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ARBITRARY PRECISION FREQUENCIES OF A FREE RECTANGULAR THIN PLATE

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A variational method developed by the authors and named whole element method (WEM) is used to find the arbitrary precision frequencies of a rectangular thin plate (within the Germain–Lagrange theory) having its four borders free of constraint. WEM consists is proposing an adequate functional and a sequence representing the plate transversal displacement $w(x, y)$. Such a sequence is made of a linear combination of functions belonging to a complete set in L_2. The sequence, and not each co-ordinate function, is required to satisfy the essential or geometric conditions. The sequence generation is systematic and no analysis of the classical natural modes of the plate is needed. In particular, trigonometric functions which *a priori* belong to a complete set in the domain are used in the present analysis. The solving equations involving very simple sums arise from the minimization of the functional. WEM is based on theorems which show the ultimate exactness of the eigenvalues and the uniform convergence of the essential functions of the problem. To the authors knowledge this problem has no classical solution.

1. INTRODUCTION

The variational methodology which is used to solve the problem referred to in the title has been developed, and applied by the authors to a large variety of boundary value problems (even non-linear) in one, two- and three-dimensional domains. Also ordinary differential equations with initial conditions as well as partial differential equations have been successfully tackled with this method (see for instance reference [1–6]). Basically, whole element method (WEM) consists first in the statement of a proper functional. An extremizing sequence is then introduced. Such a sequence is a linear combination of functions belonging to a complete set in L_2. The satisfaction of only the essential (or geometric) boundary conditions is required for the sequence (not for each co-ordinate function). That is if the problem is governed by a differential equation of order $2k$, the essential conditions or functions are those involving derivatives of order $\leqslant (k-1)$. The authors have stated and demonstrated the theorems and corollaries that assure the exactness of the eignvalues and the uniform convergence of the essential functions (here modal

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shapes and the respective first derivatives [6]). In particular, in the problem under study, the free vibration of thin rectangular plates with free borders, all the conditions are natural whereby no conditions need to be imposed on the sequence. Consequently, in this very particular case, WEM would be a Ritz method in which a trial function linearly combines elements of a complete set. However, achieving such a function for a two-dimensional problem or even higher order ones (see, for instance, Filipich *et al.* [5]) is probably one of the main contributions of WEM. The extremizing sequences are systematically generated even with diverse problems. There is no need to find *good* trial functions in each case. Even more, WEM is able to handle problems in which the functional does not exist in the classical sense. In effect, the authors have shown through theorems that the procedure ends with the statement on a *pseudo*-virtual work in these particular sequences. Diverse problems such as linear or not, conservative or not, boundary value, initial-boundary value problems have been dealt with using WEM.

In this work, the natural vibrations of free thin rectangular plates are studied using the Germain–Lagrange formulation. The generation of the extremizing sequence is detailed in the next section. The statements of the necessity and sufficiency conditions that theoretically found WEM are stated and the demonstrations are briefly presented in Appendix A.

Values of the frequency parameter for the free plate are obtained for various aspect ratios. Particular boundary conditions derived from the same model are also included. The results are compared with those obtained with the Ritz method by Leissa [7] and others using a general-purpose FEM code. Another well-known tool is the superposition method. A representative and related work is that of Gorman [8].

2. THEORETICAL CONCEPTS

In this section the differential problem is stated and also, as required by WEM, a proper functional. The generation of the extremizing sequence is explained. The extremizing sequence which satisfies eventual geometric conditions is named *WEM solution.*

2.1. GOVERNING DIFFERENTIAL EQUATION AND CORRESPONDING FUNCTIONAL

As is known, the study of natural vibrations leads to eigenvalue problems. In the case of rectangular thin plates of dimensions a and b in x and y plane directions, respectively, the domain is R^2 (once non-dimensionalized $\{R^2: 0 \leqslant x \leqslant 1,\ 0 \leqslant y \leqslant 1\}$). The governing differential equation, after assuming normal modes, is

$$w'''' + 2\alpha^2 \bar{\bar{w}}'' + \alpha^4 \bar{\bar{\bar{\bar{w}}}} - \Omega^2 w = 0 \tag{1}$$

with $\alpha = a/b$ and where $\partial(\cdot)/\partial x \equiv (\cdot)'$, $\partial(\cdot)/\partial y \equiv (\bar{\cdot})$, etc, If $w \in C^4$ and satisfies simultaneously the differential equation (1) and all the constraints of the problem, it is called a *classical solution.* Ω are the eigenvalues of the problem (proportional to the natural frequencies of vibration),

$$\Omega = \omega a^2 \sqrt{\frac{\rho h}{D}}, \tag{2}$$

where ω is the circular frequency, ρ is the density, h is the plate thickness, $D = Eh^3/12(1 - v^2)$ is the flexural rigidity of the plate, E is the modulus of elasticity and v is the Poisson ratio. It will be shown in Appendix A that proposing a function with certain requirements that belongs to a set wider than the one of the classical solution, permits one to attain uniform convergence of the essential modal functions and the exactness of their eigenvalues. The demonstrations of the necessary and sufficient conditions that give justification of the WEM solution for the vibration of thin plates will be briefly discussed in Appendix A.

2.2. AN EXTREMIZING SEQUENCE

The requirements to be satisfied by the sequences are herein presented. Such sequences are named *extremizing* since, in general, they conduce to the stationarity condition for a certain functional $\mathscr{F}[\cdot]$. In the case of rectangular plates, the functional is

$$\mathscr{F}[u] = \Omega^2[u] \equiv \Omega^2 = \frac{\| u'' + \alpha^2 \bar{\bar{u}} \|^2 + 2\alpha^2(1 - v)[\| \bar{u}' \|^2 - (u'', \bar{\bar{u}})]}{\| u \|^2}, \tag{3}$$

where functional analysis notation was introduced: the inner product $(f, g) = \int_D fg \,\mathrm{d}x\,\mathrm{d}y$; the norm $\| f \|^2 = \int_D f^2 \,\mathrm{d}x\,\mathrm{d}y$. Since $\mathscr{F}$ is symmetric and positive definite for this particular problem, the sequences are minimizing.

The Fourier series that may be used in rectangular domains are of the form $\sum\sum f_1 f_2$ in which $f_1 f_2$ are any of the following combinations $s_i s_j$, $s_i c_j$, $c_i s_j$, $c_i c_j$. The following notations have been introduced:

$\alpha_i \equiv i\pi$, $\beta_j \equiv j\pi$, $s_i \equiv \sin \alpha_i x$, $s_j \equiv \sin \beta_j y$, $c_i \equiv \cos \alpha_i x$, $c_j \equiv \cos \beta_j y$, $(i, j) = 0,1,2, \ldots$

Such series guarantee, as is known, the convergence in the mean of any square integrable function. However, WEM requires the uniform convergence of the essential continuous functions, which in our problem are w, w' and $\bar{w}$. Next, the way the bidimensional series for a continuous function are generated will be briefly shown.

A continuous function, say $\phi = \phi(x, y)$, of two variables for which uniform convergence in R^2 is required, may be represented by one of the following two expansions:

$$\phi_M(x, y) = \sum_i^M B_i(y) \sin \alpha_i x + xB_0(y) + bb_0(y), \tag{4a}$$

$$\phi_M(x, y) = \sum_i^M A_i(y) \cos \alpha_i x + A_0(y), \tag{4b}$$

with $M \to \infty$. From equations (4a, b) and in order to have uniform convergence (from Fourier theory and with the support function) it will suffice that

$$B_i(y) = 2 \int_0^1 [\phi(\eta, y) - \eta B_0(y) - bb_0(y)] \sin \alpha_i \eta \, d\eta \tag{5}$$

$$B_0(y) = \phi(1, y) - \phi(0, y), \qquad bb_0(y) = \phi(0, y). \tag{6}$$

On the other hand,

$$A_i(y) = 2 \int_0^1 \phi(\eta, y) \cos \alpha_i \eta \, d\eta, \quad A_0(y) = \int_0^1 \phi(\eta, y) \, d\eta. \tag{7}$$

Equation (4b), as well as equation (4a), with the support function, $xB_0(y) + bb_0(y)$, gives (as may easily be demonstrated) uniform convergence for ϕ. It should be noted that even though such support functions, in some cases, would arise naturally after integration they are one of the bases for the demonstration of the WEM solution validity. Now, if any of the functions of y in equations (4a, b) is expanded in an analogous way in the variable y formally equal to equations (4) all possible combinations will be considered. That is, if one starts from equation (4a), and for instance

$$B_i(y) = \sum_j^N A_{ij} \sin \beta_j y + y A_{i0} + a_i, \tag{8}$$

$$B_0(y) = \sum_j^N A_{0j} \sin \beta_j y + y A_{00} + a_0, \tag{9}$$

$$bb_0(y) = \sum_j^N b_j \sin \beta_j y + y b_0 + k, \tag{10}$$

the following may be written:

$$\phi_{MN}(x, y) = \sum_{i=1}^M \sum_{j=1}^N A_{ij} s_i s_j + F(x, y) \tag{11}$$

and the bidimensional support function $F(x, y)$ is written as

$$F(x, y) \equiv x\left(a_0 + \sum^N A_{0j} s_j\right) + y\left(b_0 + \sum^M A_{i0} s_i\right) + A_{00}\, xy + \sum^N b_j s_j + \sum^M a_i s_i + k. \tag{12}$$

(An analogous derivation may be done by starting from the cosine expansion (4b)). As observed, uniform convergence is attained when starting from equation (4a). This is also achieved with equation (4b). Other combinations are evidently valid and possible. The most suitable one must be selected.

Consequently, the function ϕ_{MN}, equation (11), which will be used in this work, is a minimizing sequence of the functional $\mathscr{F}$ since the second derivatives of ϕ_{MN}

converge, at least, in L_2 towards the corresponding second derivatives. All second derivatives will contain at least one term of the form $\sum\sum f_1 f_2$ (as may be easily verified), with convergence in L_2.

The unknown are the constants $\{A_{ij}\}$, $\{a_i\}$, $\{b_j\}$ and k with $i, j = 0, 1, 2, \ldots$. In particular, the requirement of satisfaction of eventually essential boundary conditions (those involving ϕ_{MN}, ϕ'_{MN} or $\bar{\phi}_{MN}$), reduces the number of unknowns. For instance, the plate with two consecutive simple supported border and the other two free (SSFF) would conduce to $\{a_i\} = \{b_j\} = k = 0$.

Then it is concluded that, due to the generation procedure, the uniform convergence of the essential functions (here ϕ', $\bar{\phi}$ and ϕ) is assured. Also $\mathscr{F}\,[\phi_{MN}] \to \mathscr{F}\,[\phi]$ as $M, N \to \infty$. Again it should be pointed out that the classical essential functions ought to be continuous in order for the uniform convergence to be attained.

3. FREE RECTANGULAR PLATE

The next extremizing sequence is proposed for the transverse displacement of the plate:

$$w_{MN}(x, y) = \sum_{i=1}^{M} \sum_{j=1}^{N} \frac{A_{ij}}{\beta_i \gamma_j} s_i s_j + x\left(a_0 + \sum_{j=1}^{N} \frac{A_{0j} s_j}{\gamma_j}\right) + y\left(b_0 + \sum_{i=1}^{M} \frac{A_{i0} s_i}{\beta_i}\right)$$
$$+ A_{00}\, xy + \sum_{j=1}^{N} \frac{b_j s_j}{\gamma_j} + \sum_{i=1}^{M} \frac{A_{i0} s_i}{\beta_i} + k. \tag{13}$$

It is formally coincident with equation (11). The free borders give rise to natural conditions that need not be fulfilled by the extremizing sequence. After its replacement in the functional (3), the minimization gives rise to the solution.

First let us distinguish the rigid-body modes ($\Omega = 0$). They consist of a constant displacement $w = k$ and the rotations $w = a_0(x - 1/2)$ and $w = b_0(y - 1/2)$. The studied modal shapes for the case $\Omega \neq 0$ may be grouped according to their symmetry features. That is, antisymmetric in x- and y-axis, symmetric in x and y, antisymmetric in x and symmetric in y and finally antisymmetric in y and symmetric in x. In particular and for the sake of brevity, the algorithm is detailed for the first symmetry case; in the other cases the proposed sequences are shown. The corresponding expressions may be found in reference [6]. Each subset is also complete in L_2.

3.1. MODE SHAPE: ANTISYMMETRIC MODAL IN x AND y

Proposing the minimizing sequence (with $E = 2, 4, 6, \ldots$)

$$w_{1MN}(x, y) = \sum_{i=E} \sum_{j=E} \frac{A_{ij} s_i s_j}{\beta_i \gamma_j} + A_{00}\left(x - \frac{1}{2}\right)\left(y - \frac{1}{2}\right) + \left(x - \frac{1}{2}\right) \sum_{j=E} \frac{A_{0j} s_j}{\gamma_i}$$
$$+ \left(y - \frac{1}{2}\right) \sum_{i=E} \frac{A_{i0} s_i}{\beta_i}, \tag{14}$$

which verifies that $w_{MN}(x, y) = -\, w_{MN}(1 - x, y) = -\, w_{MN}(x, 1 - y)$, the resulting equations obtained from the application of WEM (equation (A.18)) are

$$A_{0j}\left[\Delta_j - \sum_{p=E}\frac{U_{pj}m_{pj}}{D_{pj}}\right] + \sum_{p=E} A_{p0}\left[r_{pj} - \frac{V_{pj}m_{pj}}{D_{pj}}\right] = -A_{00}\left\{k_j - \sum_{p=E}\frac{W_{pj}m_{pj}}{D_{pj}}\right\}, \tag{15}$$

$$\sum_{q=E} A_{0q}\left[r_{iq} - \frac{n_{iq}U_{iq}}{D_{iq}}\right] + A_{i0}\left[\delta_i - \sum_{q=E}\frac{V_{iq}n_{iq}}{D_{iq}}\right] = -A_{00}\left\{k_i - \sum_{q=E}\frac{W_{iq}n_{iq}}{D_{iq}}\right\}, \tag{16}$$

$$\sum_{q=E} A_{0q}\frac{\mu_q}{\gamma_q}\left\{\sum_{p=E}\frac{U_{pq}\lambda_p}{\beta_p D_{pq}} - \frac{1}{12}\right\}\sum_{p=E} A_{p0}\frac{\lambda_p}{\beta_p}\left\{\frac{V_{pq}\mu_p}{\gamma_q D_{pj}} - \frac{1}{12}\right\}$$

$$= -A_{00}\left\{\frac{2\alpha^2(1-\nu) - \Omega^2/144}{\Omega^2} + \sum_{p=E}\sum_{q=E}\frac{W_{pq}\lambda_p\mu_q}{\beta_p\gamma_q D_{pj}}\right\}. \tag{17}$$

The following general notation was introduced (both for even and odd indices):

$$D_{ij} = [(\beta_i^2 + \alpha^2\gamma_j^2)^2 - \Omega^2], \quad U_{ij} = 2\lambda_i\beta_i(\alpha^4\gamma_j^4 + \alpha^2\nu\beta_i^2\gamma_j^2 - \Omega^2),$$

$$V_{ij} = 2\mu_j\gamma_j(\beta_j^4 + \alpha^2\nu\beta_i^2\gamma_j^2 - \Omega^2), \qquad W_{ij} = 4\lambda_i\mu_i\beta_i\gamma_j\Omega^2,$$

$$\lambda_i = I_i - L_i/2, \qquad \mu_j = J_j - M_j/2, \quad I_p = (x, s_p) = (-1)^{p+1}/\beta_p,$$

$$J_q = (y, s_q) = \frac{(-1)^{q+1}}{\gamma_q}, \quad L_p = (1, s_p) = \frac{1-(-1)^p}{\beta_p}, \quad M_q = (1, s_q) = \frac{1-(-1)^q}{\gamma_q},$$

$$\Delta_j = \frac{\gamma_j^2\alpha^4}{24} + \alpha^2(1-\nu) - \frac{\Omega^2}{24\gamma_j^2}, \quad m_{pj} = -\frac{\lambda_p}{2}\left[\frac{-\gamma_j^2\alpha^4}{\beta_p} - \nu\alpha^2\beta_p + \frac{\Omega^2}{\beta_p\gamma_j^2}\right],$$

$$r_{pj} = \nu\alpha^2\lambda_p\mu_j\beta_p\gamma_j + \Omega^2\frac{\lambda_p\mu_j}{\beta_p\gamma_j}, \qquad k_j = -\frac{\Omega^2\mu_j}{12\gamma_j},$$

$$n_{iq} = -\frac{\mu_q}{2}\left[\frac{-\beta_i^2}{\gamma_q} - \nu\alpha^2\gamma_q + \frac{\Omega^2}{\beta_i^2\gamma_q}\right], \qquad \delta_i = \frac{\beta_i^2}{24} + \alpha^2(1-\nu) - \frac{\Omega^2}{24\beta_i^2}.$$

3.2. MODE SHAPE: SYMMETRIC IN x AND y

The next minimizing sequence is proposed ($O = 1, 3, 5, \ldots$):

$$w_{2_{MN}}(x, y) = \sum_{i=O}\sum_{j=O}\frac{A_{ij}s_i s_j}{\beta_i\gamma_j} + \sum_{i=O}\frac{a_i s_i}{\beta_i} + \sum_{j=O}\frac{b_j s_j}{\gamma_j} + k. \tag{18}$$

3.3. MODE SHAPE: ANTISYMMETRIC IN x AND SYMMETRIC IN y

The minimizing sequence is ($O = 1, 3, 5, \dots$ and $E = 2, 4, 6, \dots$)

$$w_{3_{MN}}(x, y) = \sum_{i=E}\sum_{j=O} \frac{A_{ij}s_i s_j}{\beta_i \gamma_j} + \sum_{i=E} \frac{a_i s_i}{\beta_i} + \left(x - \frac{1}{2}\right)\left[a_0 + \sum_{j=O} \frac{A_{0j}s_j}{\gamma_j}\right]. \qquad (19)$$

3.4. MODE SHAPE: ANTISYMMETRIC IN y AND SYMMETRIC IN x

The results are obtained by exchanging i for j and *vice versa*, in equation (19).

4. NUMERICAL RESULTS AND COMMENTS

Tables 1, 2 and 3 depict the values of the natural frequency parameters for the free rectangular plate and for aspect ratios $\alpha = a/b = 1$, 0·4 and 2/3 found with WEM using $M = N = 30$. The values found by Leissa [7] applying the Ritz method with beam functions are reported. Also, frequency results obtained by Leissa and Narita [9] are depicted in Table 1. The latter work deals with the vibration of free shallow shells and the numerical results are obtained as a limit case for the plane thin plate.

TABLE 1

Natural frequency parameters of a free square plate ($\alpha = 1$): $\nu = 0{\cdot}3$, $O = 1, 3, 5, \dots, E = 2, 4, 6, \dots$. *$n$ is the mode order. WEM Solution with* $M = N = 30$

	$\alpha = 1$				
n	Mode	WEM $M=N=30$	[7] R-R 6 beam function	[9] Limit case shallow shell	FEM 50×50
1	$i = E, j = E$ $A_{00} = 1$	13·47	13·49	13·468	13·46
2	$i = O, j = O$ $k = 0$ $a_m = -b_m$	19·61	19·79	19·596	19·56
3	$i = O, j = O$ $k = 1$	24·28	24·43	24·271	24·25
4	$i = E, j = O$ $a_0 = 1$	34·82	35·02	34·801	34·77
5	$i = O, j = E$ $a_0 = 1$	34·82	35·02	34·801	34·77
6	$i = E, j = O$ $a_0 = 1$	61·13	61·53	61·111	60·98
7	$i = O, j = E$ $a_0 = 1$	61·13	61·53	61·111	60·98
8	$i = O, j = O$ $k = 1$	63·72	—	—	63·62

TABLE 2

Natural frequency parameters of a free rectangular plate ($\alpha = 0{\cdot}4$): $\nu = 0{\cdot}3$, $O = 1, 3, 5, \ldots, E = 2, 4, 6, \ldots$. *n is the mode order. WEM solution with* $M = N = 30$

	$\alpha = 0{\cdot}4$		
n	Mode	WEM	[7]
1	$i = O, j = O$ $k = 1$	3·435	3·463
2	$i = E, j = E$ $A_{00} = 1$	5·278	5·288
3	$i = E, j = O$ $a_0 = 1$	9·547	9·622
4	$i = E, j = O$ $a_0 = 1$	11·33	11·44
5	$i = O, j = O$ $k = 1$	18·64	18·79
6	$i = E, j = E$ $A_{00} = 1$	18·93	19·10

TABLE 3

Natural frequency parameters of a free rectangular plate ($\alpha = 2/3$): $\nu = 0{\cdot}3$, $O = 1, 3, 5, \ldots, E = 2, 4, 6, \ldots$. *n is the mode order. WEM Solution with* $M = N = 30$

	$\alpha = 2/3$		
n	Mode	WEM	[7]
1	$i = E, j = E$ $A_{00} = 1$	8·932	8·946
2	$i = O, j = O$ $k = 1$	9·524	9·602
3	$i = E, j = O$ $a_0 = 1$	20.61	20.74
4	$i = O, j = O$ $k = 1$	22·19	22·35
5	$i = E, j = O$ $a_0 = 0$	25·67	25·87
6	$i = E, j = O$ $a_0 = 1$	29·81	29·97

In particular, in the case of the square plate, the general-purpose FEM code by ALGOR [10] was used to find other comparison values. The first eight mode shapes are shown in Figures 1–12 along with a plane cut at $z = 0$ showing the nodal lines. It should be noted that mode shapes experimentally found by Waller are

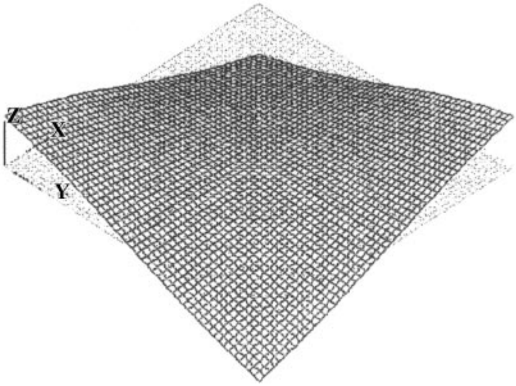

Figure 1. Free square plate: first vibration mode.

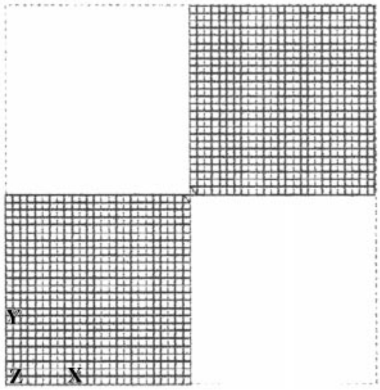

Figure 2. Free square plate: first vibration mode; plane cut $z = 0$.

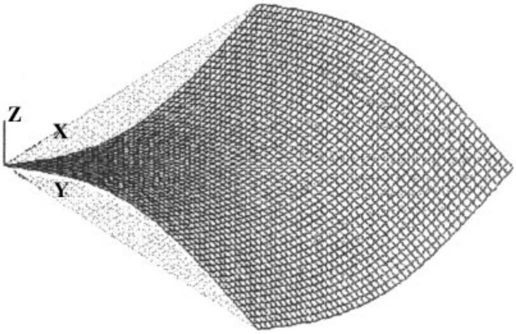

Figure 3. Same as Figure 1: second mode.

reported in reference [11]. Regarding the sixth one (equal to the seventh), the frequency value appears correct and it is coincident with Leissa's results but the mode shape published in reference [11] as the sixth actually corresponds to the eight mode (there are two missing mode shaped in Figure 11–6 of reference [11];

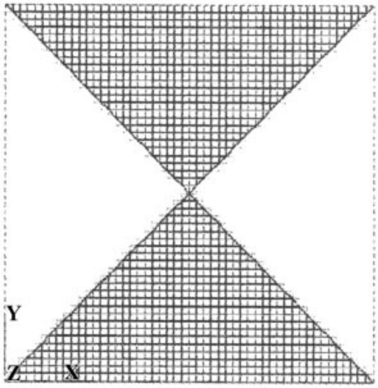

Figure 4. Same as Figure 2: second mode.

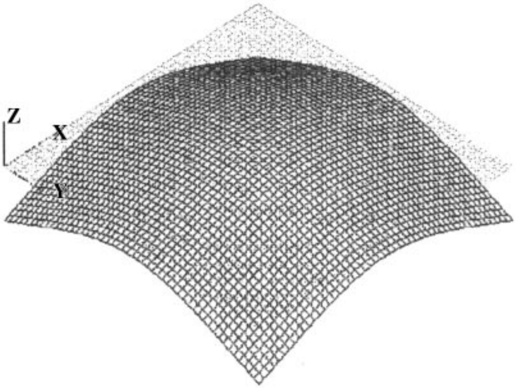

Figure 5. Same as Figure 1: third mode.

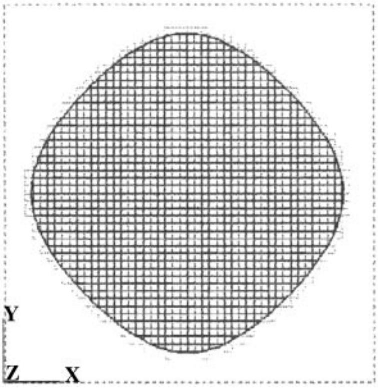

Figure 6. Same as Figure 2: third mode.

the authors could not read the original by Waller and so it could be a reproduction mistake). The sixth and seventh correct modes are shown in Figures 9 and 10. The eight frequency and mode are not reported by Leissa in reference [7].

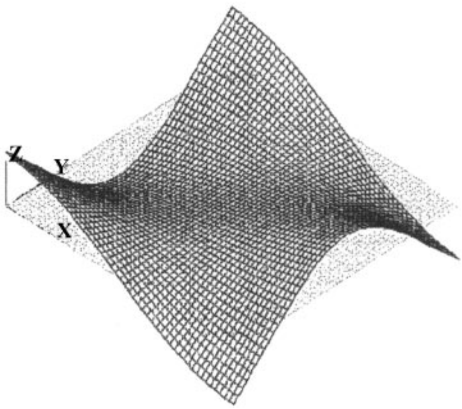

Figure 7. Same as Figure 1: fourth mode (equal to fifth mode shifted 90°).

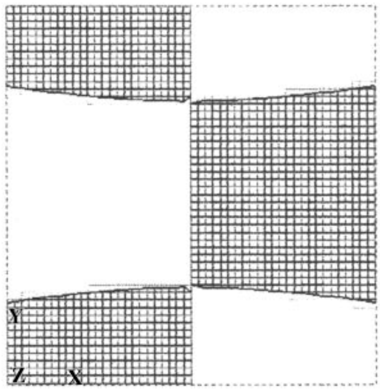

Figure 8. Same as Figure 2: fourth mode (equal to fifth mode shifted 90°).

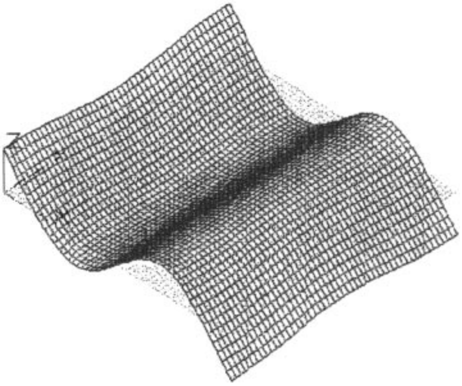

Figure 9. Same as Figure 1: sixth mode (equal to seventh mode shifted 90°).

The values in reference [7] obtained with Ritz method using six beam eigenfunctions are upper bounds. The authors have not found in the literature the demonstration of the completeness of such a set in the domain R^2 without which

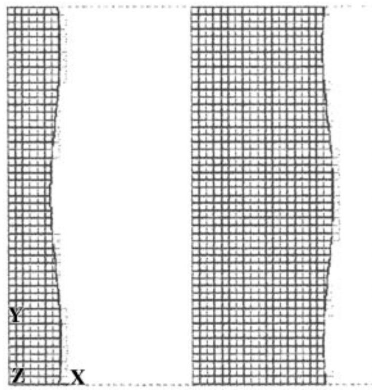

Figure 10. Same as Figure 2: sixth mode (equal to seventh mode shifted 90°).

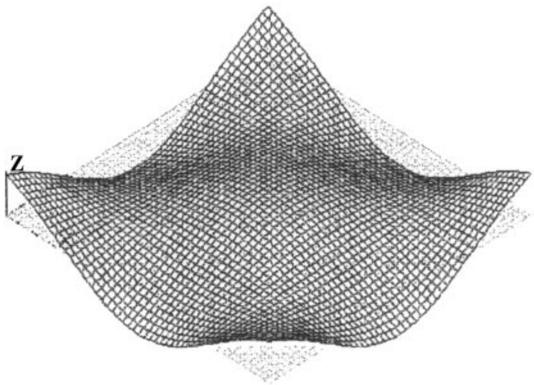

Figure 11. Same as Figure 1: eighth mode.

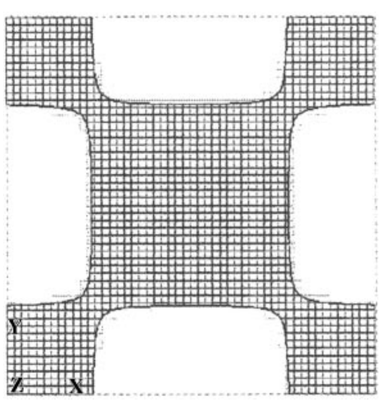

Figure 12. Same as Figure 2: eighth mode.

the frequency values are upper bound and not exact frequencies; furthermore they could converge to non-exact frequencies. On the other hand, as is known, it is possible to "lose" some eigenvalues when using incomplete sets. Leissa and Shihada

[12] analyzed (though only through a numerical study) this important concept, many years after Leissa's relevant paper on plates [7].

Instead these disadvantages are not present in the WEM solution. However, if more exact digits are required in the results, the values of M and N must be increased. It may be observed that the numerical values found by Leissa and Narita [9] are very accurate, even lower than the ones found by WEM. Two comments may be made in this regard: WEM results may be numerically improved by increasing the number of terms (it was shown that they converge to the exact solution) and the results from reference [9] arise from a limit case (when the curvature tends to zero) of a shell theory statement and not from an *ad hoc* theory for plates.

The plate element used in the FEM code ALGOR is that of Fraeijs De Veuvecke (5 degrees of freedom d.o.f.s in each node). The plane motions in xy were restricted (u, v and rotation around z) yielding three remaining d.o.f. in each node. These are the only restrictions to be considered since the plate borders are free. The reported FEM results were found using a 50×50 mesh, i.e., dividing the complete plate into 2500 elements. Due to the fact that ALGOR algorithm makes use of a lumped mass approach the resulting eigenvalues for this particular boundary conditions (free borders) are lower bounds.

Let us make some comments regarding the mode shapes. Both the experimental results reported in reference [11] and the ALGOR FEM code modes yield some modal shapes which show nodal lines other than the ones herein observed, for instance the mode shapes numbered 4 and 5 here. In effect, both experimental and ALGOR mode shapes show a diagonal nodal line (null displacement); meanwhile the WEM solution conduces to a line located at $x = 1/2$ (or $y = 1/2$) as may be observed in equation (19). WEM renders modes with symmetry and antisymmetry combinations as detailed in Section 2.2. The difference arises from adding or substructing rigid-body displacements and rotations. Leissa reported these and other mode combinations in his basic work [13].

Additionally, and using the free plate equations, other support conditions may be handled as particular cases. For instance, the SSFF plate (simply supported in two consecutive borders and the other two free) yields from the mode shapes of a quarter of a plate (antisymmetric in x and y). In this way, the first two frequencies of the square SSFF plate with WEM are found to be 3·3675 and 17·3407; reference [7] reports 3·369 and 17·41 respectively. Analogously, the square plate simply supported at its corners yielded with WEM 7·118 (using $M = N = 30$), whereas Leissa [13] reported 7·12 previously found by Reed. In particular, an arbitrary precision WEM result of 7·11091 was obtained by increasing the number of terms until desired accuracy was attained. This particular boundary condition was analyzed by taking $k = 0$ in equation (18) with which the plate is no more free. In effect, when one is dealing with free borders the inertial resultants ($\int\int w\,\mathrm{d}x\,\mathrm{d}y$, $\int\int wx\,\mathrm{d}x\,\mathrm{d}y$ and $\int\int wy\,\mathrm{d}x\,\mathrm{d}y$ in our case) should vanish. The second mode of the free square plate (see Table 1) required also $k = 0$, but in order for the inertial resultant to be null an additional condition was needed; i.e., $k = 0$ in equation (18) implies null displacements at the four corners. However to attain a free plate behavior (null inertial force) the condition $a_m = -b_m$ with m odd must be imposed. In the other case (plate supported at its four corners) $a_m = b_m$. A similar behavior is found when an experimental test is carried out in a free plate since a certain type of restriction is unavoidable. It is then possible to find modal shapes other than the one corresponding to the free plate.

5. CONCLUSIONS

The natural vibration problem of a free plate is presented here. A variational method, so-called WEM, previously developed by the authors, is applied to a bidimensional problem. The proposed solution is a linear combination of a complete set of functions in L_2. The theory assures the exactness of the obtained natural frequencies. Also, the uniform convergence of the essential eigenfunctions is guaranteed. The complete set also prevents "lost" frequencies. The application of the method is systematic: the sequences are always the same regardless of the problem, except for the domain. An example of a three-dimensional solution may be found in reference [5]. The way of generation assures the validity of the above-stated theorems and results. The authors have also shown that the method is equivalent to the statement of a *pseudo*-virtual work in the extremizing sequences [6]. As mentioned in the introduction, in this very particular case WEM would be like a Ritz method in which the trial function linearly combines elements of a complete set. The first eight values of frequencies and mode shapes of the square plate are reported. The number of exact digits may be increased using larger M and N (number of terms in the series). Obviously, the computational effort is consequently augmented. The first six values of frequencies of rectangular plates with aspect ratios of $\alpha = a/b = 0{\cdot}4$, 2/3 are also reported. Comparison is made with values found by Leissa [7, 13] using the Ritz method with beam functions. Additionally, numerical results obtained by Leissa and Narita [9] as a limit case of a shallow shell theory are depicted. A FEM algorithm using Veuvecke plate elements yields—for this particular boundary condition—lower bounds.

Having accurate values of frequencies allows one to verify approximate methods. On the other hand, the advantage of the usage of WEM was shown: a systematic statement of the solution not only makes this method a convenient tool but what is more relevant, it assures the completeness of the set and consequently that the limit is within it. Furthermore, no special consideration has to be made regarding eventual non-desired restrictions: WEM—as a direct method—yields automatically null global inertial reactions and moments when dealing with a free plate.

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REFERENCES

1. C. P. Filipich and M. B. Rosales 1994 *Journal of Sound and Vibration* **170**, 263–269. Beams and arcs: exact frequencies via a generalized solution.
2. C. P. Filipich, M. B. Rosales and V. H. Cortinez 1996 *Latin American Applied Research* **26,** 71–77. A generalized solution for exact frequencies of a Timoshenko beam.
3. C. P. Filipich and M. B. Rosales 1997 *Applied Mechanics Review* **50**, S50–S55. A variational solution for an initial conditions problem.
4. C. P. Filipich and M. B. Rosales 1999 *Proceedings of the 4th World Congress on Computational Mechanics (WCCM), Buenos Aires, Argentina*. CD-ROM 16 pp. Dynamic behavior of a uniform linear beam supported with non-linear rotational springs.
5. C. P. Filipich and M. B. Rosales and P. M. Bellés 1998 *Journal of Sound and Vibration* **212**, 599–610. Natural vibration of rectangular plates considered as tridimensional solids.
6. M. B. Rosales 1997 *Ph.D. Thesis, Universidad Nacional de Sur, Argentina*. A non-classical variational method and its application to statics and dynamics of structural elements (in Spanish).
7. A. W. Leissa 1973 *Journal of Sound and Vibration* **31**, 257–293. The free vibration of rectangular plates.
8. D. J. Gorman 1978 *Journal of Sound and Vibration* **57**, 437–447. Free vibration analysis of the completely free rectangular plate by the method of superposition.
9. A. W. Leissa and Narita 1984 *Journal of Sound and Vibration* **96**, 207–218. Vibrations of completely free shallow shells of rectangular planform.
10. *ALGOR Software*. Version 4 1997 Pittsburgh, USA: Algor Inc.
11. R. D. Blevins 1979 *Formulas for Natural Frequency and Mode Shape*. New York: Van Nostrand Reinhold Company Inc.
12. A. W. Leissa and S. M. Shihada 1995 *Proceedings Applied Mechanics in the Americas, PACAM IV, Buenos Aires, Argentina*, II, 99–103. Convergence consideration for the Ritz method.
13. A. W. Leissa 1969 *NASA SP-160*. Vibration of plates.

APPENDIX A: THEOREMS AND DEMONSTRATIONS

In this appendix the necessary and sufficient conditions are stated and demonstrated in the domain R^2.

Condition 1 (Necessary Condition: N). *If w is the classical solution of the governing equation, the functional $\mathscr{F}[w]$ assumes an extreme value among the $\mathscr{F}[w_{MN}]$ as $M, N \to \infty$ where w_{MN} is an extremizing sequence, i.e.,*

$$\mathscr{F}[w_{MN}] - \mathscr{F}[w] = 0, \quad M, N \to \infty. \tag{A.1}$$

Condition 2 (Sufficient Condition: S). *If the functional $\mathscr{F}[w]$ attains an extreme value among the functionals $\mathscr{F}[w_{MN}]$ as $M, N \to \infty$ where w_{MN} is the WEM solution, i.e.,*

$$\mathscr{F}[w_{MN}] - \mathscr{F}[w] = 0, \quad M, N \to \infty, \tag{A.2}$$

then w should be the classical solution that satisfies the governing differential equation and its boundary conditions.

Demonstrations. In what follows, functional analysis notation will be introduced: the inner product $(f, g) = \int_D fg\,\mathrm{d}x\,\mathrm{d}y$; the norm $\| f \|^2 = \int_D f^2\,\mathrm{d}x\,\mathrm{d}y$.

Necessary condition: Let us write the functional

$$\mathscr{F}[w] = \Omega^2[w] \equiv \Omega^2 = \frac{\{ \| w'' \|^2 + \alpha^4 \| \bar{\bar{w}} \|^2 + 2\alpha^2 \nu (w'', \bar{\bar{w}}) + 2(1-\nu)\alpha^2 \| \bar{w}' \|^2 \}}{\| w \|^2} \tag{A.3}$$

which is totally equivalent to expression (3). Alternatively,

$$\begin{aligned}\mathscr{F}[w_{MN}] \| w_{MN} \|^2 = \Omega^2_{MN} \| w_{MN} \|^2 = \| \Delta''_{MN} \|^2 + [w'', (2w''_{MN} - w'')] \\ + \alpha^4 \{ \| \bar{\bar{\Delta}}_{MN} \|^2 + [\bar{\bar{w}}, (2\bar{\bar{w}}_{MN} - \bar{\bar{w}})] \} \\ + 2\nu\alpha^2 \left\{ (\Delta''_{MN}, \bar{\bar{\Delta}}_{MN}) + \left[\frac{\bar{\bar{w}}}{2}, (2w''_{MN} - w'') \right] \right. \\ \left. + \left[\frac{w''}{2}, (2\bar{\bar{w}}_{MN} - \bar{\bar{w}}) \right] \right\} + 2(1-\nu)\alpha^2 \{ \| \bar{\Delta}'_{MN} \|^2 \\ + [\bar{w}', (2\bar{w}'_{MN} - \bar{w}')] \}.\end{aligned} \tag{A.4}$$

The following notation is introduced: $\Delta_{MN} \equiv w_{MN} - w$, $\Delta'_{MN} \equiv w'_{MN} - w'$, $\bar{\Delta}_{MN} \equiv \bar{w}_{MN} - \bar{w}$, etc.

Now, if the Green–Ampere (plane divergence) theorem is applied, after denoting $\Omega^2_{MN} = \text{Num}/\text{Den}$ and Ω as the exact frequencies, one finds

$$\begin{aligned}\text{Num} = \| \Delta''_{MN} \|^2 + \alpha^4 \| \bar{\bar{\Delta}}_{MN} \|^2 + 2\nu\alpha^2 \ (\Delta''_{MN}, \bar{\bar{\Delta}}_{MN}) \\ + 2(1-\nu)\alpha^2 \| \bar{\Delta}'_{MN} \|^2 + \mathscr{C} + \Omega^2 [w, (2w_{MN} - w)],\end{aligned} \tag{A.5}$$

$$\text{Den} = \| \Delta_{MN} \|^2 + [w, (2w_{MN} - w)], \tag{A.6}$$

where $\mathscr{C}$ indicates the boundary conditions,

$$\begin{aligned}\mathscr{C} = \frac{1}{D} \left\{ - \oint M_x (2w'_{MN} - w')\,dy + \oint M_y (2\bar{w}_{MN} - \bar{w})\,\mathrm{d}x \right. \\ - \oint M_{xy} (2\bar{w}_{MN} - \bar{w})\,\mathrm{d}y + \oint M_{xy} (2w'_{MN} - w')\,\mathrm{d}x \\ \left. + \oint Q_x (2w_{MN} - w)\,\mathrm{d}y - \oint Q_y (2w_{MN} - w)\,\mathrm{d}x \right\},\end{aligned} \tag{A.7}$$

with the internal forces

$$M_x = -D(w'' + \alpha^2 v\bar{\bar{w}}), \quad M_y = -D(\alpha^2\bar{\bar{w}} + vw''),$$

$$M_{xy} = M_{yx} = -(1-v)\, D\alpha\bar{w}',$$

$$Q_x = -D(w''' + \alpha^2\bar{\bar{w}}'), \quad Q_y = -D(\alpha^3\bar{\bar{\bar{w}}} + \alpha\bar{w}'').$$

Triangular inequality applied to expression (5) renders

$$|Num| \leqslant \|\Delta''_{MN}\|^2 + \alpha^4\|\bar{\bar{\Delta}}_{MN}\|^2 + 2v\alpha^2|(\Delta''_{MN}, \bar{\bar{\Delta}}_{MN})|$$
$$+ 2(1-v)\alpha^2\|\bar{\Delta}'_{MN}\|^2 + |\mathscr{C}| + \Omega^2|[w, (2w_{MN} - w)]|. \qquad \text{(A.8)}$$

Making use of the Cauchy–Schwarz inequality the following relationship stands:

$$|(\Delta''_{MN}, \bar{\bar{\Delta}}_{MN})| \leqslant \|\Delta''_{MN}\|\,\|\bar{\bar{\Delta}}_{MN}\|, \qquad \text{(A.9)}$$

and then a bound for the frequency parameter may be obtained,

$$\Omega^2_{MN} = \frac{\text{Num}}{\text{Den}} \leqslant \frac{\mathscr{N}}{\text{Den}}, \qquad \text{(A.10)}$$

where

$$\mathscr{N} \equiv \|\Delta''_{MN}\|^2 + \alpha^4\|\bar{\bar{\Delta}}_{MN}\|^2 + 2v\alpha^2\|\Delta''_{MN}\|\,\|\bar{\bar{\Delta}}_{MN}\|$$
$$+ 2(1-v)\alpha^2\|\bar{\Delta}'_{MN}\|^2 + |\mathscr{C}| + \Omega^2|[w, (2w_{MN} - w)]|. \qquad \text{(A.11)}$$

Since the WEM solution satisfies (at least) convergence in the mean as follows,

$$\|\Delta''_{MN}\| \to 0, \ \|\bar{\bar{\Delta}}_{MN}\| \to 0, \ \|\bar{\Delta}'_{MN}\| \to 0,$$

$$|\Delta'_{MN}| \to 0 \ \Rightarrow \ \|\Delta'_{MN}\| \to 0,$$

$$|\bar{\Delta}_{MN}| \to 0 \ \Rightarrow \ \|\bar{\Delta}_{MN}\| \to 0,$$

$$|\Delta_{MN}| \to 0 \ \Rightarrow \ \|\Delta_{MN}\| \to 0,$$

and $\mathscr{C}$ is verified by w and its derivatives (that satisfy all the boundary conditions) and w_{MN} must only verify the essential boundary conditions, it is shown that

$$\Omega^2_{MN} \leqslant \frac{\Omega^2\,[w,\ (2w_{MN} - w)]}{[w,\ (2w_{MN} - w)]} = \Omega^2. \qquad \text{(A.12)}$$

Since the general (Rayleigh's quotient)

$$\Omega^2_{MN} \geqslant \Omega^2, \tag{A.13}$$

one concludes that

$$\Omega^2_{MN} = \Omega^2, \tag{A.14}$$

always, as M, $N \to \infty$.

Sufficient condition: The sufficient conditions is expressed by (γ is a scalar constant)

$$\bar{\delta}\Omega^2 \equiv \left.\frac{\partial \Omega^2 [w + \gamma v]}{\partial \gamma}\right|_{\gamma=0} = 0, \tag{A.15}$$

with Ω^2 from (A.3). Once the extreme is stated, and after the application of the Green–Ampere theorem, the next expression is found:

$$\left.\frac{\partial \Omega^2}{\partial \gamma}\right|_{\gamma=0} = 0 \;\Rightarrow\; \{[w'''' + 2\alpha^2 \bar{\bar{w}}'' + \alpha^4 \bar{\bar{\bar{\bar{w}}}} - \Omega^2 w], v\} + \mathscr{C} = 0, \tag{A.16}$$

in which

$$\Omega^2 |_{\gamma=0} = \Omega^2. \tag{A.17}$$

Then the functional Ω^2 is an extreme if in equation (A.16) it is taken into account that w is the classical solution of the problem (i.e. satisfies the differential equation and all the boundary conditions) and, $v \equiv w_{MN}$ is the WEM solution (verifies the essential boundary conditions).

Finally the following theorems will be stated without demonstration.

Corollary 1. *If the functional* $\mathscr{F}[w]$ *assumes an extreme value among the* $\mathscr{F}[w_{MN}]$ *as* M, $N \to \infty$, $\mathscr{F}[w_{MN}]$ *will also be an extreme with the adequate selection of the constants.*

This is a direct consequence of Condition 1 and conduces to the methodology employed in the practical usage of the WEM:

$$\frac{\partial \mathscr{F}[w_{MN}]}{\partial \sigma} = 0, \tag{A.18}$$

where σ stands for any of the constants $\{A_{ij}\}$, $\{a_i\}$, $\{b_j\}$ and k with $i, j = 0, 1, 2, \ldots$

Theorem 1 (TN). *If* w *is the classical solution of the differential problem the series* w_{MN} *found by* (A.18) *is an extremizing sequence.*

Theorem 2 (TS). *If w_{MN} is an extremizing sequence for w, in order for* (A.18) *to be satisfied w must be the classical solution of the differential problem.*

The last two theorems assure the uniform convergence of the essential functions. The demonstrations are not included since they are rather cumbersome (though simple) and beyond the scope of the present work.