in the pressure, they do not discuss the possibility that the variations could be due to changes in the packing fraction itself. In a more recent work, Janda *et al.* studied velocity and packing fraction profiles at the outlet, and they obtained a new expression, independent of k, for the granular flow rate [33].

In the present article, we study the discharge of monodisperse acrylic rings, driven through an orifice, at a constant velocity, by a horizontal conveyor belt. For various initial packing fractions, we report simultaneous measurements of the grains velocity, packing fraction, and flow rate throughout the discharge process.

II. SETUP AND PROTOCOL

The experimental setup (Fig. 1) consists of a conveyor belt made of black paper (width 11 cm, length 34.5 cm) above which a confining cardboard frame (inner width 9 cm, length 20 cm) is maintained at a fixed position in the frame of the laboratory. A motor drives the belt at a constant velocity V. The granular material is made of acrylic rings of thickness $e = (2.00 \pm 0.01)$ mm and external diameter $D = (4.00 \pm 0.01)$ mm.

Downstream, the confining frame exhibits, at the center, a sharp aperture of width A. The aperture width can be tuned up to 9 cm, but we shall report data obtained for a single width $A = (4.1 \pm 0.1)$ cm. The aperture size A is of about 10 times the grain diameter D, so that the condition insuring the continuous flow, $A/D \geqslant 6$, is satisfied [27].

The grains are imaged from top by means of a digital scanner (Canon, CanoScan LIDE200) placed upside down above the frame. In order to focus on the top of the grains without mechanical contact (gap of about 1 mm) and thus avoid friction between the grains and the scanner window, the latter has been replaced by a thinner one. The use of a scanner has the advantage of avoiding optical aberrations and makes it possible to obtain, cheaply, homogeneously lighted images with a high resolution (12 pixels/mm, the grain diameter being thus of the order of 50 pixels).

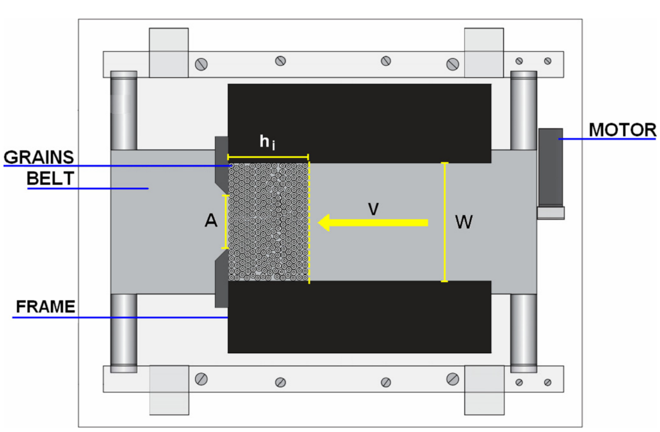

FIG. 1. (Color online) Sketch of the experimental setup.

Before the flow is started, the initial state of the system is obtained by placing inside the confining frame, in a disordered manner, $N_0 = 350$ grains, which initially cover the surface area $S = Wh_i$, where W is the inner width of the frame ($W = 9$ cm), and, thus, h_i the length in the flow direction that is initially covered with grains. We prepare systems with different initial packing fractions: $\langle C_i \rangle = 0.81 \pm 0.02$ ($h_i \simeq 6$ cm), $\langle C_i \rangle = 0.66 \pm 0.02$ ($h_i \simeq 7.5$ cm), $\langle C_i \rangle = 0.46 \pm 0.03$ ($h_i \simeq 10.5$ cm), and $\langle C_i \rangle = 0.38 \pm 0.06$ ($h_i \simeq 13.0$ cm). The homogeneity of the initial packing throughout the system is controlled by measuring the packing fraction along the flow direction in successive layers of width W and thickness $2\,D$. Grains are locally rearranged if the packing fraction is not within 10% of the chosen average $\langle C_i \rangle$.

The discharge is then initiated by setting the belt velocity V to a chosen value. Experiments were performed using six different values of V: (3.6 ± 0.2) mm/s [$0.9\,D\,\mathrm{s}^{-1}$]; (8.7 ± 0.3) mm/s [$2.2D\,\mathrm{s}^{-1}$]; (9.6 ± 0.2) mm/s ($2.4\,D\,\mathrm{s}^{-1}$); (11.3 ± 0.3) mm/s [$2.8\,D\,\mathrm{s}^{-1}$]; and (13.4 ± 0.6) mm/s [$3.3\,D\,\mathrm{s}^{-1}$]. The evolution of the discharge process is assessed by repetitively moving the belt at the chosen constant velocity V during a time interval $dt = 0.1$ s and by recording an image from the scanner while the belt is at rest.

For the present study the image analysis is used to determine the packing fraction, C, and the number of grains, N_{in}, that remain inside the confining frame at time t. To do so, an intensity threshold is used to convert each image into binary: white is assigned to the rings (grains), and black is assigned to the background. Therefore, black disks at the center of each grain are isolated from one another, which makes it easy to detect them and to compute the number of grains remaining in the frame, N_{in}, or, equivalently, the number of disks that flowed out the system at time t, $N \equiv N_0 - N_{\mathrm{in}}$. The instantaneous flow rate (averaged over $dt = 0.1$ s, because of the acquisition rate) is defined as $Q = dN/dt$.

The packing fraction C is, by definition, the fraction of the surface area covered by the grains. In order to measure C, the black disk at the center of the rings is filled with white in order to obtain white disks. The number of white pixels over the total number of pixels in the region of interest is a direct measurement of C.

The reproducibility of the experiments has been checked by repeating the procedure up to three times for each set of the control parameters ($\langle C_i \rangle$, V).

III. EXPERIMENTAL RESULTS

A. Flow rate

The discharge process is analyzed as long as grains fill a distance of $2D$ upstream of the outlet. We report the number of grains that flowed out the system, N as a function of time t. Two types of behavior are observed (Fig. 2):

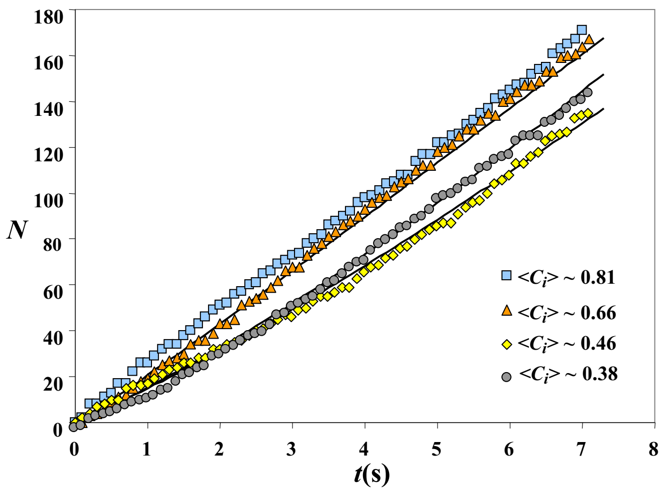

FIG. 2. (Color online) Number of grains $N(t)$ vs time t for $V = 9.6$ mm/s and different initial packing fractions C_i. The number $N(t)$ is linear in t for initially dense systems ($\langle C_i \rangle \sim 0.81$) indicating a constant flow rate. For initially loose systems, $N(t)$ exhibits a nonlinear dependence on time t, which is explained by the increase of the packing fraction in the outlet region. Solid lines correspond to fitting curves obtained with Eq. (7). $\langle C_i \rangle \sim 0.66$ is fitted with $C_i = 0.65 \pm 0.01$, $\alpha = (0.90 \pm 0.06)$ s^{-1} leading to $\lambda = (1.1 \pm 0.1)$ cm and $\beta/V = (31 \pm 1)$ cm^{-1}. $\langle C_i \rangle \sim 0.46$ is fitted with $C_i = 0.49 \pm 0.01$, $\alpha = (0.34 \pm 0.1)$ s^{-1} leading to $\lambda = (2.8 \pm 0.2)$ cm and $\beta/V = (30 \pm 1)$ cm^{-1}. $\langle C_i \rangle \sim 0.38$ is fitted with $C_i = 0.40 \pm 0.01$, $\alpha = (0.31 \pm 0.04)$ s^{-1} leading to $\lambda = (3.5 \pm 0.5)$ cm and $\beta/V = (31 \pm 1)$ cm^{-1}.

(1) For initially dense systems, N increases linearly with the time t. The flow rate Q is constant.

(2) For initially loose systems, N does not increase linearly with time t. The flow rate Q varies during the discharge.

The difference can be easily understood by considering that, for initially loose systems, the grains are progressively piling against the downstream wall (Fig. 3). The discharge process can be thus described in two stages:

(1) First stage (*transient*): The grains are piling progressively and the flow rate Q depends on time.

(2) Second stage (*steady*): The system has reached a steady packing fraction C_∞ slightly smaller than the maximum possible packing fraction $C_{\max} = 0.82$ (corresponding to the close packing), and the flow rate Q remains constant.

B. Packing fraction

We expect the flow rate to be influenced by the packing fraction near the outlet. Therefore, we measure the packing fraction upstream of the aperture, in a region of width A and thickness $2\,D$. The region under analysis is highlighted by a solid box in each of the images in Fig. 3.

During the discharge process, the grains pile progressively against the downstream wall until a steady state is reached. Accordingly, we observe that the packing fraction C increases up to the asymptotic value, $C_\infty \sim 0.8$, slightly smaller than the value $C_{\max} = 0.82$ corresponding to the close packing (Figs. 4 and 5).

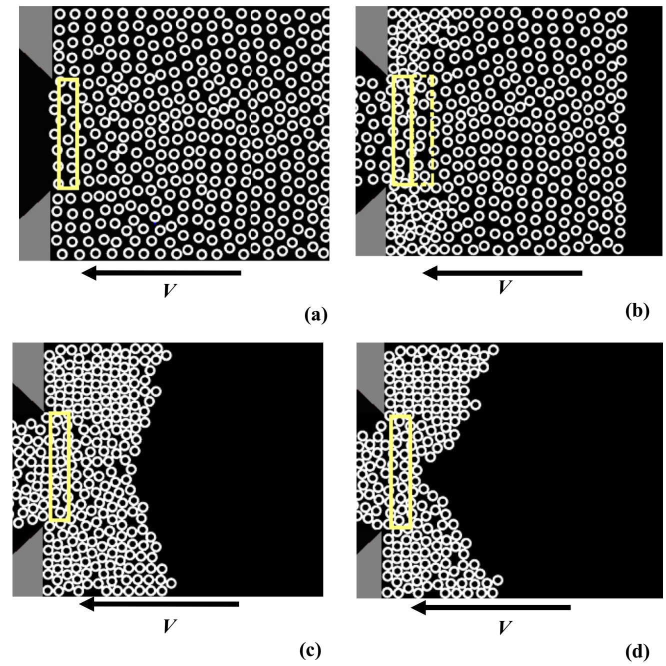

FIG. 3. (Color online) Snapshots of the system during the discharge process for a system with $\langle C_i \rangle = 0.46 \pm 0.03$ driven at $V = (8.7 \pm 0.3)$ mm/s. The arrows indicate the flow direction. Panels (a) and (b) correspond to $t = 0$ s and $t = 2$ s, the first stage of the process (transient stage): the grains are piling progressively, and the flow rate depends on time. Panels (c) and (d) correspond to $t = 9.3$ s and $t = 12.8$ s, the second stage of the process: the system has reached a steady packing fraction, and the flow rate Q remains constant. The solid box in each image encloses the region of surface area $2\,D\,A$ upstream of the outlet (of size A) in which the packing fraction is measured. Note that (d) corresponds to the last image considered for the analysis. The dashed box in (b) indicates the upstream region in which we define $C_{\text{vic.}}$ in the model.

We observe that the temporal evolution of the packing fraction strongly depends on the initial packing fraction (Fig. 4), and, as expected, the asymptotic value is reached

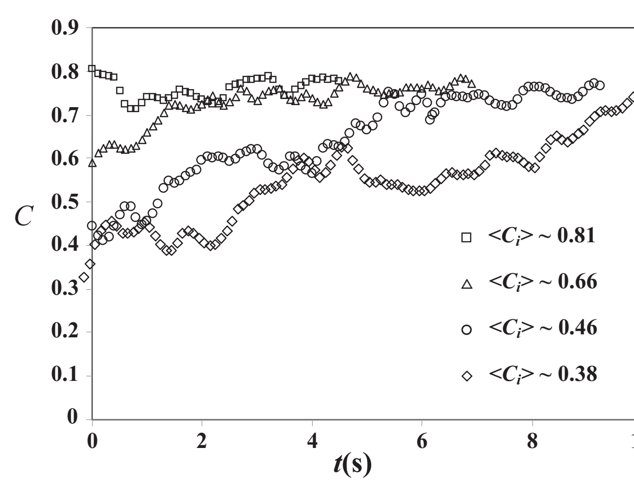

FIG. 4. Packing fraction C in the outlet region vs time t. We observe that the temporal evolution of C depends strongly on its initial value C_i ($V = 9.6$ mm/s).

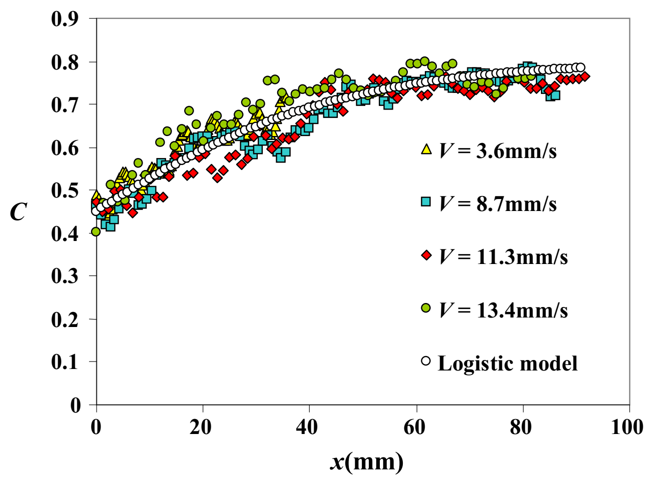

FIG. 5. (Color online) Packing fraction C in the outlet region as a function of the distance traveled by the belt $x = V\,t$. A nice collapse of the experimental results is observed. The dotted line corresponds to the logistic model [Eq. (6)] with $C_\infty = 0.8$, $C_i = 0.45$, and $\lambda = (2.5 \pm 0.5)$ cm.

faster for larger belt velocities, V. Indeed, for a given initial $\langle C_i \rangle$, all curves collapse when C is reported against $x = V\,t$, the distance traveled by the belt at time t (Fig. 5).

IV. DISCUSSION AND CONCLUSIONS

We aim here at accounting for the temporal evolution of the packing fraction in region close to the outlet, $C(t)$.

On the one hand, it is expected that the packing fraction increase, due to grains that enter the outlet region from the upstream region, at a rate $r_{\rm in}$ which should be proportional to:

(1) The belt velocity V: the higher the value of V the larger the income of grains from the upstream region

(2) The packing fraction in the vicinity $C_{\rm vic.}$ upstream of the outlet, i.e., the region enclosed in the dashed box in Fig. 3(b): a larger packing fraction indicates a larger amount of grains accessing from the upstream region

(3) The available space, thus to the difference between the C and its maximum accessible value, $C_{\max}$: more available space allows a larger income of grains from the upstream region.

In addition, as can be observed in Fig. 3(b), we can further assume that the packing fraction in the vicinity of the outlet does not differ significantly from that in the outlet region and we take $C_{\rm vic.} \simeq C$. We thus write

$$r_{\rm in} = \beta_{\rm in}\, V\, (C_{\max} - C)\, C. \tag{2}$$

On the other hand, C is expected to decrease, due to the grains that flow out through the aperture, at a rate $r_{\rm out}$ proportional to:

(1) V: The higher the value of V the larger the outflow from the system

(2) The local packing fraction C: a larger packing fraction at the outlet indicates a larger amount of grains leaving the system.

Therefore,

$$r_{\rm out} = -\beta_{\rm out}\, V\, C. \tag{3}$$

Collecting Eqs. (2) and (3), we obtain the net variation of the packing fraction in the form

$$\frac{dC}{dt} = \alpha\, C\, (1 - C/C_\infty), \tag{4}$$

where $C_\infty = C_{\max} - \beta_{\rm out}/\beta_{\rm in}$ and $\alpha = \beta_{\rm in}\, C_\infty\, V$.

Taking into account the initial condition that $C(0) = C_i$, the solution of Eq. (4) can be written in the form

$$C(t) = \frac{C_\infty}{1 + \frac{C_\infty - C_i}{C_i}\, e^{-\alpha\, t}}. \tag{5}$$

We point out that the prefactor α is proportional to the belt velocity V, which provides the only time scale of the problem. This assertion is compatible with the observation of a nice collapse of the experimental data observed when the packing fraction in the outlet region is reported as a function of the distance traveled by the belt $x = V\,t$ (Fig. 5). Therefore Eq. (5) can be rewritten as

$$C(x) = \frac{C_\infty}{1 + \frac{C_\infty - C_i}{C_i}\, e^{-x/\lambda}} \tag{6}$$

with λ a characteristic travel distance which is thus independent of the velocity. The measurements of C (Fig. 5) are satisfactorily described by Eq. (6). For instance, the interpolation of the experimental data for all velocity V leads to $\lambda = (2.5 \pm 0.5)$ cm ($\sim 0.6\,A$) and $C_\infty = 0.80$ for $C_i = 0.45$. We indeed observe that the steady value of the packing fraction C_∞ is smaller than $C_{\max}$ as expected from our simple description of the problem.

Later we will discuss the meaning of this characteristic length λ and its dependence with the initial packing fraction. But, now, it is particularly interesting to analyze the potential effects of the changes in the local packing fraction C on the flow rate. To do so, we consider that the flow rate Q is proportional to C and V.

We report in Fig. 6 the average velocity, $V_{\rm g}$, of the grains in the region upstream the outlet (Fig. 3). We display the

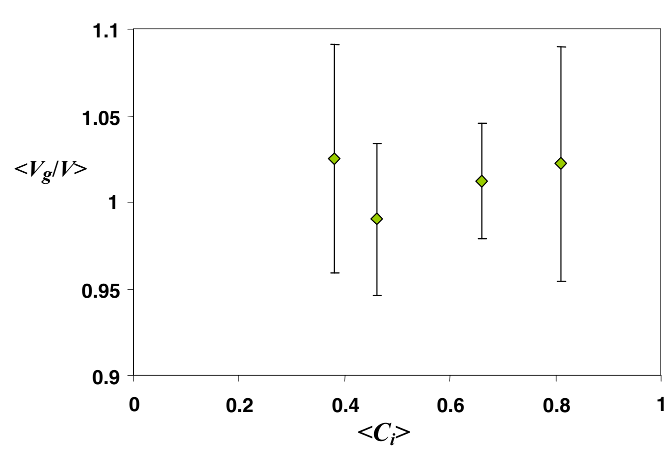

FIG. 6. (Color online) Velocity $V_{\rm g}$ of the grains upstream of the outlet vs. C_i. The velocity is averaged over the duration of the discharge and normalized with the belt velocity. Even if the packing fraction increases during the discharge, i.e., for systems with $C_i < 0.8$, grain velocities oscillate within 7% of the velocity of the conveyor belt.

average over the duration of the discharge. We observe that V_g almost equals the belt velocity (to within the experimental uncertainty). No systematic dependence is observed as a function of C_i, which indicates that this average is not altered by the presence or the absence of a transient. Therefore we can state that the characteristic velocity of the grains at the outlet remains approximately constant and equal to the belt velocity during the entire discharge. Moreover, we have observed that the instantaneous velocity, even if the measurements are noisier, does not significantly deviate from V. Thus, the variations of the flow rate can only be attributed to the changes in the local packing fraction C.

With the above statement in mind, we can replace the constant packing fraction in Eq. (1) by the time-dependent packing fraction given by Eq. (5). Doing so, we get the number of grains that left the system at time t in the form

$$N(t) = C_\infty \beta \left\{ t - \frac{1}{\alpha} \ln\left[\frac{C(t)}{C_i}\right]\right\} \tag{7}$$

with $\beta = \frac{4V}{\pi D^2}(A - k\,D)$. A good agreement of experimental data with Eq. (7) (solid lines in Fig. 2) is observed. We found that $\langle k\rangle = 0.8 \pm 0.4$ ($\langle \frac{\beta}{V}\rangle = (30 \pm 2)$ cm^{-1}) and, as will be explained below, we also observed that values of $\lambda = V/\alpha$ depend on C_i. The agreement confirms that the typical velocity of the grains at the outlet to be considered in Berverloo's law is not altered by the local packing fraction C. It should also be noted that for initially dense systems the second term in Eq. (7) vanishes and Beverloo's law [Eq.(1)] with a constant $C = C_\infty$ is retrieved: $Q = \frac{dN}{dt} = C_\infty\beta$. Actually, in this case, $\alpha \to \infty$ and $\lambda \to 0$ and a linear regression corroborates that $\langle \frac{\beta}{V}\rangle = (30 \pm 1)$ cm^{-1}.

As for the meaning of the characteristic length λ, it corresponds to the travel distance over which the system reaches the steady state [Eq. (6)]. It can be estimated by considering that the packing fraction, in a region above the downstream wall of typical height $A/2$ (which corresponds to the typical height of the arch that forms above the outlet), must have reached its steady-state value (of about $C_{\max}$) for $x \sim \lambda$. In order to get a crude estimate, neglecting the outflow, one can assume that a region of height $A/2 + \lambda$ and packing fraction C_i is compacted in a region of height $A/2$ and packing fraction $C_{\max}$, which leads to $\lambda \sim (A/2)(C_{\max} - C_i)/C_i$. This estimate is compatible with the increase of λ when C_i is decreased (see Fig. 4) and with the absence of significant transitory for $C_i \sim C_{\max}$. In our experimental configuration, the outflow cannot be neglected as the width of the system W is not much larger than the aperture size A and the maximum packing fraction that can be reached is $C_\infty \cong 0.8$. In order to take into account the grains that escape the system, one can add a correction factor and write

$$\lambda(A/2)(C_\infty - C_i)/[C_i(1 - A/W)] \tag{8}$$

with parameters α and C_i obtained from fitting the experimental data with Eq. (7); mean values of $\lambda = \frac{V}{\alpha}$ as a function of $\langle C_i\rangle$ are shown in Fig. 7 and are in agreement with values of λ obtained with Eq. (8).

In summary, we have simultaneously measured the flow rate and the packing fraction in the outlet region of a discharging 2D silo. We have observed that, for initially loose systems, the packing fraction in the outlet region evolves during the discharge and that, at the same time, the flow rate is not constant. We proposed that the flow rate is directly altered by the variations of the local density of the granular material and not by variations of the typical velocity at the outlet. This assertion is supported by a, simplistic, logistic model, accounting for the temporal evolution of both the packing fraction and the flow rate, which proved to be in agreement with our experimental data.

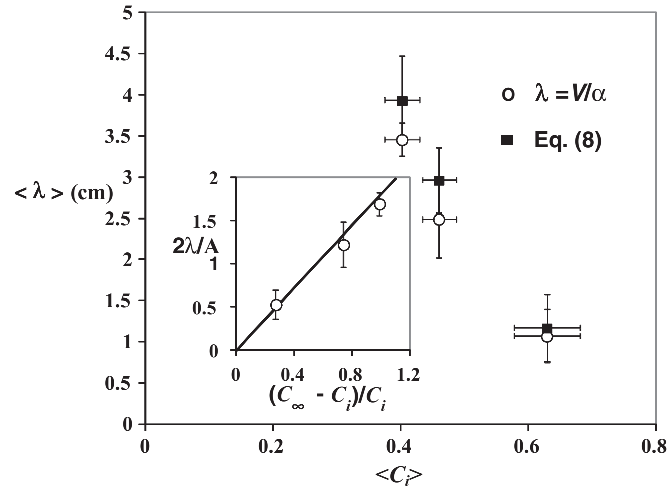

FIG. 7. $\langle\lambda\rangle$ as a function of $\langle C_i\rangle$ is presented. (○) corresponds to values of $\lambda = \frac{V}{\alpha}$ obtained with α values from fitting experimental data with Eq. (7) and (■) corresponds to λ values obtained from Eq. (8). Values of $\langle C_i\rangle$ are mean values obtained from fitting experimental data with Eq. (7). Inset: Experimental values $\lambda = \frac{V}{\alpha}$ are fitted with Eq. (8), slope is found to be 1.7 ± 0.1 in accordance with the expected value $\frac{W}{W-A} = 1.8 \pm 0.1$ (solid line), it can be observed that for $C_i = C_\infty$ effectively $\lambda = 0$.

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APPENDIX: DERIVATION OF EQ. (7)

As explained in Sec. IV, we can replace the constant packing fraction in Eq. (1) by the time-dependent packing fraction given by Eq. (5):

$$Q \equiv dN/dt = C(t)\left(\frac{4}{\pi D^2}\right) V\,(A - k\,D) \tag{A1}$$

Therefore $N(t)$ can be obtained by integrating the above expression between 0 and t:

$$N(t) = N(0) + \left(\frac{4}{\pi D^2}\right) V\,(A - k\,D) \int \frac{C_\infty}{1 + \frac{C_\infty - C_i}{C_i} e^{-\alpha t}} dt. \tag{A2}$$

The following substitution can be made $y = Ae^{-\alpha t}$ with $A = \frac{C_\infty - C_i}{C_i}$ leading to

$$\int C(t)\,dt = \frac{1}{\alpha}\ln\left(\frac{1+Ae^{-\alpha t}}{Ae^{-\alpha t}}\right), \tag{A3}$$

which evaluated between 0 and t is

$$\int C(t)\,dt = \frac{1}{\alpha}\,ln\left[\left(\frac{1+Ae^{-\alpha t}}{Ae^{-\alpha t}}\right)\left(\frac{A}{1+A}\right)\right]. \tag{A4}$$

Considering that $1 + A = \frac{C_\infty}{C_i}$ and $1 + Ae^{-\alpha t} = \frac{C_\infty}{C(t)}$:

$$\int C(t)\,dt = t - \frac{1}{\alpha}\ln\left[\frac{C(t)}{C_i}\right]. \tag{A5}$$

So we finally arrive to Eq. (7) by considering $N(0) = 0$; i.e., there are no disks flowing out of the system at $t = 0$:

$$N(t) = C_\infty\beta\left\{t - \frac{1}{\alpha}\ln\left[\frac{C(t)}{C_i}\right]\right\}. \tag{A6}$$

[1] W. A. Beverloo, H. A. Leninger, and J. van de Valde, Chem. Eng. Sci. **15**, 260 (1961).
[2] L. P. Kadanoff, Rev. Mod. Phys. **71**, 435 (1999).
[3] P. G. de Gennes, Rev. Mod. Phys. **71**, S374 (1999).
[4] V. Trappe, V. Prasad, L. Cipelletti, P. N. Serge, and D. A. Weitz, Nature (London) **411**, 772 (2001).
[5] H. M. Jaeger, S. R. Nagel, and R. P. Behringer, Rev. Mod. Phys. **68**, 1259 (1996).
[6] J. Duran, *Sands, Powders and Grains* (Springer, New York, 2000).
[7] G. H. Ristow, *Pattern Formation in Granular Materials* (Springer, New York, 2000).
[8] R. M. Nedderman, U. Tüzün, S. B. Savage, and G. T. Houlsby, J. Chem. Eng. Sci. **37**, 1597 (1982).
[9] U. Tüzün, G. T. Houlsby, R. M. Nedderman, and S. B. Savage, J. Chem. Eng. Sci. **37**, 1691 (1982).
[10] S. B. Savage, R. M. Nedderman, U. Tüzün, and G. T. Houlsby, J. Chem. Eng. Sci. **38**, 189 (1983).
[11] B. P. Tighe and M. Sperl, Granular Matter **9**, 141 (2007).
[12] C. Mankoc, A. Janda, R. Arévalo, J. M. Pastor, I. Zuriguel, A. Garcimartín, and D. Maza, Granular Matter **9**, 407 (2007).
[13] C. Mankoc, A. Garcimartín, I. Zuriguel, D. Maza, and L. A. Pugnaloni, Phys. Rev. E **80**, 011309 (2009).
[14] F. Ulissi, Graduate thesis, engeneering Faculty, University of Buenos Aires, Argentina, 2008.
[15] R. L. Brown and J. C. Richards, *Principles of Powder Mechanics* (Pergamon Press, Oxford, 1970).
[16] R. M. Nedderman and C. Laohakul, Powder Technol. **25**, 91 (1980).
[17] J. Y. Zhang and V. Rudolph, Ind. Eng. Chem. Res. **30**, 1977 (1991).
[18] H. G. Sheldon and D. J. Durian, Granular Matter **12**, 579 (2010).
[19] K. To, P.-Y. Lai, and H. K. Pak, Phys. Rev. Lett. **86**, 71 (2001).
[20] I. Zuriguel, L. A. Pugnaloni, A. Garcimartín, and D. Maza, Phys. Rev. E **68**, 030301(R) (2003).
[21] I. Zuriguel, A. Garcimartín, D. Maza, L. A. Pugnaloni, and J. M. Pastor, Phys. Rev. E **71**, 051303 (2005).
[22] A. Janda, I. Zuriguel, A. Garcimartín, L. A. Pugnaloni, and D. Maza, Euro. Phys. Lett. **84**, 44002 (2008).
[23] Bao De-Song, Zhang Xun-Sheng, Xu Guang-Lei, Pan Zheng-Quan, Tang Xiao-Wei, and Lu Kun-Quan, Phys. Rev. E **67**, 062301 (2003).
[24] A. Guariguata, D. T. Wu, C. A. Koh, A. K. Sum, and D. Sloan, in *Proceedings of the 6th International Conference on Micromechanics of Granular Media*, edited by M. Nakagawa and S. Luding (AIP, Golden, CO, 2009), Vol. 1145, pp. 543–546.
[25] A. Guariguata, M. A. Pascall, M. W. Gilmer, A. K. Sum, E. D. Sloan, C. A. Koh, and D. T. Wu, Phys. Rev. E **86**, 061311 (2012).
[26] P. G. Lafond, M. W. Gilmer, C. A. Koh, E. D. Sloan, D. T. Wu, and A. K. Sum, Phys. Rev. E **87**, 042204 (2013).
[27] M. A. Aguirre, J. G. Grande, A. Calvo, L. A. Pugnaloni, and J.-C. Géminard, Phys. Rev. Lett. **104**, 238002 (2010).
[28] M. A. Aguirre, J. G. Grande, A. Calvo, L. A. Pugnaloni, and J.-C. Géminard, Phys. Rev. E **83**, 061305 (2011).
[29] C. Perge, M. A. Aguirre, P. A. Gago, L. A. Pugnaloni, D. Le Tourneau, J.-C. Géminard, Phys. Rev. E **85**, 021303 (2012).
[30] D. Huang, G. Sun, and K. Lu, Phys. Rev. E **74**, 061306 (2006).
[31] D. Huang, G. Sun, and K. Lu, Phys. Lett. A **375**, 3375 (2011).
[32] H. Ahn, Z. Başaranoğu, M. Yilmaz, A. Buğutekin, and M. Zafer Gül, Powder Technol. **186**, 65 (2008).
[33] A. Janda, I. Zuriguel, and D. Maza, Phys. Rev. Lett. **108**, 248001 (2012).